2023 GRAPH ALGEBRAS @ Będlewo

Conjugacy and continuous orbit equivalence of graphs

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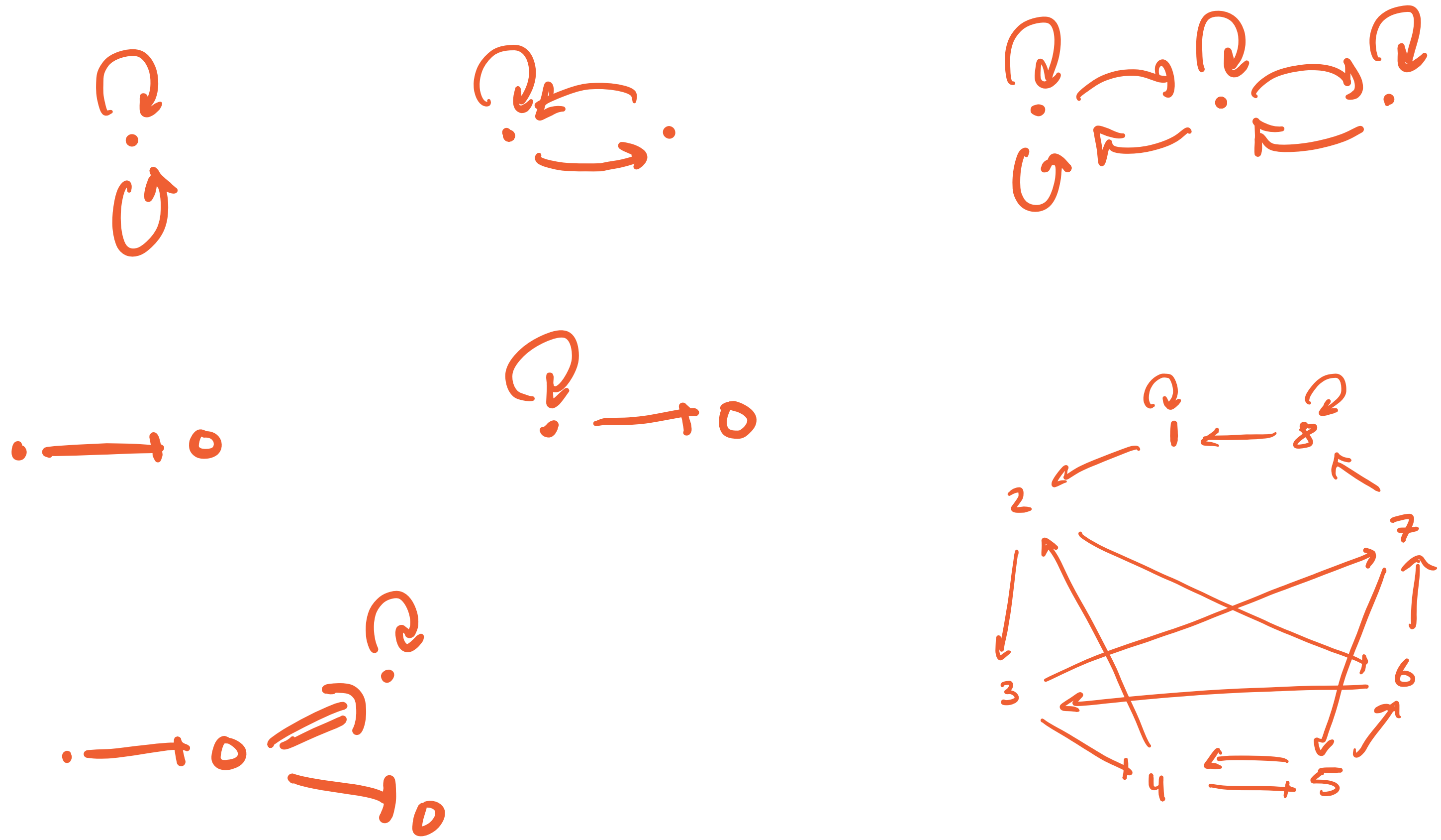
- Directed graphs give rise to
  1) Dynamics
  2) Groupoids
  3) C^*-algebras
- Conjugacy and continuous orbit equivalence
- Perspectives: → local homeomorphisms
  → Sofic shifts

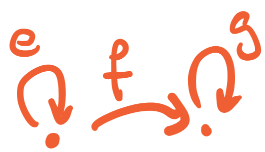
A **directed graph** is (E^0, E^1, r, s) where $r, s : E^1 \to E^0$ are range and source maps.



1: Dynamics appear as infinite path space

$$E^\infty = \{ e_0 e_1 e_2 \cdots : e_i \in E^1,\ r(e_i) = s(e_{i+1})\ \forall i \in \mathbb{N} \}$$

with the shift $\sigma_E(e_0 e_1 e_2 \cdots) = e_1 e_2 \cdots$.

