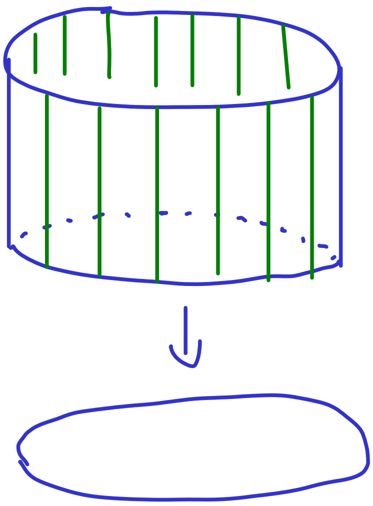
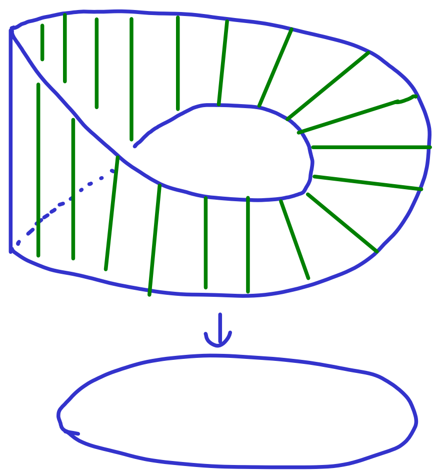
Cohomology, bundles and representations.

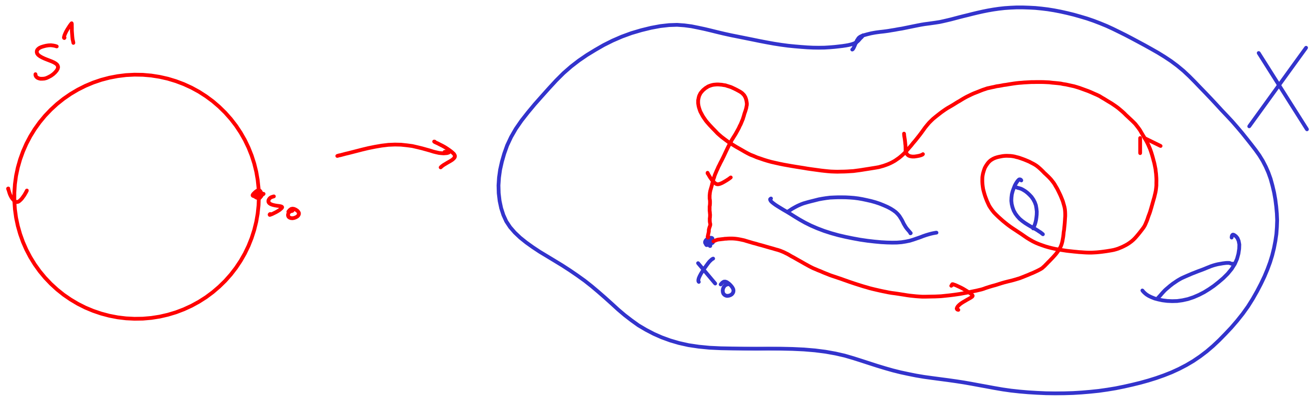
Problem: how can bundles twist over a space?



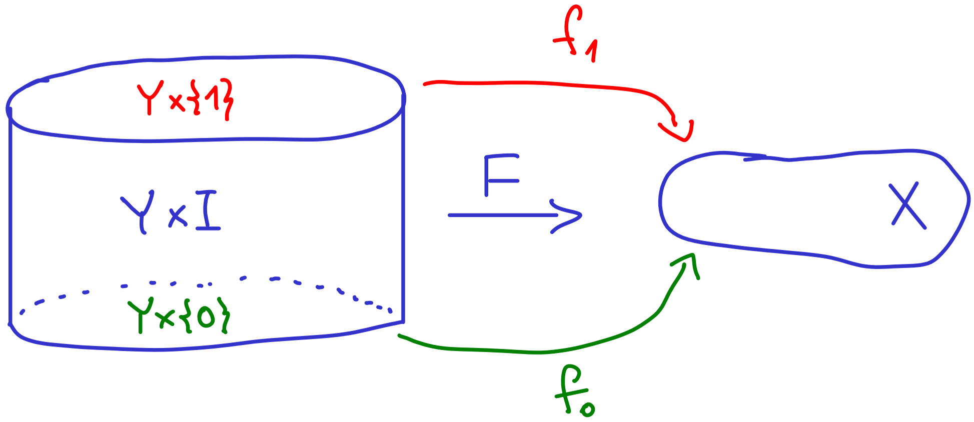


I. Fundamental group.

We consider loops in X (continuous maps $S^1 \to X$, $s_0 \mapsto x_0$) up to homotopy.



