Algebraic Kirchberg-Phillips and twisted Katsura algebras

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Graph Algebras
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Let $p \neq q \geq 1$, $2 \leq n_1, \dots, n_p, m_1, \dots, m_q \leq \infty$.

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Proposition

The Morita class of $C^*(E)$ is invariant under both standard graph moves and Cuntz' splice.

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Theorem (Johansen-Sørensen, [JS16])

There is no $*$-algebra isomorphism $L_2(\mathbb{Z}) \to L_{2^-}(\mathbb{Z})$.

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Theorem (C-Montero, [CM20])

E, F spi, $(\mathfrak{BF}(E), [1_E]) \cong (\mathfrak{BF}(F), [1_F]) \Rightarrow \exists\ \phi : L(E) \longleftrightarrow L(F) : \psi$ unital, $\psi \circ \phi \sim \mathrm{id}_{L(E)}$, $\phi \circ \psi \sim \mathrm{id}_{L(F)}$.

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Katsura algebras

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$C^*_{A,B}$ in KK

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\downarrow \cong & & \downarrow \cong & & \\
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Recall $KK(\mathbb{C}[-1], \mathbb{C}[-1]) = KK(\mathbb{C}, \mathbb{C}) = K_0(\mathbb{C}) = \mathbb{Z}$.

$$f = \begin{bmatrix} I - A^t & 0 \\ 0 & I - B^t \end{bmatrix}$$

$C^*_{A,B}$ in KK, continued

We have a KK-triangle

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$\mathcal{O}_{A,B}$ in kk

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$$g = \begin{bmatrix} M & P \\ 0 & N \end{bmatrix} \text{ with } M, N \in \mathbb{Z}^{n\times m} \text{ and } P \in (\ell^*)^{n\times m}.$$

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Thank you!