Split extensions & KK-equivalences for quantum projective spaces

Graph Algebras 2023

Joint work with Sophie E. Zegers (Prague → Delft)
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Motivation: Non-commutative topology. Understand the K-theory of Vaksman–Soibelman QPS's through KK-theory.

Goal: construct an explicit KK-equivalence between

$C(\mathbb{C}P^n_q)$ and $C(\mathbb{C}P^n)$

1. The VS quantum proj. spaces (as graph algebras)
2. KK-theory & KK-equivalences (split extensions & KK)
3. Construction of a splitting & proof of KK-equivalence

① VS QPS's $C(\mathbb{CP}^n_q)$ $n \geq 1$

quantum homogeneous space structure

- $U(1)$ action on the VS odd spheres

$$C(\mathbb{CP}^n_q) \simeq C(S^{2n+1}_q)^{U(1)}$$

$$C(S^{2n+1}_q) \simeq \frac{SU_q(n+1)}{SU_q(n)}$$

- $U_q(n) \curvearrowright SU_q(n+1)$ (more on this later)

$q \in (0,1)$ $C(S^{2n+1}_q)$ is the universal C^*-algebra gen by $n+1$ elements $z_0, \dots, z_n$

$$z_i z_j = q^{-1} z_j z_i \quad i<j \qquad\qquad z_i z_j^* = q z_j^* z_i \quad i \neq j$$

$$\sum_{i=0}^{n} z_i z_i^* = 1 \qquad (*)$$

$$z_i^* z_i = z_i z_i^* + \sum_{j=i+1}^{n} (1-q^2) z_j z_j^* \qquad (*)$$

$C(\mathbb{C}P^n_q) := C(S_q^{2n+1})^{U(1)}$ $\qquad z_i \mapsto w \cdot z_i \quad \forall w \in U(1)$

generated by elements $p_{ij} = z_i^* z_j$ + relations deduced from $(*)$

p_{ij} are entries of a $(n+1)\times(n+1)$ projection

$$\sum p_{ij} p_{jk} = p_{ik} \quad \& \quad p_{ij}^* = p_{ji}$$

$n = 1$ $\qquad C(\mathbb{C}P^1_q) = C(S^2_q)$

2002 Hong-Szymański the quantum spheres $C(S_q^{2n+1})$ & the quantum proj. spaces $C(\mathbb{C}P^n_q)$ are graph C^*-algebras.

<u>Recall</u>: A <u>directed graph</u> $E = (E^0, E^1, r, s)$

E^0, E^1 countable sets of vertices & edges

$r, s : E^1 \to E^0$ are the range & source maps

$$v \xrightarrow{e} w \qquad r(e) = w \quad s(e) = v$$

A vertex $v \in E^0$ is called regular if $s^{-1}(v) = \{e \in E^1 \mid s(e) = v\}$ is finite & non-empty

if $s^{-1}(v) = \emptyset$ $\quad v$ is a sink

$|s^{-1}(v)| = \infty$ $\quad V$ is an infinite emitter

<u>Def</u> E directed graph, $C^*(E)$ is the universal C^* algebra generated by projections $\{P_v \mid v \in E^0\}$ & partial isometries $\{S_e \mid e \in E^1\}$ s.t.

1. $P_v P_w = 0 \quad v \neq w$
2. $S_e^* S_f = 0 \quad e \neq f$
3. $S_e^* S_e = P_{r(e)}$
4. $S_e S_e^* \leq P_{s(e)}$
5. $P_v = \sum_{s(v)=e} S_e S_e^*$ for all v regular