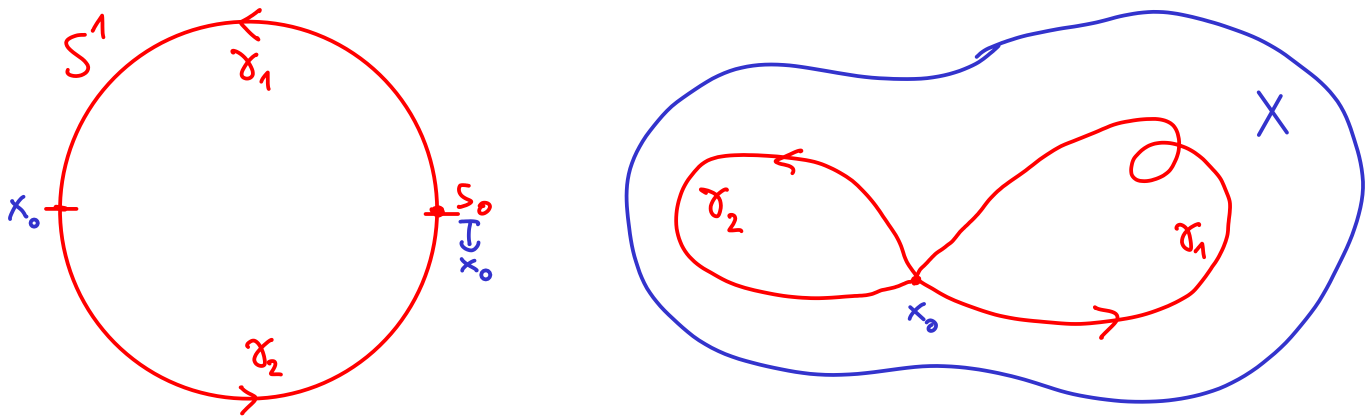
Maps $f_0, f_1 : Y \to X$ are homotopic, if there exists
$F : Y \times [0,1] \to X$ such that $F(y,0) = f_0(y)$, $F(y,1) = f_1(y)$.



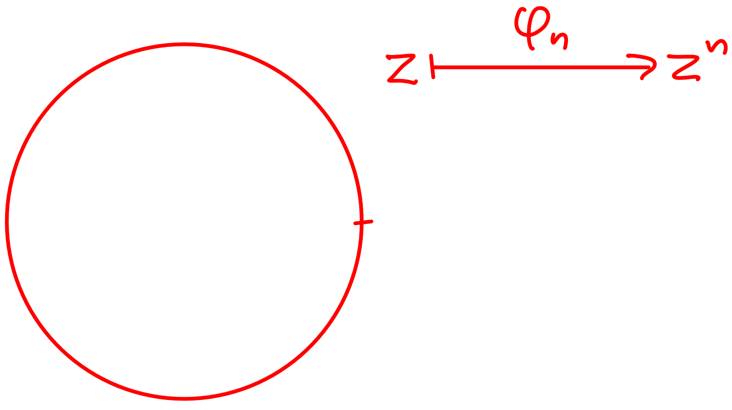
Definition.

The fundamental group of X – $\pi_1(X)$ – is the set of homotopy classes of loops in X.

With the following multiplication this set is a group:

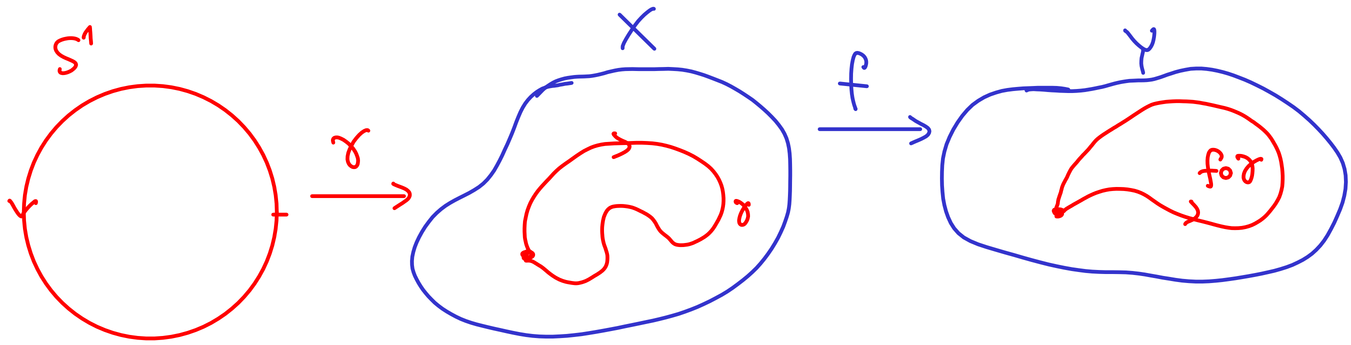


Example: $\pi_1(S^1)$.



Fact: Every map $S^1 \to S^1$ is homotopic to exactly one φ_n. $\pi_1(S^1) \cong \mathbf{Z}$.

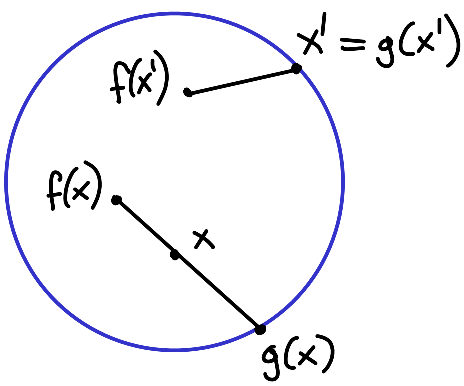
A map $X \to Y$ induces a homomorphism $\pi_1(X) \to \pi_1(Y)$.



Application: Brouwer's fixed point theorem for D^2.

