The generation conjecture for regular graphs

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Content

Outline

Singular and regular vertices

Graphs $E = (E^0, E^1, r, s)$ are given with $r, s : E^1 \to E^0$. Both E^0, E^1 must be countable.

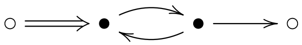
Definitions

Let E be a graph and $v \in E^0$.

- v is a *sink* if $|s^{-1}(\{v\})| = 0$
- v is an *infinite emitter* if $|s^{-1}(\{v\})| = \infty$

Definition

v is *singular [○]* if v is a sink or an infinite emitter. v is *regular [•]* if it is not singular. A graph is *regular* when all its vertices are.



Graph C^*-algebras

Definition

The *graph C^*-algebra* $C^*(E)$ is given as the universal C^*-algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges subject to the Cuntz–Krieger relations

1. $s_e^* s_e = p_{r(e)}$
2. $s_e s_e^* \leqslant p_{s(e)}$
3. $p_v = \sum_{s(e)=v} s_e s_e^*$ for every regular v

$C^*(E)$ is unital precisely when E has finitely many vertices.

Observation

$$\gamma_z(p_v) = p_v \qquad \gamma_z(s_e) = z s_e$$

induces a **gauge action** $\mathbb{T} \mapsto \mathrm{Aut}(C^*(E))$

Definition

$$\mathfrak{D}_E = \overline{\mathrm{span}}\{s_\alpha s_\alpha^* \mid \alpha \text{ path of } E\}$$

Note that $\mathfrak{D}_E$ is commutative and that

$$\mathfrak{D}_E \subseteq \mathfrak{F}_E = \{a \in C^*(E) \mid \forall z \in \mathbb{T} : \gamma_z(a) = a\}$$

Definition

With y, z $\in \{0,1\}$ we write

$$(E,F) \in \overline{\underline{\mathsf{1yz}}}$$

when there exists a $*$-isomorphism $\varphi : C^*(E) \to C^*(F)$ with

- $\varphi \circ \gamma_z = \gamma_z \circ \varphi$ when y $= 1$
- $\varphi(\mathfrak{D}_E) = \mathfrak{D}_F$ when z $= 1$

and

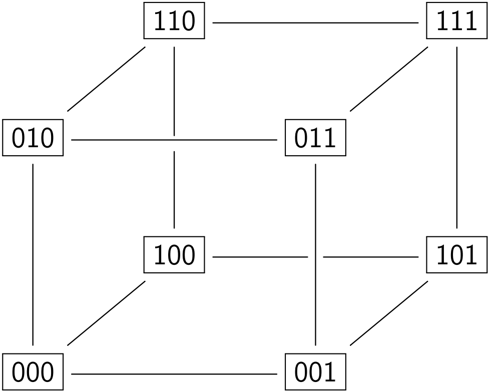
$$(E,F) \in \overline{\underline{\mathsf{0yz}}}$$

when there exists a $*$-isomorphism $\varphi : C^*(E) \otimes \mathbb{K} \to C^*(F) \otimes \mathbb{K}$ which additionally satisfies

- $\varphi \circ (\gamma_z \otimes \mathsf{Id}) = (\gamma_z \otimes \mathsf{Id}) \circ \varphi$ when y $= 1$
- $\varphi(\mathfrak{D}_E \otimes c_0) = \mathfrak{D}_F \otimes c_0$ when z $= 1$

Eight kinds of isomorphism

[Exact, gauge, diagonal]



Goals

Interpret geometrically. Connect to dynamics.
Find complete invariants. Decide decidability.

