On the geometry of simply connected wandering domains

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We consider the dynamical system given by the iterates of an entire function $f : \mathbb{C} \to \mathbb{C}$ (not linear).

- **Fatou set** $\mathcal{F}_f = \{z \in \mathbb{C} \mid (f^n) \text{ is normal in a nbh of } z\}$
- **Fatou component** is a connected component of $\mathcal{F}_f$
- A Fatou component Ω is **pre-periodic** if there are non-negative integers $n \neq m$ such that $f^n(\Omega) \cap f^m(\Omega) \neq \emptyset$.
- A Fatou component which is not pre-periodic is called a wandering Fatou component or a **wandering domain**.

One of the main goals in complex dynamics is to obtain a complete classification of all possible Fatou components for a given class of maps in terms of their dynamical behaviour and their geometry.

Introduction

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Classification in terms of internal dynamical behaviour

For entire functions we have a complete description of:

- Pre-periodic Fatou components (Fatou & examples of Siegel and Baker)
- Multiply connected wandering domains (Bergweiler-Rippon-Stallard '13)
- Simply connected wandering domains in terms of the hyperbolic distance between orbits of points and in terms of convergence to the boundary
  - (Benini-Evdoridou-Fagella-Rippon-Stallard '19) *All nine types can be realized by escaping wandering domains.*
  - (Evdoridou-Rippon-Stallard '20) *only six of these types can be realized by oscillating wandering domains.*

External point of view

- **Escaping** wandering domain - the orbit leaves every compact,
- **Oscillating** wandering domain - there is a subsequence of the orbit that leaves every compact, and another subsequence that is bounded,
- **Dynamically bounded** wandering domains - the orbit is bounded.

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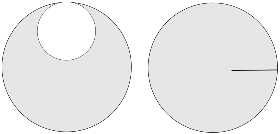
Question

Does ther exist an entire function with a dynamically bounded wandering domain?

Theorem (B.T. 2021)

Let $\Omega \subset \mathbb{C}$ be a bounded connected regular open set whose closure has a connected complement. There exists an entire function f for which Ω is an escaping (oscillating) wandering domain and the iterates $f^n|_\Omega$ are univalent.

Recall that an open set U is called **regular** if and only if $U = \text{Int}(\overline{U})$ and notice that the conditions of the theorem imply that Ω is simply connected.



Corollary

Every simply connected Jordan domain is a wandering domain of some entire function.

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Necessity of conditions

The condition that Ω is a regular open set is not only sufficient but also necessary.

- Ω wandering domain $\Rightarrow \mathcal{F}_f$ is disconnected.
- f is continuous and open map $\Rightarrow f^n(\text{Int}(\overline{\Omega})) \subset \text{Int}(\overline{f^n(\Omega)}) \subset \text{Int}(\overline{U_n})$ for all $n \geq 0$. Here U_n is a Fatou component.
- If Ω is not regular $\Rightarrow \text{Int}(\overline{\Omega}) \cap \mathcal{J}_f \neq \emptyset \Rightarrow \cup_{n=0}^{\infty} f^n(\text{int}(\overline{\Omega}))$ covers the whole plane with at most one exception.
- This is now a contradiction since

$$\bigcup_{n=0}^{\infty} f^n(\text{int}(\overline{\Omega})) \subset \bigcup_{U \text{ Fat.Comp.}} \text{int}(\overline{U}) \subsetneq \mathbb{C} \backslash \{point\}.$$

If Ω is regular then $\partial\Omega = \partial(\mathbb{C}\backslash\overline{\Omega})$ hence there exists a sequence of points $(x_n) \subset \mathbb{C}\backslash\overline{\Omega}$ that accumulates everywhere on $\partial\Omega$.

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The other two conditions in our theorem, namely that Ω is bounded and that $\mathbb{C}\backslash\overline{\Omega}$ is connected, are needed for the application of the following stronger version of the well-known Runge's Approximation Theorem.