Every (continuous) map $f\colon D^2 \to D^2$ has a fixed point.

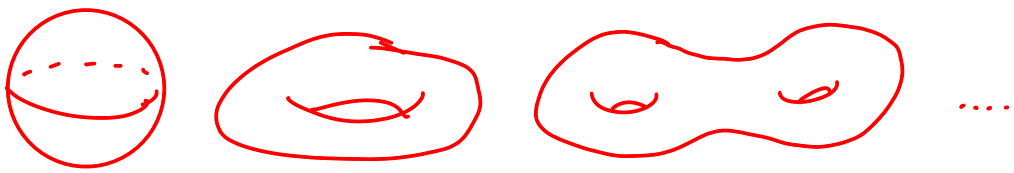
If not:



$$\begin{tikzcd} S^1 \arrow[rr, hook] \arrow[dr, "\mathrm{Id}"'] & & D^2 \arrow[dl, "g"] \\ & S^1 & \end{tikzcd} \quad \overset{\pi_1}{\rightsquigarrow} \quad \begin{tikzcd} \mathbb{Z} \arrow[rr] \arrow[dr, "\mathrm{Id}"'] & \arrow[d, phantom, "\lightning"] & 0 \arrow[dl] \\ & \mathbb{Z} & \end{tikzcd}$$

II. Homology

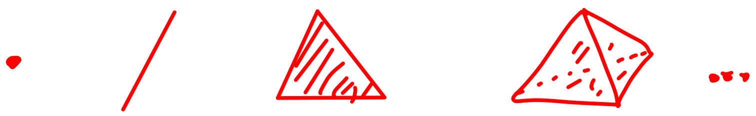
Higher dimensional cycles in X may have different shapes:

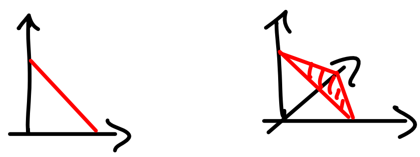


Idea:

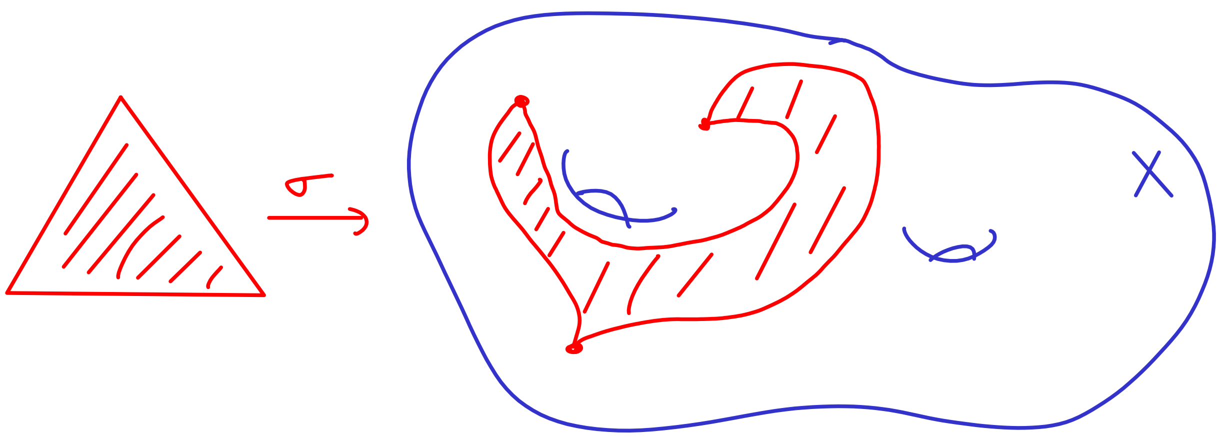
- do not specify shape a priori;
- use simplices to build all possible shapes.

- Simplex: $\Delta^n = \mathrm{Conv}(e_0, e_1, \ldots, e_n)$ in $\mathbf{R}^n$





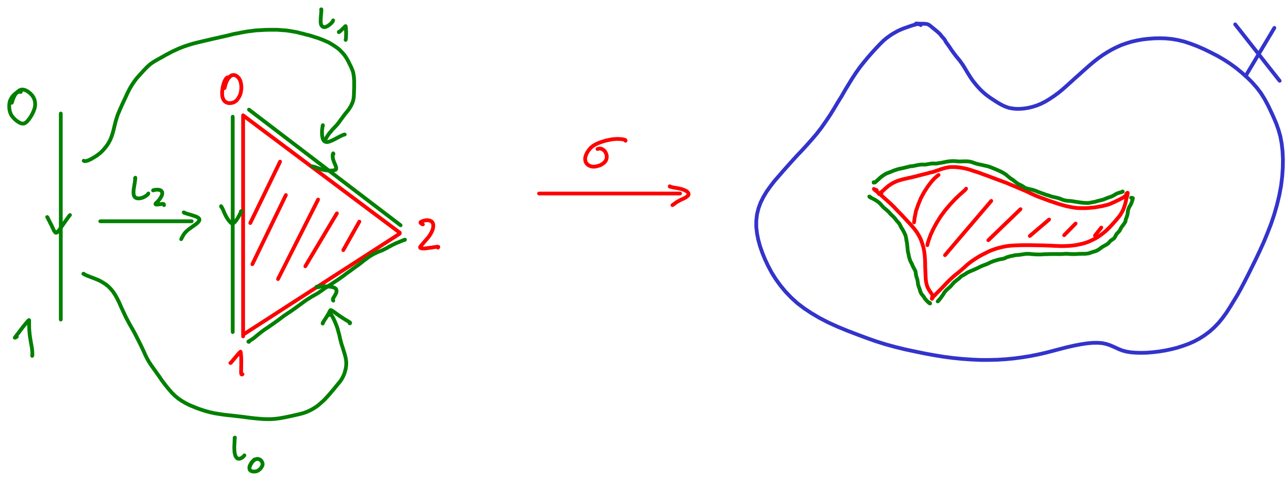
- Singular simplex in X: map $\sigma \colon \Delta^n \to X$.