Report card

[Exact, gauge, diagonal]

For regular graphs:

xyz	GEO	DYN	INV	DEC	m
000	$\surd$	$\div$	$\surd$	$\surd$	4
001	$\surd$	$\surd$	$\surd$	$\surd$	6
010	$\div$	$\surd$	$(\surd)$	$(\surd)$	7
011	$\surd$	$\surd$	$\div$	$\div$	9
100	$\surd$	$\div$	$\surd$	$\surd$	3
101	$(\surd)$	$\surd$	$(\surd)$	$(\surd)$	7
110	$\div$	$\surd$	$(\surd)$	$\div$	5
111	$\surd$	$\surd$	$\div$	$\div$	6

Outline

To a finite graph $E = (E_0, E_1, r, s)$ we associate X_E defined as

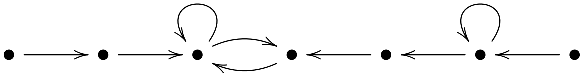
$$\mathsf{X}_E = \{(e_n) \in (E_1)^{\mathbb{Z}} \mid r(e_n) = s(e_{n+1})\}$$

Note that X_E is closed in the topology of $(E_1)^{\mathbb{Z}}$ and comes equipped with a shift map $\sigma : \mathsf{X}_E \to \mathsf{X}_E$ which is a homeomorphism. We call X_E a **shift space** (2-sided, of finite type) over the **alphabet** E_1.

Definition

Shift spaces X and Y are *conjugate* (written $X \simeq Y$) if there is a shift-invariant homeomorphism $\varphi : X \to Y$.

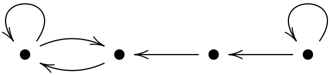
Essentiality



$$\mathsf{X}_E = \{(e_n) \in (E_1)^{\mathbb{Z}} \mid r(e_n) = s(e_{n+1})\}$$

Definition

The *essential part* E_{ess} of a regular graph is obtained by deleting all sources repeatedly. We say the graph is *essential* when $E = E_{\text{ess}}$.



Note that $\mathsf{X}_E = \mathsf{X}_{E_{\text{ess}}}$.

Rigidity

[Exact, gauge, diagonal]

Theorem [Carlsen-Rout]

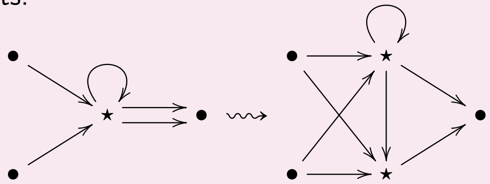
For regular essential graphs E and F, the following are equivalent

1. $(E, F) \in \overline{011}$
2. $\mathsf{X}_E \simeq \mathsf{X}_F$.

Outsplitting

Move (O)

Outsplit at any vertex using a partition of outgoing edges into non-empty sets:



Invariance

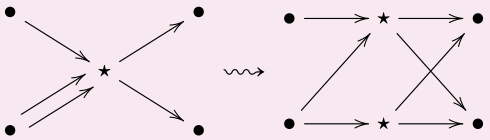
$\langle \mathbf{(O)} \rangle \subseteq \underline{\overline{111}}$

For general (non-regular) graphs, one must also require that at most one set in the partition is infinite.

Insplitting

Move (I)

Intsplit at any vertex using a partition of incoming edges into non-empty sets:



Invariance

$\langle(\mathbf{I})\rangle \subseteq \overline{011}$

For general (non-regular) graphs one must also require that the vertex is regular.

Essential rigidity

[Exact, gauge, diagonal]

Theorem [Carlsen-Rout, Williams]

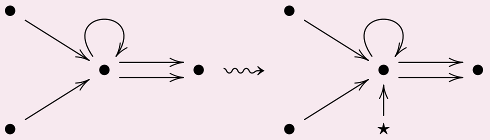
For regular essential graphs E and F, the following are equivalent

1. $(E, F) \in \overline{011}$
2. $\mathsf{X}_E \simeq \mathsf{X}_F$
3. $(E, F) \in \langle \mathbf{(O)}, \mathbf{(I)} \rangle$

Add source

Move (S)

Add a new source anywhere



General invariance

$\langle \textbf{(S)} \rangle \subseteq \overline{001}$

Specialized invariance

$\langle \textbf{(S)} \rangle \subseteq \overline{011}$ within the class of regular graphs.

Note that $(E, E_{\text{ess}}) \in \langle \textbf{(O)}, \textbf{(S)} \rangle$ follows directly from the definition.

