On the dimension of points which escape to infinity at given rate under exponential iteration

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Escaping set and Julia set

Let

$$f : \mathbb{C} \to \mathbb{C}$$

be a transcendental entire map.

The **escaping set** is defined as

$$I(f) = \{z \in \mathbb{C} : |f^n(z)| \to \infty \text{ as } n \to \infty\},$$

while the **Julia set** $J(f)$ is

$$J(f) = \{z \in \mathbb{C} : \{f^n\}_{n=1}^{\infty} \text{ is not a normal family in any nbhd of } z\}.$$

- $J(f) = \partial I(f)$ (Eremenko 1989)
- $J(f) = \overline{I(f)}$ for $f \in \mathcal{B}$ (Eremenko and Lyubich 1992)

$$\mathcal{B} = \{\text{maps with bounded set of critical and asymptotic values}\}$$

Dimension of escaping set and Julia set

The **exponential map** is defined as

$$E_\lambda(z) = \lambda e^z, \qquad z \in \mathbb{C}, \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

- The Julia sets of exponential maps have Hausdorff dimension 2 (Mcmullen 1987)
- Since then, many results on the dimension of $J(f)$, $I(f)$ and their dynamically defined subsets (Bergweiler, Bishop, Karpińska, Kotus, Mayer, Osborne, Pawelec, Peter, Rempe-Gillen, Rippon, Rottenfußer, Rückert, Schleicher, Schubert, Sixsmith, Stallard, Urbański, Waterman, Zdunik, Zheng, Zimmer,...)

Various kinds of escaping

- **Fast escaping set** (Bergweiler and Hinkkanen 1999)

  $$A(f) = \{z \in I(f) : |f^{n+l}(z)| \geq M_f^n(R), n \in \mathbb{N}, \text{ for some } l \geq 0\}$$

  for $M_f(r) = \max_{|z|=r} |f(z)|$.

- **Moderately slow escaping set** (Rippon and Stallard 2011)

  $$M(f) = \{z \in I(f) : \limsup_{n\to\infty} \frac{1}{n} \log\log |f^n(z)| < \infty\}$$

- **Slow escaping set** (Rippon and Stallard 2011)

  $$L(f) = \{z \in I(f) : \limsup_{n\to\infty} \frac{1}{n} \log |f^n(z)| < \infty\}$$

Theorem (Bergweiler, Karpińska, Stallard 2009, Rippon and Stallard 2014)

Fast escaping set has Hausdorff dimension 2 for $f \in \mathcal{B}$ of finite order or 'not too large' infinite order.

Sets with prescribed escape rate

For sequences $\underline{a} = (a_n)_{n=1}^{\infty}$, $\underline{b} = (b_n)_{n=1}^{\infty}$ with $0 < a_n \leq b_n$ let

$$\begin{aligned} I_{\underline{a}}(f) &= \{z : |f^n(z)| \geq a_n \text{ for large } n \in \mathbb{N}\}, \\ I^{\underline{b}}(f) &= \{z : |f^n(z)| \leq b_n \text{ for large } n \in \mathbb{N}\}, \\ I_{\underline{a}}^{\underline{b}}(f) &= \{z : a_n \leq |f^n(z)| \leq b_n \text{ for large } n \in \mathbb{N}\}. \end{aligned}$$

Remark

To guarantee that the sets are consideration are not empty, one usually assumes that the sequence $\underline{a}$ is **admissible**, which roughly means $a_{n+1} < M_f(a_n)$.

Some results on $I_{\underline{a}}^{\underline{b}}(f)$

- $I_{\underline{a}}^{\underline{b}}(E_\lambda) \neq \emptyset$ for every admissible sequence $\underline{a} = (a_n)_{n=1}^{\infty}$ with $a_n \to \infty$ and $b_n = ca_n$, $c > 1$ (Rempe 2006)
- The same holds for arbitrary transcendental entire (or meromorphic) maps f (Rippon and Stallard 2011)
- $\dim_H(I(f) \cap I^{\underline{b}}(f)) \geq 1$ for every transcendental entire map f in the class $\mathcal{B}$ and $b_n \to \infty$ (Bergweiler and Peter 2013)

Remark

The Julia sets of exponential maps contain **hairs** (Devaney and Krych 1984, Devaney and Tangerman 1986, Schleicher and Zimmer 2003). For exponential maps with an attracting fixed point the Julia set is the union of hairs together with their endpoints (Aarts and Oversteegen 1993). The hairs without endpoints are contained in the fast escaping set (Rempe, Rippon and Stallard 2010).

