Triply factorised groups and the structure of skew left braces [1]

Adolfo Ballester-Bolinches[1]

Departament de Matemàtiques
Universitat de València

Bedlewo, July, 2022

[1] Joint work with R. Esteban-Romero

Introduction

Aim

Main aim

We present some results that can be regarded as a contribution to the study of the algebraic structure of skew left braces. Such structure has proved to be useful as a source of set-theoretic solutions of the Yang-Baxter equation.

Introduction

Aim

Approach

Description of a skew left brace in terms of a triply factorised group obtained from the action of the multiplicative group on the additive group.

Skew left braces

Basic concepts

Definition

A skew left brace is a set B with two binary operations, $+$ and $\cdot$, such that $(B, +)$ is a group, $(B, \cdot)$ is a group, and

$$a(b + c) = ab - a + ac \qquad \text{for all } a, b, c \in B. \qquad (*)$$

L. Guarnieri and L. Vendramin.
Skew-braces and the Yang-Baxter equation.
Math. Comp., 86(307):2519–2534, 2017.

Skew left braces are extremely useful to produce and study bijective non-degenerate set-theoretic solutions of the Yang-Baxter equation.

Skew left braces

Basic concepts

- An example: B is a trivial skew left brace if $(B,+)$ is a group and the operations $+$ and $\cdot$ coincide.
- If $\mathcal{X}$ is a class of groups, we shall say that a skew left brace is of $\mathcal{X}$-type of $(B,+)$belongs to $\mathcal{X}$.
- Rump's left braces are exactly the skew left braces of abelian type.

W. Rump.
Braces, radical rings, and the quantum Yang-Baxter equation.
J. Algebra, **307**, 153–170 (2007).

Skew left braces

Basic concepts

Proposition

If B is a skew left brace, we have an action $\lambda\colon (B,\cdot) \longrightarrow \mathrm{Aut}(B,+)$ defined by $\lambda(a) = \lambda_a$, where

$$\lambda_a(b) = -a + ab \qquad \text{for all } a, b \in B.$$

This is called the lambda map of B.

Note that

- $a + \lambda_a(b) = ab$ for all $a, b \in B$,
- $a + b = a \cdot \lambda_{a^{-1}}(b) = a \cdot \lambda_a^{-1}(b)$ for all $a, b \in B$.

Skew left braces

Skew left braces and derivations

Suppose that a group $(C, \cdot)$ acts on a group $(K, +)$ by means the group homomorphism $\lambda\colon C \longrightarrow \mathrm{Aut}(K)$.
A derivation associated to λ is a map $\delta\colon C \longrightarrow K$ satisfying the following equation:

$$\delta(ce) = \delta(c) + \lambda_c(\delta(e)), \qquad c, e \in C.$$

If B is a brace, the identity map $id_B\colon (B, \cdot) \longrightarrow (B, +)$ is a derivation associated to lambda map.

Skew left braces

Skew left braces and derivations

Theorem

Suppose that there exists an action $\lambda\colon (C,\cdot) \longrightarrow \mathrm{Aut}(K,+)$ *and that* $\delta\colon (C,\cdot) \longrightarrow (K,+)$ *is a bijective derivation with respect to* λ*. Then we can define an addition on* C *via* $b + c = \delta^{-1}(\delta(b) + \delta(c))$ *and* $(C,+,\cdot)$ *becomes a skew left brace.*

Skew left braces

Skew left braces and derivations

Lemma

Let $(C, \cdot)$ and $(K, +)$ be two groups.
Suppose that $\delta\colon C \longrightarrow K$ is a derivation associated to an action λ of C on K and that L is a C-invariant subgroup of K (for instance, this happens when L is a characteristic subgroup of K).
Then $\delta^{-1}(L)$ is a subgroup of C.

Skew left braces

Skew left braces and derivations

Assume that $(K, +)$ is finite and nilpotent and δ is bijective. Then

- For every set of primes π, K has a characteristic Hall π-subgroup K_π.
- Then $C_\pi = \delta^{-1}(K_\pi)$ is a Hall π-subgroup of $C = (B, \cdot)$.
- Therefore C is soluble.