Theorem

Let $K_1, \ldots, K_n \subset \mathbb{C}$ be pairwise disjoint compact sets whose complements $\mathbb{C}\backslash K_j$ are connected. Let $L_k \subset K_k$ be a finite set of points and $h_k : K_k \to \mathbb{C}$ a holomorphic map for every $1 \leq k \leq n$. For every $\epsilon > 0$ there exists an entire function f satisfying:

- $\|h_k - f\|_{K_k} < \epsilon$
- $f(x) = h_k(x)$ *for all* $x \in L_k$
- $f'(x) = h_k'(x)$ *for all* $x \in L_k$

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This theorem is a combination of several results due to Benhke-Stein 49', Florack 48', Royden 67'. A very similar approximation result was presented by Eremenko-Lyubich 87' (Main Lemma).

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Question 1

Is the condition that $\mathbb{C}\backslash\overline{\Omega}$ is connected a necessary condition? In particular is the complement of the closure of a bounded simply connected Fatou component always connected?

The positive answer would imply that we have achieved a complete description of all possible geometries of bounded simply connected wandering domains.

Question 2

If f has more than two Fatou components, can two of these components share the same boundary?

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Advertisement

The answer to the above questions will be reviled in the following talk given by James Waterman. Stay tuned !

Proof

Let Ω be an open set as in our theorem and choose $R > 0$ and a strictly increasing sequence of integers (m_n) such that disks $\Delta(m_n, R)$ are pairwise disjoint and disjoint from Ω. Finally choose a sequence of points $(x_n) \in \mathbb{C} \backslash \overline{\Omega}$ which accumulates everywhere on $\partial\Omega$.

The idea is to construct an entire function f with the following properties:

1. $f^n(\Omega) \subset \Delta(m_n, R)$, for all $n \geq 1$,
2. $f(\zeta) = \zeta \in \mathbb{C} \backslash \overline{\Omega}$ and $f'(\zeta) = \frac{1}{2}$
3. $f^n(x_n) = \zeta$ as for all $n \geq 1$,
4. $f^n|_\Omega$ is univalent for all $n \geq 1$

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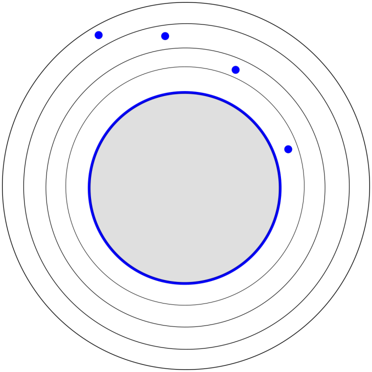
How to construct f for the unit disk

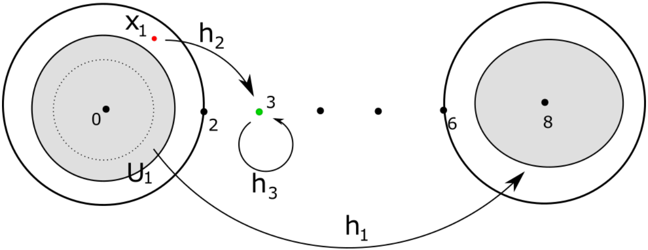
The function f is obtained as the uniform limit of the sequence of entire functions (f_n) that is constructed inductively by using Runge's theorem.

Let us choose:

- $U_{n+1} \subset \text{int}(U_n) \subset \Delta(0, 2)$ for all $n \geq 0$,
- $\overline{\Delta}(0, 1) = \bigcap_{n \geq 0} U_n$.
- $x_n \in \text{int}(U_{n-1}) \backslash U_n$ for all $n \geq 1$,
- (x_n) accumulates everywhere on $\partial\Delta(0, 1)$.



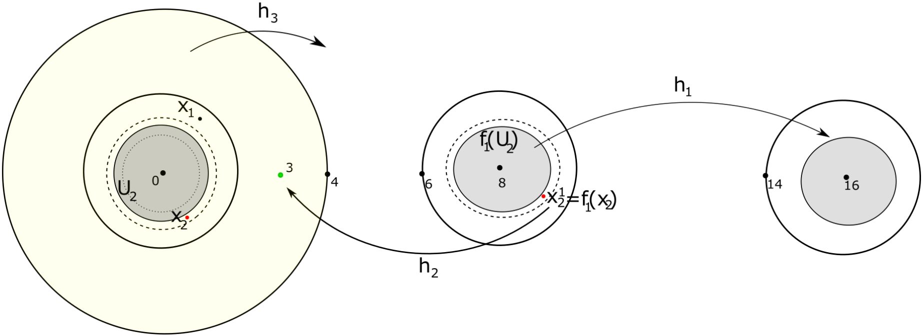
Let $K_1 = U_1$, $K_2 = \{x_1\}$, $K_3 = \{3\}$ and $h_1(z) = z + 8$, $h_2(z) = 3$, $h_3(z) = \frac{1}{2}(z - 3) + 3$.



