Relative Cuntz–Pimsner algebras: Classification and Applications

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Bedlewo workshop:

Operator algebras that one can see

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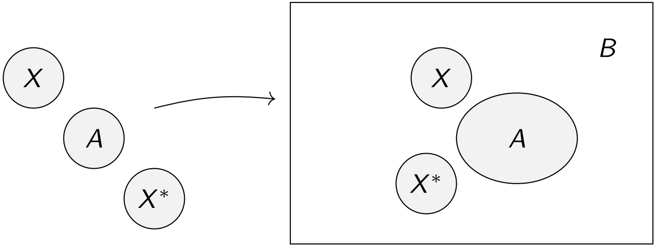
Correspondence = bimodule + pairing:

$$\big(\ X = \text{linear space} \ \big| \ A = \text{operator alg} \ \big) :$$

$$X^* X = \langle X | X \rangle \subseteq A, \qquad AX \subseteq X, \qquad XA \subseteq X.$$

Question: various realisations in ambient algs ?



$\Longrightarrow$ Which ones and how many of such exist?

$\Longrightarrow$ Can we classify / parametrise them?

Why: Insight into ideal structures!

Systematic approach: classify kernel / cokernel morphism:

$$0 \longrightarrow (XK, K) \longrightarrow (X, A) \longrightarrow \left(\tfrac{X}{XK}, \tfrac{A}{K}\right) \longrightarrow 0$$

Faithful embedding:

$$K = \ker(A \to B): \qquad (X, A) \longrightarrow \left(\tfrac{X}{XK}, \tfrac{A}{K}\right) \subseteq (Y, B)$$

General morphisms: $(\tau, \varphi) : (X, A) \longrightarrow (Y, B)$:

$$\varphi(x^* x) = \tau(x)^* \tau(y) \qquad \varphi(a)\tau(x) = \tau(ax) \qquad \tau(x)\varphi(a) = \tau(xa)$$

Question following: which additional relations ?

$$X^*X \to Y^*Y \qquad XA \to YB \qquad AX \to BY \qquad XX \to YY \qquad \ldots$$

$$XX^* \to YY^* \qquad XX^*X \to YY^*Y \qquad X^*XX^* \to Y^*YY^* \qquad \ldots$$

Question arising: covariant representation?

$$\begin{array}{ccc} XX^* & \longrightarrow & YY^* \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array}$$

$$A \cap XX^* = \left\{ \, a \in A \;\middle|\; (a = K) \in XX^* : \; (a - K)X \equiv 0 \, \right\} :$$

$$(a - K)X \equiv 0 \quad \Longrightarrow \quad \varphi(a)Y \equiv \tau(K)Y \; ??$$

Amount of covariance: $\mathrm{cov}(X \to Y) \trianglelefteq A$:

$$\mathrm{cov}(X \to Y) := \ker\Big[(A \cap XX^* \to B \to YY^*) \; - \; (A \cap XX^* \to XX^* \to YY^*)\Big]$$

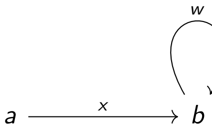
Observation by Katsura: $\mathrm{cov}\big(X \cup A \subseteq B\big) \perp \ker(A \curvearrowright X)$ (obstruction!)

Maximal covariance: $\max(X, A) = \ker(A \curvearrowright X)^{\perp} \cap XX^*$ (intrinsic!)

Intrinsic characterisation: kernel–covariance pairs =

$$\Big(\text{ invariant ideal } \Big| \text{ bounded ideal }\Big) = \Big(K \trianglelefteq A : \; X^*KX \subseteq K \;\Big|\; J \trianglelefteq A/K : \; J \subseteq \max(\tfrac{X}{XK}) \Big)$$

Graph correspondence: $\Big(X = \mathbb{C}x \oplus \mathbb{C}w \;\Big|\; A = \mathbb{C}a \oplus \mathbb{C}b \Big)$:



$$x^*x = s(x) = a \quad w^*w = s(w) = b$$

$$xa = x = bx \qquad wb = w = bw$$

Kernel ideals (invariant): $\big(X^*KX \subseteq K \iff KX \subseteq XK\big)$:

$$K = (0) \qquad K = (a) \qquad \sout{K = (b)} \qquad K = (a \cup b)$$

Covariance ideals (bounded): $\max(X, A) = \mathbb{C}\big(\text{regular ver}\big)$:

$$0 \subseteq J \subseteq \max(X, A) : \qquad X : \quad J = 0 \quad J = (b) \quad \sout{J = (a \cup b)} \qquad \frac{X}{XK} : \quad \ldots \quad ;$$

