CW structures in Noncommutative Geometry

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03/07/2023

Based on:

F. D'Andrea, P. M. Hajac, T. Maszczyk, A. Sheu, B. Zielinski, arXiv:2002.09015.

Why CW-complexes?

Combinatorial nature that allows for homotopy computations.

(CW-complexes : homotopy $=$ graph C*-algebras : K-theory)

Homotopy equivalence:
$$\begin{tikzcd} X \arrow[loop left, "\mathrm{id}_X"] \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "g"] \arrow[loop right, "\mathrm{id}_Y"'] \end{tikzcd}$$

$\Downarrow \qquad \not\Uparrow$

Weak homotopy equivalence: $X \xrightarrow{f} Y$ that induces iso in homotopy.

Whitehead Theorem

- Every weak homotopy equivalence between CW-complexes is a (strict) homotopy equivalence.

CW-approximation Theorem

- For every topological space X there is a CW-complex Z and a weak homotopy equivalence $f : Z \to X$.

We are interested in finite CW-complexes $\implies$ compact Hausdorff spaces.

(Notations. CHAUS $:=$ category of compact Hausdorff spaces.)

The category of quantum spaces

Gelfand-Naimark functor from CHAUS to commutative unital C*-algebras:

$$\left\{ \begin{array}{c} X \\ f \big\downarrow \\ Y \end{array} \right\} \longmapsto \left\{ \begin{array}{c} C(X) \\ f^* \big\uparrow \\ C(Y) \end{array} \right\}$$

We define:

$$\text{QSPACE} := \left\{\text{Unital C*-algebras}\right\}^{\text{op}}$$

Objects in QSPACE are denoted by $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, etc. The corresponding (not necessarily commutative) C*-algebras by $C(\mathbf{X}), C(\mathbf{Y}), C(\mathbf{Z})$, etc. In this category, a morphism

$$\mathbf{X} \longrightarrow \mathbf{Y} \qquad \text{means a unital *-hom} \qquad C(\mathbf{X}) \longleftarrow C(\mathbf{Y})$$

⚠ Not a concrete category!

CHAUS embeds as a full subcategory in QSPACE.

Goal: develop a theory of CW-complexes in QSPACE.

What generalized (co)homology?

Exercise

Let $n \geqslant 1$. Compute the K-theory:

$$K(n \text{ points}) = \mathbb{Z}^n \qquad K(\mathbb{C}P^{n-1}) = \mathbb{Z}[x]/(x^n)$$

$\implies K(n \text{ points}) \cong K(\mathbb{C}P^{n-1})$ as (graded) abelian groups, but $\not\cong$ as rings.

Where does the ring structure come from? Let A be a unital C*-algebra.

If (and only if) A is commutative,

then $m : A \otimes A \to A$ is a *-homomorphism

$\implies m$ induces a group hom $K(A) \otimes K(A) \to K(A \otimes A) \xrightarrow{K(m)} K(A)$.

A way out? Let X, Y be in CHAUS and $\varphi : K(Y) \to K(X)$ an iso of abelian groups.

If φ is induced by a cont. map $X \to Y$, then it is an isomorphism of rings.

In general

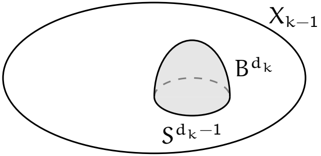
$$\text{K-equivalence} \implies \text{iso of K-groups}$$

CW-complexes

Let X be in CHAUS. A (finite) CW-structure on X is a filtration “by skeleta”

$$X_0 \hookrightarrow X_1 \hookrightarrow \ldots \hookrightarrow X_{n-1} \hookrightarrow X_n = X$$

s.t. X_0 is finite discrete and each X_k is obtained from X_{k-1} by attaching closed balls.



In categorical terms, for each $1 \leqslant k \leqslant n$ we have a pushout diagram

$$\begin{tikzcd}
Y & & \\
& X_k \arrow[ul, dashed] & X_{k-1} \arrow[l] \arrow[ull, bend right] \\
& B^{d_k} \arrow[u] \arrow[uul, bend left] & S^{d_k-1} \arrow[l, hook', "\partial"'] \arrow[u, "a"']
\end{tikzcd}$$

a = attaching map, ∂ = boundary map.