By the approximation theorem for every $\epsilon_1 > 0$ there is an entire function f_1, such that

- $\|f_1 - h_j\|_{K_j} < \epsilon_1$ for $j = 1, 2, 3$
- $f_1(x_1) = 3$
- $f_1(3) = 3$ and $f_1'(3) = \frac{1}{2}$.

Let $K_1 = f_1(U_2)$, $K_2 = \{x_2^1\}$, $K_3 = \overline{\Delta}(0,4)$ and $h_1(z) = z + 8$, $h_2(z) = 3$, $h_3(z) = f_1(z)$.



Given $\epsilon_2 > 0$ there is an entire function f_2, such that

- $\|f_2 - h_j\|_{K_j} < \epsilon_2$ for $j = 1, 2, 3$
- $f_2(x_1) = f_1(x_1)$
- $f_2(x_2) = f_1(x_2) =: x_2^1$
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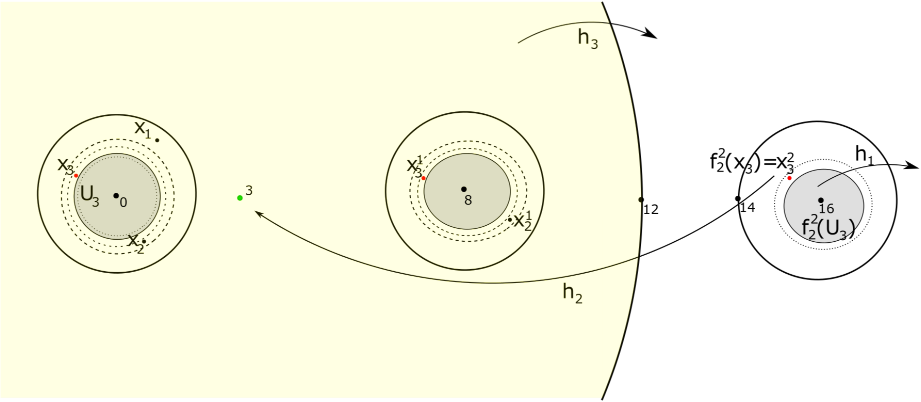
- $f_2(x_2^1) = 3$

Let $K_1 = f_2^2(U_3)$, $K_2 = \{x_3^2\}$, $K_3 = \overline{\Delta}(0, 12)$ and $h_1(z) = z + 8$, $h_2(z) = 3$, $h_3(z) = f_2(z)$.



Given $\epsilon_3 > 0$ there is an entire function f_3, such that

- $\|f_3 - h_j\|_{K_j} < \epsilon_3$ for $j = 1, 2, 3$
- $f_3(x_1) = f_2(x_1)$
- $f_3^j(x_2) = f_2^j(x_2)$ for $j = 1, 2$
- $f_3^j(x_3) = f_2^j(x_3) =: x_3^j$ for $j = 1, 2$
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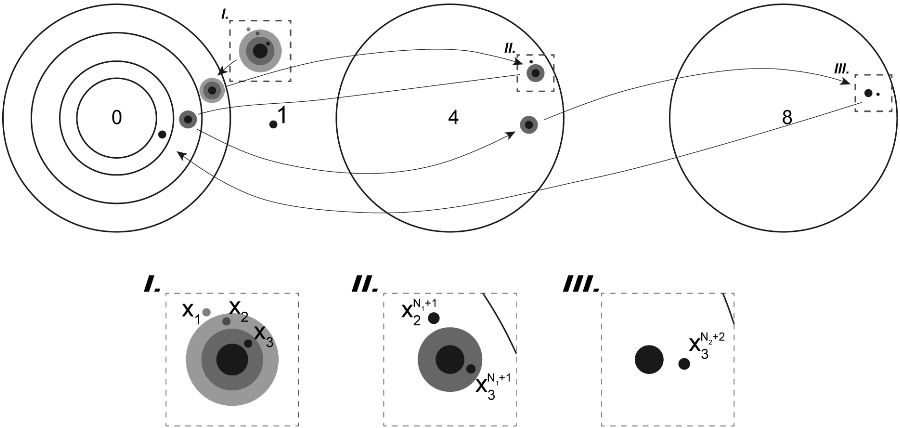
- $f_3(x_3^2) = 3$