Results on $I_{\underline{a}}^{\underline{b}}(E_\lambda)$

In 2016 Sixsmith proved that $\dim_H I_{\underline{a}}^{\underline{b}}(E_\lambda) \leq 1$ for admissible sequences $\underline{a} = (a_n)_{n=1}^{\infty}$ with $a_n \to \infty$ and $b_n = ca_n$ for $c > 1$. Moreover, he showed $\dim_H I_{\underline{a}}^{\underline{b}}(E_\lambda) = 1$ in the following cases:

(a) $a_n = c_1 R^n$ and $b_n = c_2 R^n$, $c_1, c_2 > 0$, $R > 1$

(b) $a_n = n^{(\log^+)^p(n)}$ and $b_n = R^n$, where $(\log^+)^p$ denotes the p-th iterate of $\log^+$, for $p \in \mathbb{N}$, $R > 1$,

(c) $a_n = e^{n(\log^+)^p(n)}$ and $b_n = e^{e^{pn}}$ for $p \in \mathbb{N}$,

(d) $\frac{\log a_{n+1}}{\log(a_1 \cdots a_n)} = 0$, $b_n = ca_n$ for large $c > 1$.

Remark

In the cases (a)–(b) the sets $I_{\underline{a}}^{\underline{b}}(E_\lambda)$ are contained in the slow escaping set, while in the cases (c)–(d) they are in the moderately slow escaping set.

Remarks

Points with bounded and unbounded trajectories

Let

$$K(f) = \{z \in J(f) : \{f^n(z)\}_{n=1}^{\infty} \text{ is bounded}\}.$$

- $\dim_H(K(E_\lambda)) > 1$ (Karpińska 1999)
- $\dim_H(J(E_\lambda) \setminus I(E_\lambda)) \in (1,2)$ for hyperbolic exponential maps (Urbański and Zdunik 2003)
- $\dim_H(K(f)) > 1$ for $f \in \mathcal{B}$ (B, Karpińska and Zdunik 2009)
- $\dim_H(J(f) \setminus (I(f) \cup K(f)) > 1$ for $f \in \mathcal{B}$ (Osborne and Sixsmith 2016)

Symbolic itineraries

In 2006 Karpińska and Urbański computed the Hausdorff dimension of subsets of $A(E_\lambda)$ consisting of points whose symbolic itineraries (describing the imaginary part of $f^n(z)$) grow to infinity at a given rate. Possible values of dimension cover $[1,2]$.

Setup

We consider **non-autonomous exponential iteration**

$$E_{\underline{\lambda}} = (E_{\lambda_n} \circ \cdots \circ E_{\lambda_1})_{n=1}^{\infty}$$

for $\underline{\lambda} = (\lambda_n)_{n=1}^{\infty} \subset \Lambda^{\mathbb{N}}$, where $\Lambda \subset \mathbb{C} \setminus \{0\}$. We assume that $\overline{\Lambda}$ is a compact set in $\mathbb{C} \setminus \{0\}$ and set

$$\lambda_{min} = \inf_{\lambda \in \Lambda} |\lambda|, \qquad \lambda_{max} = \sup_{\lambda \in \Lambda} |\lambda|.$$

For $\underline{a} = (a_n)_{n=1}^{\infty}$, $\underline{b} = (b_n)_{n=1}^{\infty}$ with $0 < a_n \leq b_n$ we consider

$$I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = \{z : a_n \leq |E_{\lambda_n} \circ \cdots \circ E_{\lambda_1}(z)| \leq b_n \text{ for large } n \in \mathbb{N}\}.$$

Remark

The sequences $\underline{a}$ and $\underline{b}$ need not tend to infinity and need not be increasing. We only assume $(a_1 \cdots a_n)^{\frac{1}{n}} \to \infty$.

Definition

A sequence $(a_n)_{n=1}^{\infty}$ is **admissible**, if $a_n > 100\lambda_{max}$ and $a_{n+1} \leq |\lambda_{n+1}|e^{qa_n}$ for large n and a constant $q < 1$.
If $a_n \to \infty$, then the condition reduces to $a_{n+1} \leq e^{qa_n}$, $q < 1$.