Applications? Get K-theory recursively (Mayer-Vietoris).

Gluing and pullbacks

A commutative diagram (of sets):

$$\begin{tikzcd} P & Y \arrow[l, "j"'] \\ X \arrow[u, "i"] & Z \arrow[l, "g"] \arrow[u, "f"'] \end{tikzcd}$$

is a pushout diagram if the map

$$P \xleftarrow{i \sqcup j} \frac{X \sqcup Y}{f(z) \sim g(z)} \quad \text{is iso.}$$

We get P by "gluing" X and Y along Z.

A commutative diagram of C^*-algebras:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

is a pullback diagram if the map

$$A \xrightarrow{i \times j} \{(b, c) \in B \times C : f(b) = g(c)\}$$

$$\|$$

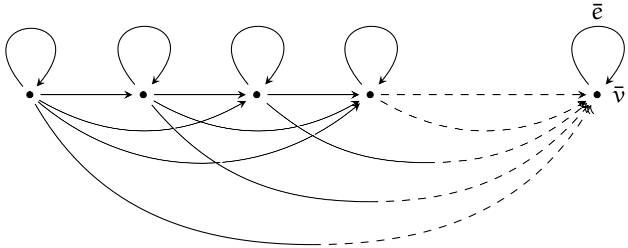
$$B \times_D C$$

is an isomorphism.

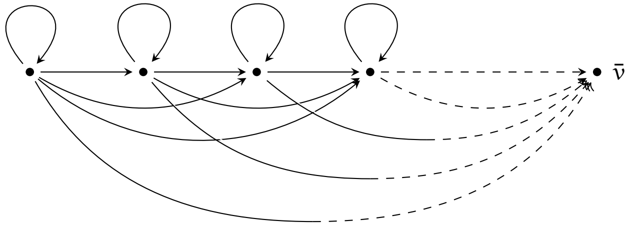
In QSPACE, Eilers-Loring-Pedersen NCCW-complexes (1998): tensor $C(B^d)$ and $C(S^{d-1})$ with finite-dimensional C*-algebras (finite q-spaces). Not enough for our purposes!

q-spheres and balls

[Hong & Szymański, 2002 & 2008]



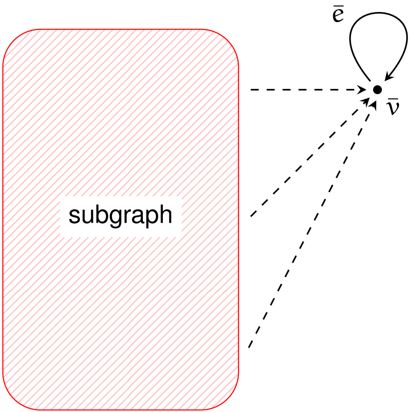
S_q^{2n+1}



B_q^{2n}

Trimmable graphs

[Arici, D'Andrea, Hajac, Tobolski, 2018]



A graph with a distinguished vertex $\bar{v}$ is called $\bar{v}$-trimmable if:

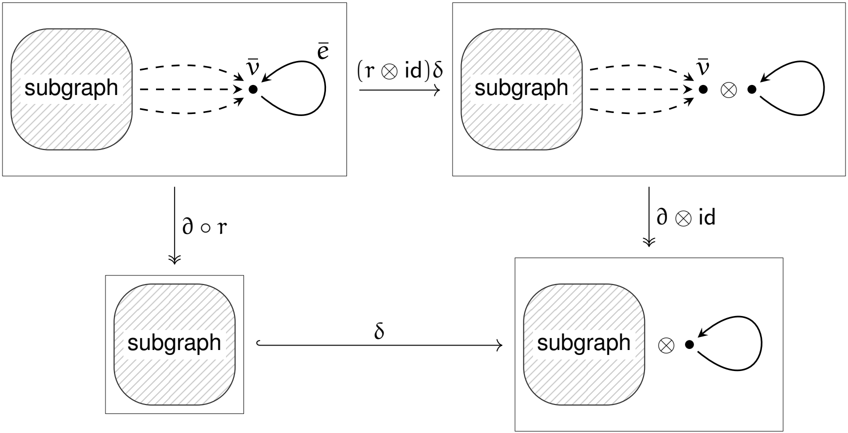
1. $\bar{v}$ emits one loop $\bar{e}$ and no other edges;
2. $\bar{v}$ is the target of other edges, besides $\bar{e}$;
3. if a vertex of the subgraph emits arrow(s) ending in $\bar{v}$, it must also emit other arrow(s) not ending in $\bar{v}$.

