Algebras associated with one-sided subshifts over arbitrary alphabets

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Cuntz-Krieger algebras and some generalisations

- Cuntz algebras (1977) - Simple infinite C*-algebras.
- Cuntz-Krieger algebras (1980) - C*-algebras for topological Markov chains.
- Graph C*-algebras (Kumjian, Pask, Raeburn, Renault - 1997).
- C*-Algebras for two-sided subshifts (Matsumoto 1997, Carlsen-Matsumoto 2004)
- Exel-Laca algebras for infinite matrices of 0-1 (1999). [1]
- Ultragraph C*-algebras (Tomforde 2003).
- C*-algebras of labelled graphs (Bates, Pask 2007).
- C*-algebras for one-sided subshifts (Carlsen 2008).
- C*-algebras of Boolean dynamical systems (Carlsen, Ortega, Pardo - 2017).
- C*-algebras of generalised Boolean dynamical systems (Carlsen, Kang - 2020).

[1] It can be presented with infinite sum relations (de C., Boava - 2022).

Tools for defining and studying these algebras

- Representations.
- Universal C*-algebras.
- Groupoids.
- Partial actions.
- Crossed products.
- C*-correspondences (Cuntz-Pimsner algebras).
- Inverse semigroups.

The purely algebraic setting

- Leavitt rings/algebras (Late 1950s, early 1960s) - rings without the IBN property.
- Purely algebraic analogue of Cuntz-Krieger algebras (Ara, Gonzáles-Barros, Goodearl, Pardo - 2004).
- Leavitt path algebras for graphs (Ara, Pino - 2005).
- Algebras for Boolean dynamical systems (Clark, Exel, Pardo - 2018).
- Leavitt path algebras for ultragraphs (Imanfar, Pourabbas, Larki - 2020).
- Leavitt path algebras for labelled graphs (Boava, de C., Gonçalves, van Wyk - 2023).
- Algebras associated with one-sided subshifts over arbitrary alphabets (Boava, de C., Gonçalves, van Wyk - 2022*).

C*-algebra of graphs

Let $\mathcal{E}$ be a graph. The graph C*-algebra $C^*(\mathcal{E})$ is the universal C*-algebra generated by mutually orthogonal projections $\{p_v\}_{v\in\mathcal{E}^0}$ and partial isometries with mutually orthogonal final projections $\{s_e\}_{e\in\mathcal{E}^1}$ satisfying the relations:

CK1 $s_e^*s_e = p_{r(e)}$ for all $e \in \mathcal{E}^1$,

CK2 $s_es_e^* \leq p_{s(e)}$, for all $e \in \mathcal{E}^1$,

CK3 $p_v = \sum_{e\in s^{-1}(v)} s_es_e^*$, for all $v \in \mathcal{E}^0_{rg}$.

If we do not consider relation CK3, we have the Toeplitz graph C*-algebra $\mathcal{T}C^*(\mathcal{E})$.

Partial actions

Definition

A partial action of G on X is a pair

$$\theta = \left(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\right)$$

consisting of a collection $\{D_g\}_{g\in G}$ of subsets of X, and a collection $\{\theta_g\}_{g\in G}$ of maps $\theta_g : D_{g^{-1}} \to D_g$, such that

1. $D_1 = X$, and $\theta_1 = Id_X$,
2. $\theta_g \circ \theta_h \leq \theta_{gh}$, for all g and h in G.

A topological partial action is a partial action such that X is a (LCH) topological space, each D_g is open and each θ_g is continuous.

A C*-algebraic partial action is a partial action such that X is a C*-algebra, each D_g is a closed two-sided ideal and each θ_g is a *-homomorphism.

C*-algebra of graphs - the partial action picture

Let $P_{\mathcal{E}} = \mathcal{E}^* \cup \mathcal{E}^\infty$. For each $\alpha \in \mathcal{E}^*$ and $F \subseteq s^{-1}(r(\alpha))$ finite, define

$$Z(\alpha, F) = \{\alpha\beta \in P_{\mathcal{E}} : \beta \in P_{\mathcal{E}}, r(\alpha) = s(\beta) \text{ and } \beta_1 \notin F \text{ if } |\beta| \geq 1\}.$$

For simplicity, we write $Z(\alpha) = Z(\alpha, \emptyset)$.