Intrinsic characterisation: kernel–covariance pairs =

$$\Big(\text{invariant ideal} \;\Big|\; \text{bounded ideal} \Big) = \Big(K \trianglelefteq A : \quad X^* K X \subseteq K \;\Big|\; J \trianglelefteq A/K : \quad J \subseteq \max(\tfrac{X}{XK}) \Big)$$

Parametrisation: Kernel Lattice + Covariance Lattice:

$$\begin{array}{ccccc} X & \longrightarrow & \frac{X}{XK} & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow \\ \frac{X}{XL} & \longrightarrow & \dots & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & \dots & \longrightarrow & X = 0 \end{array} \quad \rightsquigarrow \quad \begin{array}{ccccc} \mathcal{T}(K) = \mathcal{O}(K,0) & \longrightarrow & \mathcal{O}(K,I) & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}(K,J) & \longrightarrow & \dots & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & \dots & \longrightarrow & \mathcal{O}(K,\max) = \mathcal{O}(K \end{array}$$

Parametrisation: $\Longrightarrow$ every equivariant repr !!

First classification goal: (kernel–covariance) pairs = exhausting ?

Gauge theorem: the general version [Fre23] build on [Kak16]

$$\left(K = \ker(A \to B) \;\middle|\; J = \operatorname{cov}\left(\tfrac{X}{XK} \to B\right) \right) : \qquad (X, A) \longrightarrow \mathcal{O}(K, J) == C^*(X \cup A) \subseteq B.$$

Second classification goal: (kernel–covariance) pairs = parametrising ?

Kernel and covariance: can be recovered? [Fre23] extended from [Kat04]

$$\ker\left(A \longrightarrow \mathcal{O}(K|J)\right) = K \qquad \textit{and} \qquad \operatorname{cov}\left(\frac{X}{XK} \longrightarrow \mathcal{O}(K|J)\right) = J.$$

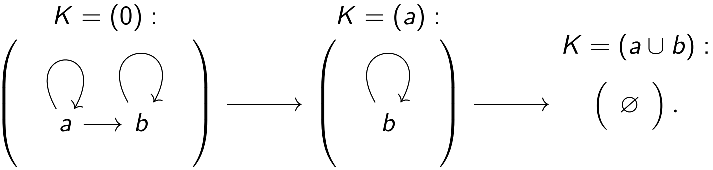
[Fre23, theorem 8.1] Order isomorphism

$$\left(K \subseteq L \;\middle|\; I \subseteq J \right) \iff \mathcal{O}(K, I) \leq \mathcal{O}(L, J)$$

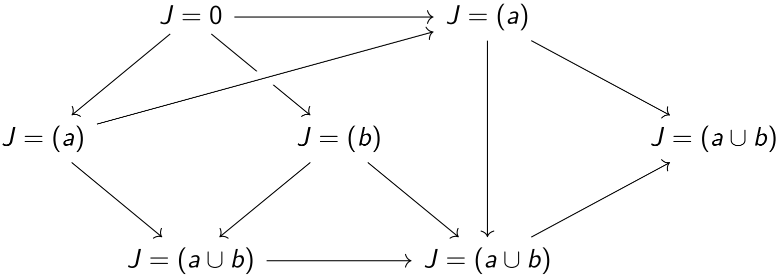
Lattice of equivariant repr:

$\implies$ may now be read off entirely from the correspondence ✓

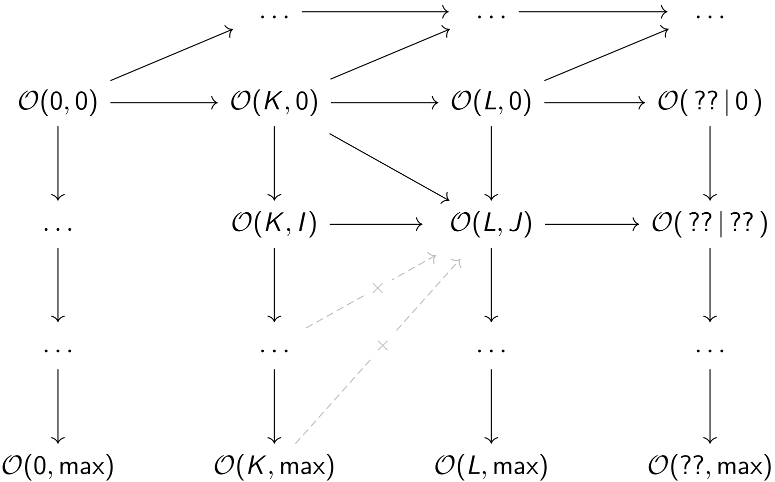
Illustration: graph correspondence:



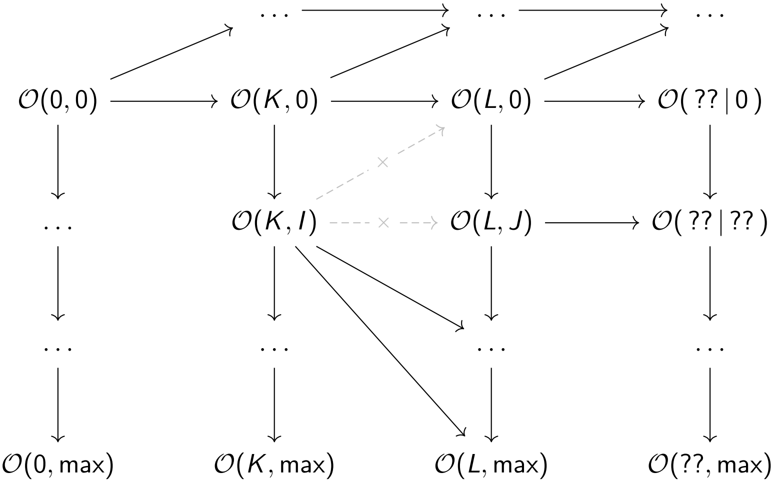
Lattice structure: equivariant repr's:



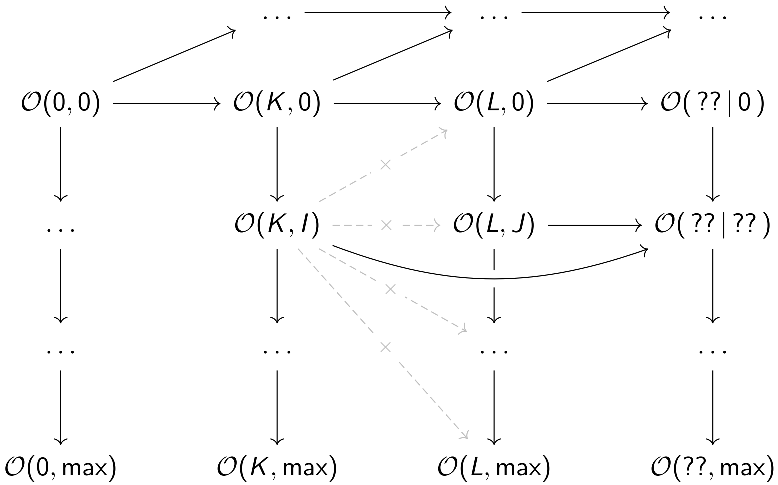
Lattice structure: may be quite interwoven!



Lattice structure: may be quite interwoven!

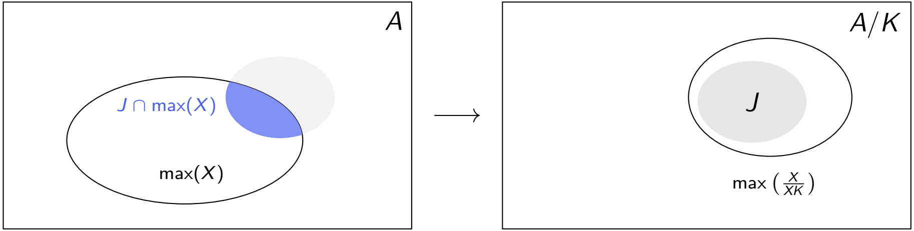


Lattice structure: may be quite interwoven!



Question 1: from up to which covariance?

$$
\begin{array}{ccccc}
\mathcal{O}(X \mid 0) & \longrightarrow & \mathcal{O}(X \mid \mathrm{cov} = ?) & \longrightarrow & \mathcal{O}(X \mid \max(X)) \\
\downarrow & \searrow & \downarrow & & \\
\mathcal{O}\left(\frac{X}{XK} \mid 0\right) & \longrightarrow & \mathcal{O}\left(\frac{X}{XK} \mid J\right) & \longrightarrow & \mathcal{O}\left(\frac{X}{XK} \mid \max\right)
\end{array}
$$



Meet operation: $\bigwedge_s (\, K_s \mid I_s \,) = \left(K = \bigcap K_s \;\middle|\; I = \bigcap I_s \cap \max\left(\frac{X}{XK}\right) \right)$

Question 2: possible to which next kernel strand?