CK relations

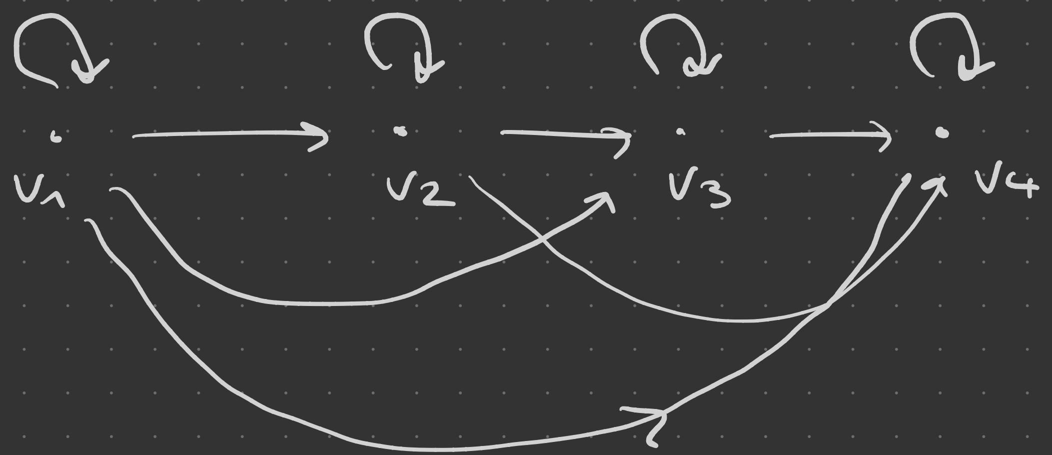
Canonical $U(1)$ action (the GAUGE action) defined

$P_v \mapsto P_v$ & $S_e \mapsto \omega S_e$ $\forall \omega \in S^1$

Hong & Szymański $C(S_q^{2n+1}) \simeq C^*(L_{2n+1})$

L_{2n+1} has $n+1$ vertices $v_1 \dots v_{n+1}$ & 1 edge from v_i to v_j $i \le j$

L_7



$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ 0 & & 1 & 1 \\ & & & 1 \end{bmatrix}$$

incidence matrix

* all isomorphic $q \in (0, 1)$

under $C^*(L_{2n+1}) \simeq C(S_p^{2n+1})$

gauge action $\longrightarrow$ $U(1)$-action described above

$$C(\mathbb{C}P^n_p) \simeq C^*(L_{2n+1})^{\gamma} = C^*(F_n)$$

$$F_3 \qquad \begin{tikzcd} w_1 \arrow[r, "(\infty)"] \arrow[rr, bend right, "(\infty)"'] \arrow[rrr, bend right=60, "(\infty)"'] & w_2 \arrow[r, "(\infty)"] \arrow[rr, bend right, "(\infty)"'] & w_3 \arrow[r, "(\infty)"] & w_4 \end{tikzcd}$$

F_n has $n+1$ vertices

∞ many edges from $v_i \to v_j$ $i<j$

F_n not row-finite
all vertices are singular

w_{n+1} is a sink
& w_i's $i<n+1$ ∞ emitters

we can compute the k-theory of $C(\mathbb{C}P^n_p)$ as kernel & cokernel of a certain map.

$$K_E : \mathbb{Z}E^0_{reg} \longrightarrow \mathbb{Z}E^0$$

$$K_E(v) = \left(\sum_{\substack{e \in E' \\ s(e)=v}} r(e)\right) - v$$

$$K_E : 0 \longrightarrow \mathbb{Z}^{n+1}$$

$$K_1(C(\mathbb{C}P^n_q)) \simeq 0$$

$$K_0(C(\mathbb{C}P^n_p)) \simeq \mathbb{Z}^{n+1}$$

agree with classical k-theory groups of $C(\mathbb{C}P^n)$

Ideal structure given a hereditary & saturated subset $\subseteq E^0$ we get a gauge invariant ideal I_H

$$C^*(E)/I_H \simeq C^*(E/H)$$

Def $H \subseteq E^0$ is hereditary & saturated if

① $v \in H$, $w \in E^0$ if $\exists$ path from v to $w \Rightarrow w \in H$;

② $w \in E^0$ $0 < |s^{-1}(w)| < \infty$ $\forall e \in E^1$ with $s(e) = w$, $r(e) = H$ $\Longrightarrow$ $w \in H$

for F_n $\{w_{n+1}\} \subseteq F_n^0$ is both hereditary & saturated

$$F_n / \{w_{n+1}\} = F_{n-1}$$

$$
\begin{array}{ccccccccc}
0 & \to & K & \to & C(\mathbb{CP}_q^n) & \to & C(\mathbb{CP}_q^{n-1}) & \to & 0 \\
 & & \wr\! \| & & \wr\! \| & & \wr\! \| & & \\
 & & I_{w_{n+1}} & & C^*(F_n) & & C^*(F_n / \{w_{n+1}\}) & &
\end{array}
$$

$n=1$

$$
\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & C(\mathbb{CP}_q^1) \arrow[r] & \mathbb{C} \arrow[r] \arrow[l, bend right, "s"'] & 0
\end{tikzcd}
$$

the sequence splits.

