Thermodynamic formalism for transcendental maps

Krzysztof Barański

University of Warsaw

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Setup

$f : \mathbb{C} \to \overline{\mathbb{C}}$ **transcendental entire** or **meromorphic** map

Julia set

$J(f) = \mathbb{C} \setminus \{z : \{f^n\}_{n>0}$ is defined and normal in a nbhd of $z\}$

Singular set $\text{Sing}(f) = \{z : f^{-1}$ has a singularity at $z\}$

$= \{$critical and asymptotic values of $f\}$

Post-singular set $\mathcal{P}(f) = \bigcup_{n=0}^{\infty} f^n(\text{Sing}(f))$

Hyperbolicity and conformal repellers

Definition

f is **hyperbolic**, if $\overline{\mathcal{P}(f)}$ is bounded and disjoint from $J(f)$.

Remark

Hyperbolic maps are in the Eremenko–Lyubich class
$\mathcal{B} = \{f : \mathrm{Sing}(f) \text{ is bounded}\}$

Definition

A set $X \subset J(f)$ is a **conformal expanding repeller**, if it is compact, forward-invariant and $|(f^n)'||_X \geq cQ^n$ for every $n > 0$, where $c > 0$, $Q > 1$.

Remark

If a rational map f is hyperbolic, then $J(f)$ is a topologically transitive conformal expanding repeller. In the transcendental case, $J(f)$ is not compact in $\mathbb{C}$ and the hyperbolicity of f does not always imply that f is expanding on $J(f)$.

Topological pressure

Definition

Let $X \subset J(f)$ be a transitive conformal expanding repeller. Then the function

$$P(f|_X, t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{w \in f^{-n}(z)\cap X} |(f^n)'(w)|^{-t}$$

for $t \in \mathbb{R}$, $z \in X$ is called **topological pressure function** (for **geometric potential** $-t \log |f'|$).

Conformal measure

Definition

A Borel probability measure ν on an invariant set $X \subset J(f)$ is ***t*-conformal** for $t > 0$, if

$$\nu(f(A)) = \int_A |f'(z)|^t d\nu(z)$$

for every Borel set $A \subset X$ on which f is injective.

Proposition

If ν is a t-conformal measure on $X = J(f)$, then ν is either positive on non-empty open sets in $J(f)$ or it is supported on the set of (at most two) exceptional values of f.

Example

For $f(z) = ze^z$, the value 0 is the unique finite exceptional value of f, with $f^{-1}(0) = \{0\}$, $f(0) = 0$ and $f'(0) = 1$. Consequently, $0 \in J(f)$ and the Dirac measure at 0 is t-conformal for every $t > 0$.

Classical thermodynamic formalism

Theorem (Bowen, Ruelle, Walters...)

Let $X \subset J(f)$ be a transitive conformal expanding repeller. Then the topological pressure function is well-defined and does not depend on the initial point $z \in X$.

We have $P(f|_X, t) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_t^n(\mathbb{1})$*, where* $\mathbb{1} \equiv 1$ *and*

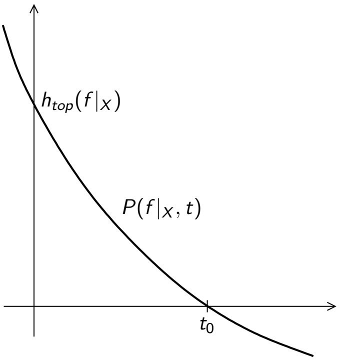
$$\mathcal{L}_t : C(X) \to C(X), \qquad \mathcal{L}_t(\phi)(z) = \sum_{w\in f^{-1}(z)\cap w\in X} \phi(w)|f'(w)|^{-t}$$

is the **Perron–Frobenius (transfer) operator**. *Moreover, for $t > 0$ there exist a t-conformal measure m_t on X, which is an eigenmeasure of the dual operator $\mathcal{L}_t^*$, and an f-invariant Gibbs measure equivalent to m_t, with good ergodic properties (EDC, CLT, LIL,...).*

Classical Bowen's formula

Theorem (Bowen 1979)

Let $X \subset J(f)$ be a transitive conformal expanding repeller. Then $\mathbf{dim}_H\, \boldsymbol{X} = \boldsymbol{t_0}$, *where t_0 is the unique zero of the pressure function $t \mapsto P(f|_X, t)$ and* $\dim_H$ *denotes the Hausdorff dimension.*



Thermodynamic formalism for transcendental maps

Aim

Establish elements of thermodynamic dynamic formalism on $J(f)$ for transcendental entire or meromorphic maps f.

