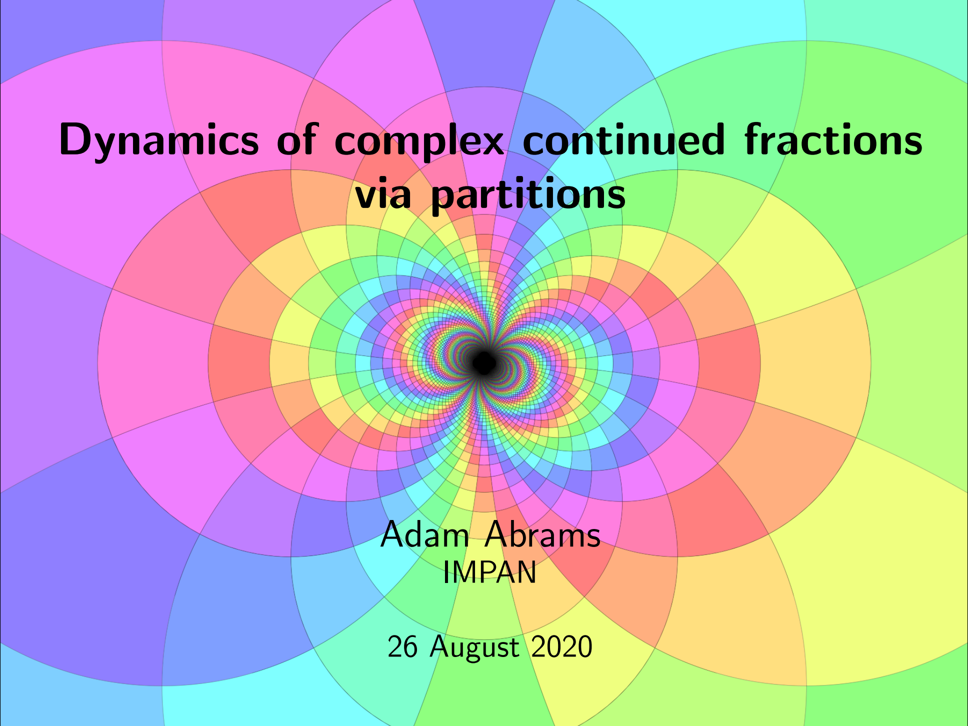

Dynamics of complex continued fractions via partitions

Adam Abrams
IMPAN

26 August 2020

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On geometric complexity of Julia sets II

Continued fractions

- Minus continued fractions are of the form

$$a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots}}}$$

  with
  - $a_i \in \mathbb{Z}$ for Real CF.
  - $a_i \in \mathbb{Z}[\mathbf{i}] = \mathbb{Z} + \mathbb{Z}\mathbf{i}$ for Complex CF.

- There are multiple algorithms to generate a digit sequence $\{a_i\}$ for a given $x \in \mathbb{R}$ or $z \in \mathbb{C}$.

References/timeline

1887 A. Hurwitz. "Über die entwicklung complexer grössen in kettenbruc̈he." *Acta Mathematica*.

1902 J. Hurwitz. "Über die reduction der binären quadratischen formen mit complexen coefficienten und variabeln." *Acta Mathematica*.

1985 Tanaka. "A complex continued fraction transformation and its ergodic properties." *Tokyo Journal of Mathematics*.

2013 Dani, Nogueira. "Continued fractions for complex numbers and values of binary quadratic forms." *Trans. American Math. Society*.

2019 Ei, Ito, Nakada, Natsui. "On the construction of the natural extension of the Hurwitz complex continued fraction map." *Monatshefte für Mathematik*.

2020 Abrams. "Finite partitions for several complex continued fraction algorithms." *Experimental Mathematics*.

Choice functions

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[1] Dani and Nogueira, 2013.

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- *Non*-example: floor $\lfloor z \rfloor = \lfloor \operatorname{Re} z \rfloor + \lfloor \operatorname{Im} z \rfloor \mathbf{i}$ does not work since $\lfloor 0.9 + 0.9\mathbf{i} \rfloor = 0$ is distance 1.273 from $0.9 + 0.9\mathbf{i}$.

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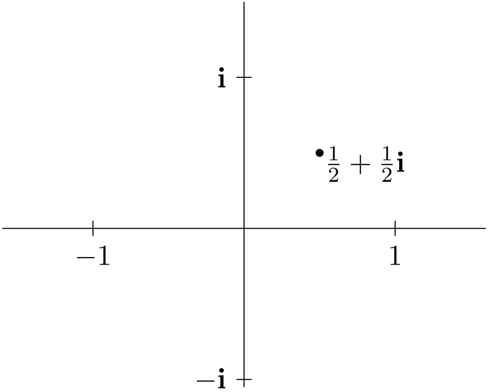

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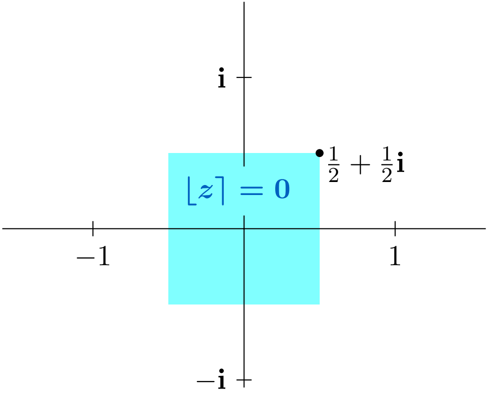

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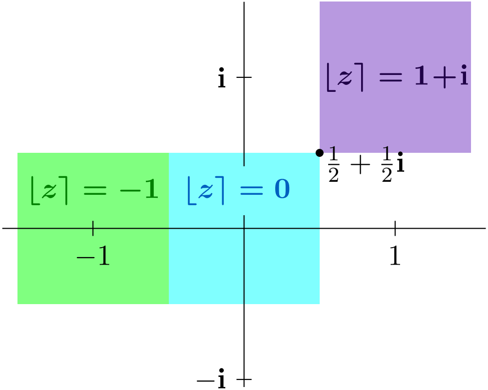
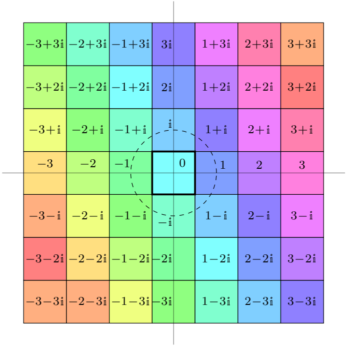


Examples



Nearest integer or Hurwitz algorithm[2]

[1] Dani and Nogueira, 2013.

[2] Adolf Hurwitz, 1887.

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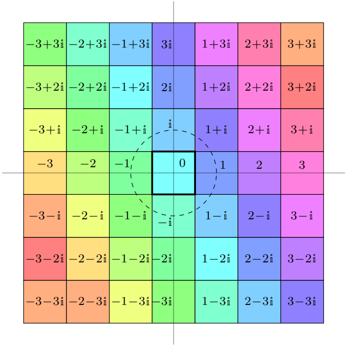

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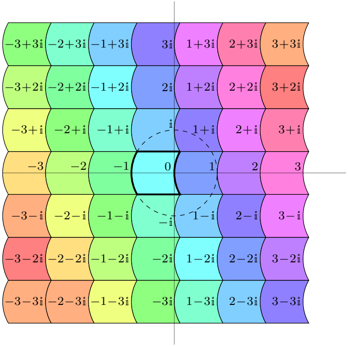
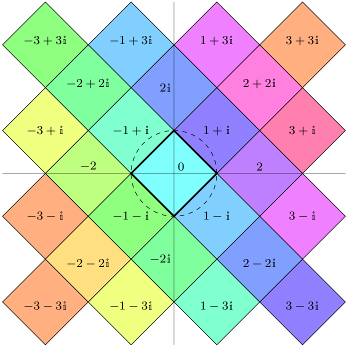


Shifted Hurwitz[1] algorithm

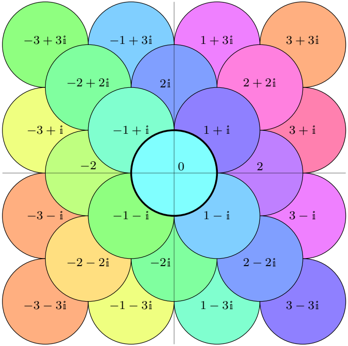
[1] Dani and Nogueira, 2013.

[2] Adolf Hurwitz, 1887.

Examples



Nearest even algorithm[2]



Disk algorithm[3]

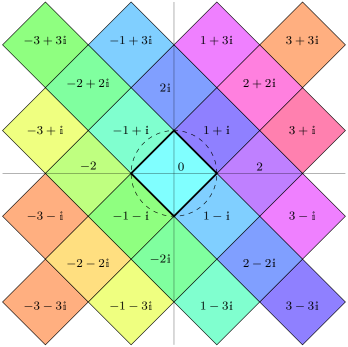
- For both, digits are in $\{\, x + y\mathbf{i} \in \mathbb{Z}[\mathbf{i}] \; : \; x + y \text{ even} \,\}$.

---

[2] Julius Hurwitz, 1902.

[3] Shigeru Tanaka, 1985.

Examples



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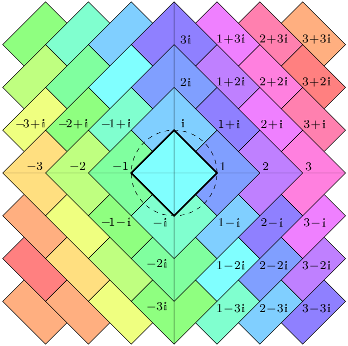


Diamond algorithm

Gauss maps

The fundamental set for an algorithm is

$$K = \overline{\{\, z - \lfloor z \rceil \;:\; z \in \mathbb{C} \,\}}.$$

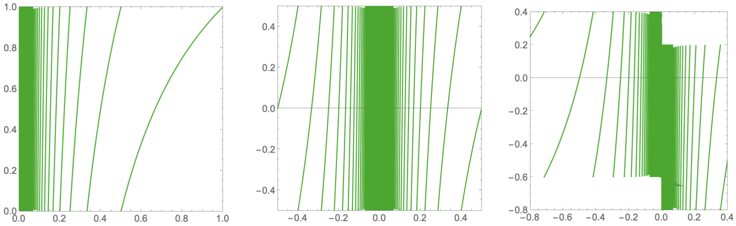
- Example: For the nearest integer algorithm, K is the unit square centered at the origin.
- Note $K \subseteq \overline{B(0,1)}$ by the definition of a choice function.

The Gauss map $g : K \to K$ is given by $g(0) = 0$ and

$$g(z) = \frac{-1}{z} - \left\lfloor \frac{-1}{z} \right\rceil.$$

Gauss maps

Real-valued Gauss maps:



$$\frac{-1}{x} - \left\lfloor \frac{-1}{x} \right\rfloor \qquad \frac{-1}{x} - \left\lfloor \frac{-1}{x} \right\rceil_{\text{Hurwitz}} \qquad \frac{-1}{x} - \left\lfloor \frac{-1}{x} \right\rceil_{\text{Zagier}}$$

$$K = [0,1] \qquad K = [\tfrac{-1}{2}, \tfrac{1}{2}] \qquad K = [\tfrac{-4}{5}, \tfrac{2}{5}]$$

Gauss maps

Theorem (Dani-Nogueira for + c.f.)

Let $\lfloor \cdot \rceil$ be such that $K \subset B(0,1)$. For any $z \in \mathbb{C} \setminus \mathbb{Q}[\mathbf{i}]$, set

$$a_0 = \lfloor z \rceil \quad \text{and} \quad a_n = \left\lfloor \frac{-1}{g^{n-1}(z-a_0)} \right\rceil \ \forall\, n \geq 1.$$

Then the value of

$$a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \frac{1}{a_n}}}}$$

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$$w_0 = z, \qquad a_n = \lfloor w_n \rceil, \qquad w_{n+1} = \frac{-1}{w_n - a_n}.$$

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Gauss maps

The natural extension of the Gauss map

$$g(z) = \frac{-1}{z} - \left\lfloor \frac{-1}{z} \right\rceil$$

is the function

$$F(u,v) = \left(\frac{-1}{u} - a, \frac{-1}{v} - a \right) \quad \text{where } a = \left\lfloor \frac{-1}{v} \right\rceil$$

or, after change of variables $(z,w) = (v, -1/u)$,

$$G(z,w) = \left(\frac{-1}{z} - a, \frac{-1}{w-a} \right) \quad \text{where } a = \left\lfloor \frac{-1}{z} \right\rceil.$$

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Modular and Fuchsian results[4]

Real (a,b)-continued fractions:

- Family of algorithms with parameters $a, b \in \mathbb{R}$.
- Natural extension is $G(x,y) = (\frac{-1}{x} - n, \frac{-1}{y-n})$ where n is the "(a,b)-generalized integer part" of x.

Co-compact Fuchsian setting:

- Instead of $\mathrm{PSL}(2,\mathbb{Z})$ we use $\Gamma = \langle T_1, \dots, T_m \rangle$.
- Natural extension is $F(x,y) = (T_i(x), T_i(y))$ where i depends only on x.

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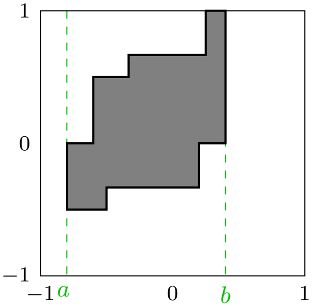
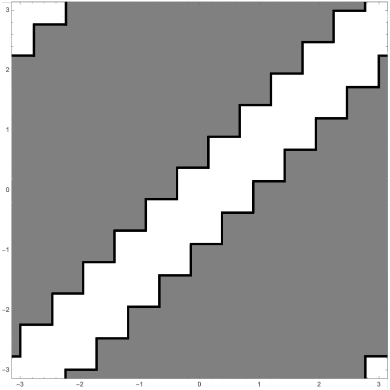


Real C.F.

$\Omega_{a,b} \subset \mathbb{R} \times \mathbb{R}$



Fuchsian

$\Omega_{\bar{A}} \subset S^1 \times S^1$

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Philosophy

For classical ($X = \mathbb{R}$) and Fuchsian (with $X = S^1$), we have

1. One-variable $f : X \to X$ is not injective.

2. Two-variable $F : X \times X \to X \times X$ of the form
$$F(x, y) = \big(\rho_x(x), \rho_x(y)\big)$$
is not injective on $X \times X$, but

3. restricting F to its global attractor $\Omega \subset X \times X$ does give an a. e. bijective function.

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Finite product structure

What is a good analogue in $\bar{\mathbb{C}} \times \bar{\mathbb{C}}$ for the 2-real-dimensional finite rectangular structure?