Theorem (see Etingof, Schedler, Soloviev)

The multiplicative group $(B, \cdot)$ of a finite brace of nilpotent type is soluble.

Skew left braces

Skew left braces and derivations

P. Etingof, T. Schedler, A. Soloviev.
Set-theoretical solutions to the quantum Yang-Baxter equation
Duke Math. J., **100**, 169–209 (1999).

Triply factorised groups

Skew left braces and trifactorised groups

Suppose that a group $(C, \cdot)$ acts on a group $(K, +)$ by means the group homomorphism $\lambda\colon C \longrightarrow \mathrm{Aut}(K)$.
Let us consider the corresponding semidirect product

$$G = [K]C = \{(k, c) \mid k \in K, c \in C\}.$$

Note that $(k_1, c_1)(k_2, c_2) = (k_1 + \lambda_{c_1}(k_2), c_1 c_2)$, k_1, $k_2 \in K$, c_1, $c_2 \in C$.

Triply factorised groups

Skew left braces and trifactorised groups

Assume that $\delta\colon C \longrightarrow K$ be is a bijective derivation associated to λ. Consider

$$D = \{(\delta(c), c) \mid c \in C\}.$$

Lemma

The set D is a subgroup of $G = [K]C$ such that $G = KD = DC$ and $K \cap D = D \cap C = 1$.

We obtain that G is a trifactorised group.
Note that $\alpha\colon C \longrightarrow D$ given by $\alpha(c) = (\delta(c), c)$, $c \in C$, is a group isomorphism.

Triply factorised groups

Skew left braces and trifactorised groups

Theorem

Suppose that C is a finite nilpotent group. Then K is soluble.

- Since $C \cong D$, C and D are nilpotent
- By a result of Kegel and Wielandt, $G = CD$ is soluble.
- Hence $K \leqslant G$ is soluble.

Triply factorised groups

Skew left braces and trifactorised groups

Theorem

Assume that $\mathcal{F}$ is a saturated formation of finite groups and $G = KC = KD = DC$ is a finite group. Suppose further that K is a normal nilpotent subgroup of G. Then C and D belongs to $\mathcal{F}$ if and only if G belongs to $\mathcal{F}$.

Triply factorised groups

Skew left braces and trifactorised groups

Theorem

Suppose that $G = [K]C = KD = DC$ is a trifactorised group such that $K \cap D = D \cap C = \{(0, 1)\}$. Then there exists a bijective derivation $\delta \colon C \longrightarrow K$ associated with the action of C on K such that $D = \{(\delta(c), c) \mid c \in C\}$.

Triply factorised groups

Skew left braces and trifactorised groups

In the following, we will use multiplicative notation in the semidirect product. Given $k, l \in K$ and $c \in C$,

- $(k + l, 1) \mapsto kl$,
- $(-k, 1) \mapsto k^{-1}$,
- $(\lambda_c(k), 1) \mapsto ckc^{-1} = k^{c^{-1}}$ (here $u^g = g^{-1}ug$).

Triply factorised groups

Skew left braces and trifactorised groups

Theorem

Let $G = [K]C = KD = DC$ with $D \leqslant G$, $K \cap D = D \cap C = \{1\}$, $\delta \colon C \longrightarrow K$, the corresponding derivation. Suppose that $E \leqslant C$ and $L = \delta(E) \trianglelefteq G$. Then the following are equivalent:

1. $E \trianglelefteq C$.
2. $[E, C] \subseteq E$.
3. $[K, E] \subseteq L$.

Triply factorised groups

Skew left braces and trifactorised groups

Lemma

Let $G = [K]C = KD = DC$ with $D \leqslant G$, $K \cap D = D \cap C = \{1\}$, and let $\delta \colon C \longrightarrow K$ be the corresponding derivation. Suppose that H is a subgroup of K such that H is normalised by C. Let $c, e \in C$, $k = \delta(c)$, $l = \delta(e)$. Suppose that three of the elements $[k, e]$, $[k, l]$, $[c, l]$ and $\delta([c, e])$ belong to H. Then so does the other one.