F: 

$$F^\infty = \{e^\infty\} \cup \{e^n f g^\infty : n \in \mathbb{N}\} \cup \{g^\infty\}$$

E: 

$$E^\infty = \{e, f\}^{\mathbb{N}}$$

$$G^\infty = \emptyset,\quad \partial G = \{e, v\}$$

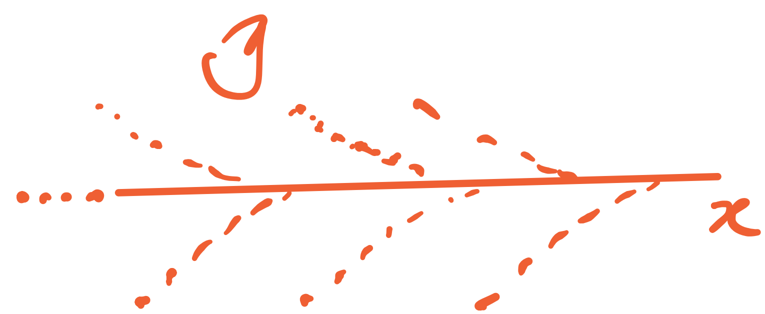
Boundary path space:

$$\partial E = E^\infty \cup \{\text{finite paths ending in a singular vertex}\}$$

The orbit of $x \in \partial E$ is

$$\mathrm{orbit}(x) = \bigcup_{k,l \in \mathbb{N}} \sigma_E^{-l} \sigma_E^{k}(x)$$

So $y \in \mathrm{orbit}(x)$ precisely if there are $k, l \in \mathbb{N}$ s.t. $\sigma^k(x) = \sigma^l(y)$.



2: The graph groupoid of $(\partial E, \sigma_E)$ is

$$G_E = \left\{ (x, k-l, y) : \begin{array}{l} x, y \in \partial E,\ k, l \in \mathbb{N}, \\ \sigma^k(x) = \sigma^l(y) \end{array} \right\}$$

with unit space $G_E^{(0)} = \{(x, 0, x) : x \in \partial E\} = \partial E$,

and

- $(x, p, y)(y, q, z) = (x, p+q, z)$

- $(x, p, y)^{-1} = (y, -p, x)$

3: The graph C^*-algebra $C^*(E)$ is the groupoid C^*-algebra $C^*(G_E)$:

a C^*-completion of $C_c(G_E)$ with

- $$f * g(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)\, g(\beta)$$

- $$f^*(\gamma) = \overline{f(\gamma^{-1})}$$

$Z(U,k,l,V)$, $U,V \subseteq \partial E$

$(x, k-l, y)$

NOTE: $D(E) \overset{def}{=} C_0(\partial E) \subseteq C_c(G_E) \subseteq C^*(G_E)$

A homeomorphism $h: \partial E \to \partial F$ is a **conjugacy** if

- $h \circ \sigma_E(x) = \sigma_F \circ h(x)$ for $x \in \partial E^{\geq 1}$
- $h^{-1} \circ \sigma_F(y) = \sigma_E \circ h^{-1}(y)$ for $y \in \partial F^{\geq 1}$

Example:

→  and  are <u>not</u> conjugate

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A conjugacy $h: \partial E \to \partial F$ induces

- a groupoid isomorphism $\psi: G_E \to G_F$

  $(x, p, y) \longmapsto (h(x), p, h(y))$

  s.t. $\psi|_{G_E^{(0)}} = h$, and

- a $*$-isomorphism $\phi: C^*(E) \to C^*(F)$ s.t.

  $\to \phi(f) = f \circ \psi^{-1}$ for $f \in C_c(G_E)$

  $\to \phi(g) = f \circ h^{-1}$ for $g \in C_0(\partial E)$

Thm (Matsumoto, Matsumoto-Matui, Carlsen-Eilers-Ortega-Restorff, Brownlowe-Carlsen-Whittaker)
(+CRST)

Let E and F be nice directed graphs.
The following are equivalent:

1) E and F are continuously orbit equivalent;

2) G_E and G_F are isomorphic as topological groupoids;

3) $(C^*(E), D(E))$ and $(C^*(F), D(F))$ are diagonally $*$-isomorphic.

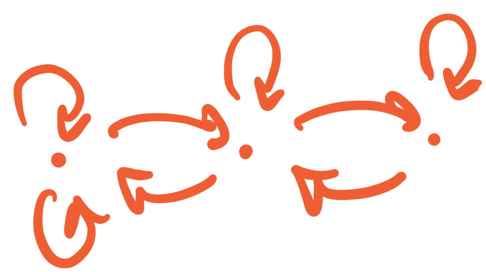
A homeomorphism $h : \partial E \to \partial F$ is a continuous orbit equivalence if there are continuous cocycles $k_E, l_E : \partial E \to \mathbb{N}$ and $k_F, l_F : \partial F \to \mathbb{N}$ s.t.