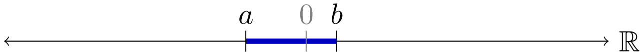
Definition

A set $\Omega \subset \bar{\mathbb{C}} \times \bar{\mathbb{C}}$ has finite product structure if there exist $N \in \mathbb{N}$ and sets $K_1, \ldots, K_N \subset \bar{\mathbb{C}}$ and $L_1, \ldots, L_N \subset \bar{\mathbb{C}}$ each connected on the Reimann sphere such that

$$\Omega = \bigcup_{i=1}^{N} K_i \times L_i.$$

Cycle property

Cycle property: the orbits of $\frac{-1}{a}$ and $a+1$ under $f : \mathbb{R} \to \mathbb{R}$ intersect, and the orbits of $\frac{-1}{b}$ and $b-1$ also intersect.



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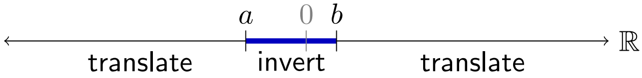

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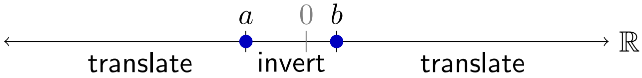

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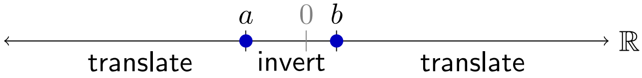

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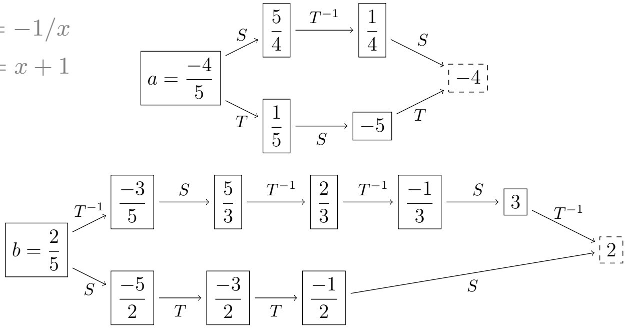

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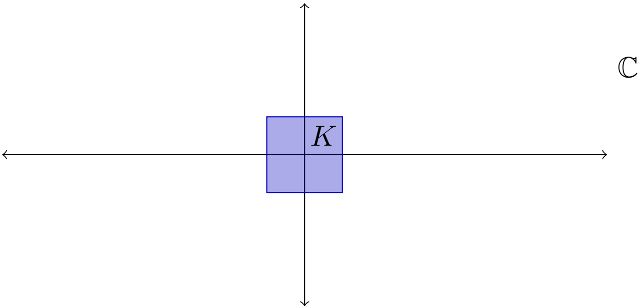

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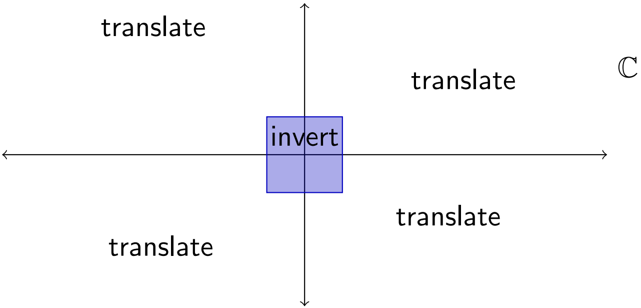

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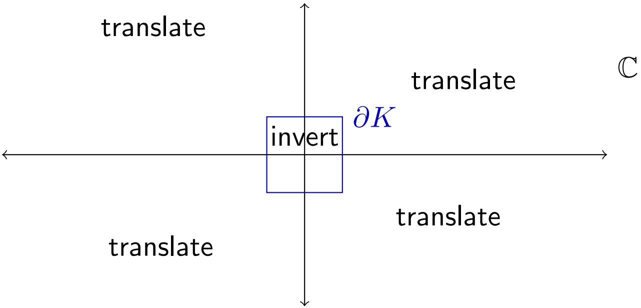

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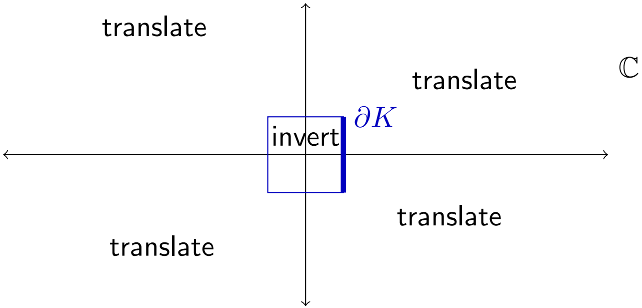

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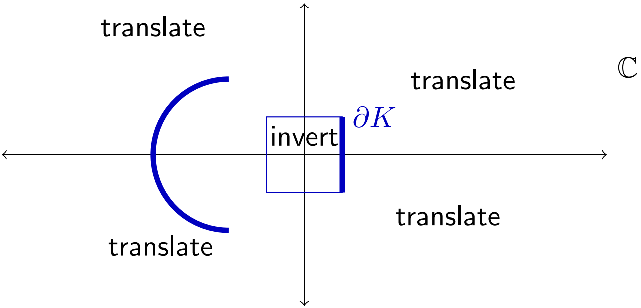

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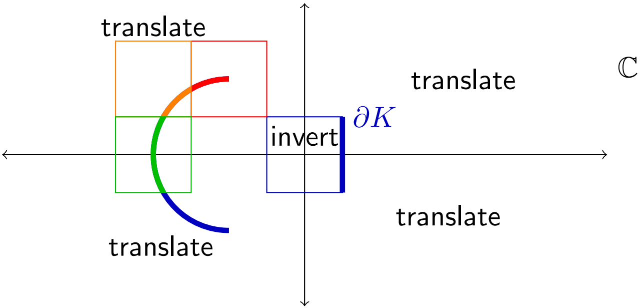

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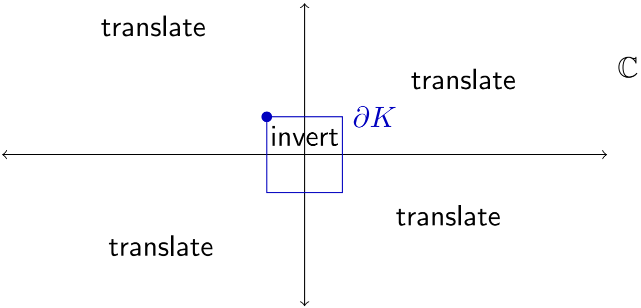

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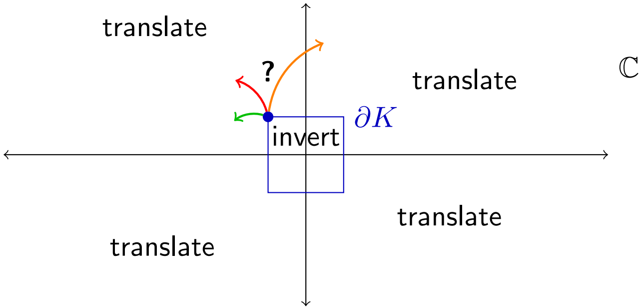

Finite building property

The following replaces the cycle property:

Definitions

- Let $\mathcal{C}$ be a collection of sets. A set is called buildable from $\mathcal{C}$ if it is equal to some union of elements of $\mathcal{C}$.
- A continued fraction algorithm with Gauss map $g : K \to K$ satisfies the finite building property if there exists a finite partition $\mathcal{P} = \{K_1, \ldots, K_N\}$ with $N > 1$ such that each $g(K_i)$ is buildable from $\mathcal{P}$.

"Partition" here means the elements of $\mathcal{P}$ have disjoint interiors.

Finite building property

We want each $g(K_i)$ to be some $\bigcup_j K_j$.

Define the maps

$$S(z) := -1/z$$
$$T^a(z) := z + a \qquad \text{for any } a \in \mathbb{Z}[\mathbf{i}]$$

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$$g(z) = T^{-a}Sz \quad \text{where} \quad a = \lfloor Sz \rceil,$$

but for $X \subset \mathbb{C}$, $g(X)$ might NOT be of the form $T^{-a}SX$ because $\lfloor \cdot \rceil$ will not generally be constant on SX.

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  but for $X \subset \mathbb{C}$, $g(X)$ might NOT be of the form $T^{-a}SX$ because $\lfloor \cdot \rceil$ will not generally be constant on SX.
- In practice, we need to decompose K_i further into sets for which g acts the same.

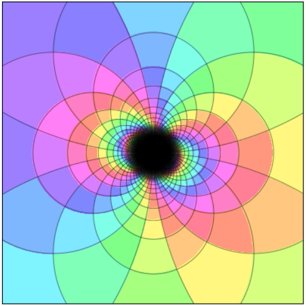
Cells

Some new notation will be useful.

- For $a \in \mathbb{Z}[\mathbf{i}]$, we have the cell

$$\langle a \rangle := \{\, z \in K \;:\; \lfloor -1/z \rceil = a \,\}\,.$$

Note that $g\big|_{\langle a \rangle}$ is exactly $T^{-a}S$.



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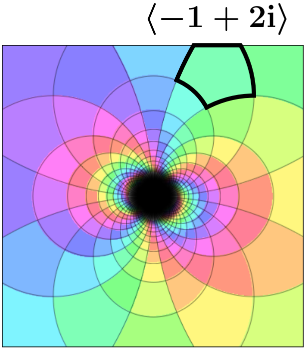

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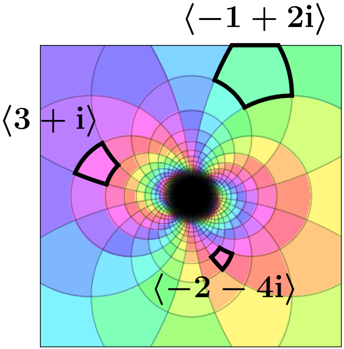

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  Then we have

  $$K_i = \bigcup_{a \in \mathbb{Z}[\mathbf{i}]} K_{i,a}$$

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$$g(\ K_i\ ) = g\left(\bigcup_{a \in \mathbb{Z}[\mathbf{i}]} K_{i,a}\right) = \bigcup_{a \in \mathbb{Z}[\mathbf{i}]} g(K_{i,a})$$

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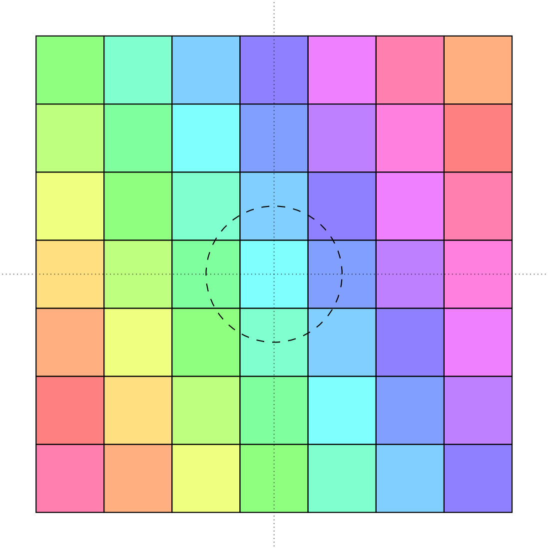
Partitions

- We actually have three partitions of K now.
  1. $\{\langle a\rangle\}$ indexed by $a \in \lfloor\mathbb{C}\rceil \subset \mathbb{Z}[\mathbf{i}]$.
  2. $\{K_i\}$ indexed by $i \in \{1,\ldots,N\}$.
  3. $\{K_{i,a}\}$ indexed by (i,a) for which $K_i \cap \langle a\rangle \neq \emptyset$.
- The partition into $\langle a\rangle$ is based on the algorithm only.
- The partition $\mathcal{P} = \{K_1,\ldots,K_N\}$ is finite.
  How do we construct these sets?

Hurwitz



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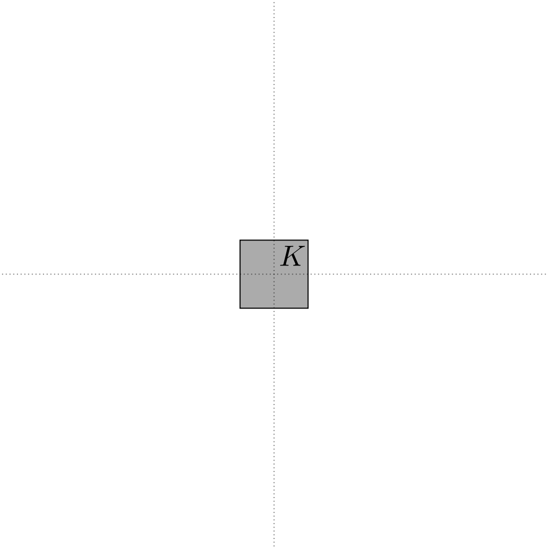

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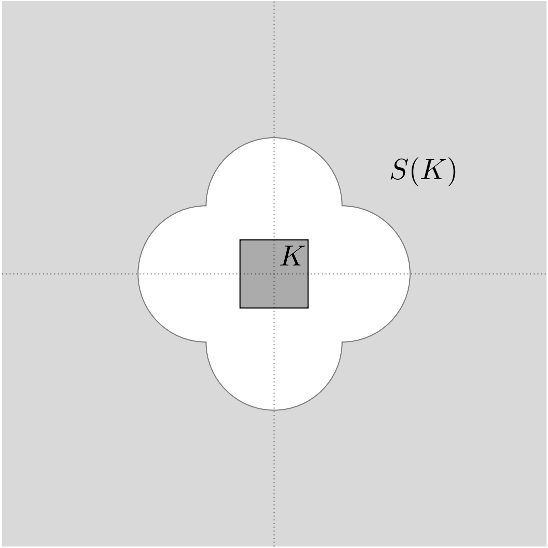

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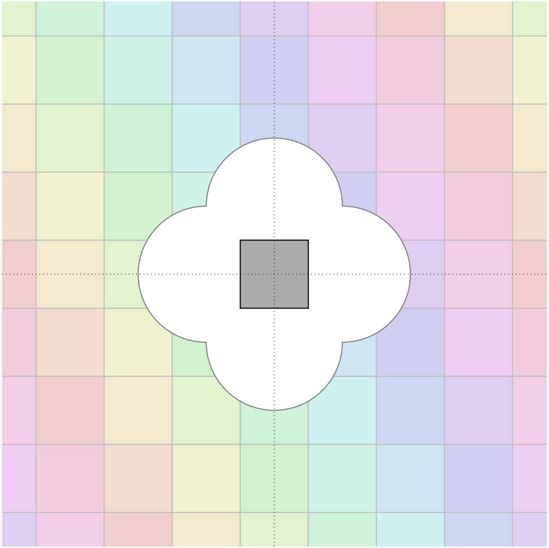

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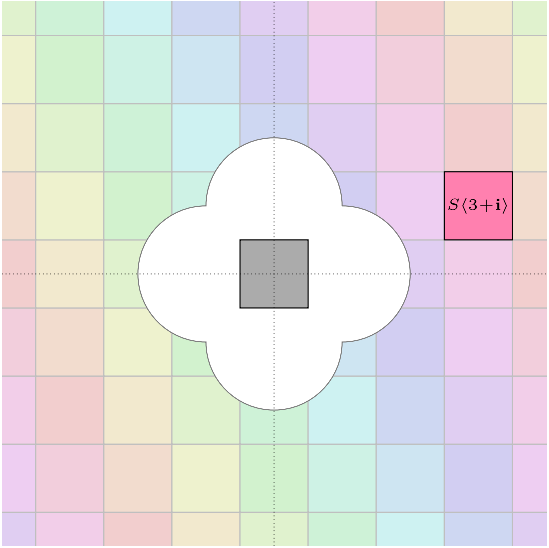