The family of sets of the form $Z(\alpha, F)$ is a basis for a topology on $P_{\mathcal{E}}$ making it a totally disconnected, locally compact Hausdorff space.

For each $e \in \mathcal{E}^1$, the map

$$\begin{array}{rccc} \varphi_e : & Z(r(e)) & \longrightarrow & Z(e) \\ & \alpha & \longmapsto & e\alpha \end{array}$$

is a homeomorphism between open subsets of $P_{\mathcal{E}}$. This is enough to define a topological partial action φ of the free group $\mathbb{F}$ generated by $\mathcal{E}^1$ on $P_{\mathcal{E}}$.

Considering the inverse semigroup $\mathcal{I}(P_{\mathcal{E}})$ of partially defined bijections on $P_{\mathcal{E}}$, we see that:

- $Id_{Z(v)} \circ Id_{Z(w)} = \emptyset$ for $v, w \in \mathcal{E}^0$ such that $v \neq w$ (mutually orthogonal projections),
- $Id_{Z(e)} \circ Id_{Z(f)} = \emptyset$ for $e, f \in \mathcal{E}^1$ such that $e \neq f$ (mutually orthogonal final projections),
- $\varphi_e^{-1} \circ \varphi_e = Id_{Z(r(e))}$ for $e \in \mathcal{E}^1$ (CK1),
- $\varphi_e \circ \varphi_e^{-1} = Id_{Z(e)} \leq Id_{Z(s(e))}$ for $e \in \mathcal{E}^1$ (CK2).

What about (CK3)?

It is not true that, for $v \in \mathcal{E}^0_{rg}$,

$$Id_{Z(v)} = \bigvee_{e \in s^{-1}(v)} Id_{Z(e)}$$

because $v \in Z(v)$, but $v \notin \bigcup_{e \in s^{-1}(v)} Z(e)$.

In order to find an analogue of (CK3) we need to restrict to the boundary path space of $\mathcal{E}$:

$$\partial\mathcal{E} = \{\alpha \in \mathcal{E}^* : r(\alpha) \in \mathcal{E}^0_{sg}\} \cup \mathcal{E}^\infty$$

as a subspace of $P_\mathcal{E}$. (It can be found using point-free topology by imposing the relation $Z(v) = \bigcup_{e \in s^{-1}(v)} Z(e)$ on the topology of $P_\mathcal{E}$ - de C., 2021).

One can then restrict the partial action φ to $\partial\mathcal{E}$.

Theorem (Carlsen, Larsen - 2016)

If $\widehat{\varphi}$ is the dual partial action on $C_0(\partial\mathcal{E})$ (resp. $C_0(P_\mathcal{E})$), then $C^*(\mathcal{E}) \cong C_0(\partial\mathcal{E}) \rtimes_{\widehat{\varphi}} \mathbb{F}$ (resp. $\mathcal{T}C^*(\mathcal{E}) \cong C_0(P_\mathcal{E}) \rtimes_{\widehat{\varphi}} \mathbb{F}$).

Remark

A similar construction was done independently by Gonçalves, Royer (2014) in the context of Leavitt path algebras.

Subshifts over finite alphabets

Let $\mathcal{A}$ be a non-empty finite set with the discrete topology. The set $\mathcal{A}^{\mathbb{N}}$ is a compact Hausdorff space. For an element $x \in \mathcal{A}^{\mathbb{N}}$ a block of x is a finite subsequence $x_i \cdots x_j$ for $i, j \in \mathbb{N}$ (if $i > j$, it is the empty block ω). The shift map is given by

$$\begin{array}{rccc} \sigma : & \mathcal{A}^{\mathbb{N}} & \longrightarrow & \mathcal{A}^{\mathbb{N}} \\ & x_0x_1x_2\ldots & \longmapsto & x_1x_2x_3\ldots \end{array}.$$

A one-sided subshift is a closed subset $X \subseteq \mathcal{A}^{\mathbb{N}}$ that is invariant by the shift map, that is, $\sigma(X) = X$.

If $\mathcal{F}$ is a family of forbidden blocks, we can define $X_{\mathcal{F}}$ as the subset of $\mathcal{A}^{\mathbb{N}}$ such that no element of $\mathcal{F}$ appears as a block of an element of $X_{\mathcal{F}}$.