- n-chain group of X: free abelian group with the set of all singular n-simplices in X as basis.

$$S_n X = \{\sum_{\text{finite}} a_i \sigma_i \mid \sigma_i \colon \Delta^n \to X\}$$

- Boundary $\partial\sigma$ of $\sigma\colon \Delta^n \to X$:
  Δ^n has $n+1$ facets; restrict σ to them and form formal sum.

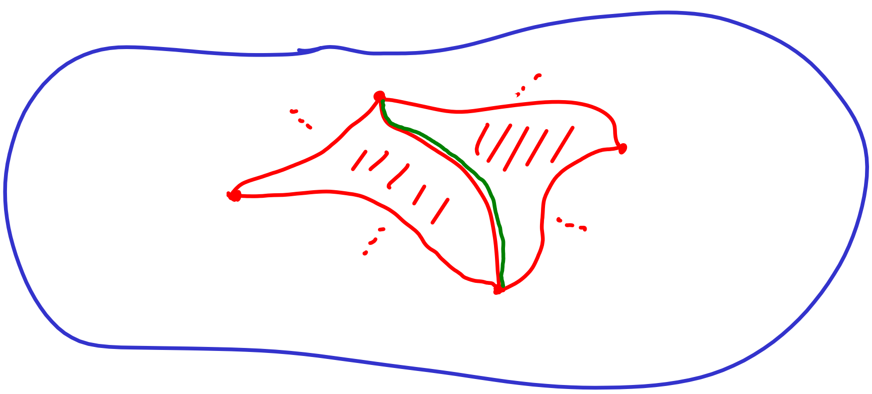


$$\partial\sigma = \sigma \circ \iota_0 - \sigma \circ \iota_1 + \sigma \circ \iota_2 \in S_{n-1}X$$

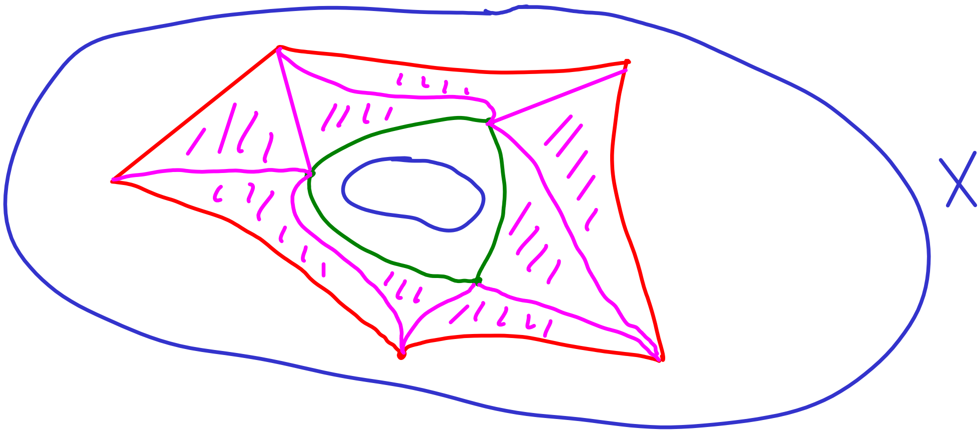
- Extend to a homomorphism $\partial = \partial_n \colon S_n X \to S_{n-1}X$.

$$\partial(\sum a_i \sigma_i) = \sum a_i \partial(\sigma_i)$$

- An n-cycle is a $c \in S_n X$ satisfying $\partial c = 0$.



- Two cycles "detect the same hole" if their difference is a boundary.



Cycles $c, c' \in S_nX$ are called homologous if $c - c' = \partial u$ for some $u \in S_{n+1}X$.

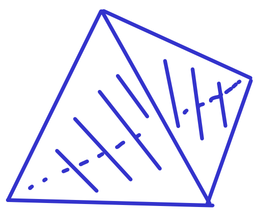
- The n-th homology group of X:

$$H_nX = \text{cycles}/\text{boundaries} = \ker \partial_n / \operatorname{im} \partial_{n+1}$$

- If X is a simplicial complex (spaces glued from simplices along faces), then the simplices from the complex suffice to compute homology.

  Example. S^2

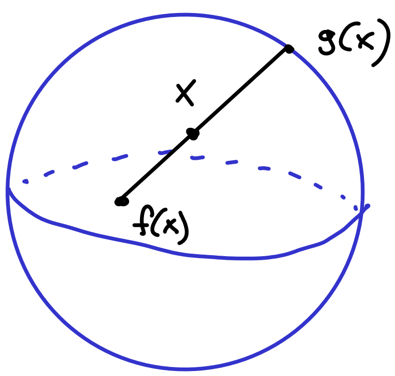
$S^2 \cong \partial\Delta^3$



n:	0	1	2	3	4 ...
H_nS^2:	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0 ...