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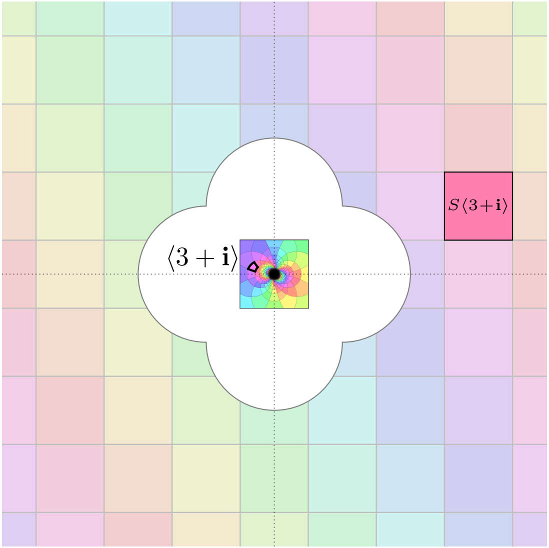

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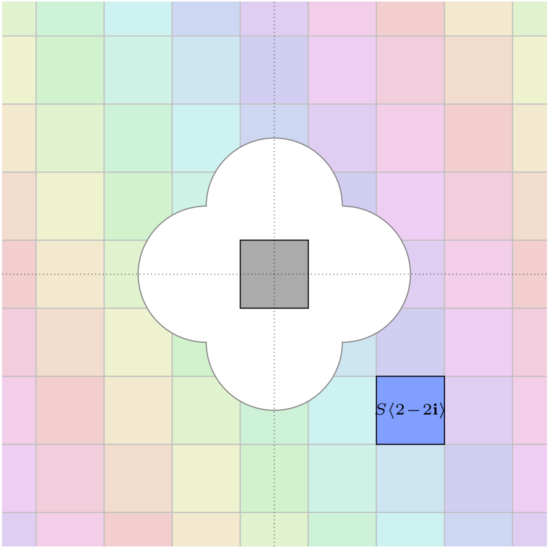

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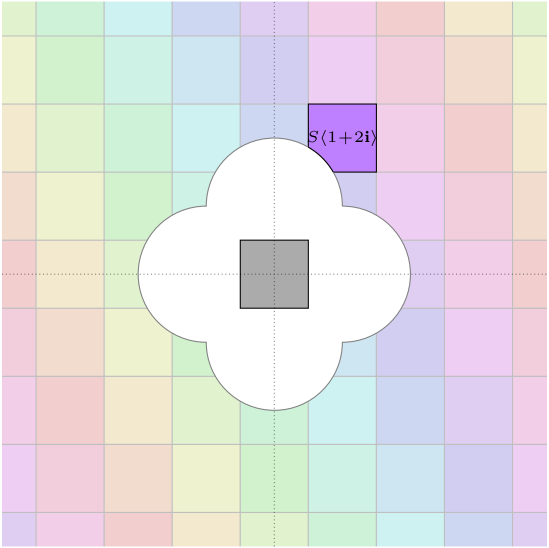

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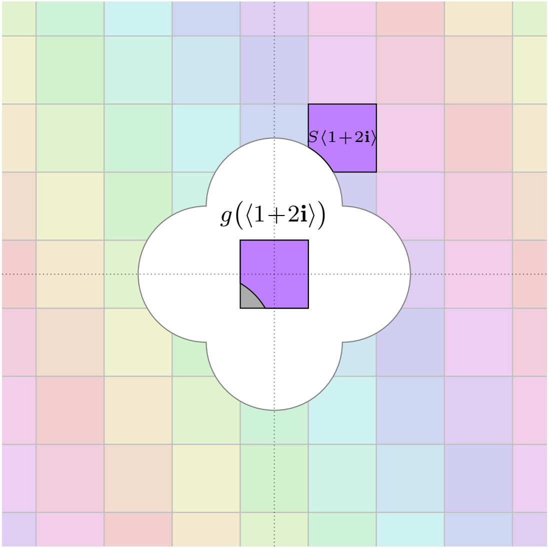
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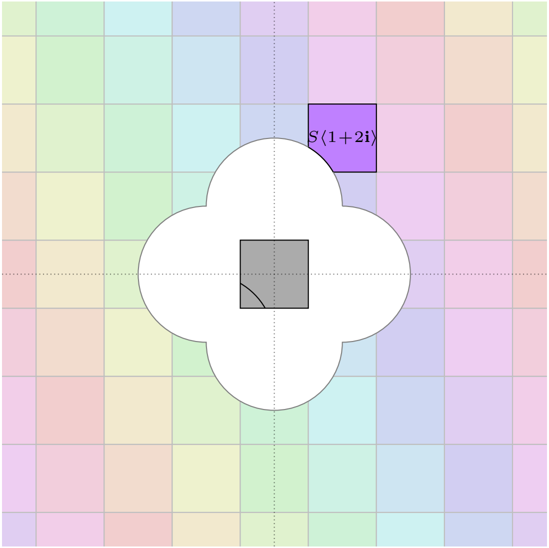
$g\big(\langle 1+2\mathbf{i}\rangle\big)$

Hurwitz



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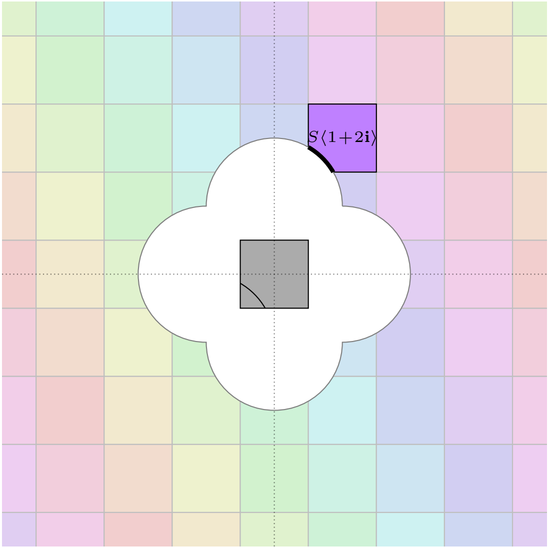

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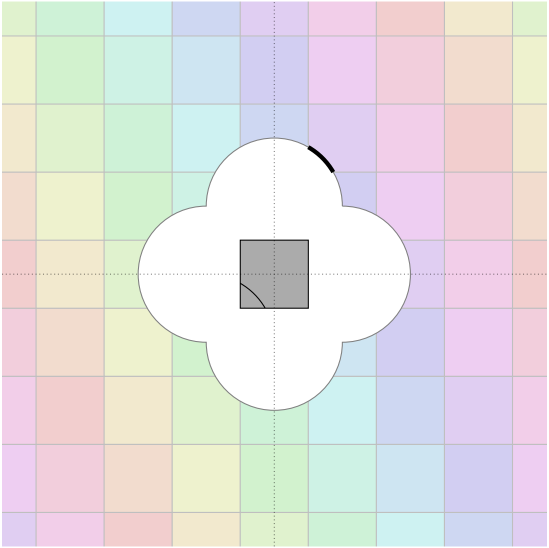

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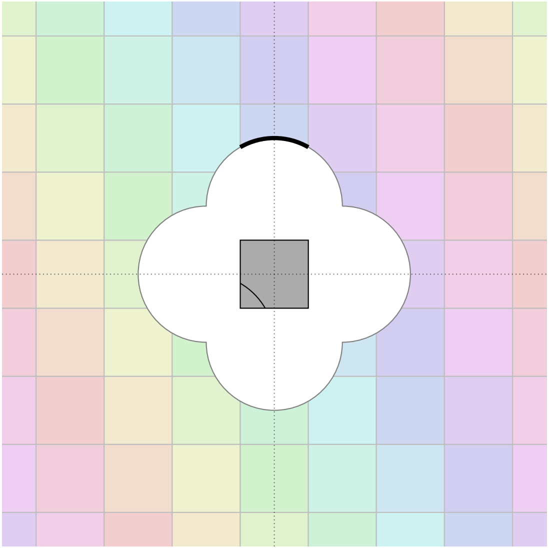

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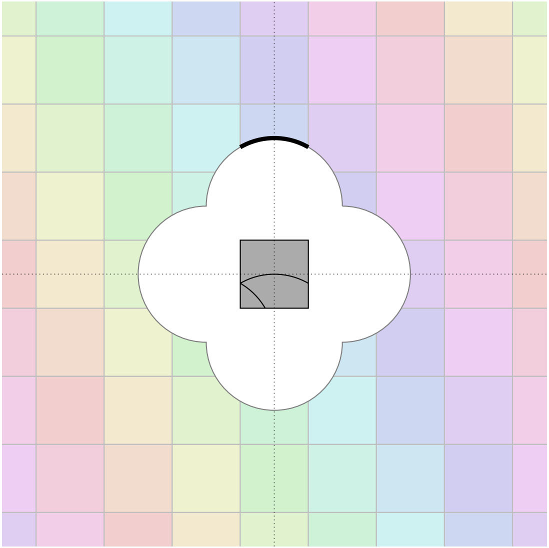

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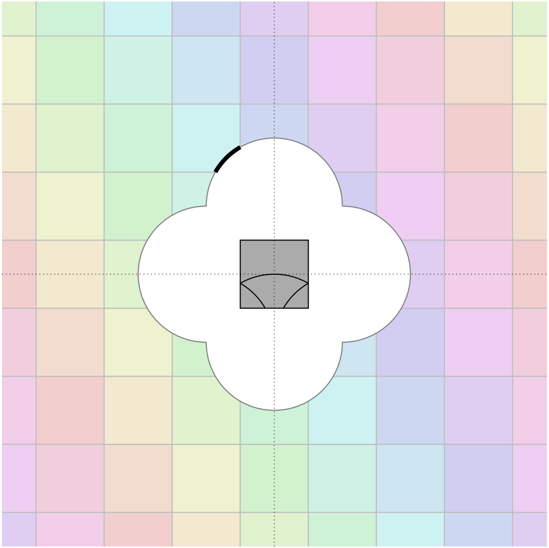

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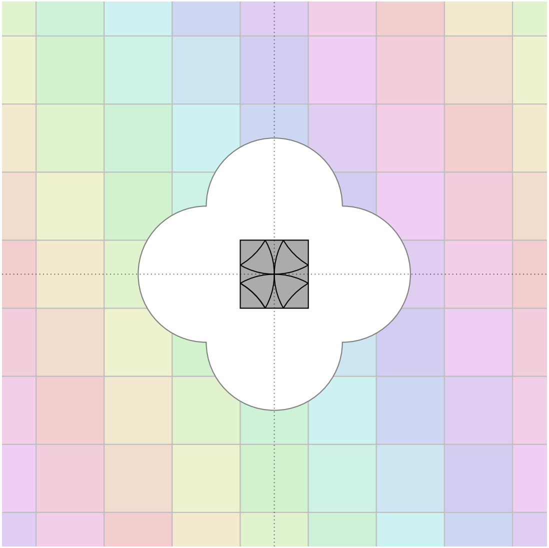

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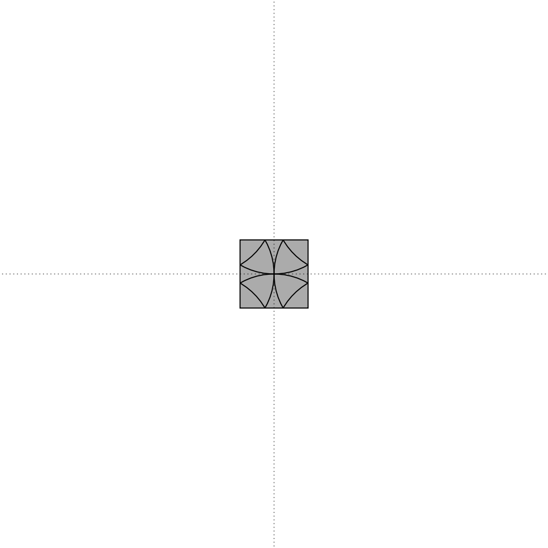

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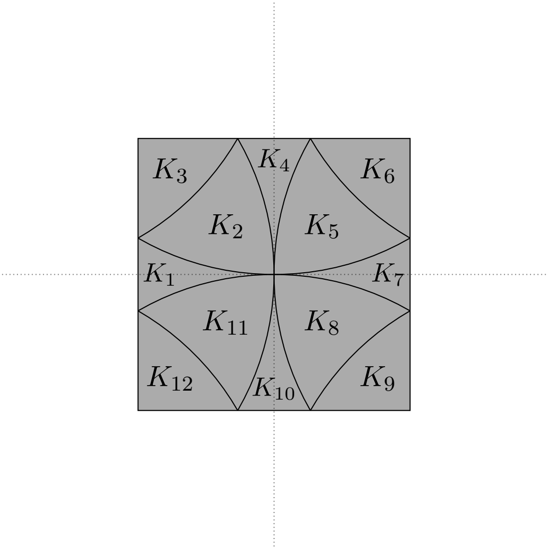

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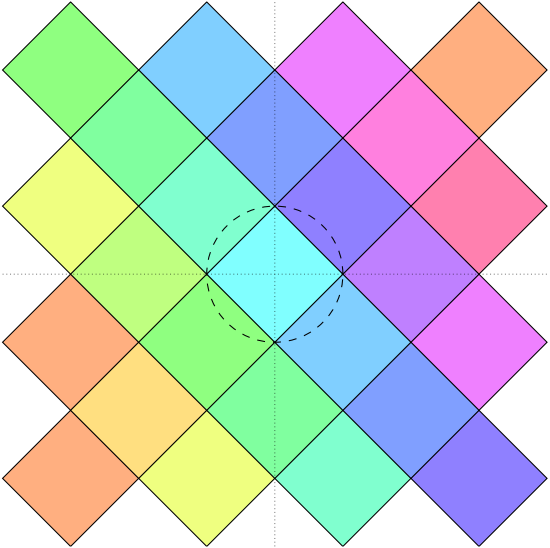

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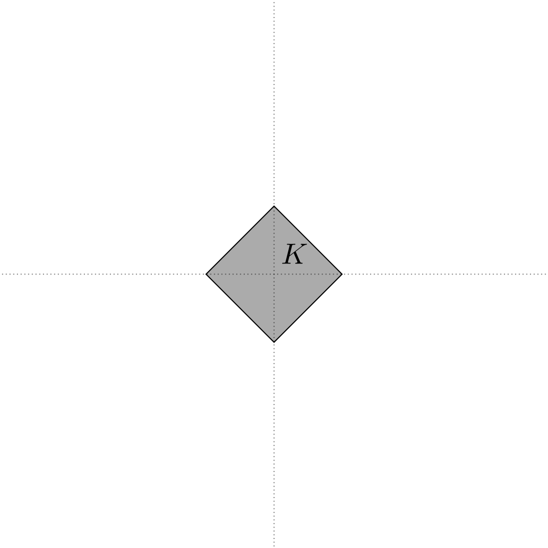