Theorem

For any family $\mathcal{F}$, $X_{\mathcal{F}}$ is a subshift. Reciprocally, every subshift is of the form $X_{\mathcal{F}}$ for some family $\mathcal{F}$.

Subshifts over arbitrary alphabets

Now let $\mathcal{A}$ be any non-empty set. As before, for $\mathcal{F}$, a family of forbidden blocks, we define $X_{\mathcal{F}}$ as the subset of $\mathcal{A}^{\mathbb{N}}$ such that no element of $\mathcal{F}$ appears as a block of an element of $X_{\mathcal{F}}$. We define one-sided subshift as subset X of $\mathcal{A}^{\mathbb{N}}$ that is equal to $X_{\mathcal{F}}$ for some family $\mathcal{F}$.

Remark

In general, a one-sided subshift X is not a locally compact space, seen as a subspace of $\mathcal{A}^{\mathbb{N}}$ with the natural topology. For this reason, for now, we look at X simply from a combinatorial perspective.

Given a subshift X, the language of X, denoted by $\mathcal{L}_X$, is the set of all words that appear as a block of an element of X.

A partial action

Let $\mathbb{F}$ be the free group generated by $\mathcal{A}$. For each $a \in \mathcal{A}$, consider the sets

$$F_a = \{x \in X : ax \in X\},$$

$$Z_a = \{x \in X : x_0 = a\}.$$

Define also the function

$$\begin{array}{cccc} \theta_a : & F_a & \longrightarrow & Z_a \\ & x & \longmapsto & ax \end{array},$$

which is a bijection. As in the graph case, this is enough to define a partial action θ of $\mathbb{F}$ on X.

Remark

Even in the finite alphabet case, in general, θ is not a topological partial action because F_a may fail to be an open subset of X.

For each $\alpha, \beta \in \mathcal{L}_X$, let

$$C(\alpha, \beta) = \{\beta x \in X : \alpha x \in X\},$$

so, in particular, $Z_\alpha = C(\omega, \alpha)$ and $F_\alpha = C(\alpha, \omega)$ for every $\alpha \in \mathcal{L}_X$, and $C(\omega, \omega) = X$. If $\alpha\beta^{-1}$ is in reduced form in $\mathbb{F}$, then $\theta_{\alpha\beta^{-1}}$ is such that

$$\begin{array}{rccc} \theta_{\alpha\beta^{-1}}: & C(\alpha, \beta) & \longrightarrow & C(\beta, \alpha) \\ & \beta x & \longmapsto & \alpha x \end{array}.$$

In general,

$$\theta_\beta \circ \theta_{\alpha^{-1}} \circ \theta_\alpha \circ \theta_{\beta^{-1}} = Id_{C(\alpha,\beta)}.$$

Generalised Boolean algebras

We can define a (generalised) Boolean algebra as relatively complemented distributive lattice with a minimum. Algebraically, it is a set $\mathcal{B}$ with operations $\vee, \wedge, \setminus$ and an element 0 satisfying several axioms.

Example

- If X is a Hausdorff space, then set of compact-open sets is a Boolean algebra with order given by inclusion.
- If A is a commutative C*-algebra, then the set of projections of A is a Boolean algebra with the usual order: $p \leq q$ if and only if $pq = p$.

The two examples are connected in the following way: if X is a LCH space and $A = C_0(X)$, then for every compact-open set $U \subseteq X$, we have that 1_U is a projection in $C_0(X)$. And in fact, it is a Boolean algebra isomorphism.

For a graph $\mathcal{E}$, there is a very important commutative subalgebra of $C^*(\mathcal{E})$, namely, the diagonal subalgebra:

$$D(\mathcal{E}) = \overline{\text{span}}\{s_\alpha s_\alpha^* : \alpha \in \mathcal{E}^*\} \cong C_0(\partial\mathcal{E}).$$

Relation (CK3) is just part of the isomorphism of the Boolean algebra of compact-open sets of $\partial\mathcal{E}$ (which is generated by cylinder sets $Z(\alpha)$ for $\alpha \in \mathcal{E}^*$) and projections in $D(\mathcal{E})$.