Corollary

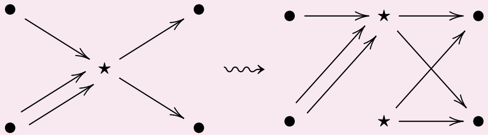
For regular graphs E and F, the following are equivalent

1. $(E, F) \in \overline{011}$
2. $\mathsf{X}_E \simeq \mathsf{X}_F$
3. $(E, F) \in \langle \textbf{(O)}, \textbf{(I)}, \textbf{(S)} \rangle$

Generalized insplitting

Move (I–)

Insplit at any vertex using a partition of incoming edges into possibly empty sets:



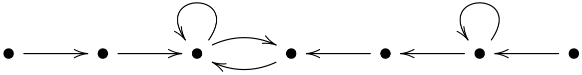
Invariance

$\langle(\mathbf{I-})\rangle \subseteq \overline{\underline{011}}$

For general (non-regular) graphs one must also require that the vertex is regular.

Lemma [E-Ruiz]

Within the class of regular graphs, $\langle \textbf{(S)} \rangle \subseteq \langle \textbf{(O)}, \textbf{(I–)} \rangle$



Theorem 011

For regular graphs E and F, the following are equivalent

1. $(E,F) \in \overline{\underline{011}}$
2. $\mathsf{X}_E \simeq \mathsf{X}_F$
3. $(E,F) \in \langle \textbf{(O)}, \textbf{(I–)} \rangle$

Report card 011

Interpret geometrically. $\surd$ | $\surd$ Connect to dynamics.

Find complete invariants. $\div$ | $\div$ Decide decidability.

To a finite graph $E = (E_0, E_1, r, s)$ we associate X_E^+ defined as

$$\mathsf{X}_E^+ = \{(e_n) \in (E_1)^{\mathbb{N}} \mid r(e_n) = s(e_{n+1})\}$$

We call X_E^+ a **one-sided shift space** (of finite type).

Definition

One-sided shift spaces X_+ and Y_+ are *conjugate* if there is a shift-invariant homeomorphism $\varphi : X_+ \to Y_+$.

Definition [Matsumoto]

One-sided shift spaces X_+ and Y_+ are *eventually conjugate* if there is a homeomorphism $h : X_+ \to Y_+$ and continuous maps $k : X_+ \to \mathbb{Z}$, $k' : Y_+ \to \mathbb{Z}$ so that

$$\sigma^{k(x)+1}(h(x)) = \sigma^{k(x)}(h(\sigma(x)))$$
$$\sigma^{k'(y)+1}(h^{-1}(y)) = \sigma^{k'(y)}(h^{-1}(\sigma(y)))$$

Theorem [Matsumoto, Carlsen-Rout]

For regular graphs E and F, the following are equivalent

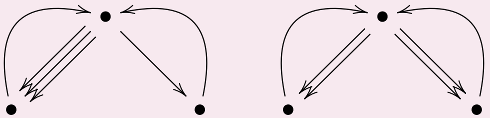
1. $(E, F) \in \overline{111}$
2. X_E^+ is eventually conjugate to X_F^+.

Theorem [Williams]

For essential regular graphs E and F, the following are equivalent

1. $(E, F) \in \langle \mathbf{(O)} \rangle$
2. X_E^+ is conjugate to X_F^+.

Key example [Brix-Carlsen]

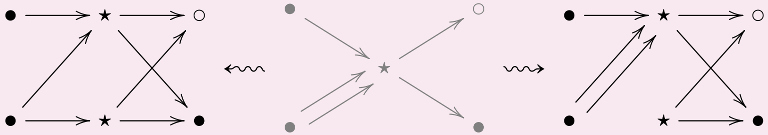


are a pair of graphs in $\overline{111} \setminus \langle \mathbf{(O)} \rangle$.

Insplitting

Move (I+)

Insplit at any vertex using a partition of incoming edges into possibly empty sets, but with the same number of sets:



Invariance

$\langle (\mathbf{I+}) \rangle \subseteq \overline{111}$

Theorem 111 [Matsumoto, Carlsen-Rout, Brix]

For regular graphs E and F, the following are equivalent

1. $(E,F) \in \overline{\underline{111}}$
2. X_E^+ is eventually conjugate to X_F^+
3. $(E,F) \in \langle \mathbf{(O)}, \mathbf{(I+)} \rangle$

Report card 111

Interpret geometrically. $\surd$ — $\surd$ Connect to dynamics.
Find complete invariants. $\div$ — $\div$ Decide decidability.