Examples:

- Vaksman-Soibelman quantum spheres S_q^{2n+1},
- ...

Pullback structure of trimmable graph C^*-algebras

A $U(1)$-equivariant (cf. gauge action) pullback diagram:



In the case of q-spheres the $U(1)$-invariant part is the pullback diagram:

$$
\begin{tikzcd}
C(\mathbb{C}\mathbf{P}_q^n) \arrow[r] \arrow[d] & C(\mathbb{C}\mathbf{P}_q^{n-1}) \arrow[d] \\
C(B_q^{2n}) \arrow[r, "\partial^*"] & C(S_q^{2n-1})
\end{tikzcd}
$$

or the pushout in QSPACE:

$$
\begin{tikzcd}
\mathbb{C}\mathbf{P}_q^n & \mathbb{C}\mathbf{P}_q^{n-1} \arrow[l] \\
B_q^{2n} \arrow[u] & S_q^{2n-1} \arrow[l, "\partial"] \arrow[u]
\end{tikzcd}
$$

Mayer-Vietoris

If $\{X, Y\}$ is an open cover of a smooth n-manifold P, one has the pushout diagram:

$$\begin{tikzcd} P & Y \arrow[l] \\ X \arrow[u] & Z := X \cap Y \arrow[l, hook'] \arrow[u] \end{tikzcd}$$

a short exact sequence of k-forms, and a long exact sequence in de Rham cohomology

$$\begin{tikzcd} 0 \arrow[r] & H^0(P) \arrow[r] & H^0(X) \oplus H^0(Y) \arrow[r] & H^0(Z) \arrow[dll, rounded corners, to path={[pos=0.25] -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\ & H^1(P) \arrow[r] & H^1(X) \oplus H^1(Y) \arrow[r] & H^1(Z) \arrow[dll, rounded corners, to path={[pos=0.25] -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\ & H^2(P) \arrow[r] & \cdots\cdots\cdots\cdots \arrow[r] & H^n(Z) \arrow[r] & 0 \end{tikzcd}$$

This holds for more general (co)homology theories (e.g. singular) and one-injective pushout diagrams (e.g. of CW-complexes). It holds in QSPACE for K-theory.

In QSPACE we want to attach cells with “injective” boundary map and prescribed K-theory.

What kind of cells?

In QSPACE, the opposite of a surjective *-hom. $C(\mathbf{Y}) \twoheadrightarrow C(\mathbf{X})$ is depicted $\mathbf{X} \hookrightarrow \mathbf{Y}$.

A morphism $\partial : \boldsymbol{S}^{d-1} \hookrightarrow \mathbf{B}^d$ is called a boundary map from a K-sphere to a K-ball if it induces one of the following short exact sequences in K-theory. Here $K^*(\mathbf{X}) := K_*(C(\mathbf{X}))$.

d even:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^0(B^d) & \longrightarrow & K^0(S^{d-1}) & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \wr\| & & \wr\| & & & & \\ & & \mathbb{Z} & & \mathbb{Z} & & & & \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^1(B^d) & \longrightarrow & K^1(S^{d-1}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \wr\| & & \wr\| & & & & \\ & & 0 & & \mathbb{Z} & & & & \end{array}$$

d odd:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^0(B^d) & \longrightarrow & K^0(S^{d-1}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \wr\| & & \wr\| & & & & \\ & & \mathbb{Z} & & \mathbb{Z} \oplus \mathbb{Z} & & & & \end{array}$$

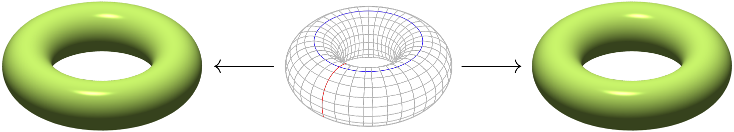
$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^1(B^d) & \longrightarrow & K^1(S^{d-1}) & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \wr\| & & \wr\| & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Examples:

- Vaskman-Soibelman quantum spheres and balls (described by graph C*-algebras).
- Heegaard quantum spheres and polydisks [Hajac, Kaygun, Nest, Pask, Sims, Zieliński].