- $\sigma_F^{l_E(x)} \circ h(x) = \sigma^{k_E(x)} \circ h \circ \sigma_E(x)$, for $x \in \partial E^{\geq 1}$

- $\sigma_E^{l_F(y)} \circ h^{-1}(y) = \sigma^{k_F(y)} \circ h^{-1} \circ \sigma_F(y)$ for $y \in \partial F^{\geq 1}$

Example:

→  and  are cont. orb. eq.

→  and  are <u>not</u> cont. orb. eq.

¿ How do we characterise conjugacy ?

B-Carlsen:

For a finite (essential) graph E, there is completely positive map $\tau_E : C^*(E) \to C^*(E)$

$$\tau_E(x) = \sum_{f,e \in E^1} s_e x s_f^*$$

and $(C^*(E), D(E), \tau_E)$ determines (E^∞, σ_E) up to conjugacy.

Need new idea to generalise!

A conjugacy $h: \partial E \to \partial F$ induces

- a groupoid isomorphism $\psi: G_E \to G_F$

$$(x, p, y) \longmapsto (h(x), p, h(y))$$

s.t. $c_F \circ \psi = c_E$, $c_E: (x, p, y) \mapsto p$.

- a $*$-isomorphism $\phi: C^*(E) \to C^*(F)$ s.t.

$$\phi \circ \gamma_z^E = \gamma_z^F \circ \phi \quad \text{for } z \in \mathbb{T}$$

The canonical cocycle $c_E : G_E \to \mathbb{Z}$

$$c_E(x, k-l, y) = k - l$$

induces the canonical gauge action $\mathbb{T} \curvearrowright C^*(E)$.

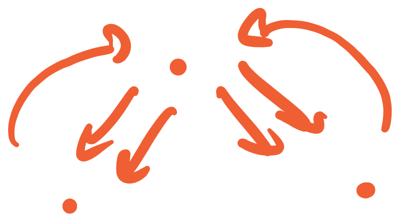
If $f \in C_0(\partial E, \mathbb{Z})$, then $c_f : G_E \to \mathbb{Z}$

$$c_f(x, k-l, y) = \sum_{i=0}^{k-1} f(\sigma_E^i(x)) - \sum_{j=0}^{l-1} f(\sigma_E^j(y))$$

induces a gauge action $\mathbb{T} \curvearrowright C^*(E)$ weighted by f.

Example (B-Carlsen):

E



F

Here, E and F are not conjugate, but

$$\left(C^*(E), D(E), \gamma^E \right) \cong \left(C^*(F), D(F), \gamma^F \right).$$

can. gauge action

Theorem (Matsumoto, Armstrong-B-Carlsen-Eilers)

The graphs E and F are conjugate iff

there is $*$-isomorphism $C^*(E) \to C^*(F)$ that intertwines all gauge actions weighted by $C_0(\partial E, \mathbb{Z})$ and $C_0(\partial F, \mathbb{Z})$.

$f \in C_0(\partial F, \mathbb{Z})$

$$\xi \in C_c(G_E): \quad \gamma_z^{E,f}(\xi)(\gamma) = z^{c_f(\gamma)}\xi(\gamma), \; z \in \mathbb{T}, \; \gamma \in G_E$$

PERSPECTIVE 1:

Theorem (Armstrong-B-Carlsen-Eilers)

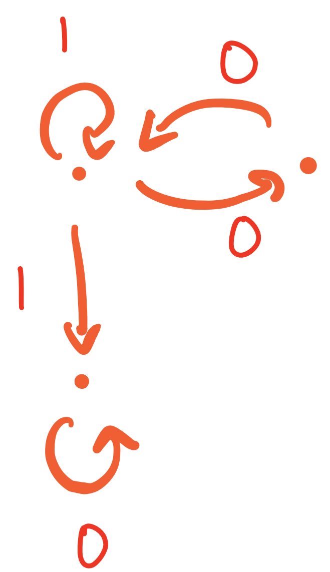
Two local homeomorphisms $T: X \to X$ and $S: Y \to Y$ are conjugate iff

there is $*$-isomorphism $C^*(T) \to C^*(S)$ that intertwines all gauge actions weighted by $C_0(X, \mathbb{R})$ and $C_0(Y, \mathbb{R})$.

PERSPECTIVE 2 :

Sofic shifts are modelled by finite labelled graphs, and C^*-algebras are graph C^*-algebras!

Even:



Odd:

B-Carlsen:

The even and odd shifts are continuously orbit equivalent.

Theorem (B)

Conjugacy of sofic shifts has similar characterisation

Theorem (B-Carlsen)

Continuous orbit eq. of sofic shifts implies diagonal $*$-isomorphism.

¿ Is the converse true ?

Summary:

- Graphs define interesting dynamical systems (symbolic dynamics)

- Continuous orbit eq. is diagonal $*$-isomorphism
- Conjugacy is diagonal $*$-isomorphism that intertwines a family of gauge actions.

- Many results extend to general dynamical systems