Triply factorised groups

Substructures of skew left braces

Rump noted that two-side braces of abelian type correspond to radical rings. Hence braces can be regarded as generalisations of radical rings and techniques of ring theory may be applied to some extend.

Triply factorised groups

Substructures of skew left braces

Definition

Let B be a brace.
Define $a * b = -a + ab - b = \lambda_a(b) - b$, $a, b \in B$.

If a is regarded as an element of $C = (B, \cdot)$ and b is regarded as an element of $K = (B, +)$, then $a * b$ corresponds in $G = [K]C$ to

$$aba^{-1}b^{-1} = [a^{-1}, b^{-1}] \in [C, K] \subseteq K.$$

Triply factorised groups

Substructures of skew left braces

Definition

If X, $Y \subseteq B$, $X * Y$ is the subgroup of K generated by $\{x * y \mid x \in X, y \in Y\}$.

If X corresponds to a subgroup E of C and Y to a subgroup H of K, this can be identified with the subgroup

$$\langle \{[e^{-1}, h^{-1}] \mid e \in E, h \in H\} \rangle = [E, H] \leqslant K.$$

Triply factorised groups

Substructures of skew left braces

Theorem

Let B be a brace of abelian type. Then B a two-sided brace if and only if $(B, +, *)$ is a radical ring.

Triply factorised groups

Substructures of skew left braces

Definition

A subgroup I of K is said to be a left ideal if $\lambda_a(I) \subseteq I$ for all $a \in B$, or equivalently, if $B * I$ is a subgroup of I.
Moreover, the left ideal I is called a strong left ideal if I is a normal subgroup of K.

A. Konovalov, A. Smoktunowicz, and L. Vendramin.
On skew braces and their ideals.
Exp. Math., p. 110 (2018).

E. Jespers, L. Kubat, A. Van Antwerpen, and L. Vendramin.
Factorizations of skew braces.
Math. Ann., **375**, 1649–1663 (2019).

Triply factorised groups

Substructures of skew left braces

- If I is a left ideal of B, corresponding to $L \leqslant K$, then L is C-invariant and so $E = \delta^{-1}(L) \leqslant C$ and $[L, C] \subseteq L$.
- If I is a strong left ideal of B, then $L \trianglelefteq G$.

Triply factorised groups

Substructures of skew left braces

Definition

An ideal of B is a left ideal I of B such that $aI = Ia$ and $a + I = I + a$ for all $a \in B$.

L. Guarnieri and L. Vendramin.
Skew-braces and the Yang-Baxter equation.
Math. Comp., 86(307):2519–2534, 2017.

Triply factorised groups

Substructures of skew left braces

- Ideals of skew left braces are true analogues of normal subgroups in groups and ideals in rings. In fact, if I is an ideal of B, we can construct the quotient skew left brace B/I.
- Suppose that the left ideal I corresponds to $L \leqslant K$ and to $E = \delta^{-1}(L) \leqslant C$.
  Then I is an ideal of B if, and only if, $LE \trianglelefteq G$.

Triply factorised groups

Nilpotency

We have seen that the star operation on a brace B can be considered as the commutator operation on the trifactorised group associated with B.
As nilpotency in groups can be defined in terms of iterated commutators, it seems natural to try to define nilpotency and some generalisations of nilpotency in braces in terms of iterated star operations.

Triply factorised groups

Nilpotency

We define inductively:

$$\mathrm{L}_0(X,Y) = Y; \qquad \mathrm{L}_n(X,Y) = X * \mathrm{L}_{n-1}(X,Y) \quad (n \geqslant 1);$$
$$\mathrm{R}_0(X,Y) = X, \qquad \mathrm{R}_n(X,Y) = \mathrm{R}_{n-1}(X,Y) * Y \quad (n \geqslant 1);$$

We have that

$$\mathrm{L}_n(X,Y) = [[Y,X],X],\ldots,X] = [Y,X,\ldots,X] \quad (X \text{ appears } n \text{ times})$$

in the semidirect product $G = [K]C$, where X is regarded as a subgroup of C and Y as a subgroup of K.
We also note that $\mathrm{L}_n(B,B) = B^{n+1}$ for all n, the radical series of B defined by Rump in his 2007 paper.