Oscillating



The *polynomially-convex hull* of a compact set $K \subset \mathbb{C}^m$ is defined as

$\widehat{K} = \{z \in \mathbb{C}^m : |p(z)| \leq \sup_K |p| \text{ for all holomorphic polynomials } p\}$

We say that K is **polynomially convex** if $\widehat{K} = K$.

A compact set $K \subset \mathbb{C}$ is polynomially convex if and only if $\mathbb{C} \backslash K$ is connected.

Theorem (B.T. 2020)

Let $\Omega \subset \mathbb{C}^{m\geq 2}$ be a bounded regular open set whose closure is polynomially convex. There exists an automorphism of $\mathbb{C}^m$ with an escaping (oscillating) wandering domain equal to Ω.

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Automorphisms of $\mathbb{C}^{m\geq 2}$

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Examples of such domains

- Any bounded convex domain in $\mathbb{C}^n$.
- Topologically non-trivial examples: It is known that any totally real compact manifold $M \subset \mathbb{C}^n$ of dimension $k < n$ can be smoothly perturbed so that its perturbation M' is totally real compact manifold which is polynomially convex, in particular M' has the same topology as M. By taking an appropriate tubular neighbourhood of M' we obtain an open set with desired properties.

This shows that there is rich variety of wandering domains which are topologically non-equivalent.

About the proof

- The idea of the proof is essentially the same, but its complexity increases due to the restrictive nature of automorphisms.
- Trouble: Uniformly convergent sequence of automorphisms may not converge to an automorphism (surjectivity can be lost, e.g Fatou-Bieberbach domains).
- The key ingredient: Approximation result of the Andersén–Lempert theory

Theorem

Let $K_1, K_2, \ldots, K_n$ be pairwise disjoint compact sets in $\mathbb{C}^m$ such that all but one are starshape. Let $F_j \in Aut(\mathbb{C}^m)$ $(j = 1, \ldots, n)$ be such that the images $K'_j = F_j(K_j)$ are pairwise disjoint.
If the sets $A = \cup_{j=1}^n K_j$ and $B = \cup_{j=1}^n K'_j$ are polynomially convex, then for every $\epsilon > 0$ there exists $G \in Aut(\mathbb{C}^m)$ such that $\|G - F_j\|_{K_j} < \epsilon$ for all $j = 1, \ldots, m$.
In particular the automorphism G can be chosen so that its finite order jets agree with the corresponding jets of F_j at any given finite set of points in K_j, for $1 \leq j \leq m$.

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- The key ingredient: Approximation result of the Andersén–Lempert theory

Theorem

Let $K_1, K_2, \ldots, K_n$ be pairwise disjoint compact sets in $\mathbb{C}^m$ such that all but one are starshape. Let $F_j \in Aut(\mathbb{C}^m)$ $(j = 1, \ldots, n)$ be such that the images $K_j' = F_j(K_j)$ are pairwise disjoint.
If the sets $A = \cup_{j=1}^n K_j$ and $B = \cup_{j=1}^n K_j'$ are polynomially convex, then for every $\epsilon > 0$ there exists $G \in Aut(\mathbb{C}^m)$ such that $\|G - F_j\|_{K_j} < \epsilon$ for all $j = 1, \ldots, m$.
In particular the automorphism G can be chosen so that its finite order jets agree with the corresponding jets of F_j at any given finite set of points in K_j, for $1 \leq j \leq m$.

THANK YOU