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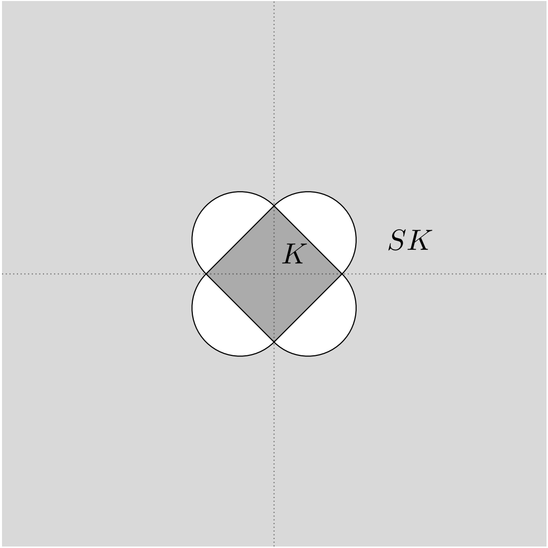

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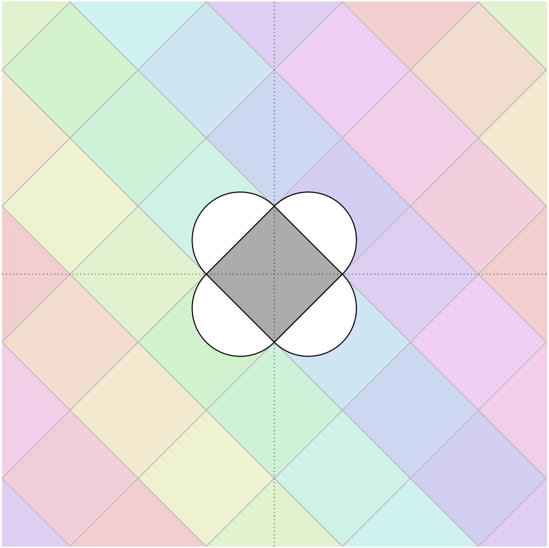

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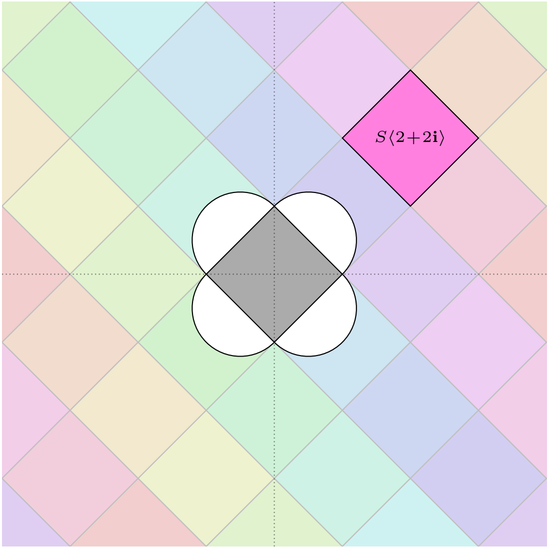

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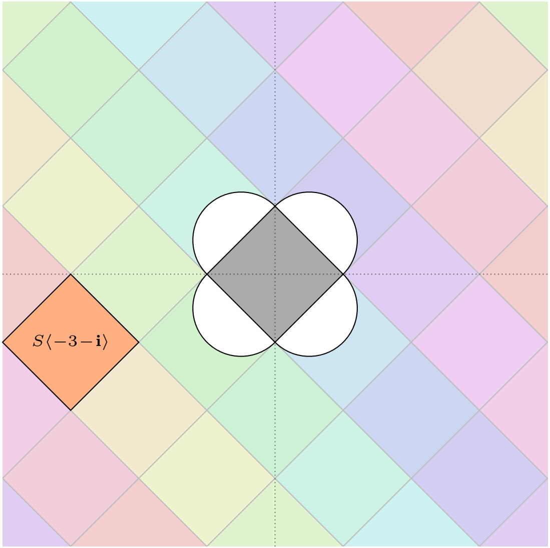

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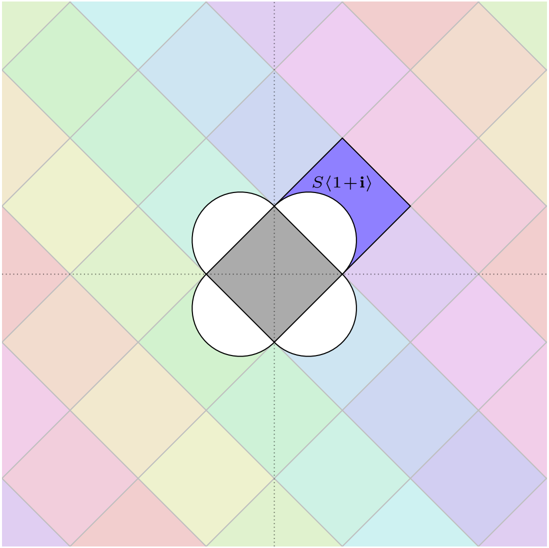

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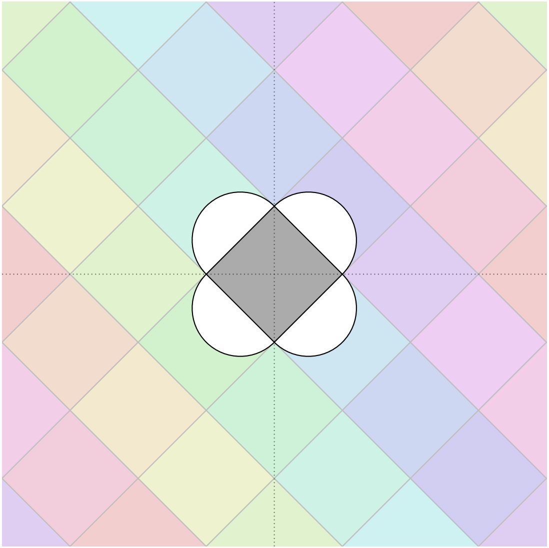

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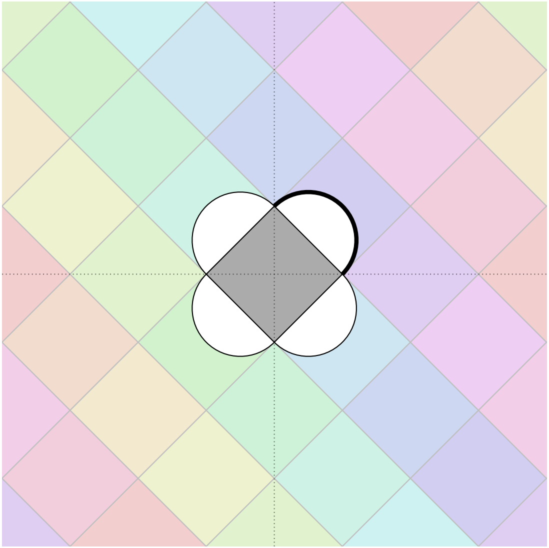

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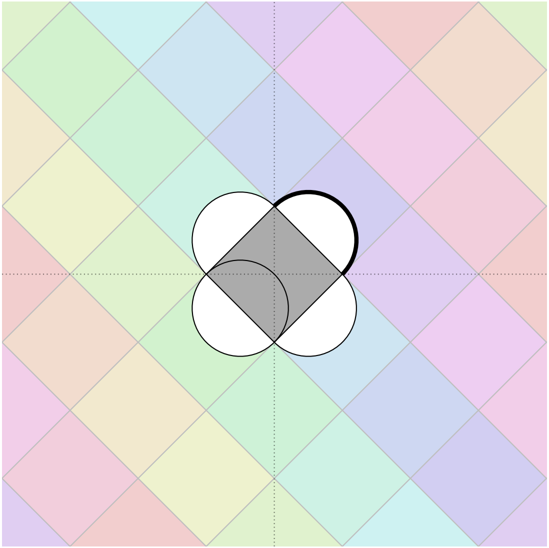

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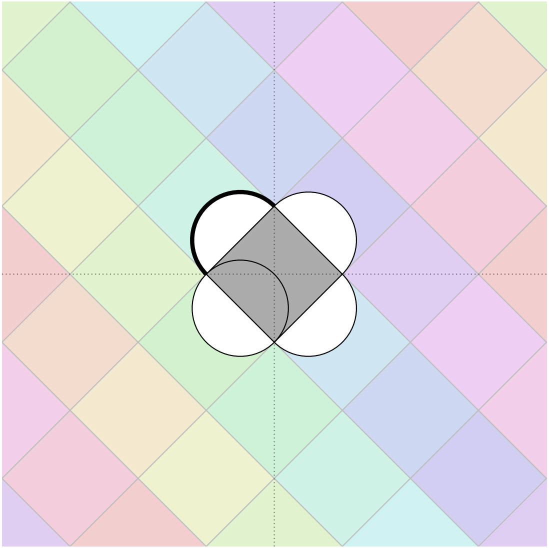

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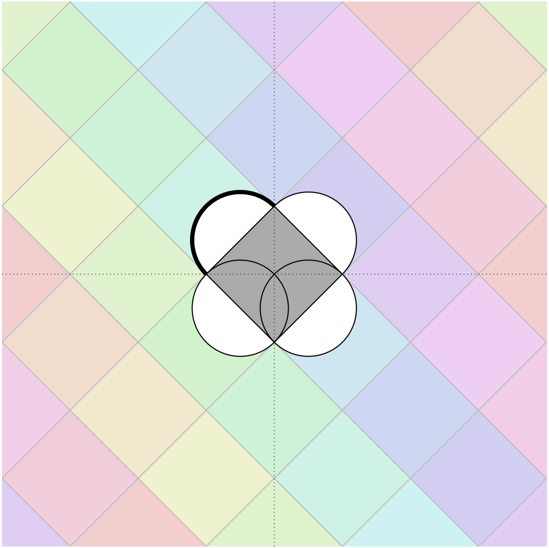

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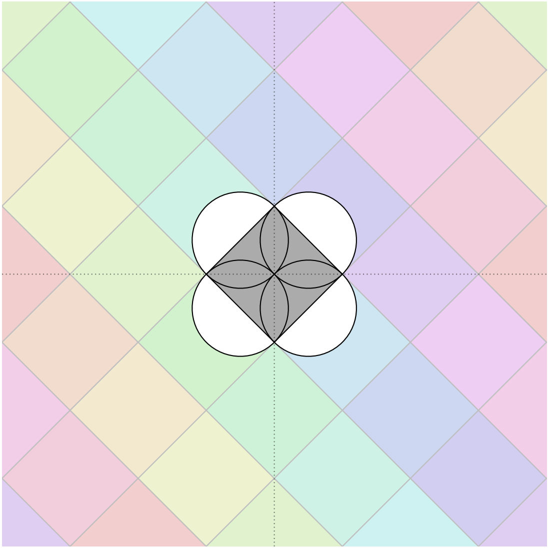

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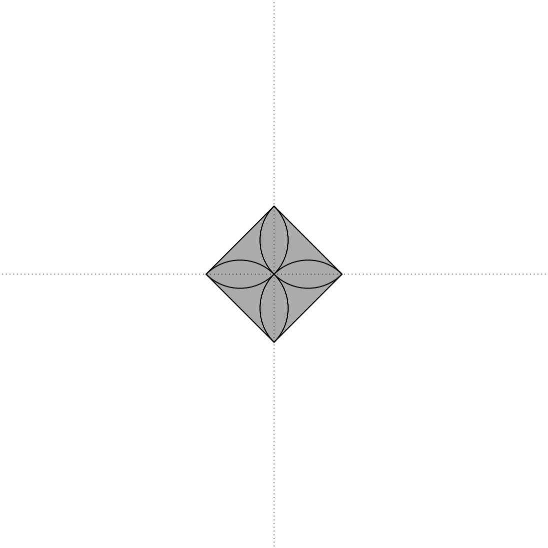

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The partitions $\mathcal{P} = \{K_1, \ldots, K_N\}$ just shown for the nearest even and nearest integer algorithms are constructed to satisfy

$$g(\langle a \rangle) = T^{-a} S \langle a \rangle = \bigcup_{j \in J(a)} K_j \qquad \text{for some } J(a).$$

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- Recall the lemma: if all $g(K_{i,a})$ are buildable from $\mathcal{P}$, then the finite building property is satisfied.

Sufficient conditions

Lemma (A.)

If each $\langle a \rangle$ *is contained in some* K_i *and each* $S\big(\langle a \rangle\big)$ *can be written as a union of sets of the form* $a + K_j$ *, then the algorithm satisfies the finite building property.*

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Proof. Let $J(a)$ be such that $S\langle a \rangle = \bigcup_{j \in J(a)} (a + K_j)$. Then

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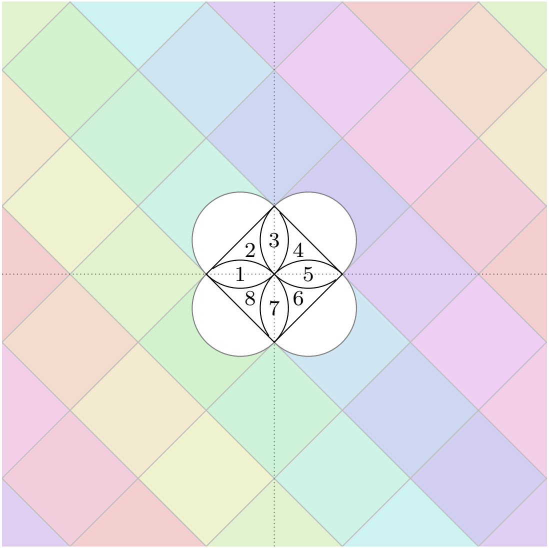
Nearest even



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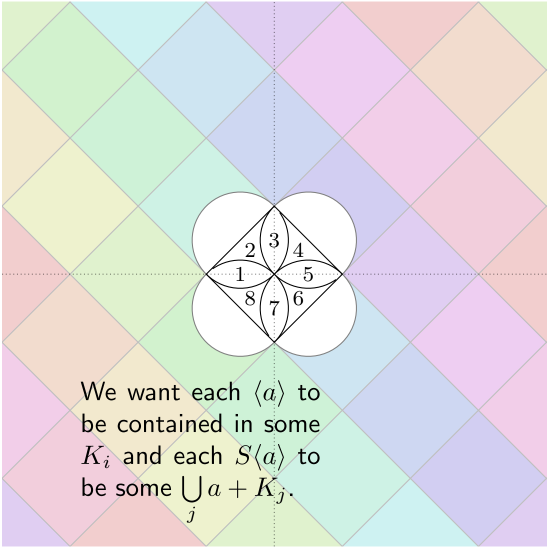
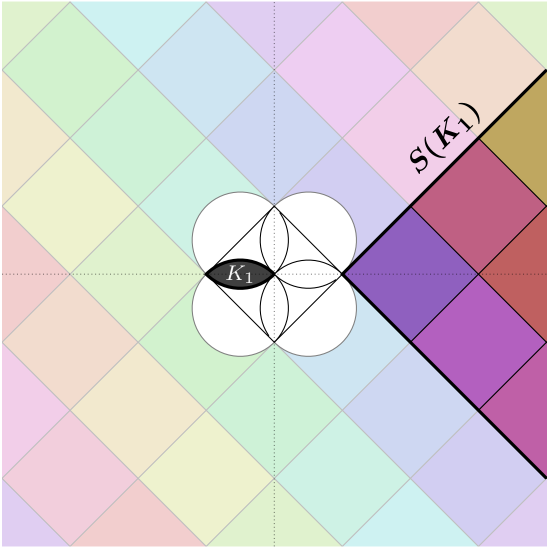


We want each $\langle a \rangle$ to be contained in some K_i and each $S\langle a \rangle$ to be some $\bigcup_j a + K_j$.