Triply factorised groups

Nilpotency

Definition

A left brace B is called left nilpotent if $L_n(B, B) = 0$ for some n.

Theorem (Smoktunowicz)

A finite left brace of abelian type is left nilpotent if and only if the multiplicative group $(B, \cdot)$ is nilpotent.

A. Smoktunowicz.
On Engel groups, nilpotent groups, rings, braces and Yang-Baxter equation.
Trans. Amer. Math. Soc., **370**(9), 6535–6564 (2018).

Triply factorised groups

Nilpotency

Smoktunovicz's result still holds for finite braces of nilpotent type as it was shown by Cedó, Smoktunowicz and Vendramin.

F. Cedó, A. Smoktunowicz, and L. Vendramin.
Skew left braces of nilpotent type.
Proc. London. Math. Soc., **118**(6), 1367–1392 (2019).

Triply factorised groups

Nilpotency

In the sequel, we will consider finite braces of nilpotent type.

Definition

We say that B is left π-nilpotent (π a set of primes) if for some n we have that $\mathrm{L}_n(B, B_\pi) = 0$.

Triply factorised groups

Nilpotency

Theorem

Suppose that $C = (B, \cdot)$ has a nilpotent Hall π-subgroup. Then B is left π-nilpotent if and only if C is π-nilpotent.

Triply factorised groups

Nilpotency

Case $\pi = \{p\}$, p a prime.

H. Meng, A. Ballester-Bolinches, and R. Esteban-Romero.
Left braces and the quantum Yang-Baxter equation.
Proc. Edinburgh Math. Soc., **62** (2019), 595–608.

E. Acri, R. Lutowski, L. Vendramin.
Retractability of solutions to the Yang-Baxter equation and p-nilpotency of skew braces.
Internat. J. Algebra Comput., **30** (2020), 91–115.

Triply factorised groups

Nilpotency

Theorem

Suppose that a brace of abelian type B can be decomposed as the sum of two ideals that are left nilpotent as left braces. Then B is left nilpotent.

Triply factorised groups

Nilpotency

Definition

Given a finite left brace B, the left-Fitting ideal l-F(B) of B is the largest ideal that, as a left brace, is left nilpotent. It coincides with the ideal generated by all ideals of B that, as left braces, are left nilpotent.

Triply factorised groups

Nilpotency

Definition

Let B be a skew left brace. B is called right nilpotent if $\mathrm{R}_n(B, B) = 0$ for some n.

Triply factorised groups

Nilpotency

Theorem

(B, r_B) is a multipermutation solution if, and only if, B is of nilpotent type and right nilpotent.

F. Cedó, A. Smoktunowicz and L. Vendramin.
Skew left braces of nilpotent type.
Proc. London Math. Soc., **118** (2019), 1367–1392.

Triply factorised groups

Nilpotency

Definition

If π is a set of primes, we say that B is right π-nilpotent when for some n we have that $\mathrm{R}_n(B_\pi, B) = 0$.

Triply factorised groups

Nilpotency

Theorem

Suppose that B is a skew left brace of nilpotent type, the Hall π-subgroup $G_\pi = K_\pi C_\pi$ of the trifactorised group associated with B is nilpotent, and that C_π is an abelian normal Hall π-subgroup of C. Then B is right π-nilpotent.

Triply factorised groups

Nilpotency

We have not been able to prove or disprove the existence of a right Fitting-like ideal. However, we have:

Theorem

Let B be a left brace that can be factorised as the product of an ideal I_1 that is trivial as a left brace and a strong left ideal I_2 that is right nilpotent as a left brace. Then B is right nilpotent.