Fact. For a finite simplicial complex the homology
- is finitely generated;
- vanishes above the dimension of the complex.

- Application: Brouwer for D^n
  Every (continuous) map $f\colon D^n \to D^n$ has a fixed point.



$$\begin{tikzcd} S^{n-1} \arrow[r, hook] \arrow[dr, "\mathrm{Id}"'] & D^n \arrow[d, "g"] \\ & S^{n-1} \end{tikzcd} \quad \overset{H_{n-1}}{\rightsquigarrow} \quad \begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[dr, "\mathrm{Id}"'] & 0 \arrow[d] \\ & \mathbb{Z} \end{tikzcd}$$

III. Cohomology

- n-cochains in X (with coefficients in an abelian group A):
  - maps from the set of singular n-simplices in X to A;
  - homomorphisms $S_nX \to A$

$$S^nX = \mathrm{Hom}(S_nX, A)$$

- Coboundary map: $d = d^n\colon S^nX \to S^{n+1}X$.

$$\begin{tikzcd} S_nX \arrow[d, "S^nX \ni \varphi"'] & S_{n+1}X \arrow[l, "\partial"'] \arrow[dl, "\underset{\overset{!!}{d\varphi}}{\varphi\circ\partial} \in S^{n+1}X"] \\ A & \end{tikzcd}$$

- Cohomology group: $H^n(X, A) = \ker d^n / \operatorname{im} d^{n-1}$.
- Pairing:
  $H_nX \times H^n(X, A) \to A$

$$([c], [\varphi]) \longmapsto \varphi(c) \qquad (= \langle [c], [\varphi] \rangle)$$

- Maps: $f\colon X \to Y$ induces $f^*\colon H^n(Y,A) \to H^n(X,A)$.

$$\begin{tikzcd} X \arrow[d, "f"'] & S_n X \arrow[d] \arrow[dr, "f^*\varphi"] & \\ Y & S_n Y \arrow[r, "\varphi"'] & A \end{tikzcd}$$

- Multiplication (A-ring).

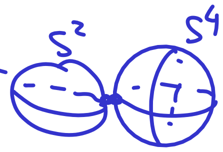
$$\alpha \in H^k(X,A), \beta \in H^\ell(X,A) \quad \longrightarrow \quad \alpha \times \beta \in H^{k+\ell}(X \times X, A)$$
$$\downarrow$$
$$\alpha \cup \beta \in H^{k+\ell}(X,A)$$

$$\Delta : \begin{array}{rcl} X & \longrightarrow & X \times X \\ x & \longmapsto & (x,x) \end{array}$$

$$\alpha \cup \beta = \Delta^*(\alpha \times \beta)$$

- Example: $\mathbf{C}P^2$ vs $S^2 \vee S^4$.

$\uparrow$
1-dim subspaces of $\mathbb{C}^2$



n:	0	1	2	3	4	5	6	...
$\mathbb{C}P^2$:	$\mathbb{Z}$	0	$\mathbb{Z}$ x	0	$\mathbb{Z}$ x^2	0	0	...
$S^2 \vee S^4$:	$\mathbb{Z}$	0	$\mathbb{Z}$ y	0	$\mathbb{Z}$ z $y^2 = 0$	0	0	...