We study the Hausdorff ($\dim_H$) and packing ($\dim_P$) dimension of the sets $I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})$.

Remark

We have

$$\dim_H \leq \dim_P .$$

Moreover,

$$\overline{\dim}_B(I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) \cap \mathbb{D}(0,r)) = \dim_P I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})$$

for every large $r > 0$, where $\overline{\dim}_B$ denotes the upper box dimension (Rippon and Stallard 2005).

Theorem (B and Karpińska 2020)

If $a_n > 100\lambda_{max}$ *for large* n *and* $\liminf_{n\to\infty} \left(\frac{\log \frac{b_{n+1}}{a_{n+1}}}{a_1 \cdots a_n} \right)^{\frac{1}{n}} = 0$, *then*

$$\dim_H I_{\frac{b}{a}}(E_{\underline{\lambda}}) \leq 1.$$

In particular, this holds provided

$$\lim_{n\to\infty} (a_1 \cdots a_n)^{1/n} = \infty \qquad \textit{and} \qquad \liminf_{n\to\infty} \frac{\log \log \frac{b_{n+1}}{a_{n+1}}}{\log(a_1 \cdots a_n)} < 1$$

or

$$\limsup_{n\to\infty} (a_1 \cdots a_n)^{1/n} = \infty \qquad \textit{and} \qquad \limsup_{n\to\infty} \frac{\log \log \frac{b_{n+1}}{a_{n+1}}}{\log(a_1 \cdots a_n)} < 1.$$

Remark

In Theorem 8 we can allow $\lambda_{min} = 0$.

Theorem (B and Karpińska 2020)

If $(a_n)_{n=1}^{\infty}$ *is admissible,* $\lim_{n\to\infty}(a_1\cdots a_n)^{1/n}=\infty$ *and* $b_n \geq c a_n$ *for* $c>1$, *then*

$$1+\inf_x \liminf_{n\to\infty} \phi_n(x) \leq \dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) \leq 1+\sup_x \liminf_{n\to\infty} \phi_n(x),$$
$$1+\inf_x \limsup_{n\to\infty} \psi_n(x) \leq \dim_P I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) \leq 1+\sup_x \limsup_{n\to\infty} \psi_n(x),$$

where $x=(x_1,x_2,\ldots)\in[a_1,b_1]\times[a_2,b_2]\times\cdots$ *and*

$$\phi_n(x)=\frac{\log\left(\min(\log\frac{b_2}{a_2},x_1)\cdots\min(\log\frac{b_n}{a_n},x_{n-1})\right)}{\log(x_1\cdots x_n)-\log\min(\log\frac{b_{n+1}}{a_{n+1}},x_n)},$$
$$\psi_n(x)=\frac{\log\left(\min(\log\frac{b_2}{a_2},x_1)\cdots\min(\log\frac{b_{n+1}}{a_{n+1}},x_n)\right)}{\log(x_1\cdots x_n)}.$$

Corollary

If $(a_n)_{n=1}^{\infty}$ *is admissible,* $\lim_{n\to\infty}(a_1\cdots a_n)^{1/n}=\infty$ *and* $b_n\geq ca_n$ *for* $c>1$*, then*

$$1\leq \dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})\leq 1+\liminf_{n\to\infty}\frac{\log\left(\log\frac{b_1}{a_1}\cdots\log\frac{b_n}{a_n}\right)}{\log(a_1\cdots a_{n-1})+\log^{+}\frac{a_n}{\log(b_{n+1}/a_{n+1})}},$$

$$1\leq \dim_P I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})\leq 1+\limsup_{n\to\infty}\frac{\log\left(\log\frac{b_1}{a_1}\cdots\log\frac{b_{n+1}}{a_{n+1}}\right)}{\log(a_1\cdots a_n)}.$$

If, additionally, $\log\frac{b_{n+1}}{a_{n+1}}\leq Ca_n$ *for* $C>1$ (*e.g. if* $b_n\leq a_n^C$) *for large* n*, then*

$$\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})=1,$$

$$\dim_P I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})\geq 1+\limsup_{n\to\infty}\frac{\log\left(\log\frac{b_1}{a_1}\cdots\log\frac{b_{n+1}}{a_{n+1}}\right)}{\log(b_1\cdots b_n)}.$$

Corollary

Suppose $(a_n)_{n=1}^{\infty}$ *is admissible,* $\lim_{n\to\infty}(a_1\cdots a_n)^{1/n}=\infty$ *and* $b_n \geq c a_n$ *for* $c>1$.