Nearest even



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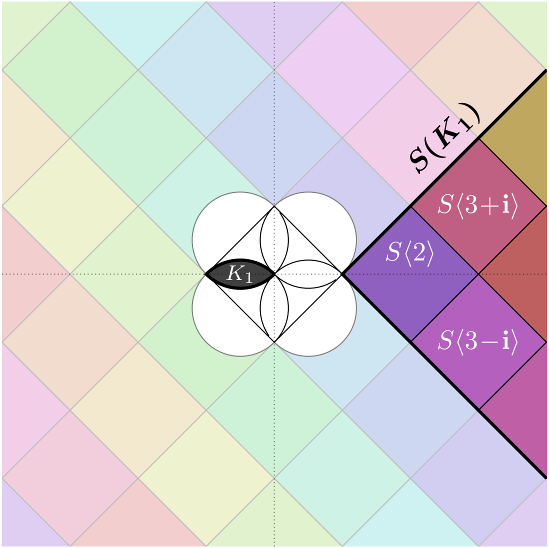

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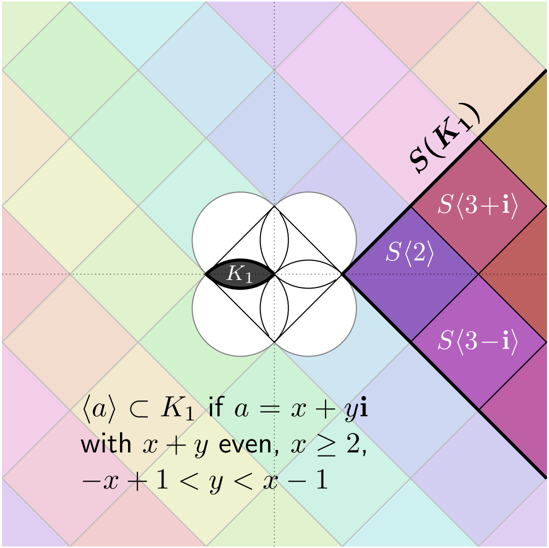
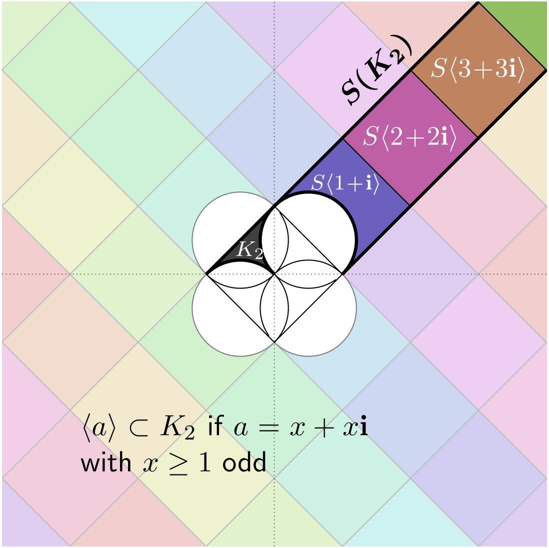


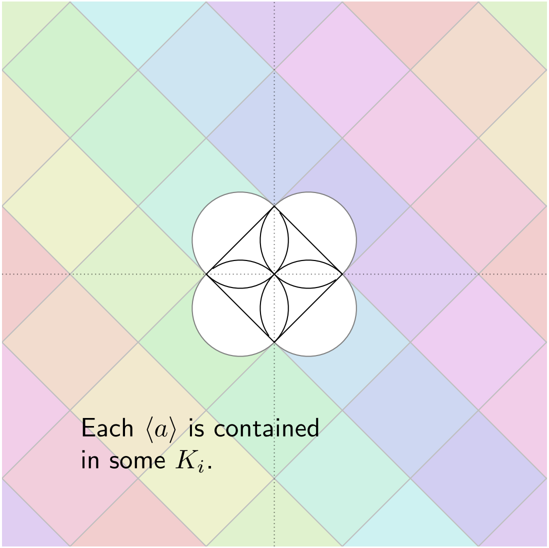
$\langle a\rangle \subset K_1$ if $a = x + y\mathbf{i}$ with $x + y$ even, $x \geq 2$, $-x + 1 < y < x - 1$

Nearest even



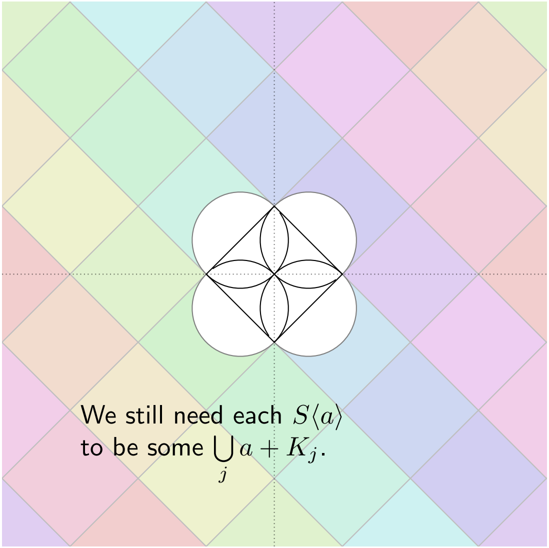
$\langle a\rangle \subset K_2$ if $a = x + x\mathbf{i}$
with $x \geq 1$ odd

Nearest even



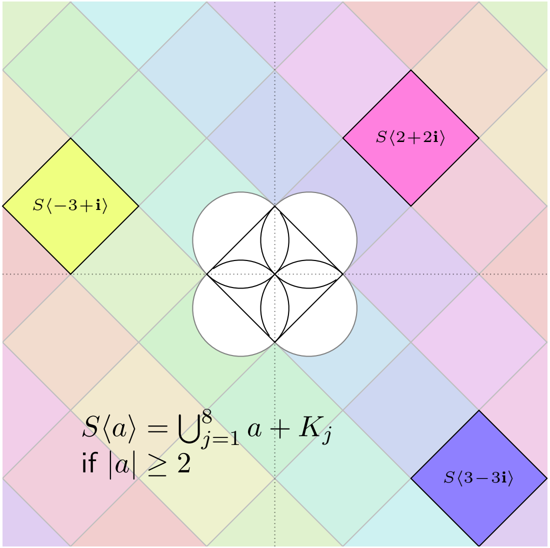
Each $\langle a \rangle$ is contained in some K_i.

Nearest even

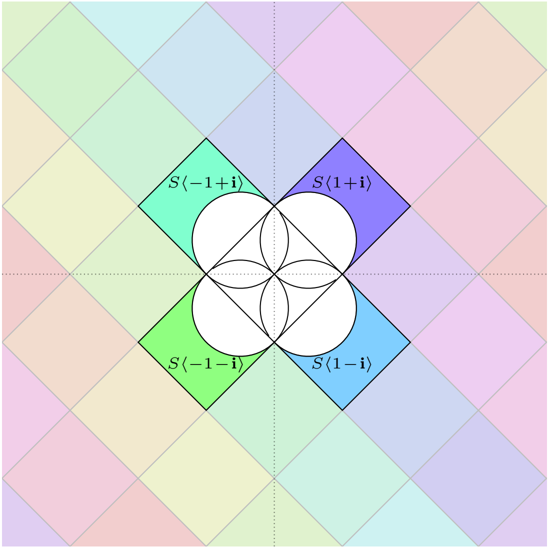


We still need each $S\langle a\rangle$ to be some $\bigcup_j a + K_j$.

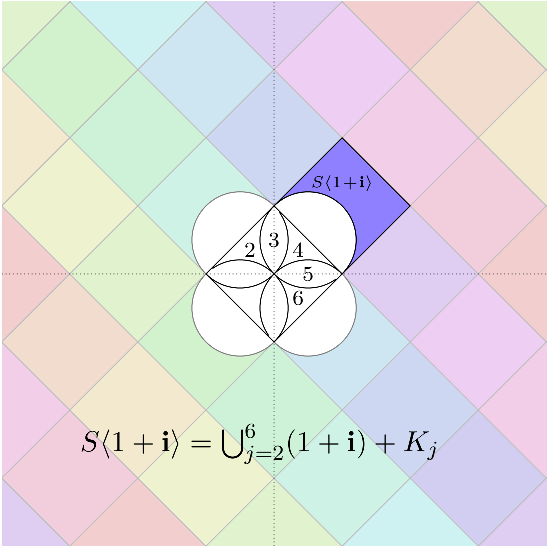
Nearest even



Nearest even



Nearest even



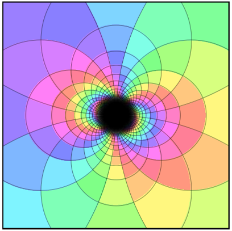
$$S\langle 1+\mathbf{i}\rangle = \bigcup_{j=2}^{6} (1+\mathbf{i}) + K_j$$

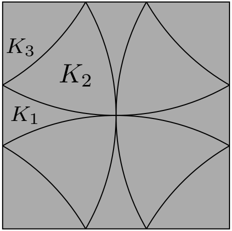
Nearest even

- For the nearest even algorithm we have sets $K_1, \ldots, K_8$ such that
  - each $\langle a \rangle$ is contained in some K_i,
  - each $S\langle a \rangle$ is some $\bigcup_j (a + K_j)$.

- Thus the finite building property is satisfied for the nearest even algorithm.

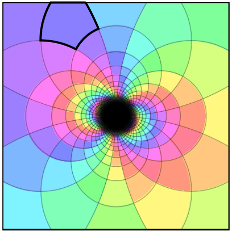
Hurwitz





Hurwitz

$\langle 1+2\mathbf{i} \rangle$



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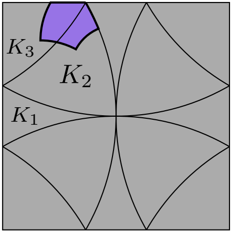
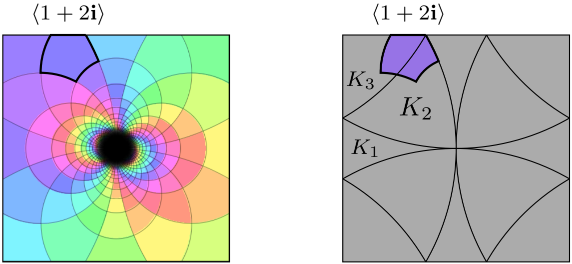


Hurwitz



It is not true that each $\langle a \rangle$ is containing in some K_i, so the second lemma cannot be used with this algorithm.

We must use the earlier lemma and show that each $K_{i,a}$ is buildable.

Hurwitz

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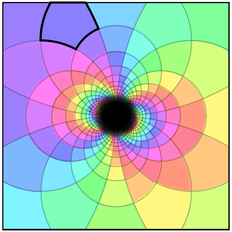
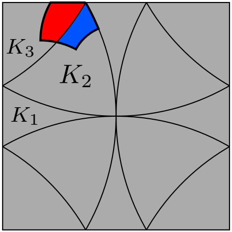


$K_{3,1+2\mathbf{i}}$ $K_{2,1+2\mathbf{i}}$



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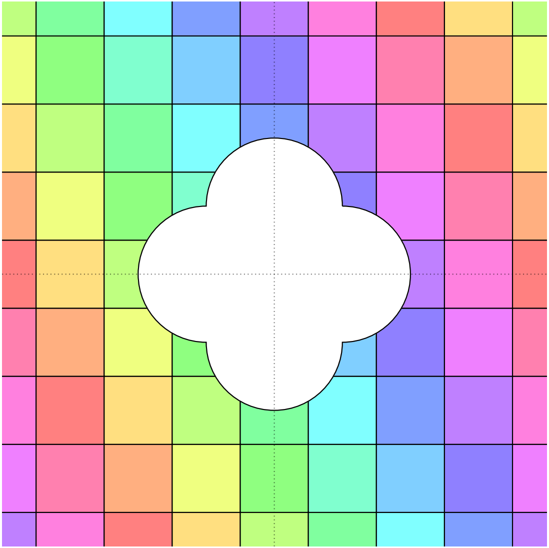
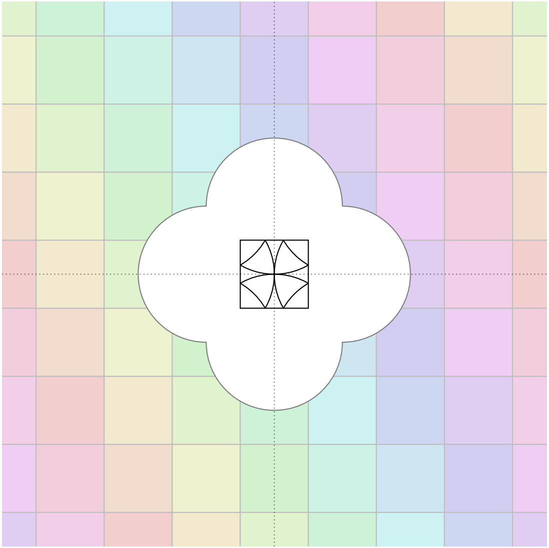
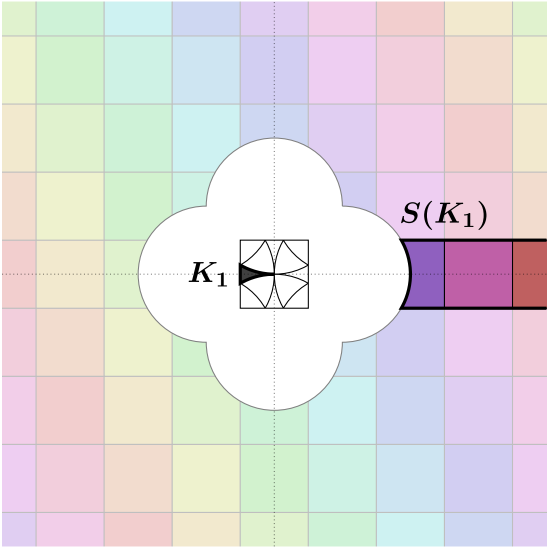
Hurwitz