<u>How about $n>1$</u> ?

② KK-theory, KK-equivalences & split extensions

Kasparov's bivariant K-theory

for any pair of sep. C^* alg. A & B _ B σ unital

Kasparov ('80): $\mathbb{Z}_2$ graded group $KK_*(A, B)$ s.t

- $KK_*(\mathbb{C}, B) \simeq K_*(B)$;
- $KK_*(A, \mathbb{C}) \simeq K^*(A)$;
- $\exists$ associative + bilinear product

$$KK_i(A, B) \times KK_j(B, C) \longrightarrow KK_{i+j}(A, C)$$

$i, j = 0, 1$

(recovers the index pairings)

- Any $x \in KK_0(A, B)$ $\quad K_i(A) \longrightarrow K_i(B)$
- $KK(B, B)$ is a ring, $K^*(B)$, $K_*(B)$ modules over this ring

Def two C^*-algebras A, B are KK-equivalent
(or K-equivalent)

if $\exists\, [\alpha] \in KK(A,B)$, $[\beta] \in KK(B,A)$

$$[\alpha] \otimes_B [\beta] = 1_{KK(A,A)}$$

$$[\beta] \otimes_A [\alpha] = 1_{KK(B,B)}$$

2 C^* alg. that are KK-equiv. cannot be distinguished by KK-theory.

Examples of KK-equivalence

- $\mathbb{C}$, $C_0(\mathbb{R}^2)$ are KK-equivalent (Bott periodicity)
- $E \to X$ complex v.b. over loc. cmpct X

  $C_0(X) \sim C_0(E)$ are KK-eq. (Thom iso)

- Morita equivalent C^*-algebras are KK-equivalence

- The UCT class & KK-equivalence

class of C^*-algebras that satisfy the universal coefficient theorem of Rosenberg & Schochet

- all the commutative C^*-algebras.
- $\mathcal{K}$ are in the UCT class
- stable under extensions

Thm (RS 87). let A and B separable in the UCT class. Suppose $K_i(A) \simeq K_i(B)$ $i = 0, 1$, then A & B are KK-equivalent.

Isom. in K-theory can be lifted to KK-equivalences.

This applies to $C(\mathbb{CP}^n_p)$ because of

$$0 \longrightarrow K \longrightarrow C(\mathbb{CP}^n_p) \longrightarrow C(\mathbb{CP}^{n-1}_q) \to 0$$

by induction on n $C(\mathbb{CP}^n_p)$ is in the UCT class

$$\Rightarrow \boxed{C(\mathbb{CP}^n_p) \underset{KK}{\sim} \mathbb{C}^{n+1} \underset{KK}{\sim} C(\mathbb{CP}^n)}$$

known by work of Neshveyev–Tuset (2011)

$(0,1) \ni q \mapsto C(G_q/K_q)$ cont. field of C^* alg

$$\overset{\displaystyle G_q/K_q}{\parallel} \atop SU_q(n+1)/U_q(n)$$

Split exact sequences and KK-theory

let us consider an extension

$$0 \to J \xrightarrow{j} E \underset{q}{\overset{s}{\rightleftarrows}} B \to 0 \quad \text{that splits}$$

$\exists$ *-hom. $s : B \to E$ s.t. $q \circ s = \mathrm{id}_B$

Fact: the element $[j] \oplus [s] \in KK(J \oplus B, E)$

$\in KK(J, E)$ $\quad$ $KK(B, E)$

is a KK-equivalence.

explicit inverse $[\alpha] \in KK(E, J \oplus B)$

One ingredient is $[q] \in KK(E, B)$

the other class is $[\pi_s] \in KK(E, J)$ splitting homom.