Outline

Definition

The **suspension flow** SX of a two-sided shift space X is $X \times \mathbb{R}/\sim$ with

$$(x,t) \sim (\sigma(x), t-1)$$

Note that SX has a canonical $\mathbb{R}$-action.

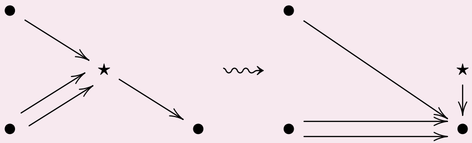
Definitions

X is flow equivalent to Y (written $X \sim_{\mathsf{fe}} Y$) if there is an orientation-preserving homeomorphism $\psi : SX \to SY$

Reduce at vertex not supporting a loop

Move (R)

Delete a regular vertex not supporting a loop, joining transitional edges



Invariance

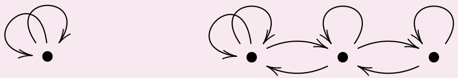
$\langle \textbf{(R)} \rangle \subseteq \overline{001}$

Theorem [Parry-Sullivan]

For regular essential graphs E and F, the following are equivalent

1. $\mathsf{X}_E \sim_{\mathsf{fe}} \mathsf{X}_F$.
2. $(E, F) \in \langle \simeq, \textbf{(R)} \rangle$

Key example [Parry-Sullivan, Bowen-Franks, Cuntz, Rørdam]



are a pair of graphs in $\overline{000} \setminus \sim_{\mathsf{fe}}$.

Theorem [Matsumoto-Matui, Carlsen-E-Ortega-Restorff]

For regular essential graphs E and F, the following are equivalent

1. $\mathsf{X}_E \sim_{\mathsf{fe}} \mathsf{X}_F$.
2. $(E, F) \in \overline{001}$

Theorem [Boyle-Huang, Boyle-Steinberg]

When (E, F) is a pair of regular essential graphs in standard form, the following are equivalent

1. $\mathsf{X}_E \sim_{\mathsf{fe}} \mathsf{X}_F$.
2. $\exists U, V \in \mathrm{SL}_{\triangledown}(\mathbb{Z}) : U(\mathsf{A}_E - 1) = (\mathsf{A}_F - 1)V$

and is a decidable property.

Theorem 001

For regular graphs E and F, the following are equivalent

1. $(E, F) \in \overline{\underline{001}}$
2. X_E is flow equivalent to X_F
3. $(E, F) \in \langle \textbf{(O)}, \textbf{(I-)}, \textbf{(R)} \rangle$

When (E, F) is in standard from, they are further equivalent to

4. $\exists U, V \in \mathrm{SL}_{\lnot}(\mathbb{Z}) : U(\mathsf{A}_E - 1) = (\mathsf{A}_F - 1)V$

which is decidable.

Report card 001

Interpret geometrically. $\surd$ $\surd$ Connect to dynamics.
Find complete invariants. $\surd$ $\surd$ Decide decidability.

Outline

Cuntz splice

Move (C)

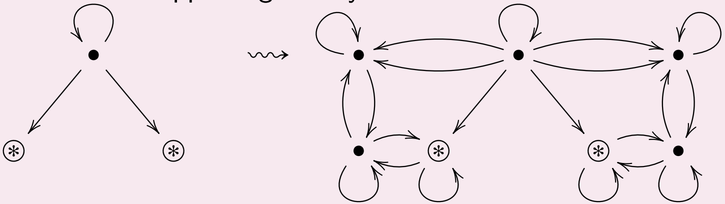
"Cuntz splice" on a vertex supporting two cycles

Invariance

$\langle \mathbf{(C)} \rangle \subseteq \overline{000}$

Move (P)

"Butterfly move" on a vertex supporting a single cycle emitting only singly to vertices supporting two cycles



Invariance

$\langle(\mathbf{P})\rangle \subseteq \overline{000}$

Theorem 000 [E-Restorff-Ruiz-Sørensen]

For regular graphs E and F, the following are equivalent

1. $(E, F) \in \overline{000}$
2. $(E, F) \in \langle \textbf{(O)}, \textbf{(I-)}, \textbf{(R)}, \textbf{(C)}, \textbf{(P)} \rangle$

When (E, F) is in standard from, they are further equivalent to

3. $\exists U, V \in \mathrm{GL}_{\triangledown}(\mathbb{Z}) : U(\mathsf{A}_E - 1) = (\mathsf{A}_F - 1)V$

which is decidable.