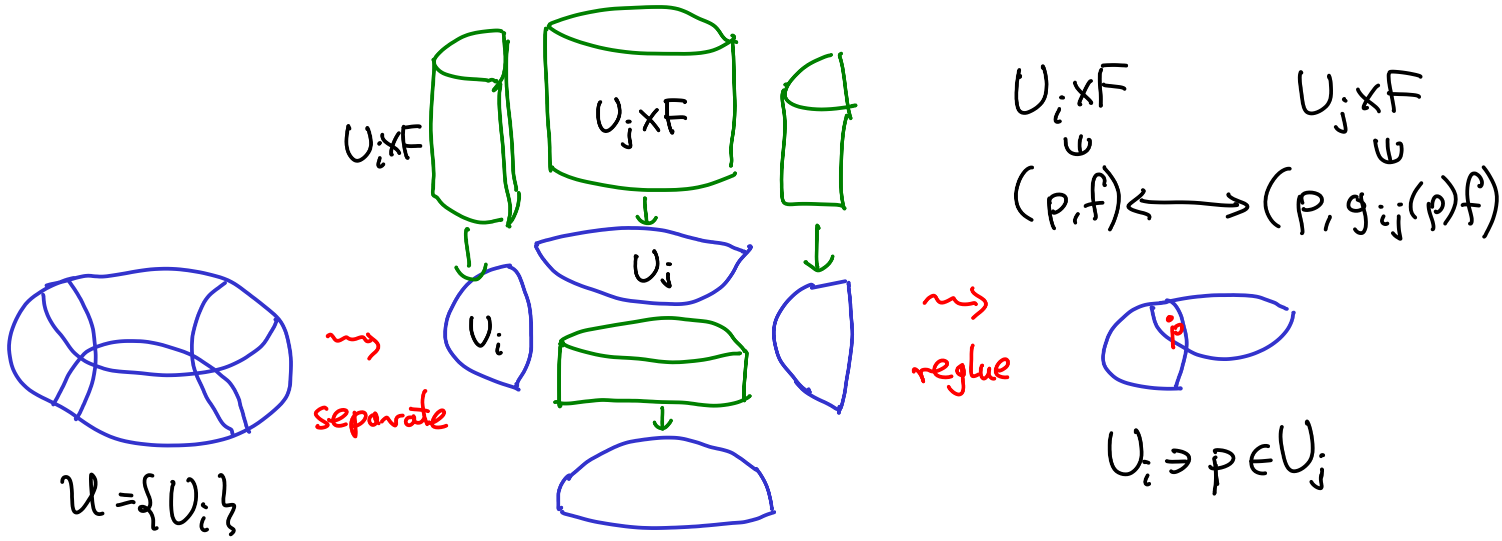
H_* or H^*

Flat bundles

How to build a flat bundle?

Ingredients:

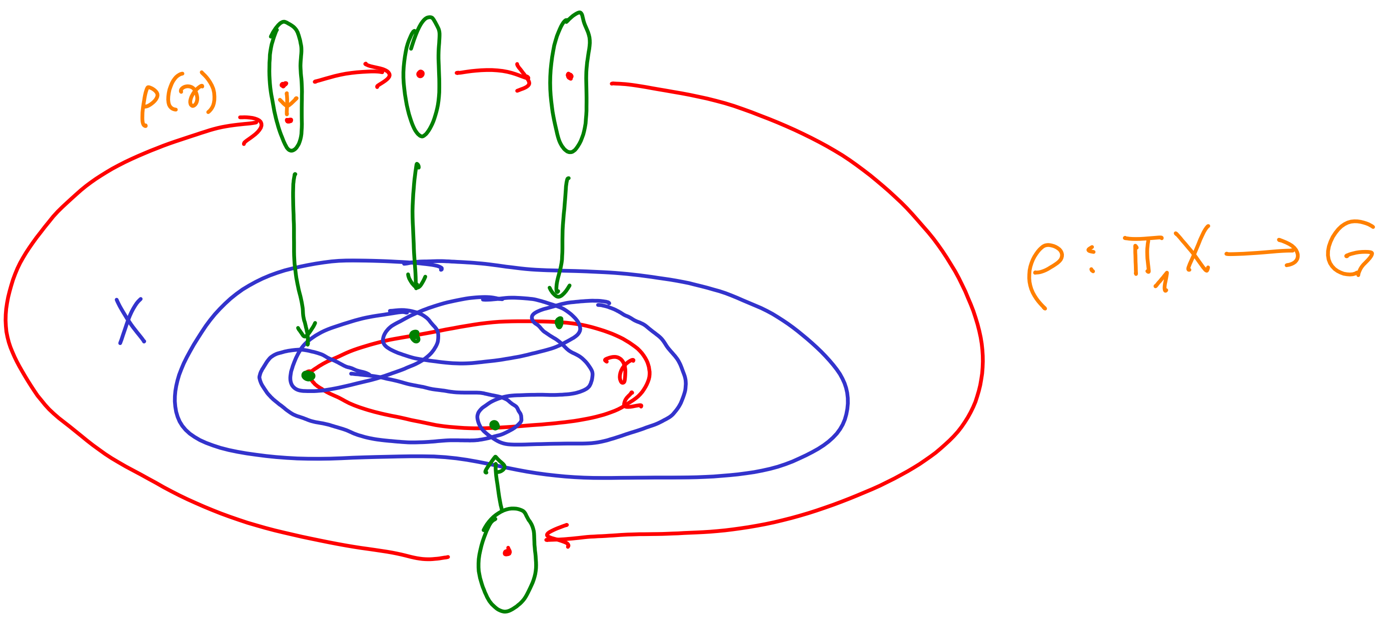
- fibre F (vector space, topological space, simplicial complex,...);
- group G acting on F (gluing group);
- base B (topological space);
- open cover $\mathcal{U} = \{U_i\}$ of B;
- locally constant functions $g_{ij}: U_i \cap U_j \to G$ (gluing functions);



- After regluing we get:

$$\begin{array}{r} \text{total space } E \\ \Big\downarrow \text{ projection map } \pi \\ \text{base } B \end{array}$$

  - for each $b \in B$ the fibre $E_b = \pi^{-1}(b)$ "is" F.
  - for b, b' close the fibres E_b, $E_{b'}$ have a privileged identification.
- Monodromy representation: $\pi_1(X) \to G$.



Fact. There is a bijection between:

- isomorphism classes of flat G-bundles over B;
- $\mathrm{Hom}(\pi_1(X), G)$/conjugation.