Cuntz's picture of KK

- $KK_h(A, B)$ cycles is a pair (ϕ_+, ϕ_-) of $*$ hom. $A \to \mathcal{M}(K \otimes B)$ s.t

$$\phi_+(a) - \phi_-(a) \in K \otimes B \quad \forall a \in A$$

In our case $\quad e_p : E \to K \otimes E$

$$e \mapsto p \otimes e \qquad p \text{ rank 1 proj.}$$

$$J \hookrightarrow E$$

$\exists$ canonical map $r_J : \mathcal{M}(K \otimes E) \to \mathcal{M}(K \otimes J)$

$$(r_J \circ e_p, \; r_J \circ e_p \circ s \cdot q) \sim (1, \; s \circ q)$$

with grading

$[\pi_s] \otimes [q] \in KK(E, J \otimes B)$ is the inverse

(3) The splitting and KK-equivalences

Our strategy is indeed to construct a splitting for the exact sequence

$$0 \to K \to C(\mathbb{CP}^n_q) \to C(\mathbb{CP}^{n-1}_q) \to 0$$

Observation: the splitting exists because of UCT & hom. algebra argument.

Lemma A separable in UCT, $K_0(A)$ free Abelian &
$K_1(A) = 0 \Rightarrow$
any $0 \to K^{\oplus m} \to E \to A \to 0$ splits

$$KK_1(A, K^{\oplus m}) \simeq \bigoplus KK_1(A, K) \overset{\text{UCT}}{=} \operatorname{Hom}(K_1(A), \mathbb{Z})^{\oplus m} \oplus \operatorname{Hom}(K_0(A), \mathbb{Z}) = 0$$

$\operatorname{Ext}(K_0(A), \mathbb{Z}) = 0$

the graph algebra is crucial in the construction of

$\delta_{\nu_i} \quad C(C\mathbb{R}P_q^{n-1}) \longrightarrow C(C\mathbb{R}P_p^n)$

$\qquad\qquad\; \cong$

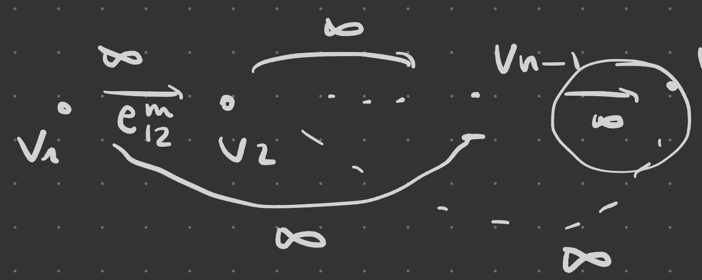
$\qquad\qquad C^*(F_{n-1}) \longrightarrow C^*(F_n)$

gen by $\quad P_{v_i} \qquad\qquad\qquad\qquad P_{w_i}$

$S_{e_{ij}^m} \quad i,j = 1 \dots n \qquad S_{f_{ij}^m} \quad i,j = 1 \dots n+1$

$m \in \mathbb{N}$

F_{n-1} vertices v_i $i = 1 \dots n$



edges with e

eg e_{12}^n $n = 0 \dots \infty$

for F_n $n+1$ vertices w_i

$f_{ij}^m \quad m \in \mathbb{N}$, $i,j = 1 \dots n+1$

- $P_{v_i} \mapsto P_{w_i} \qquad i = 1, \dots, n-1$
- $P_{v_n} \mapsto P_{w_n} + P_{w_{n+1}}$
- $S_{e_{ij}^m} \mapsto S_{f_{e_{ij}}^m} \qquad j \neq n$
- $S_{e_{in}^m} \mapsto S_{f_{in}^m} + S_{f_{i,n+1}^m}$

Thm (A. Mikkelsen '21)

the map $s_n : C(\mathbb{C}P_p^{n-1}) \longrightarrow C(\mathbb{C}P_p^n)$ defined as

above is a splitting for the s.e.s

$$
\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "j_n"'] & C(\mathbb{C}P_q^n) \arrow[r, "q_n"'] & C(\mathbb{C}P_q^{n-1}) \arrow[l, dashed, bend right, "s_n"'] \arrow[r] & 0
\end{tikzcd}
$$

$$[j_n] \oplus [s_n] \in KK(K \oplus C(\mathbb{C}P_q^{n-1}), C(\mathbb{C}P_q^n))$$

is a KK-equivalence

$\rightsquigarrow$ KK eq. between $C(\mathbb{C}P_p^n)$ & $\mathbb{C}^{n+1}$

up to m.e

Thank you!

$$KK_1(K^{\oplus m}, A) = 0$$

prove $KK_1(\mathbb{C}, A) = 0$ $\qquad A = C_0(X)$

any ext

$$0 \to K \to E \to A \to 0$$