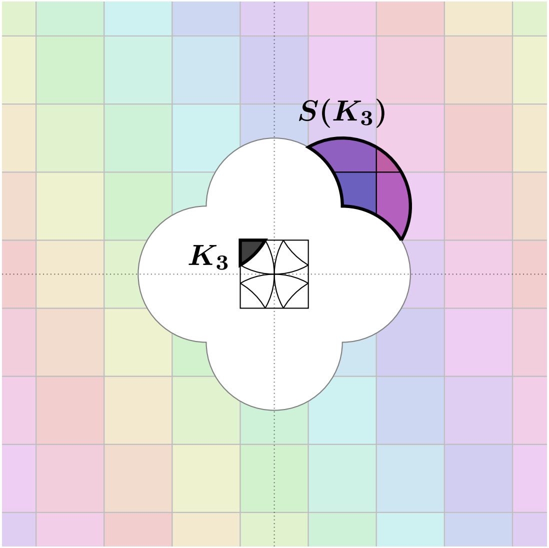
Hurwitz



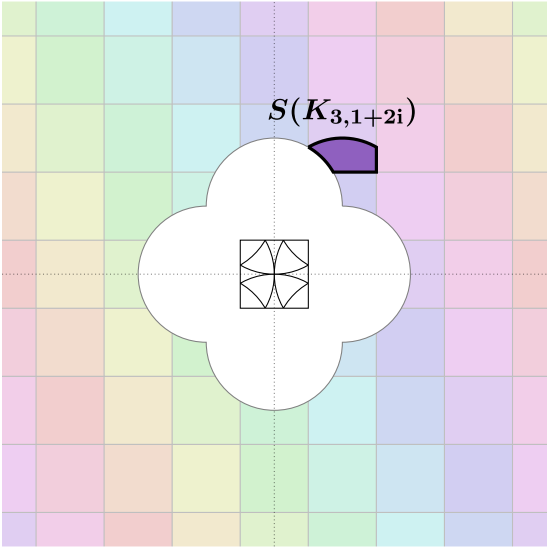
Hurwitz



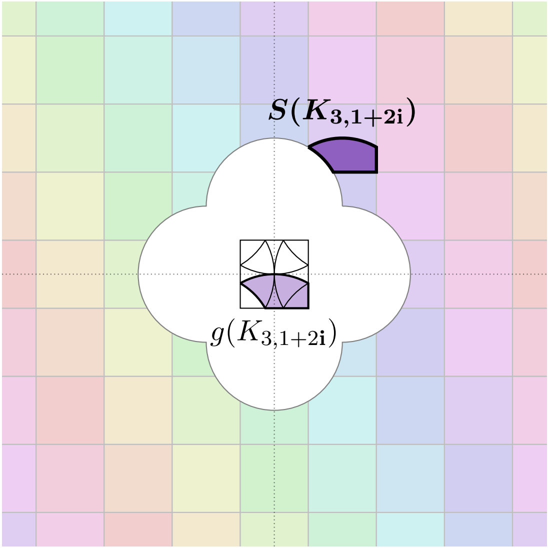
Hurwitz



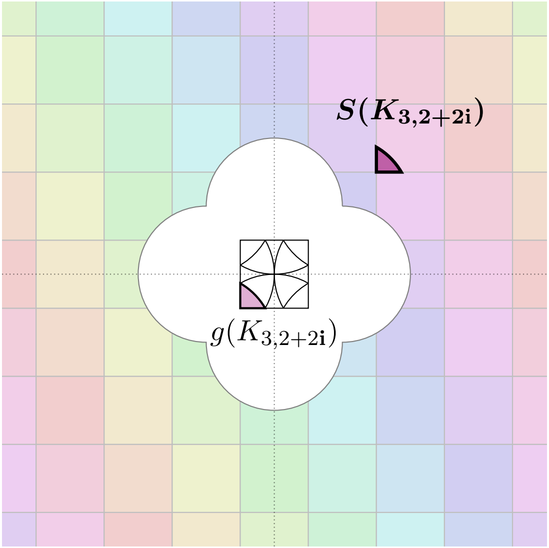
Hurwitz



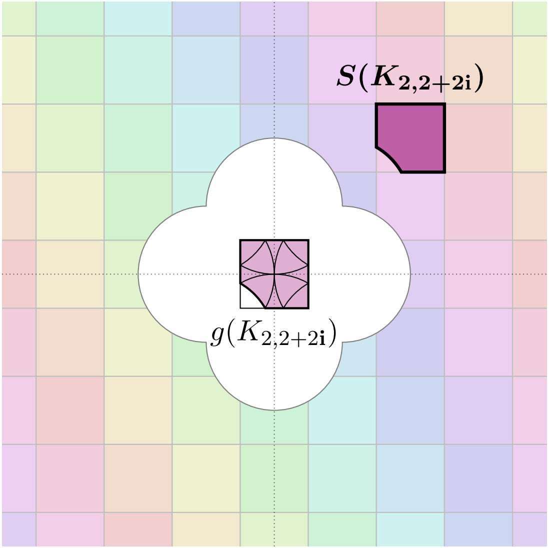
Hurwitz



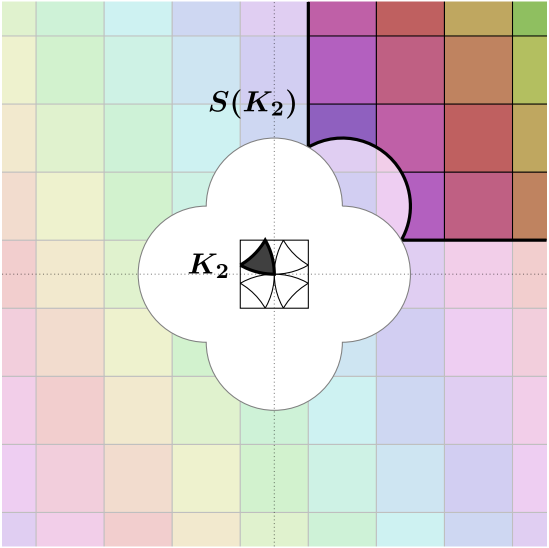
Hurwitz



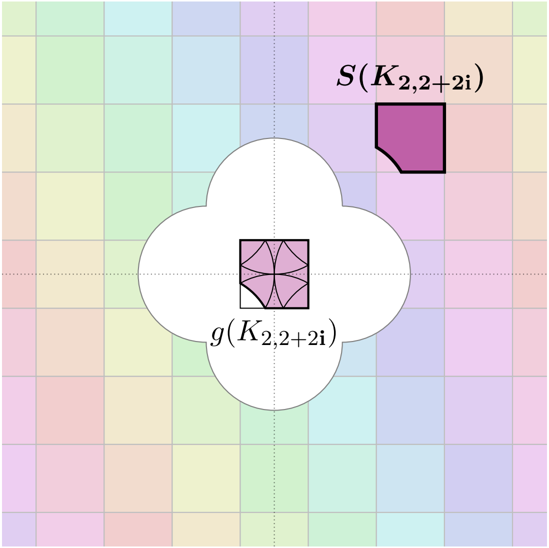
Hurwitz



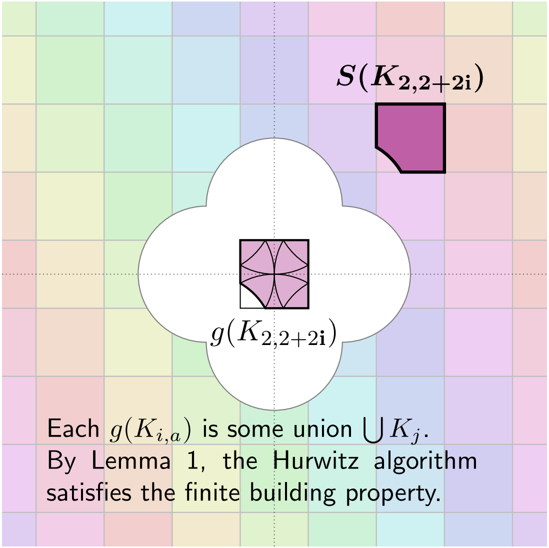
Hurwitz



Hurwitz

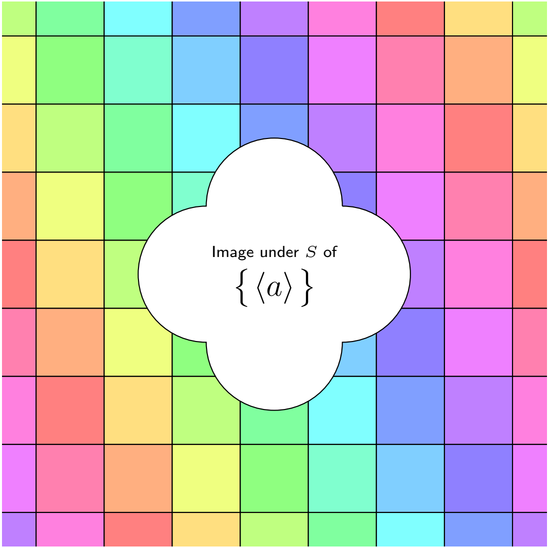


Hurwitz



Each $g(K_{i,a})$ is some union $\bigcup K_j$.
By Lemma 1, the Hurwitz algorithm satisfies the finite building property.

Hurwitz



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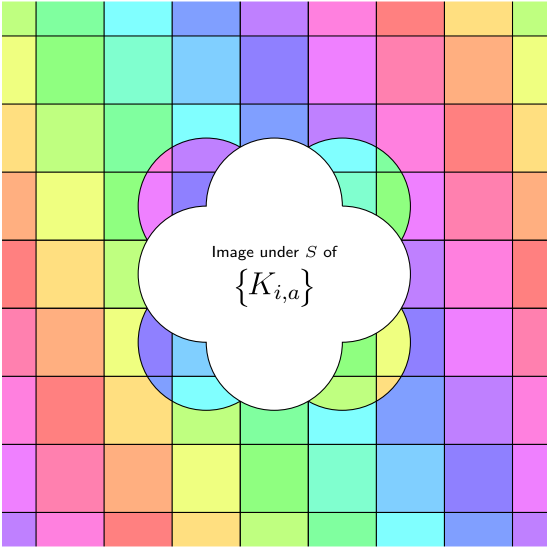
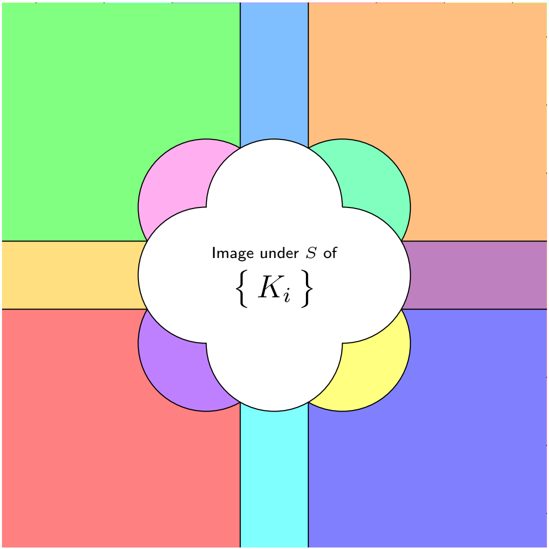
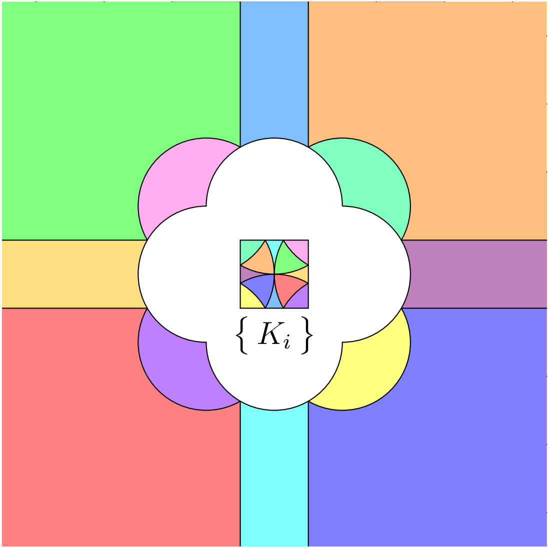


Hurwitz



Hurwitz



Additional properties

- Some algorithms have additional properties that can help make proofs easier.

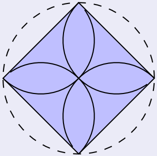
	Z-translates of K tile $\mathbb{C}$	Each $\langle a \rangle$ is contained in some K_i
Nearest integer	Yes, $Z = \mathbb{Z}[\mathbf{i}]$	No
Nearest even	Yes, $Z =$ evens	Yes
Diamond algorithm	No	No
Disk algorithm	No	Yes
Shifted Hurwitz	Yes, $Z = \mathbb{Z}[\mathbf{i}]$	No

Examples

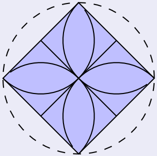
Proposition (A.)

The following algorithms satisfy the finite building property:

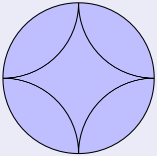
- Nearest even



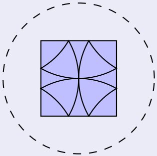
- Diamond



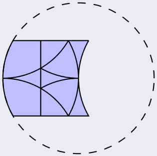
- Disk algorithm



- Nearest integer



- Shifted Hurwitz



Finite product structure

Define $G : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ by

$$G(z, w) = \left(\frac{-1}{z} - \left\lfloor \frac{-1}{z} \right\rceil, \frac{-1}{w - \left\lfloor \frac{-1}{z} \right\rceil} \right)$$

$$= (\, T^{-a} S\, z,\ S T^{-a}\, w), \quad a = \lfloor Sz \rceil$$

- One motivation for partitioning K by

  $$\mathcal{P} = \{K_1, K_2, \ldots, K_N\}$$

  is that we want to find a set

  $$\bigcup_{i=1}^{N} K_i \times L_i$$

  that is a bijectivity domain for G.

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System of equations

- For each $1 \le i \le N$, define $\mathcal{A}_i \subset \{1, \dots, N\} \times \mathbb{Z}[\mathbf{i}]$ as

$$\begin{aligned} \mathcal{A}_i &= \{ (j,a) \,:\, K_i \subset g(K_{j,a}) \} \\ &= \{ (j,a) \,:\, K_i \subset T^{-a}S(K_{j,a}) \} \\ &= \{ (j,a) \,:\, ST^a(K_i) \subset K_{j,a} \}. \end{aligned}$$

Theorem (A.)