Heegaard quantum spheres

S^3 as a pushout of (Heegaard splitting):



$$B^2 \times S^1 \xleftarrow{\ \mathrm{id}\ } S^1 \times S^1 \xrightarrow{\ \vec{z} \mapsto -\vec{z}\ } B^2 \times S^1$$

Let $\mathcal{T}$ be the Toeplitz C*-algebra. We define

- Quantum polydisk

$$C(\mathbf{D}_q^{\times n}) := \mathcal{T}^{\otimes n}$$

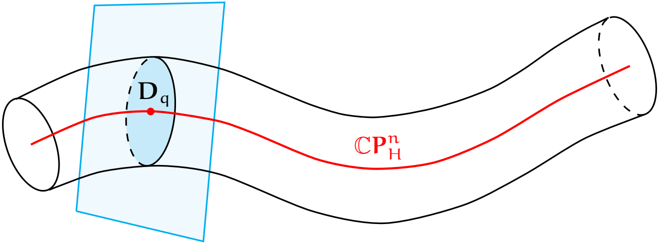
- Heegaard quantum sphere $\mathbf{S}_H^{2n-1}$ defined as a pushout similar to the one above, with the closed disk B^2 replaced by $\mathbf{D}_q$. It is the boundary of a quantum polydisk

$$C(\mathbf{S}_H^{2n-1}) \cong \mathcal{T}^{\otimes n} / \mathcal{K}^{\otimes n}$$

- Quantum projective spaces

$$C(\mathbb{C}\mathbf{P}_H^n) := C(\mathbf{S}_H^{2n-1})^{U(1)}$$

A tubular neighbourhood theorem



Theorem

The map

$$C(\mathbb{C}\mathbf{P}^n_{\mathrm{H}}) \xrightarrow{\mathrm{id}\otimes 1_{\mathcal{T}}} C\big(\mathrm{TN}(\mathbb{C}\mathbf{P}^n_{\mathrm{H}})\big) := \big(C(\mathbf{S}^{2n+1}_{\mathrm{H}}) \otimes \mathcal{T}\big)^{U(1)}$$

is a K-equivalence.

We will write

$$\mathrm{TN}(\mathbb{C}\mathbf{P}^n_{\mathrm{H}}) \xrightarrow{\sim} \mathbb{C}\mathbf{P}^n_{\mathrm{H}}$$

Almost a filtration by skeleta

$\exists$ pushout

TN thm.

$$\begin{tikzcd}
 & & \widetilde{\mathbb{C}\mathbf{P}^n_{\mathrm{H}}} & & \\
 & \mathbb{C}\mathbf{P}^{n-1}_{\mathrm{H}} \arrow[ur, hook, dashed] & & \mathbb{C}\mathbf{P}^n_{\mathrm{H}} \arrow[ul, dashed, "\sim"'] & \\
 & & \mathrm{TN}(\mathbb{C}\mathbf{P}^{n-1}_{\mathrm{H}}) \arrow[ul, "\sim"] \arrow[ur, hook] \arrow[u, phantom, "\overset{\bullet}{\vee}"] & & \mathbf{D}_q^{\times n} \arrow[ul] \\
 & & & \mathbf{S}^{2n-1}_{\mathrm{H}} \arrow[ul] \arrow[ur, hook] \arrow[uu, phantom, "\overset{\bullet}{\vee}"] &
\end{tikzcd}$$

Heegaard constr.

Idea: CW-structure up to homotopy.

What homotopy category?