$$
\begin{array}{ccccc}
\mathcal{O}(K \mid 0) & \longrightarrow & \mathcal{O}(K \mid I) & \longrightarrow & \mathcal{O}(K \mid \max) \\
\downarrow & & \downarrow & \searrow & \\
\mathcal{O}(L = ? \mid 0) & \longrightarrow & \mathcal{O}(L = ? \mid \dots) & \longrightarrow & \mathcal{O}(L =? \mid \max).
\end{array}
$$

Negative example:

$$
\begin{array}{ccccc}
\mathcal{T}\Big(X = \ell^2(a \to b) \Big) & \longrightarrow & \mathcal{T}\Big(X = \ell^2(b) \Big) & \longrightarrow & \mathcal{T}(X = 0) \\
\downarrow & \nearrow & \| & & \| \\
\mathcal{O}\Big(X = \ell^2(a \to b) \Big) & \dashrightarrow \times \dashrightarrow & \mathcal{O}\Big(X = \ell^2(b) \Big) & & \mathcal{O}(X = 0).
\end{array}
$$

First application: Pimsner dilations ?

$$(X,A) \longrightarrow \big(Y = ?, B = ? \big) \longrightarrow \mathcal{O}(Y,B) =\!=\!= \mathcal{O}(K,I)$$

Minimal equivariant embedding:

$$\mathcal{T}(Y,B) \longrightarrow \dots \longrightarrow \mathcal{O}(Y,B): \qquad (Y,B) \subseteq \mathcal{O}(Y,B)$$

Detection of gauge-inv ideals:

$$\big[\ \dots\dots\ \big] \trianglelefteq \mathcal{O}(Y,B): \qquad \big[\ \dots\dots\ \big] \cap B = 0 \quad \Longrightarrow \quad \big[\ \dots\dots\ \big] = 0$$

Covariance ideals:

$$\Big[\, \mathcal{O}(X,I) \begin{pmatrix} I' & \\ & 0 \end{pmatrix} \mathcal{O}(X,I) \,\Big] \trianglelefteq \mathcal{O}(X,I): \qquad \begin{pmatrix} I & \\ & 0 \end{pmatrix} \subseteq \begin{pmatrix} I' & \\ & 0 \end{pmatrix} \subseteq \begin{pmatrix} \max(X,A) & \\ & 0 \end{pmatrix} \subseteq \mathcal{O}(X,I)_0$$

[Fre23] Maximal dilation ✓

$$\mathcal{O}(K,I) = \mathcal{O}\Big(\, Y = \mathcal{O}(K,I)_1 \,\Big|\, B = \mathcal{O}(K,I)_0 \, \Big)$$

First application: Pimsner dilations ?

$$(X, A) \subseteq (Y, B) \subseteq \mathcal{O}(X, I) : \qquad Y \subseteq \mathcal{O}(X, I)_1 \qquad B \subseteq \mathcal{O}(X, I)_0$$

Minimal equivariant embedding:

$$\mathcal{T}(Y, B) \longrightarrow \dots \longrightarrow \mathcal{O}(Y, B) : \qquad (Y, B) \subseteq \mathcal{O}(Y, B)$$

Detection of gauge-inv ideals:

$$\big[\ \dots\dots\ \big] \trianglelefteq \mathcal{O}(Y, B) : \qquad \big[\ \dots\dots\ \big] \cap B = 0 \quad \Longrightarrow \quad \big[\ \dots\dots\ \big] = 0$$

Covariance ideals:

$$\Big[\, \mathcal{O}(X, I) \begin{pmatrix} I' & \\ & 0 \end{pmatrix} \mathcal{O}(X, I) \,\Big] \trianglelefteq \mathcal{O}(X, I) : \qquad \begin{pmatrix} I & \\ & 0 \end{pmatrix} \subseteq \begin{pmatrix} I' & \\ & 0 \end{pmatrix} \subseteq \begin{pmatrix} \max(X, A) & \\ & 0 \end{pmatrix} \subseteq \mathcal{O}(X, I)_0$$

[Fre23] Maximal dilation ✓

$$\mathcal{O}(K, I) = \mathcal{O}\Big(\ Y = \mathcal{O}(K, I)_1 \ \Big|\ B = \mathcal{O}(K, I)_0 \ \Big)$$

Pimsner dilations:

$$X \subseteq Y \subseteq \mathcal{O}(X,I)_1 \quad A \subseteq B \subseteq \mathcal{O}(X,I)_0 : \qquad Y^*Y \subseteq B \qquad YB \subseteq Y \qquad BY \subseteq Y$$

$$I \subseteq I' \subseteq \max(X,A) : \qquad B \cap \left[\mathcal{O}(X,I) \begin{pmatrix} I' & \\ & 0 \end{pmatrix} \mathcal{O}(X,I)\right] \neq 0$$