Report card 000

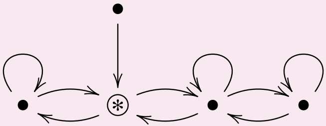
Interpret geometrically. $\surd$ $\div$ Connect to dynamics.
Find complete invariants. $\surd$ $\surd$ Decide decidability.

Exact moves

Move (C+)

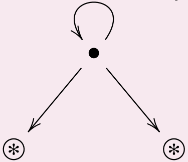
"Cuntz splice" on a vertex supporting two cycles

⇝

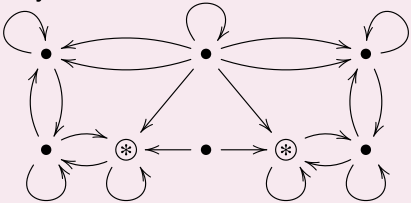


Move (P+)

"Butterfly move" on a vertex supporting a single cycle emitting only singly to vertices supporting two cycles

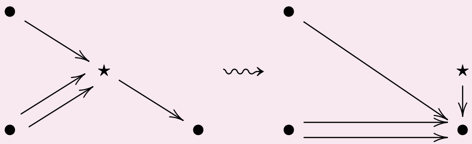


⇝



Move (R+)

Delete a regular vertex not supporting a loop, joining transitional edges and retaining outgoing edges at a new source



Invariance

$\langle(\mathbf{R+})\rangle \subseteq \overline{\underline{101}}, \langle(\mathbf{C+}), (\mathbf{P+})\rangle \subseteq \overline{\underline{100}}$

Theorem 100 [Arklint-E-Ruiz]

For regular graphs E and F, the following are equivalent

1. $(E, F) \in \overline{100}$
2. $(E, F) \in \langle \textbf{(O)}, \textbf{(I+)}, \textbf{(R+)}, \textbf{(C+)}, \textbf{(P+)} \rangle$

When (E, F) is in augmented standard from, they are further equivalent to

3. $\exists U, V \in \mathrm{GL}_{\triangledown,1}(\mathbb{Z}) : U(\mathsf{A}_E - 1) = (\mathsf{A}_F - 1)V$

which is decidable.

Report card 100

Interpret geometrically. $\surd$ — $\div$ Connect to dynamics.

Find complete invariants. $\surd$ — $\surd$ Decide decidability.

Outline

The generation conjecture for regular graphs

	x	y	z
(O)	1	1	1
(I+)	1	1	1
(I–)	0	1	1
(R+)	1	0	1
(C+)	1	0	0
(P+)	1	0	0

000	$\langle$(O), (I–), (R+), (C+), (P+)$\rangle$
001	$\langle$(O), (I–), (R+)$\rangle$
010	?
011	$\langle$(O), (I–)$\rangle$
100	$\langle$(O), (I+), (R+), (C+), (P+)$\rangle$
101	?
110	?
111	$\langle$(O), (I+)$\rangle$

Rigidity

[Exact, gauge, diagonal]

Definition [Matsumoto]

One-sided shift spaces X_+ and Y_+ are *continuous orbit equivalent* if there is a homeomorphism $h : X_+ \to Y_+$ and continuous maps $k, \ell : X_+ \to \mathbb{Z}$, $k', \ell' : Y_+ \to \mathbb{Z}$ so that

$$\sigma^{\ell(x)}(h(x)) = \sigma^{k(x)}(h(\sigma(x)))$$
$$\sigma^{\ell'(y)}(h^{-1}(y)) = \sigma^{k'(y)}(h^{-1}(\sigma(y)))$$

Theorem [Matsumoto-Matui, Carlsen-E-Ortega-Restorff]

For regular graphs E and F, the following are equivalent

1. X_E^+ is continuous orbit equivalent to X_F^+
2. $(E, F) \in \overline{101}$

Theorem [Arklint-E-Ruiz, cf. Carlsen-Ortega-Restorff]