- Bundles pull back:

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} f^*E = \{(x,e) \mid f(x) = \pi(e)\} \arrow[d] \\ X \end{tikzcd}$$

$$(f^*E)_x = E_{f(x)}$$

- Characteristic classes:

To every flat G-bundle E over any base B we assign a cohomology class $c(E) \in H^k B$, so that for every $f\colon X \to Y$ and every bundle E over Y we have

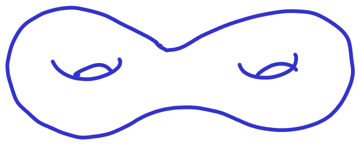
$$c(f^*E) = f^* c(E)$$

$$\begin{tikzcd} f^*E \arrow[d] & E \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

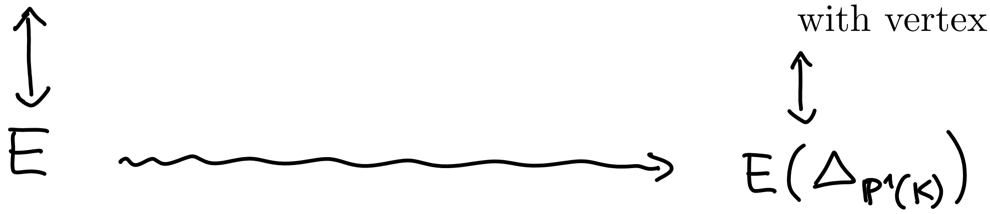
$$\begin{array}{ccc} H^k X & \xleftarrow{f^*} & H^k Y \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ c(f^*E) & \longmapsfrom & c(E) \end{array}$$

V. Construction for $SL(2, K)$

Suppose we have a flat bundle with

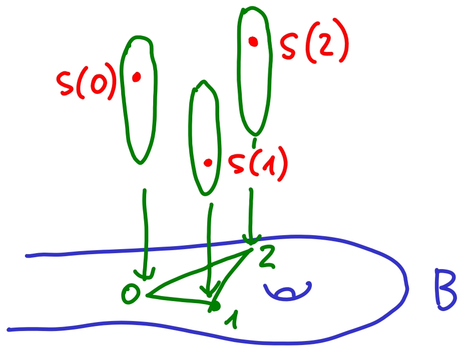
$$G = SL(2, K), \quad F = K^2, \quad B =$$ 

- Modify the fibre:
  $K^2 \longrightarrow \mathbf{P}^1(K) = \{L < K^2 \mid \dim L = 1\} \longrightarrow \Delta_{\mathbf{P}^1(K)}$ –infinite simplex with vertex set $\mathbf{P}^1(K)$.



- Section.

$s(0), s(1), s(2)$
span a triangle in $\Delta_{\mathbf{P}^1(K)}$
defined up to G-action



Let φ be a G-invariant 2-cocycle on $\Delta_{\mathbf{P}^1(K)}$; then $\varphi(s(0), s(1), s(2))$ is well-define. We define a class $s^*\varphi \in H^2(B, A)$ by

$$(s^*\varphi)(0, 1, 2) = \varphi(s(0), s(1), s(2)).$$

Fact. The cohomology class $s^*\varphi$ does not depend on the choice of s.

- How to get φ and A?
  (we put $\Delta = \Delta_{\mathbf{P}^1(K)}$; by $C_*\Delta$ we denote simplicial chains)

- $\varphi(c) = c \in C_2\Delta$ (not a cocycle: $d\varphi(t) = \varphi(\partial t) = \partial t$)
- $\varphi(c) = [c] \in C_2\Delta/\operatorname{im}(\partial_3)$ (is a cocycle, but is not G-invariant)
- $\varphi(c) = [c] \in C_2\Delta/\langle \operatorname{im}(\partial_3), c - gc\rangle_{g\in G, c\in C_2\Delta} =: A$

  We got a G-invariant cocycle φ with coefficients in A.

- What is A?
  Fact. Triangles in Δ modulo G-action correspond to elements of $\dot{K}/\dot{K}^2$.

$$v_0, v_1, v_2 \in K^2 \rightsquigarrow \langle v_0\rangle, \langle v_1\rangle, \langle v_2\rangle \in \mathbb{P}^1(K)$$

$$\Big\downarrow g$$

$$\left\langle \begin{pmatrix}1\\0\end{pmatrix}\right\rangle, \left\langle \begin{pmatrix}0\\1\end{pmatrix}\right\rangle, \left\langle \begin{pmatrix}a\\1\end{pmatrix}\right\rangle \longleftrightarrow a \in \dot{K} \to [a] \in \dot{K}/\dot{K}^2$$

$$\Big\downarrow \begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$$

$$\left\langle \begin{pmatrix}1\\0\end{pmatrix}\right\rangle, \left\langle \begin{pmatrix}0\\1\end{pmatrix}\right\rangle, \left\langle \begin{pmatrix}\lambda a\\ \lambda^{-1}\end{pmatrix}\right\rangle \longleftrightarrow \lambda^2 a \in \dot{K} \nearrow$$

$$\left\langle \begin{pmatrix}\lambda a\\ \lambda^{-1}\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}\lambda^2 a\\ 1\end{pmatrix}\right\rangle$$

Relation from the boundaries of tetrahedra:

$$[a] + [b] = [a+b] + [ab(a+b)], \qquad a, b, a+b \in \dot{K}.$$

$$\langle v_0\rangle, \langle v_1\rangle, \langle v_2\rangle, \langle v_3\rangle \longrightarrow \left\langle \begin{pmatrix}1\\0\end{pmatrix}\right\rangle, \left\langle \begin{pmatrix}0\\1\end{pmatrix}\right\rangle, \left\langle \begin{pmatrix}a\\1\end{pmatrix}\right\rangle, \left\langle \begin{pmatrix}b\\1\end{pmatrix}\right\rangle$$

It is convenient to impose an extra (alternation) relation $[-a] = -[a]$. Then we get

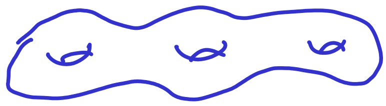
$$A \cong \mathbf{Z}[\dot{K}/\dot{K}^2]/\langle [a] + [b] = [a+b] + [ab(a+b)], [-a] = -[a]\rangle.$$

This is Witt's description of the Witt ring of quadratic forms $W(K)$...