Combinatorial dilation:

$$B = A + \begin{pmatrix} I \subseteq I' \subseteq \max(X,A) & \\ & 0 \end{pmatrix} \rightsquigarrow Y = X + (XB = 0) + BX \rightsquigarrow Y^*Y = B$$

[Fre23] Maximal covariance $\implies$ Combinatorial dilation

$$\mathcal{O}(X,I) = \mathcal{O}\left(Y = XB \;\middle|\; B = A \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} \max(X,A) & \\ & 0 \end{pmatrix} \right)$$

[Fre23] Intrinsic description

$$\mathcal{O}(X,I) = C^*\left(X \cup A \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \cup \begin{pmatrix} \max(X,A) & \\ & 0 \end{pmatrix} \;\middle|\; \begin{pmatrix} I & \\ & 0 \end{pmatrix} = 0 \right)$$

Uncovered: $\implies$ Combinatorial dilation = Katsura construction [Kat07]

[Fre23]: Combinatorial dilation = outsplitting

$$\mathcal{O}(F) = \mathcal{O}\big(E, R \subseteq \mathrm{reg}\big) : \quad \begin{aligned} \mathrm{ver}(F) &= \mathrm{sing}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \cup \begin{pmatrix} 0 & \\ & \mathrm{reg} \end{pmatrix} \cup \begin{pmatrix} \mathrm{reg}\setminus R & \\ & 0 \end{pmatrix} \\ \mathrm{edges}(F) &= E\left[\mathrm{sing}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \cup \begin{pmatrix} 0 & \\ & \mathrm{reg} \end{pmatrix} \cup \begin{pmatrix} \mathrm{reg}\setminus R & \\ & 0 \end{pmatrix}\right] \end{aligned}$$

Graph correspondences: Combinatorial dilation = always minimal ✓

Graph correspondences: Minimal dilation = unique ??

Dynamical systems: Combinatorial dilation ≠ always minimal !!

[Fre23] Correspondences over spaces

Katsura dilation = minimal $\Longleftrightarrow$ *max covariance = discrete subspace*

Second application: Joint with Piotr M. Hajac and Mariusz Tobolski.

Quotient pullbacks

$$\mathcal{O}\Big((K,I) \wedge (L,J) \Big) = \mathcal{O}(K,I) \oplus_{\mathcal{O}\big((KI)\vee(LJ) \big)} \mathcal{O}(L,J).$$

For comparison [RS11, theorem 3.3]:

Lots of conditions to ensure by hand "surjectivity".

Instead purely categorical argument!

No further pullbacks

$$(K,I) \vee (K',I') < (L,J) : \qquad \mathcal{O}\Big((K,I) \wedge (K',I') \Big) \subsetneq \mathcal{O}(K,I) \oplus_{\mathcal{O}(L,J)} \mathcal{O}(K',I')$$

Failure of pullbacks: absolute Cuntz–Pimsner:

$$(1) \qquad \mathcal{O}(\, K \mid \mathsf{max}\,) \wedge \mathcal{O}(\, L \mid \mathsf{max}\,) \neq \mathcal{O}(\, K \cap L \mid \mathsf{max}\,)$$

$$(2) \qquad \mathcal{O}(\, K \mid \mathsf{max}\,) \vee \mathcal{O}(\, L \mid \mathsf{max}\,) \neq \mathcal{O}(\ \dots \mid \mathsf{max}\,)$$

Examples for failure: graph algebras !

Third application: Gauge-inv ideals = relative Cuntz–Pimsner algs ?

For comparison: Gauge-inv quotients = relative Cuntz–Pimsner algs ✓

Candidate (well-known): [FMR03, theorem 3.1]:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & XK & \longrightarrow & X & \longrightarrow & \frac{X}{XK} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{O}\big(XK \,\big|\, (J \cap \max) \cap K \big) & \sim & \dots & \longrightarrow & \mathcal{O}\big(X \,\big|\, J \cap \max \big) & \longrightarrow & \mathcal{O}\big(\frac{X}{XK} \,\big|\, J \subseteq \max \big) & & \end{array}$$

Negative example: graph algebras:

$$\mathcal{O}\big(a \Longrightarrow b\big) \neq \dots \longrightarrow \mathcal{O}\big(a \Longrightarrow b \Longrightarrow c\big) \longrightarrow \mathcal{O}\big(b \Longrightarrow c\big) \longrightarrow 0$$

Further examples: Higher tensor powers ✓

Negative example: even no finite tensor power ×

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Thank you for your attention.

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