(a) *If* $\lim_{n\to\infty}\frac{\log\log\frac{b_{n+1}}{a_{n+1}}}{\log a_n}=0$, *then* $\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})=\dim_P I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})=1$.

(b) *If* $\liminf_{n\to\infty}\frac{\log\log\frac{b_{n+1}}{a_{n+1}}}{\log a_n}<1$, *then* $\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})=1$.

(c) *If* $\liminf_{n\to\infty}\frac{\log\log\frac{b_{n+1}}{a_{n+1}}}{\log b_n}\geq 1$, *then* $\dim_P I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})=2$.

(d) *If* $\liminf_{n\to\infty}\frac{\log\log\frac{b_{n+1}}{a_{n+1}}}{\log b_n}>1$, *then* $\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})=\dim_P I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})=2$.

Remark

The assertions (b)–(c) imply that if $\dim_P I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})<2$, then $\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})=1$.

Moderately slowly escaping points

Corollary

(a) *If* $a_n > 100\lambda_{max}$ *for large* n, $\lim_{n\to\infty}(a_1 \cdots a_n)^{\frac{1}{n}} = \infty$ *and* $\liminf_{n\to\infty}(\log b_n)^{1/n} < \infty$, *then* $\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) \leq 1$.

(b) *If, additionally,* $(a_n)_{n=1}^{\infty}$ *is admissible and* $b_n \geq ca_n$ *for* $c > 1$, *then* $\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = 1$.

In particular, if $(a_n)_{n=1}^{\infty}$ *is admissible,* $b_n \geq ca_n$ *for* $c > 1$ *and* $I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}})$ *is contained in the moderately slow escaping set*

$$M(E_{\underline{\lambda}}) = \{z \in I(E_{\underline{\lambda}}) : \limsup_{n\to\infty} \frac{1}{n} \log\log |E_{\lambda_n} \circ \cdots \circ E_{\lambda_1}(z)| < \infty\},$$

then $\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = 1$.

Points with exact growth rate

Definition

We say that the iterations of a point $z \in \mathbb{C}$ under $E_{\underline{\lambda}}$ have **growth rate** $\underline{a} = (a_n)_{n=1}^{\infty}$, if $z \in I_{\underline{a}/c}^{c\underline{a}}(E_{\underline{\lambda}})$ for some constant $c > 1$, i.e.

$$\frac{a_n}{c} \leq |E_{\lambda_n} \circ \cdots \circ E_{\lambda_1}(z)| \leq c a_n$$

for large n.

Corollary

(a) *If* $\underline{a} = (a_n)_{n=1}^{\infty}$ *is admissible and* $\lim_{n\to\infty} (a_1 \cdots a_n)^{1/n} = \infty$*, then the set of points with growth rate* $\underline{a}$ *has Hausdorff dimension* 1*.*

(b) *If* $\underline{a} = (a_n)_{n=1}^{\infty}$ *is admissible and* $\lim_{n\to\infty} a_n = \infty$*, then the set of points with growth rate* $\underline{a}$ *has Hausdorff and packing dimension* 1*.*

Precise dimension formulas

Theorem

If $(a_n)_{n=1}^{\infty}$ *is admissible,* $\lim_{n\to\infty}(a_1 \cdots a_n)^{1/n} = \infty$, $b_n \geq c a_n$ *for* $c > 1$ *and* $\lim_{n\to\infty} \frac{\log b_n}{\log a_n} = 1$, *then*

$$\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = 1,$$

$$\dim_P I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = 1 + \limsup_{n\to\infty} \frac{\log\left(\log \frac{b_1}{a_1} \cdots \log \frac{b_{n+1}}{a_{n+1}}\right)}{\log(a_1 \cdots a_n)}.$$

Theorem

For every $D \in [1,2]$ *there exist* $\underline{a} = (a_n)_{n=1}^{\infty}$, $\underline{b} = (b_n)_{n=1}^{\infty}$ *with* $a_n \to \infty$, $b_n \geq a_n$, *such that*

$$\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = 1, \qquad \dim_P I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = D.$$

Proof.

