Integer ratios of factorials as Hausdorff moments

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Abstract

Consider positive integers a, b with $\gcd(a, b) = 1$. The following three ratios of factorials for $n = 0, 1, \ldots$ turn out to be integers [1] :

$$u_1(a,b,n) = \frac{[(a+b)n]!}{(an)!(bn)!}\,,\ u_2(a,b,n) = \frac{(2an)!(bn)!}{(an)!(2bn)![(a-b)n]!} \quad (\text{for } a > b)\,,$$
$$\text{and} \quad u_3(a,b,n) = \frac{(2an)!(2bn)!}{(an)!(bn)![(a+b)n]!}\,.$$

The same applies to the fourth ratio in the form [2]

$$u_4(a,b,n) = \frac{[(2a+1)n]![(b+\frac{1}{2})n]!}{[(2b+1)n]![(a+\frac{1}{2})n]![(a-b)n]!} \quad (\text{for } a > b)\,.$$

We solve exactly the Hausdorff moment problem with moments given by $u_i(a, b, n)$ for $i = 1, \ldots, 4$. We use the technique of the inverse Mellin transform and Meijer G functions to obtain the positive smooth measures $w_i(x)$ as well as their supports $(0, R_i(a, b))$. All these measures are U-shaped, are singular at the support bounds, and their singularities at $x = 0$ are of power-law type. The radii of convergence $r_i(a, b)$ of the ordinary generating functions (OGFs) of $u_i(a, b, n)$ satisfy $r_i(a, b) = [R_i(a, b)]^{-1}$ for $i = 1, \ldots, 4$. All these OGFs are algebraic [1, 2, 3]. An attempt is made to understand to what extent the proven algebraicity of the OGFs is synchronized with the possible algebraicity of the $w_i(x)$.

(Joint work with G. H. E. Duchamp and G. Koshevoy, Univ. Paris XIII)

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[2] P. Bala, *"Some integer ratios of factorials"*, in the On-Line Encyclopedia of Integer Sequences (https ://oeis.org/), sequence A276098.

[3] Fernando Rodriguez -Villegas, *Integral ratios of factorials and algebraic hypergeometric functions*, arXiv preprint math/0701362 (2007).