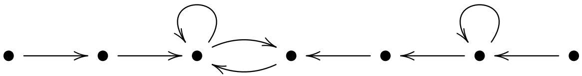
When E and F are regular graphs, the following are equivalent

1. $(E, F) \in \langle \textbf{(O)}, \textbf{(I+)}, \textbf{(R+)} \rangle$
2. There exist a pair (E', F') of graphs in standard form so that
   - $(E, E') \in \langle \textbf{(O)}, \textbf{(R+)} \rangle$
   - $(F, F) \in \langle \textbf{(O)}, \textbf{(R+)} \rangle$
   - $\exists U, V \in \mathrm{SL}_{\triangledown,1}(\mathbb{Z}) : U(\mathsf{A}_{E'} - 1) = (\mathsf{A}_{F'} - 1)V$

The gauge simple case

Observation

When E is regular, $C^*(E)$ has only trivial gauge-invariant ideals precisely when E_{ess} is strongly connected.



Theorem 101

For regular graphs E and F defining gauge simple C^*-algebras, the following are equivalent

1. $(E,F) \in \overline{\underline{101}}$
2. $(E,F) \in \overline{\underline{100}} \cap \overline{\underline{001}}$
3. X_E is continuous orbit equivalent to X_F
4. $(E,F) \in \langle \textbf{(O)}, \textbf{(I+)}, \textbf{(R+)} \rangle$

When (E,F) is in standard form, they are further equivalent to

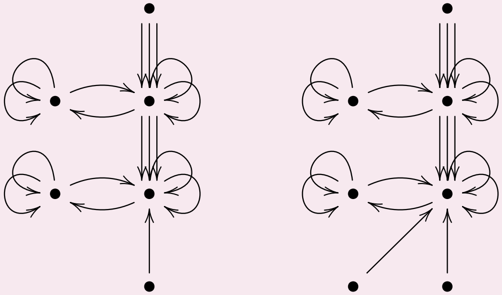
5. $\exists U,V \in \mathrm{SL}_{\triangledown,1}(\mathbb{Z}) : U(\mathsf{A}_E - 1) = (\mathsf{A}_F - 1)V$
6.
   - $\exists U',V' \in \mathrm{GL}_{\triangledown,1}(\mathbb{Z}) : U'(\mathsf{A}_E - 1) = (\mathsf{A}_F - 1)V'$
   - $\exists U'',V'' \in \mathrm{SL}_{\triangledown}(\mathbb{Z}) : U''(\mathsf{A}_E - 1) = (\mathsf{A}_F - 1)V''$

which are decidable.

Report card 101

Interpret geometrically. $(\surd)$ $\surd$ Connect to dynamics.
Find complete invariants. $(\surd)$ $(\surd)$ Decide decidability.

Key example [E-Ruiz]

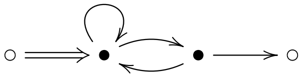


is a pair of graphs in

$$(\overline{\underline{100}} \cap \langle \mathbf{(S)} \rangle) \setminus \langle \mathbf{(O)}, \mathbf{(I+)}, \mathbf{(R+)} \rangle$$

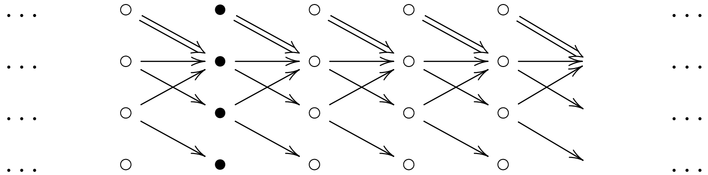
Outline

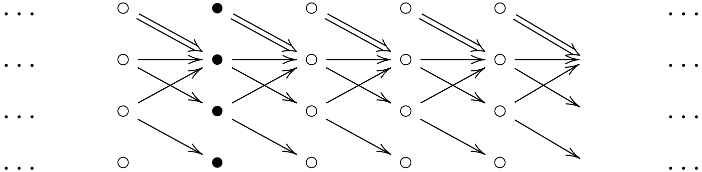
Fixed point algebra



Theorem

$\mathfrak{F}_E$ is itself a corner of a graph C^*-algebra which is AF. It is best described as $p^0 C^*(E \times_1 \mathbb{Z}) p^0$ with p^0 and $E \times_1 \mathbb{Z}$ as indicated below.