Definition. A cw-Waldhausen category is a category $\mathscr{C}$ with pushouts, with an initial and a terminal object, and two distinguished classes of morphisms, **Cof** of cofibrations depicted $\hookrightarrow$ and **Weq** of weak equivalences depicted $\xrightarrow{\sim}$, such that

- isomorphisms are both weak equivalences and cofibrations;
- a composition of morfisms in **Cof** (resp. **Weq**) is still in **Cof** (resp. **Weq**);
- for every object X, the unique morphism from the initial object to X is a cofibration;
- in a pushout diagram of the form

$$\begin{tikzcd} P & X \arrow[l, "f"'] \\ Y \arrow[u, "g"] & Z \arrow[l, hook'] \arrow[u, "h"'] \arrow[ul, phantom, "\cdot\lrcorner"] \end{tikzcd}$$

one has (i) $f \in$ **Cof**,
(ii) $g \in$ **Weq** $\iff$ $h \in$ **Weq**.

- given any commutative diagram of the form

$$\begin{tikzcd} X \arrow[d, "\wr"] & Z \arrow[l] \arrow[r, hook] \arrow[d, "\wr"] & Y \arrow[d, "\wr"] \\ \widetilde{X} & \widetilde{Z} \arrow[l] \arrow[r, hook] & \widetilde{Y} \end{tikzcd}$$

the induced map $X \sqcup_Z Y \to \widetilde{X} \sqcup_{\widetilde{Z}} \widetilde{Y}$ is a weak equivalence.

The last axiom can be rephrased as follows. Given a commutative diagram

$$
\begin{tikzcd}
& P \arrow[rrrr, dashed, bend left] & & & & \widetilde{P} & \\
X \arrow[ur] \arrow[rrrr, bend right] & & Y \arrow[ul] \arrow[rrrr, bend left] & & \widetilde{X} \arrow[ur] & & \widetilde{Y} \arrow[ul] \\
& Z \arrow[ul] \arrow[ur, hook] \arrow[uu, phantom, "\ulcorner", very near end] \arrow[rrrr, bend right] & & & & \widetilde{Z} \arrow[ul] \arrow[ur, hook] \arrow[uu, phantom, "\ulcorner", very near end] &
\end{tikzcd}
$$

where each square is a pushout, if the horizontal solid lines are weak equivalences, then the dashed one is a week equivalence as well.

Example

$$
\begin{aligned}
\mathscr{C} &= \text{QSPACE} \\
\text{initial obj} &= \text{empty space} \\
\text{terminal obj} &= \text{one-point space} \\
\textbf{Cof } (\hookrightarrow) &= \text{opposite of surjective unital } *\text{-homomorphisms} \\
\textbf{Weq } (\xrightarrow{\sim}) &= \text{opposite of K-equivalences}
\end{aligned}
$$

If $\mathscr{C}$ is a cw-Waldhausen category, $\mathrm{Ho}(\mathscr{C}) := \mathscr{C}[\mathbf{Weq}^{-1}]$ is well-defined.

Morphisms in $\mathbf{Weq}^{-1} \circ \mathbf{Cof}$, depicted as $\rightarrowtail$, are called weak cofibrations and are represented by roofs:

$$
X \rightarrowtail Y = \left[\begin{tikzcd}[ampersand replacement=\&] \& \widetilde{Y} \& \\ X \arrow[ur, hook] \& \& Y \arrow[ul, "\sim"'] \end{tikzcd}\right],
$$

Composition of weak cofibrations:

$$
\begin{tikzcd}
 & & \widetilde{\widetilde{Z}} & & \\
 & \widetilde{Y} \arrow[ur, hook, dashed] & & \widetilde{Z} \arrow[ul, dashed, "\sim"'] & \\
X \arrow[ur, hook] & & Y \arrow[ul, "\sim"'] \arrow[ur, hook] \arrow[uu, phantom, "\ulcorner" rotate=-45] & & Z \arrow[ul, "\sim"']
\end{tikzcd}
$$

Weak CW-complexes

Definition. A weak CW-structure is a (finite) sequence of weak cofibrations

$$X_0 \rightarrowtail X_1 \rightarrowtail \ldots \rightarrowtail X_{n-1} \rightarrowtail X_n$$

each represented by a roof

$$\begin{tikzcd}
 & \widetilde{X}_k & \\
X_{k-1} \arrow[ur, hook] & & X_k \arrow[ul, "\wr"']
\end{tikzcd}$$

where X_0 is a finite quantum space, each cofibration is part of a pushout of the form

$$\begin{tikzcd}
 & \widetilde{X}_k & \\
X_{k-1} \arrow[ur, hook] & & B^{d_k} \arrow[ul] \\
 & S^{d_k - 1} \arrow[ul] \arrow[ur, hook, "\partial_k"'] \arrow[u, phantom, "\ulcorner"] &
\end{tikzcd}$$

and each $\partial_k : S^{d_k - 1} \hookrightarrow B^{d_k}$ is a boundary map from a K-sphere to a K-ball.