Difficulties compared with the finite degree case

Due to the lack of compactness, the standard Perron–Frobenius operator and the pressure can be not well-defined.

Tricks

- project the map f to a cylinder or torus (for periodic or doubly periodic maps)
- consider derivative of f in a different (non-Euclidean) metric

Example – the exponential map $E(z) = \lambda e^z$, $\lambda \in \mathbb{C} \setminus \{0\}$

- The standard Perron–Frobenius–Ruelle operator on the constant function $\mathbb{1}$ is infinite for all $t > 0$:

$$\mathcal{L}_t(\mathbb{1})(z) = \sum_{w \in E^{-1}(z)} |E'(w)|^{-t} = \sum_{w \in E^{-1}(z)} \frac{1}{|z|^t} = \infty.$$

- For the quotient map $\tilde{E} : \mathbb{C}/2\pi i\mathbb{Z} \to \mathbb{C}/2\pi i\mathbb{Z}$ the modified operator $\tilde{\mathcal{L}}_t$ on the function $\mathbb{1}$ is finite for $t > 1$:

$$\tilde{\mathcal{L}}_t(\mathbb{1})(z) = \sum_{w \in \tilde{E}^{-1}(z)} |\tilde{E}'(w)|^{-t} = \sum_{k \in \mathbb{Z}} \frac{1}{|z + 2\pi i k|^t} < \infty.$$

- Alternatively, in the new metric $d\sigma = dz/|z|$, the modified operator $\mathcal{L}_{\sigma,t}$ on the function $\mathbb{1}$ is finite for $t > 1$:

$$\mathcal{L}_{\sigma,t}(\mathbb{1})(z) = \sum_{w \in E^{-1}(z)} |E'(w)|_\sigma^{-t} = \sum_{w \in E^{-1}(z)} \frac{1}{|w|^t} = \sum_{k \in \mathbb{Z}} \frac{1}{\left|\log\left|\frac{z}{\lambda}\right| + i \operatorname{Arg}\left(\frac{z}{\lambda}\right) + 2\pi i k\right|^t} < \infty.$$

Classes of transcendental maps admitting thermodynamic formalism

B. 1995

Hyperbolic periodic maps of the form $f(z) = R(e^z)$, where R is a non-polynomial rational map, e.g. $f(z) = \lambda \tan z$

Kotus–Urbański, Mayer–Urbański 2004–2005

Hyperbolic doubly periodic elliptic functions, e.g. $f(z) = \lambda\wp(z)$, where $\wp$ is the Weierstrass function

Urbański–Zdunik 2003–2004

Hyperbolic exponential maps $E(z) = \lambda e^z$

Mayer–Urbański 2005–2008

Hyperbolic maps of finite order with **rapid/balanced derivative growth** $|f'(z)| \asymp |z|^\alpha |f(z)|^\beta$ as $|z| \to \infty$, e.g. previous examples, $f(z) = P(e^{Q(z)})$, where P, Q polynomials, $f(z) = \sin(az + b)$

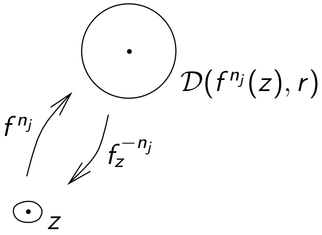
Mayer–Urbański 2017

Maps with Hölder tracts

Radial Julia set

Definition

The **radial Julia set $J_r(f)$** is the set of $z \in J(f)$ for which there exists $r > 0$ and a sequence $n_j \to \infty$, such that a branch of f^{-n_j} sending $f^{n_j}(z)$ to z is well-defined on the disc $\mathcal{D}(f^{n_j}(z), r)$ with respect to the spherical metric on $\overline{\mathbb{C}}$.