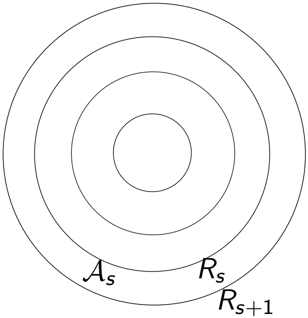
If $a_{n+1} = e^{n a_n^d}$ for $d \in [0,1)$, $b_n = a_n^{1+\frac{1}{n}}$, then
$\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = 1$, $\dim_P I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = 1 + d$.

If $a_{n+1} = e^{n a_n^{(n-1)/n}}$, $b_n = a_n^{1+\frac{1}{n}}$, then
$\dim_H I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = 1$, $\dim_P I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = 2$. □

Annular itineraries

Let $f: \mathbb{C} \to \mathbb{C}$ be a transcendental entire map, R_s, $s \geq 0$, be a sequence of positive numbers increasing to infinity and

$$\mathcal{A}_s = \{z \in \mathbb{C} : R_s \leq |z| < R_{s+1}\}.$$



Definition (Rippon and Stallard 2015)

Annular itinerary of a point $z \in \mathbb{C}$ is a sequence $\underline{s}(z) = (s_n)_{n=0}^{\infty}$ defined by the condition $f^n(z) \in \mathcal{A}_{s_n}$, $n \geq 0$.

Annular itineraries in non-autonomous iteration

$$\underline{s}(z) = (s_n)_{n=0}^{\infty} \iff E_{\lambda_n} \circ \cdots \circ E_{\lambda_1}(z) \in \mathcal{A}_{s_n}.$$

For given symbolic sequence $\underline{s} = (s_n)_{n=0}^{\infty}$ let

$$\mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = \{z \in \mathbb{C} : \underline{s}(z) = \underline{s}\}$$

Fact

$$\mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) \qquad \textit{for} \quad a_n = R_{s_n}, \quad b_n = R_{s_n+1}.$$

Definition

We say that a sequence $\underline{s} = (s_n)_{n=0}^{\infty}$ is **admissible**, if the sequence $(a_n)_{n=1}^{\infty}$, $a_n = R_{s_n}$, is admissible.

Case $R_s = R^s$

$$\mathcal{A}_s = \{z \in \mathbb{C} : R^s \leq |z| < R^{s+1}\} \qquad \text{for a large } R > 1.$$

Theorem

(a) *If* $\limsup_{n\to\infty} \frac{s_1+\cdots+s_n}{n} = \infty$, *then* $\dim_H \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) \leq 1$.

(b) *If* $\underline{s}$ *is admissible and* $\lim_{n\to\infty} \frac{s_1+\cdots+s_n}{n} = \infty$, *then* $\dim_H \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = \dim_P \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = 1$.

Proof.

$$\dim \mathcal{I}_{\underline{s}}(f_{\underline{\lambda}}) = \dim I_{\underline{a}}^{\underline{b}}(f_{\underline{\lambda}}) \qquad \text{for} \quad a_n = R^{s_n},\ b_n = R^{s_n+1}.$$ □

Remark

This answers a question from [Sixsmith 2016].

Case $R_s = R^{s^\kappa}$

$$\mathcal{A}_s = \{z \in \mathbb{C} : R^{s^\kappa} \le |z| < R^{(s+1)^\kappa}\} \quad \text{for a large } R > 1,\ \kappa > 1.$$

Theorem

(a) *If* $\limsup_{n\to\infty} \frac{s_1^\kappa + \cdots + s_n^\kappa}{n} = \infty$ *and* $\underline{s}$ *is admissible, then* $\dim_H \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) \le 1$.

(b) *If* $\lim_{n\to\infty} s_n = \infty$ *and* $\underline{s}$ *is admissible, then:*

$$\dim_H \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = 1,$$

$$\dim_P \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = 1 + \frac{\kappa - 1}{\log R} \limsup_{n\to\infty} \frac{\log s_{n+1}}{s_1^\kappa + \cdots + s_n^\kappa},$$

$$\dim_P \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) < 2 - \frac{1}{\kappa}.$$

Proof.