Examples: $W(\mathbb{R}) \simeq \mathbb{Z}$; $W(\mathbb{F}_q) \simeq \mathbb{Z}/4$ if $q \equiv 3 \bmod 4$

VI. Properties of the Witt class

g holes (genus g)

Suppose that E is an $SL(2, K)$-bundle over $B =$ 

We defined a cohomology class $W(E) \in H^2(B, W(K))$.

In H_2B we have the fundamental class $[B]$: the sum of all triangles in a triangulation.

Evaluate $W(E)$ on $[B]$:

$$w(E) := \langle W(E), [B] \rangle \in W(K).$$

In other words: to any representation of $\pi_1(B)$ in $SL(2, K)$ we have assigned an element of $W(K)$.

- What are the possible values of $w(E)$?

Fundamental ideal: $I = I(K) \subset W(K)$ is the set of classes $\sum n_i[a_i]$ for which $\sum n_i$ is even.

Fact. $w(E) \in I^2$.

Example 1.

$K = \mathbf{R}$. Then $W(K) = \mathbf{Z}$, $I = 2\mathbf{Z}$, $I^2 = 4\mathbf{Z}$ – therefore $4|w(E)$. Moreover, for a surface of genus g we have $|w(E)| \leq 4g$ (Milnor-Wood).

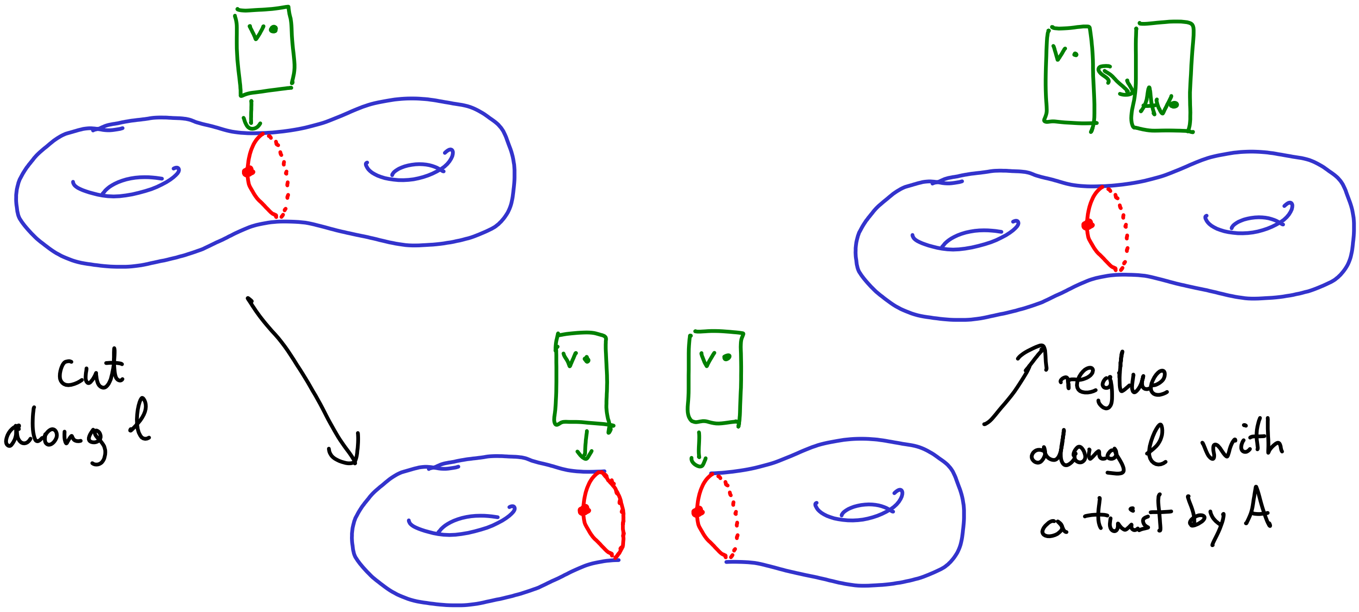
Example 2.

$K = \mathbf{Q}$. Then $I^2 = \mathbf{Z} \oplus \bigoplus_{p\,\text{prime}} \mathbf{Z}/2$.

Then $w(E)$ corresponds to an integer and a finite set of primes.

Question. What finite collections of primes do appear?

- Twists – operations that change the bundle.
  Let ℓ be a closed curve in B.
  Let $A \in SL(2, K)$ commute with the monodromy of E along ℓ.



We get a new bundle – a twist of E, denoted $\tau_{\ell,A}(E)$

Fact. $w(\tau_{\ell,A}(E)) = w(E)$.

Question. Suppose that $w(E) = w(E')$. Can E' be obtained from E by a sequence of twists?

Remark.
In some sense w is the universal twist–invariant characteristic class.