The radial Julia set in transcendental dynamics

Theorem (Urbański–Mayer–Zdunik 2003–2008)

For transcendental maps admitting thermodynamic formalism the conformal and invariant measure are supported on the radial Julia set $J_r(f)$*. The Bowen's formula has the form* $\mathbf{dim}_H\, \boldsymbol{J_r(f) = t_0}$*, where* t_0 *is the unique zero of the pressure function.*

Example

For hyperbolic exponential map E_λ $\dim_H(J(E_\lambda)) = 2$ for every $\lambda \in \mathbb{C} \setminus \{0\}$ (McMullen 1987) and $1 < \dim_H J_r(E_\lambda) < 2$ (Urbański–Zdunik 2004).

Remark

$\dim_H J_r(f) = \dim_{hyp} J(f)$
$\stackrel{\text{def}}{=} \sup\{\dim_H X : X \subset J(f) \text{ a conformal expanding repeller}\}$
(Rempe 2009).

Question

Which elements of thermodynamic formalism can be established for general classes of transcendental maps?

Convention

From now on, in the definition of the pressure and conformal measures we consider derivative in the **spherical metric**

$$ds = \frac{2\,dz}{1 + |z|^2}.$$

Pressure for maps in $\mathcal{S}$ and $\mathcal{B}$

Theorem (Karpińska, Zdunik, B. 2010)

Let f be an arbitrary map with a finite number of singularities (class $\mathcal{S}$), or a hyperbolic map. Then for every $t > 0$ the pressure $P(f,t) = P(f,t,z_0)$ exists (possibly equal to $+\infty$) and is independent of $z_0 \in \mathbb{C}$ up to a set of Hausdorff dimension zero. The following version of Bowen's formula holds:

$$\dim_H J_r(f) = \dim_{hyp} J(f) = t_0,$$

where $t_0 = \inf\{t > 0 : P(f,t) \leq 0\}$.

Remark

In fact, the theorem is valid for all maps in $\mathcal{B}$ with "minimal" hyperbolicity (**non-exceptional tame maps**).

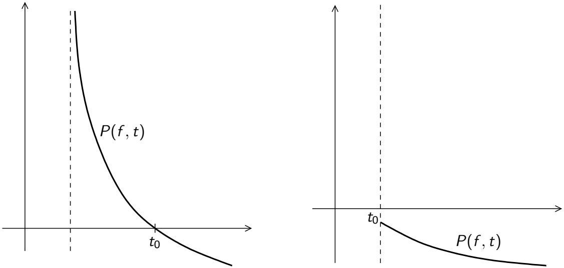
Remark

The same results (and more) were proved for rational maps by Przytycki, Rivera-Letelier and Smirnov in 2004.

Question (Mauldin 2013)

Is the condition $P(f, t) = 0$ equivalent to the existence of a t-conformal measure on $J(f)$?

Possible situations



regular case **non-regular case**

Remark

Non-regular hyperbolic transcendental maps exist (Mayer–Zdunik 2019)

Existence of the zero of the pressure

Definition

The **escaping set $I(f)$** is defined as

$$I(f) = \{z \in \mathbb{C} : f^n(z) \text{ is defined for all } n > 0 \text{ and } \lim_{n\to\infty} f^n(z) = \infty\}.$$

Theorem (Karpińska, Zdunik, B. 2018)

If a hyperbolic map f admits a t-conformal measure m_t for some $t > 0$, then $P(f,t) \leq 0$. Moreover, if $m_t(J(f) \setminus I(f)) > 0$, then $P(f,t) = 0$.