$\dim \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = \dim I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) \quad \text{for} \quad a_n = R^{s_n^\kappa},\ b_n = R^{(s_n+1)^\kappa}.$ □

Theorem

For every $D \in [1, 2 - \frac{1}{\kappa})$ *there exists a sequence* $\underline{s} = (s_n)_{n=0}^{\infty}$ *with* $s_n \to \infty$ *such that*

$$\dim_H \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = 1, \qquad \dim_P \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = D.$$

Proof.

If $s_{n+1} = R^{\frac{d}{\kappa-1} s_n^{\kappa}}$ for $d \in [0, 1 - \frac{1}{\kappa})$, then $\underline{s}$ is admissible and $\dim_H \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = 1$, $\dim_P \mathcal{I}_{\underline{s}}(E_{\underline{\lambda}}) = 1 + d$. □

Proofs of main theorems – preliminaries

For a fixed $N \geq 0$ let

$$A_n = \log \frac{a_{N+n}}{|\lambda_{N+n}|}, \qquad B_n = \log \frac{b_{N+n}}{|\lambda_{N+n}|}, \qquad \Delta_n = B_n - A_n,$$

$$S_n = \{z : A_n \leq \operatorname{Re}(z) \leq B_n\}$$

Fact

$$a_{N+n} \leq |E_{\lambda_{N+n}}(z)| \leq b_{N+n} \iff z \in S_n.$$

Consequently, $\dim I_{\underline{a}}^{\underline{b}}(E_{\underline{\lambda}}) = \lim_{N\to\infty} \dim J_N = \sup_N \dim J_N$ *for*

$$J_N = \{z : E_{\lambda_{N+n}} \circ \cdots \circ E_{\lambda_N}(z) \in S_{n+1} \textit{ for } n \geq 0\}.$$

Proofs of main theorems – notation

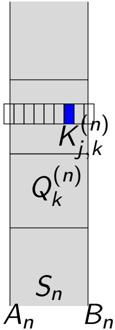
For a small fixed $\delta > 0$, $j \geq 0$, $k \in \mathbb{Z}$ let

$$K_{j,k}^{(n)} = [j\delta, (j+1)\delta) \times \Big[-\frac{\pi}{2} - \operatorname{Arg}\lambda_{N+n} + 2k\pi, \frac{\pi}{2} - \operatorname{Arg}\lambda_{N+n} + 2k\pi \Big],$$

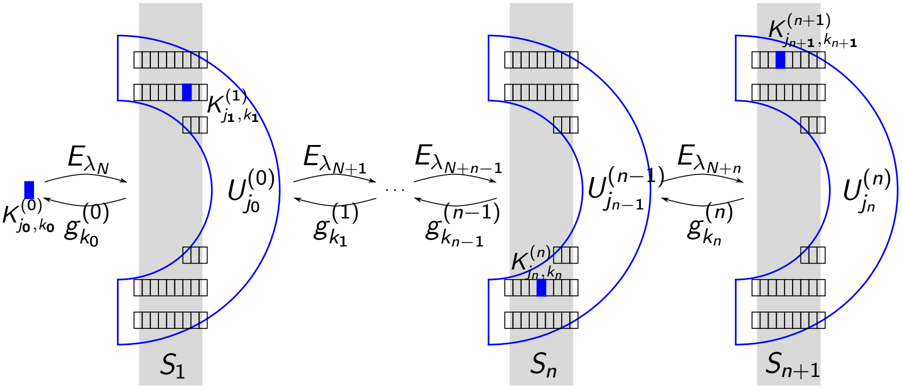
$$U_j^{(n)} = E_{\lambda_{N+n}}(K_{j,k}^{(n)}) = \{z : |\lambda_{N+n}|e^{j\delta} \leq |z| < |\lambda_{N+n}|e^{(j+1)\delta},\ \operatorname{Re}(z) \geq 0\}.$$

$$Q_k^{(n)} = \{z \in \mathbb{C} : A_n \leq \operatorname{Re}(z) \leq B_n,\ \Delta_n k \leq \operatorname{Im}(z) \leq \Delta_n(k+1)\},$$

$g_k^{(n)}$ inverse branches of $E_{\lambda_{N+n}}$.



Proofs of the main theorems – construction



Thank you for attention!