Suppose an algorithm satisfies the finite building property with partition $\{K_1, \dots, K_N\}$, and let $L_1, \dots, L_N$ be arbitrary complex sets. The map G is bijective a.e. on $\bigcup_{i=1}^N K_i \times L_i$ if and only if the following system of equalities holds:

$$S(L_i) = \bigcup_{(j,a)\in\mathcal{A}_i} T^{-a} L_j, \qquad 1 \le i \le N.$$

System of equations

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System of equations

Proof that G bijective on $\bigcup K_i \times L_i$ implies $S(L_i) = \bigcup_{(j,a)\in\mathcal{A}_i} T^{-a} L_j$.

$$\begin{aligned}
G\left(\bigcup_{i=1}^{N} K_i \times L_i\right) &= G\left(\bigcup_{\substack{1\le j\le N\\ a\in\mathbb{Z}[\mathbf{i}]}} K_{j,a} \times L_j\right) \\
&= \bigcup_{\substack{1\le j\le N\\ a\in\mathbb{Z}[\mathbf{i}]}} \left(T^{-a} S K_{j,a} \times S T^{-a} L_j\right) \\
&= \bigcup_{\substack{1\le j\le N\\ a\in\mathbb{Z}[\mathbf{i}]}} \left(\left(\bigcup_{\substack{i \text{ s.t.}\\ (j,a)\in\mathcal{A}_i}} K_i\right) \times S T^{-a} L_j\right) \\
&= \bigcup_{\substack{1\le j\le N\\ a\in\mathbb{Z}[\mathbf{i}]}} \bigcup_{\substack{i \text{ s.t.}\\ (j,a)\in\mathcal{A}_i}} \left(K_i \times S T^{-a} L_j\right)
\end{aligned}$$

System of equations

Proof that G bijective on $\bigcup K_i \times L_i$ implies $S(L_i) = \bigcup_{(j,a)\in\mathcal{A}_i} T^{-a}L_j$.

$$
\begin{aligned}
G\left(\bigcup_{i=1}^{N} K_i \times L_i\right) &= \cdots \\
&= \bigcup_{\substack{1\le j\le N \\ a\in\mathbb{Z}[\mathbf{i}]}} \; \bigcup_{\substack{i \text{ s.t.} \\ (j,a)\in\mathcal{A}_i}} \left(K_i \times ST^{-a}L_j\right) \\
&= \bigcup_{i=1}^{N} \; \bigcup_{(j,a)\in\mathcal{A}_i} \left(K_i \times ST^{-a}L_j\right) \\
&= \bigcup_{i=1}^{N} \left(K_i \times \left(\bigcup_{(j,a)\in\mathcal{A}_i} ST^{-a}L_j\right)\right)
\end{aligned}
$$

System of equations

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In order for $G(\Omega)$ to equal Ω, it must be that

$$L_i = \bigcup_{(j,a)\in\mathcal{A}_i} ST^{-a}L_j$$

for $i = 1, \ldots, N$. ☐

System of equations

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In order for $G(\Omega)$ to equal Ω, it must be that

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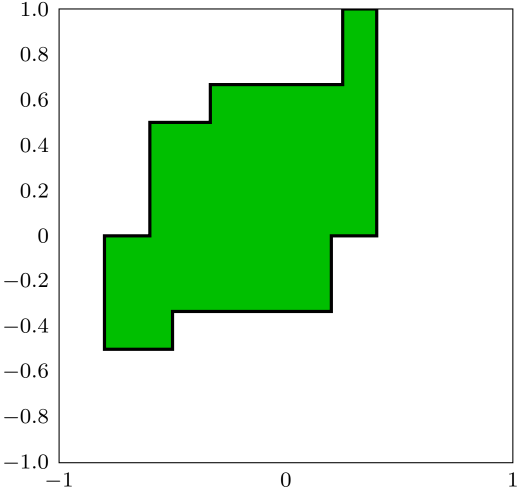
for $i = 1, \ldots, N$. □

Computer simulation

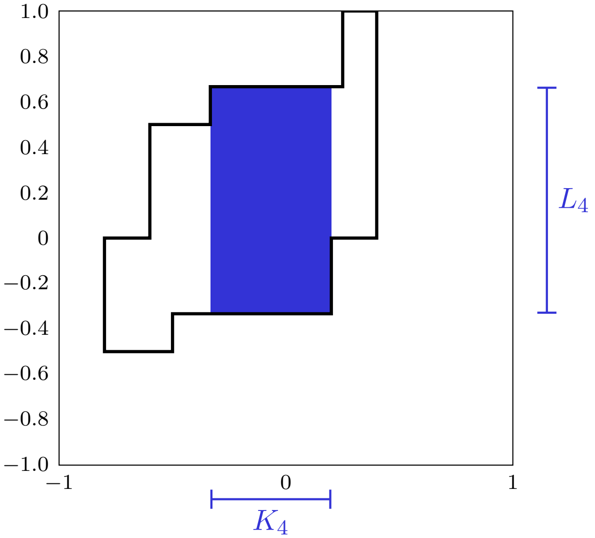
In practice, the system of union equations is not useful for constructing L_i.

- Instead, we can generate a scatter plot of points in L_i using a computer, then verify that a potential collection $L_1, \ldots, L_N$ satisfies the system.

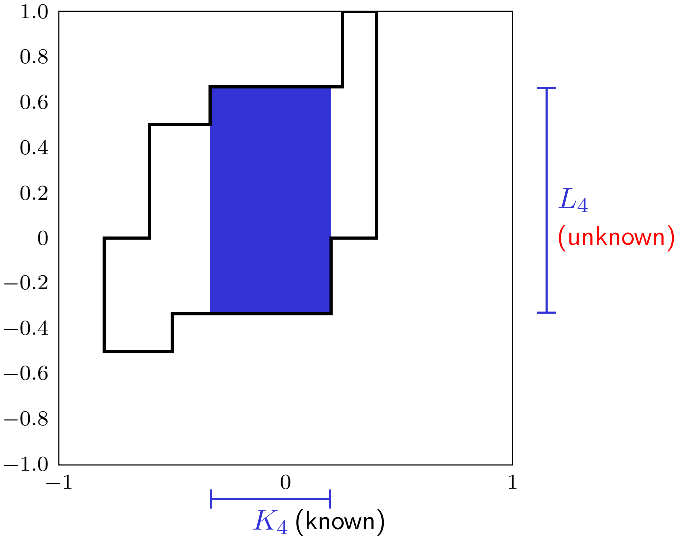
Computer simulation



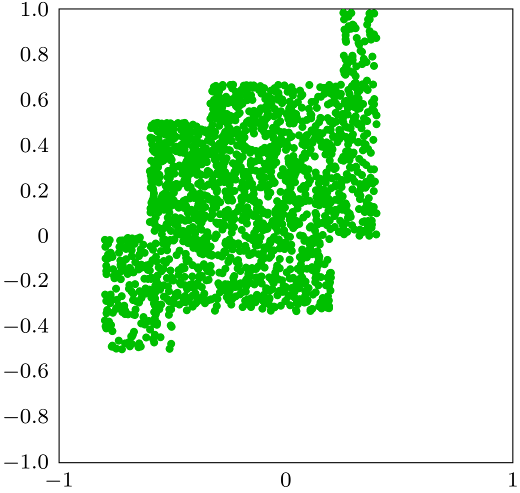
Computer simulation



Computer simulation



Computer simulation



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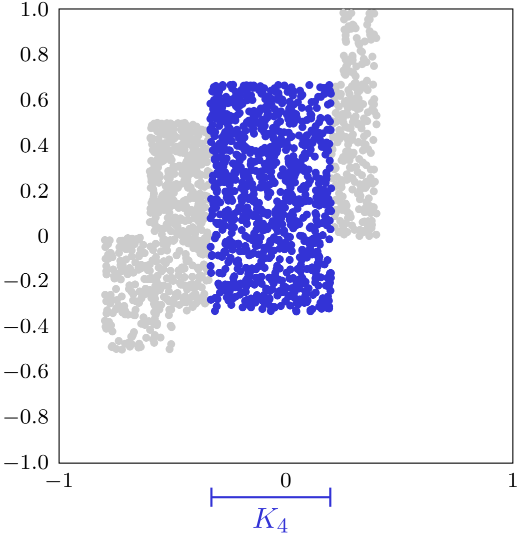

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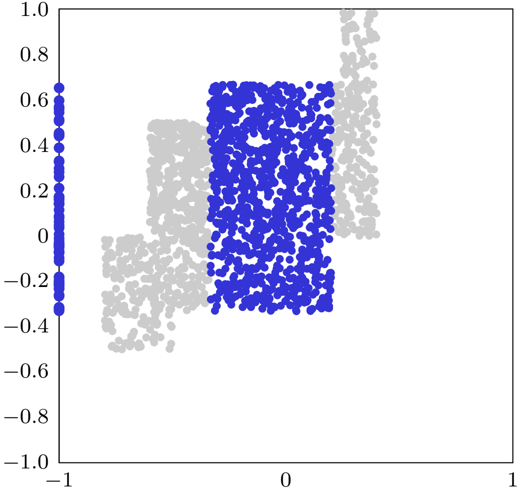

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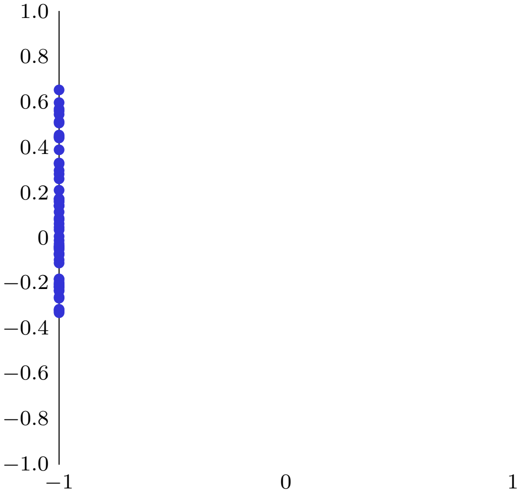

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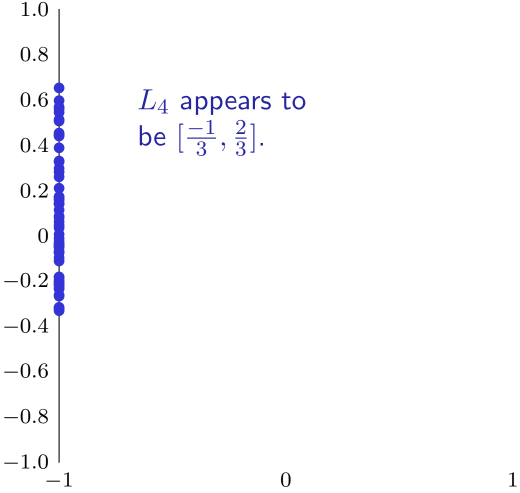

L_4 appears to be $[\frac{-1}{3}, \frac{2}{3}]$.

Computer simulation

Exact descriptions of L_i are known for

- Nearest even algorithm
- Disk algorithm
- Diamond algorithm

Computer simulation

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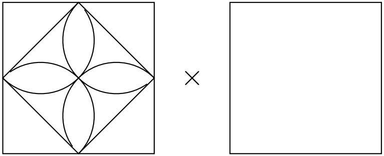

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Computer simulation

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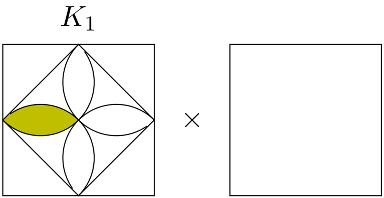

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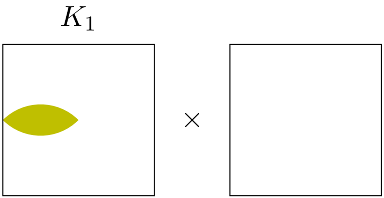

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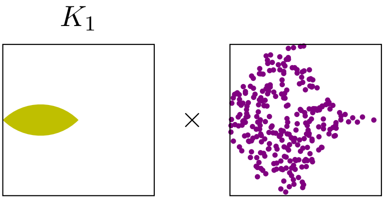

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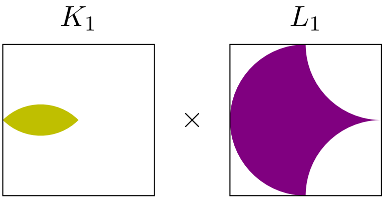
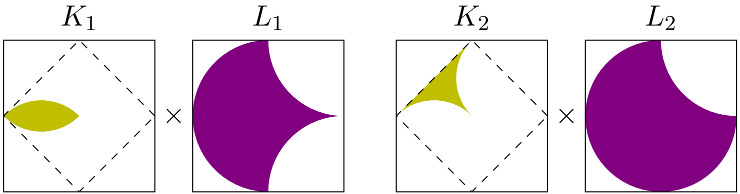
Exact descriptions of L_i are known for

- Nearest even algorithm

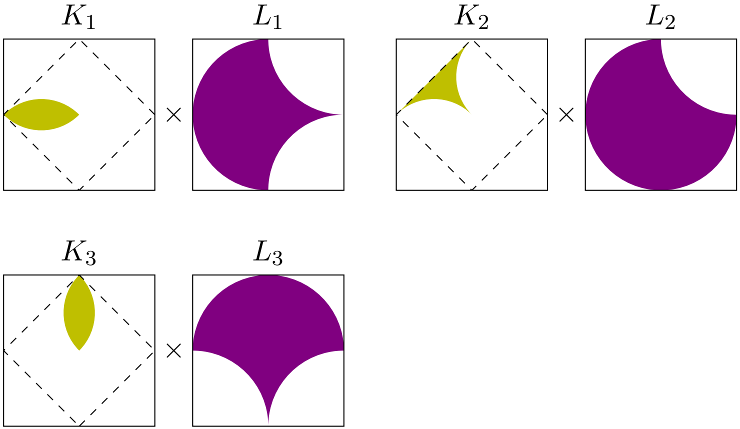


- Disk algorithm
- Diamond algorithm

Nearest even algorithm ($N = 8$)

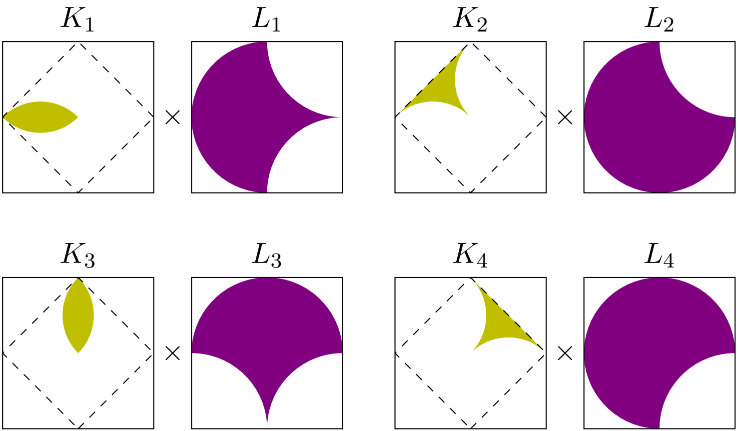


Nearest even algorithm ($N = 8$)



etc.

Nearest even algorithm ($N = 8$)



etc. etc.

System of equations

Recall the theorem that G is bijective a.e. on $\bigcup_{i=1}^{N} K_i \times L_i$ if and only if

$$S(L_i) = \bigcup_{(j,a)\in\mathcal{A}_i} T^{-a} L_j, \qquad 1 \le i \le N,$$

where

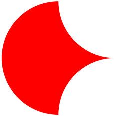
$$\begin{aligned}\mathcal{A}_i &= \{\, (j,a) \,:\, K_i \subset g\big(K_{j,a}\big) \,\} \\ &= \{\, (j,a) \,:\, K_i \subset T^{-a} S\big(K_{j,a}\big) \,\} \\ &= \{\, (j,a) \,:\, ST^{a}(K_i) \subset K_{j,a} \,\}.\end{aligned}$$

- What does this look like visually?