Example

For $f(z) = \lambda \sin z$, $\lambda \in \mathbb{C} \setminus \{0\}$, the set $I(f)$ has positive 2-dimensional Lebesgue measure (McMullen 1987), and the normalized 2-dimensional spherical Lebesgue measure on $I(f)$ is 2-conformal. If, additionally, f is hyperbolic, then $P(f,2) < 0$ (Coiculescu–Skorulski 2007).

Existence of the conformal measure

Theorem (Karpińska, Zdunik, B. 2018)

Let f be an arbitrary map with a finite number of singularities (class $\mathcal{S}$), or a hyperbolic map with a logarithmic tract over ∞. If $P(f,t)=0$ for some $t>0$, then there exists a t-conformal measure m_t on $J(f)$ such that

$$m_t(\mathbb{C}\setminus\mathbb{D}(r)) = o\left(\frac{(\ln r)^{3t}}{r^t}\right) \quad \text{as} \quad r\to\infty,$$

where $\mathbb{D}(r)=\{z\in\mathbb{C} : |z|<r\}$.

Remark

All maps with bounded set of singularities, which are entire or have a finite number of poles admit a logarithmic tract over ∞.

Thank you for attention!

Wszystkiego najlepszego!

Logarithmic tracts

Definition

An unbounded simply connected domain $U \subset \mathbb{C}$ is called a **logarithmic tract** of F over ∞, if the following are satisfied:

- ∂U is a smooth open simple arc in $\mathbb{C}$,
- $F : \overline{U} \to \mathbb{C}$ is continuous, holomorphic on U,
- $F|_U$ is a universal covering of $V = \mathbb{C} \setminus \overline{\mathbb{D}(r)}$ for some $r > 0$,
- $F(\partial U) = \partial \mathbb{D}(r)$.

Spherical Distortion Theorem for logarithmic tracts

Theorem (Karpińska, Zdunik, B. 2010)

Let $F : U \to V = \{z : |z| > R\}$ *be a logarithmic tract for some* $R > 1$, $0 \notin U$ *and let* $z_1, z_2 \in V$ *with* $|z_1| \geq |z_2| \geq LR$ *for some* $L > 1$*. If* g *is a branch of* F^{-1} *near* z_1*, then*

$$c_1 \frac{|z_1|}{|z_2|}\left(\frac{\log|z_1|}{\log|z_2|}\right)^{-3} \leq \frac{|g^*(z_1)|}{|g^*(z_2)|} \leq c_2 \frac{|z_1|}{|z_2|}\frac{\log|z_1|}{\log|z_2|},$$

for some extension of g*, where* c_1, c_2 *depend only on* R, L *(not on* F*).*

Construction of the conformal measure m_t (following Patterson, Denker–Urbański...)

Suppose $P(f, t) = 0$. Define

$$\mu_s = \frac{1}{\Sigma_s} \sum_{n=1}^{\infty} b_n e^{-ns} \sum_{w \in f^{-n}(z_0)} \frac{\delta_w}{|(f^n)^*(w)|^t},$$

where $s > 0$, $z_0 \in J(f)$, $b_n > 0$, δ_w is the Dirac measure at w, and

$$\Sigma_s = \sum_{n=1}^{\infty} b_n e^{-ns} \sum_{w \in f^{-n}(z_0)} |(f^n)^*(w)|^{-t} < \infty.$$

We can choose the sequence b_n so that

$$\lim_{n \to \infty} \frac{b_{n+1}}{b_n} = 1, \qquad \lim_{s \to 0^+} \Sigma_s = +\infty.$$

Lemma

For sufficiently large $r > 0$,

$$\mu_s(J(f) \setminus \mathbb{D}(r)) < c\frac{(\log r)^{3t}}{r^t}$$

for a constant $c > 0$ independent of s.

Corollary

The family $\{\mu_s\}_{s\in(0,1)}$ is tight. Consequently, there exists a weak limit

$$m_t = \lim_{j\to\infty} \mu_{s_j}$$

for some sequence $s_j \to 0^+$, which is a probability measure with support in $J(f)$. The measure m_t is t-conformal with respect to the spherical metric.