Computation of K-theory

Every weak CW-structure induces six-term exact sequences (one for each k):

$$\begin{tikzcd}
K^0(\mathbf{X}_k) \arrow[r] & K^0(\mathbf{X}_{k-1}) \arrow[r] & K^0(\mathbf{S}^{d_k-1})/K^0(\mathbf{B}^{d_k}) \arrow[d, "\delta_{01}"] \\
K^1(\mathbf{S}^{d_k-1})/K^1(\mathbf{B}^{d_k}) \arrow[u, "\delta_{10}"] & K^1(\mathbf{X}_{k-1}) \arrow[l] & K^1(\mathbf{X}_k) \arrow[l]
\end{tikzcd}$$

Theorem. If d_k is odd:

$$K^0(\mathbf{X}_{k-1}) \cong K^0(\mathbf{X}_k) \oplus \ker \delta_{01}$$

$$K^1(\mathbf{X}_k) \cong K^1(\mathbf{X}_{k-1}) \oplus \operatorname{im} \delta_{01} \qquad \text{(only if } K^1(\mathbf{X}_{k-1}) \text{ is free)}$$

If d_k is even:

$$K^0(\mathbf{X}_k) \cong K^0(\mathbf{X}_{k-1}) \oplus \operatorname{im} \delta_{10} \qquad \text{(only if } K^0(\mathbf{X}_{k-1}) \text{ is free)}$$

$$K^1(\mathbf{X}_{k-1}) \cong K^1(\mathbf{X}_k) \oplus \ker \delta_{10}$$

Quantum projective spaces

Assume that we have a filtration

$$\star = \mathbf{X}^0 \rightarrowtail \mathbf{X}^1 \rightarrowtail \cdots \rightarrowtail \mathbf{X}^{n-1} \rightarrowtail \mathbf{X}^n = \mathbf{X}$$

where at each step we attach a single even-dimensional cell.

Theorem

$$K^0(\mathbf{X}) = \mathbb{Z}^{n+1} \qquad K^1(\mathbf{X}) = 0$$

Both Vaksman-Soibelman and Heegaard quantum projective spaces are special cases of this class of examples.

In fact, for these we have a weak equivalence of weak filtrations by skeleta

$$\begin{tikzcd}
\mathbb{C}\mathbf{P}^0_H \arrow[r, tail] \arrow[d, "\wr"] & \mathbb{C}\mathbf{P}^1_H \arrow[r, tail] \arrow[d, "\wr"] & \dots \arrow[r, tail] & \mathbb{C}\mathbf{P}^{n-1}_H \arrow[r, tail] \arrow[d, "\wr"] & \mathbb{C}\mathbf{P}^n_H \arrow[d, "\wr"] \\
\mathbb{C}\mathbf{P}^0_q \arrow[r, hook] & \mathbb{C}\mathbf{P}^1_q \arrow[r, hook] & \dots \arrow[r, hook] & \mathbb{C}\mathbf{P}^{n-1}_q \arrow[r, hook] & \mathbb{C}\mathbf{P}^n_q
\end{tikzcd}$$

What next?

- CW structure of quantum flag manifolds?
- Matassa, Yuncken: (higher rank) graph C*-algebra description for $q = 0$.
- Brzeziński, Krähmer, Ó Buachalla, Strung: quantum flag manifolds as graph C*-algebras (MFO report 2022). E.g., they claim

$$C(F_q(1,2,3)) \cong C^*\left(\begin{tikzcd} & \bullet \arrow[r, "\infty"] & \bullet \arrow[dr, "\infty"] & \\ \bullet \arrow[ur, "\infty"] \arrow[dr, "\infty"'] & & & \bullet \\ & \bullet \arrow[r, "\infty"'] & \bullet \arrow[ur, "\infty"'] & \end{tikzcd}\right)$$

- Trimmable higher rank graph C*-algebras?
- ...

Thanks for your attention!