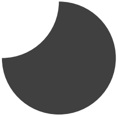
Nearest even algorithm ($N = 8$)

Goal: $S(L_1) = \bigcup_{(j,a)\in\mathcal{A}_1} T^{-a} L_j$

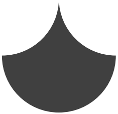
L_1



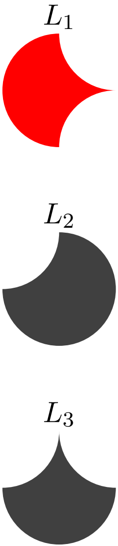
L_2



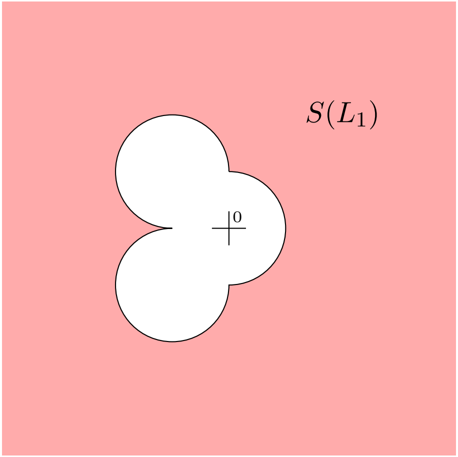
L_3



Nearest even algorithm ($N = 8$)



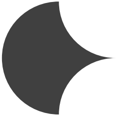
Goal: $S(L_1) = \bigcup_{(j,a)\in\mathcal{A}_1} T^{-a} L_j$



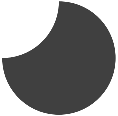
Nearest even algorithm ($N = 8$)

Goal: $S(L_1) = \bigcup_{(j,a)\in\mathcal{A}_1} T^{-a} L_j$

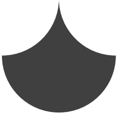
L_1

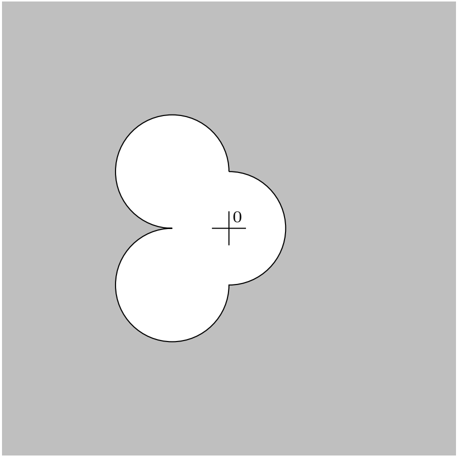


L_2

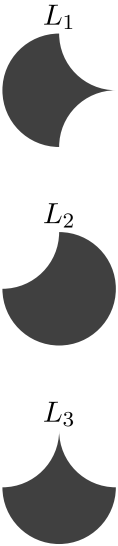


L_3

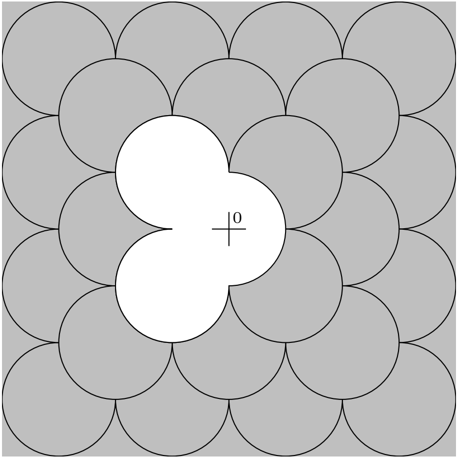




Nearest even algorithm ($N = 8$)

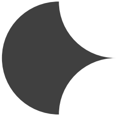


Goal: $S(L_1) = \bigcup_{(j,a)\in\mathcal{A}_1} T^{-a} L_j$

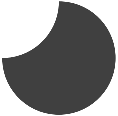


Nearest even algorithm ($N = 8$)

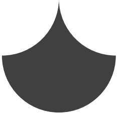
L_1



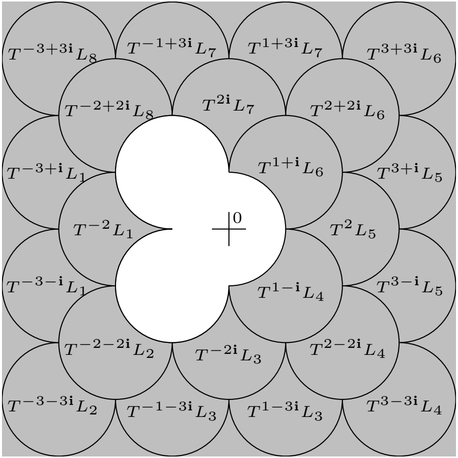
L_2



L_3

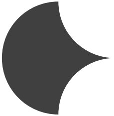


Goal: $S(L_1) = \bigcup_{(j,a)\in\mathcal{A}_1} T^{-a} L_j$

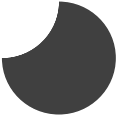


Nearest even algorithm ($N = 8$)

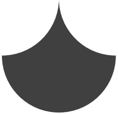
L_1



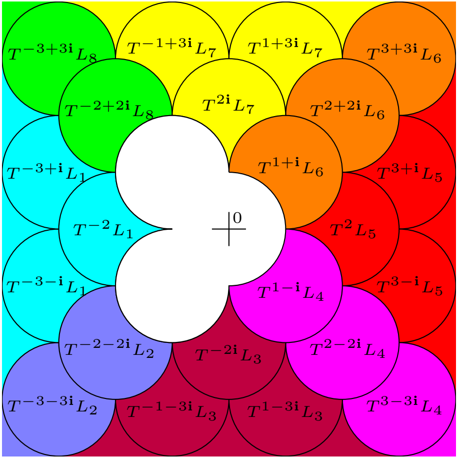
L_2



L_3

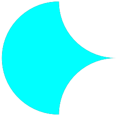


Goal: $S(L_1) = \bigcup_{(j,a)\in\mathcal{A}_1} T^{-a}L_j$

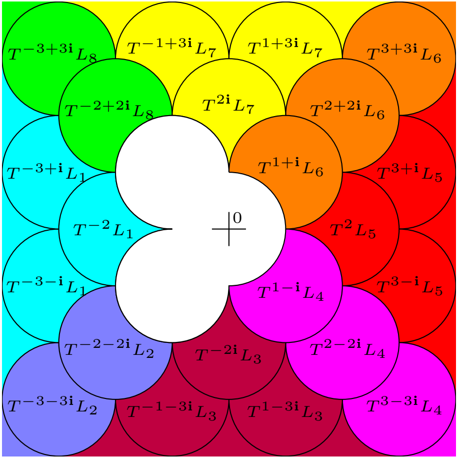


Nearest even algorithm ($N = 8$)

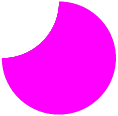
L_1



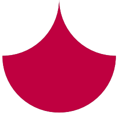
Goal: $S(L_1) = \bigcup_{(j,a)\in\mathcal{A}_1} T^{-a} L_j$



L_2



L_3



Finite product structures

For each algorithm, the set

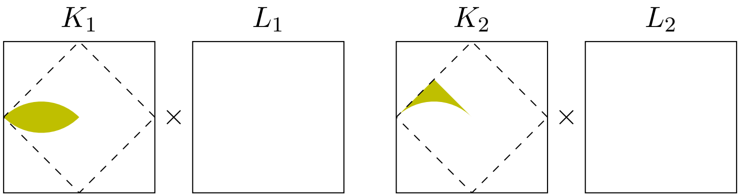
$$\Omega = \bigcup_{i=1}^{N} K_i \times L_i$$

is a bijectivity domain for $G : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$, and $G\big|_\Omega$ is the natural extension of the Gauss map $g : K \to K$.

As a conclusion, here are images of K_i and L_i for various algorithms.

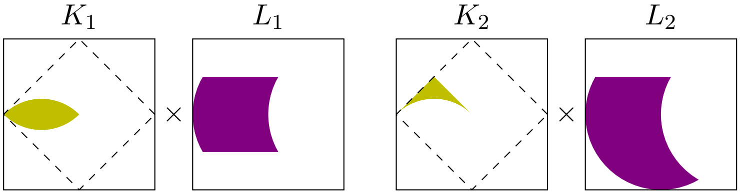
- In many cases only a few sets are shown (not all N) because the rest are rotations or reflections.

Diamond algorithm ($N = 12$)



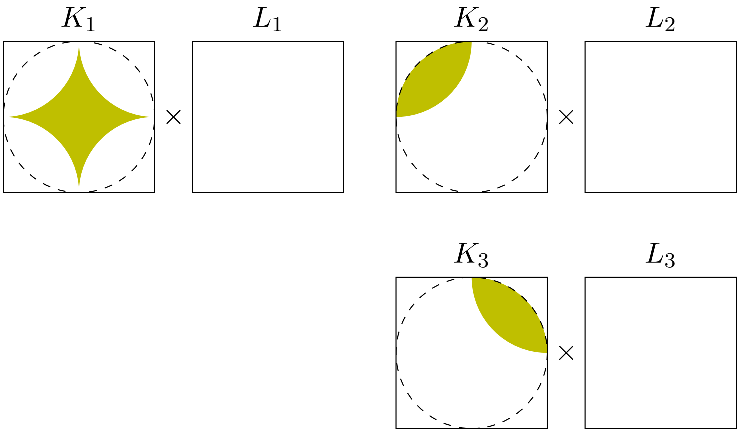
etc. etc.

Diamond algorithm ($N = 12$)



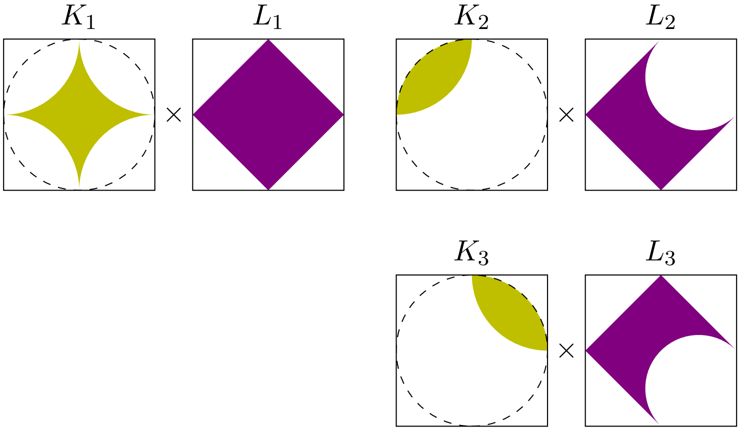
etc. etc.

Disk algorithm ($N = 5$)



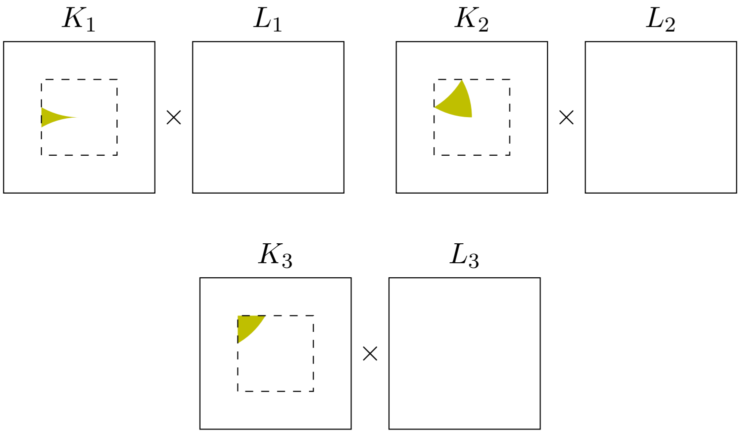
etc.

Disk algorithm ($N = 5$)



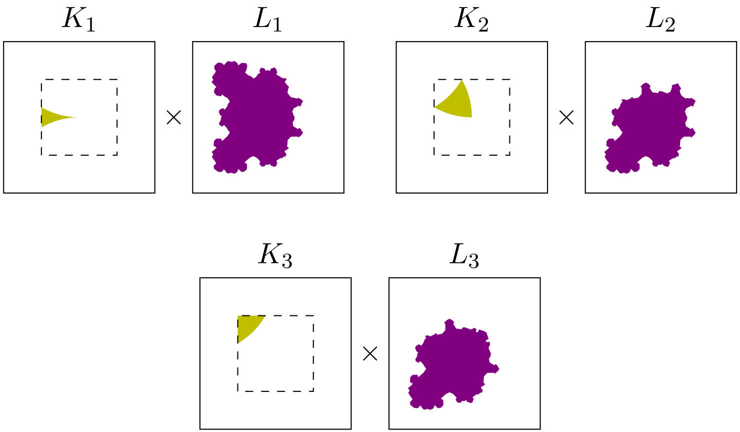
etc.

Hurwitz algorithm ($N = 12$)



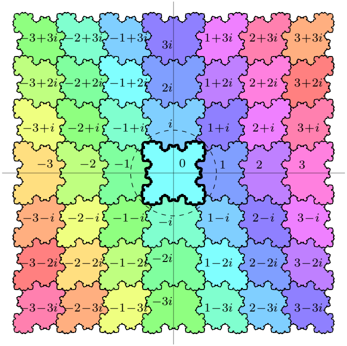
etc.

Hurwitz algorithm ($N = 12$)



etc.

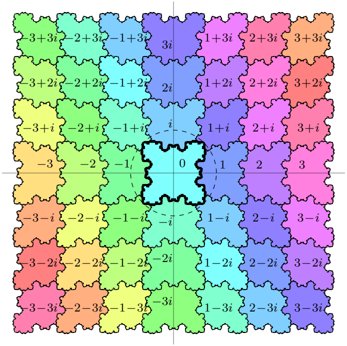
One last choice function



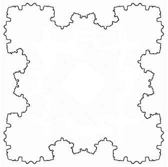
Dual-Hurwitz algorithm[5]

[5] Hiromi Ei, Shunji Ito, Hitoshi Nakada, Rie Natsui, 2019.

One last choice function

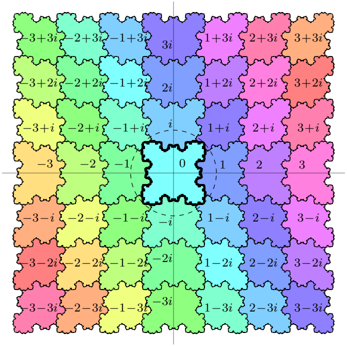


Dual-Hurwitz algorithm[5]

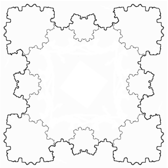


[5] Hiromi Ei, Shunji Ito, Hitoshi Nakada, Rie Natsui, 2019.

One last choice function



Dual-Hurwitz algorithm[5]



$(N = 9)$

[5] Hiromi Ei, Shunji Ito, Hitoshi Nakada, Rie Natsui, 2019.


Thank you!