Observation

Note that $C^*(E \times_1 \mathbb{Z})$ comes with a shift map $\sigma \in \mathrm{Aut}(C^*(E \times_1 \mathbb{Z}))$.

Definition [Krieger]

The *dimension triple* of E is

$$\mathcal{DT}(E) = (K_0(C^*(E \times_1 \mathbb{Z})), K_0(C^*(E \times_1 \mathbb{Z}))_+, \sigma_*)$$

Definition

The *dimension quadruple* of E is

$$\mathcal{DQ}(E) = (K_0(C^*(E \times_1 \mathbb{Z})), K_0(C^*(E \times_1 \mathbb{Z}))_+, \sigma_*, [p^0])$$

Theorem [Bratteli-Kishimoto, E-Szabó]

The following are equivalent for essential regular graphs defining gauge simple C^*-algebras

1. $(E, F) \in \overline{010}$
2. X_E and X_F are shift equivalent
3. $\mathcal{DT}(E) \simeq \mathcal{DT}(F)$

This would generalize by a *Hazrat conjecture*.

Theorems [Kim-Roush]

- There exist graphs E, F defining gauge simple C^*-algebras so that X_E and X_F are shift equivalent but not conjugate.
- $\mathcal{DT}(E) \simeq \mathcal{DT}(F)$ is a decidable property.

Theorem [Bratteli-Kishimoto, E-Szabó, Brix]

The following are equivalent for regular graphs defining gauge simple C^*-algebras

1. $(E, F) \in \overline{110}$
2. X_E^+ and X_F^+ are balanced shift equivalent
3. $\mathcal{DQ}(E) \simeq \mathcal{DQ}(F)$

This would generalize by another Hazrat conjecture.

The Krieger move

Definition

We say that E is obtained from F by a **(K+)** move when

$$\mathcal{DQ}(E) \simeq \mathcal{DQ}(F)$$

Theorem 010

The following are equivalent for regular graphs defining gauge simple C^*-algebras

1. $(E, F) \in \overline{010}$
2. X_E and X_F are shift equivalent
3. $(E, F) \in \langle \textbf{(O)}, \textbf{(I–)}, \textbf{(K+)} \rangle$

Report card 010

Interpret geometrically. $\div$

($\surd$) Connect to dynamics.

Find complete invariants. ($\surd$)

($\surd$) Decide decidability.

Theorem 110

The following are equivalent for regular graphs defining gauge simple C^*-algebras

1. $(E,F) \in \overline{\underline{110}}$
2. X_E and X_F are balanced shift equivalent
3. $(E,F) \in \langle \textbf{(O)}, \textbf{(I+)}, \textbf{(K+)} \rangle$
4. $(E,F) \in \langle \textbf{(K+)} \rangle$

Report card 110

Interpret geometrically. $\div$ | $(\surd)$ Connect to dynamics.
Find complete invariants. $(\surd)$ | $\div$ Decide decidability.

Status for regular graphs (defining simple C^*-algebras)

	x	y	z
(O)	1	1	1
(I+)	1	1	1
(I−)	0	1	1
(R+)	1	0	1
(C+)	1	0	0
(P+)	1	0	0
(K+)	1	(1)	0

000	⟨**(O)**, **(I−)**, **(R+)**, **(C+)**, **(P+)**⟩
001	⟨**(O)**, **(I−)**, **(R+)**⟩
010	(⟨**(O)**, **(I−)**, **(K+)**⟩)
011	⟨**(O)**, **(I−)**⟩
100	⟨**(O)**, **(I+)**, **(R+)**, **(C+)**, **(P+)**⟩
101	(⟨**(O)**, **(I+)**, **(R+)**⟩)
110	(⟨**(O)**, **(I+)**, **(K+)**⟩)
111	⟨**(O)**, **(I+)**⟩

We know no counterexample even in the non-regular case, but then one must add **(S)** as a $\overline{\underline{001}}$ move.