Boolean algebras for subshifts

Given a subshift X over an alphabet $\mathcal{A}$, we define two Boolean algebras inside $\mathcal{P}(X)$ (the power set of X), the main difference being if we ask X itself to be in the Boolean algebra or not:

- $\mathcal{U}$ is the Boolean algebra generated by all $C(\alpha,\beta)$ for $\alpha,\beta\in\mathcal{L}_X$.
- $\mathcal{B}$ is the Boolean algebra generated by all $C(\alpha,\beta)$ for $\alpha,\beta\in\mathcal{L}_X$ that are not simultaneously the empty word.

If $\mathcal{A}$ is finite, $\mathcal{U}=\mathcal{B}$ because $X=\bigcup_{a\in\mathcal{A}} Z_a$.

Ott-Tomforde-Willis subshifts

Let $\tilde{\mathcal{A}} := \mathcal{A} \cup \{\infty\}$ and

$$\Sigma_{\mathcal{A}} = \{(x_i)_{i\in\mathbb{N}} \in \tilde{\mathcal{A}}^{\mathbb{N}} : x_i = \infty \text{ implies } x_{i+1} = \infty\}.$$

In particular $\vec{0} := (\infty\infty\cdots) \in \Sigma_{\mathcal{A}}$. There is a topology generated by the generalised cylinder sets:

$$\mathcal{Z}(\alpha, F) = \{y \in \Sigma_{\mathcal{A}} : \ y_i = \alpha_i \, \forall\, 1 \leq i \leq |\alpha|, \ y_{|\alpha|+1} \notin F\},$$

where $\alpha \in \mathcal{A}^*$ and $F \subseteq \mathcal{A}$ is finite. With this topology, $\Sigma_{\mathcal{A}}$ is compact.

Given a subshift X, the corresponding Ott-Tomforde-Willis subshift X^{OTW} is the closure of X in $\Sigma_{\mathcal{A}}$. If $\mathcal{A}$ is finite, then $X^{OTW} = X$.

There is a natural shift map on X^{OTW} that is continuos, except at $\vec{0}$ if $\vec{0} \in X^{OTW}$.

Remark

X^{OTW} can be defined directly from a family of forbidden words.

C*-algebras associated with one-sided subshifts

Definition

We define $\widetilde{\mathcal{O}}_X$ as the universal unital C*-algebra generated by projections $\{p_A : A \in \mathcal{U}\}$ and partial isometries $\{s_a : a \in \mathcal{A}\}$ subject to the relations:

- $p_X = 1$, $p_{A\cap B} = p_A p_B$, $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ and $p_\emptyset = 0$, for every $A, B \in \mathcal{U}$;
- $s_\beta s_\alpha^* s_\alpha s_\beta^* = p_{C(\alpha,\beta)}$ for all $\alpha, \beta \in \mathcal{L}_X$, where $s_\omega := 1$ and, for $\alpha = \alpha_1 \dots \alpha_n \in \mathcal{L}_X$, $s_\alpha := s_{\alpha_1} \cdots s_{\alpha_n}$ and $s_\alpha^* := s_{\alpha_n}^* \cdots s_{\alpha_1}^*$.

There is a natural gauge action on $\widetilde{\mathcal{O}}_X$ given by $\gamma_z(p_A) = p_A$ and $\gamma_z(s_a) = z s_a$ for $z \in \mathbb{T}$, $A \in \mathcal{U}$ and $a \in \mathcal{A}$.

Definition

We define the subshift algebra $\mathcal{O}_X$ as the universal C*-algebra generated by projections $\{p_A : A \in \mathcal{B}\}$ and partial isometries $\{s_a : a \in \mathcal{A}\}$ subject to the relations:

- $p_{A\cap B} = p_A p_B$, $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ and $p_\emptyset = 0$, for every $A, B \in \mathcal{B}$;
- $s_\beta s_\alpha^* s_\alpha s_\beta^* = p_{C(\alpha,\beta)}$ for all $\alpha, \beta \in \mathcal{L}_X \setminus \{\omega\}$, where for $\alpha = \alpha_1 \ldots \alpha_n \in \mathcal{L}_X \setminus \{\omega\}$, $s_\alpha := s_{\alpha_1} \cdots s_{\alpha_n}$ and $s_\alpha^* := s_{\alpha_n}^* \cdots s_{\alpha_1}^*$;
- $s_\alpha^* s_\alpha = p_{C(\alpha,\omega)}$ for all $\alpha \in \mathcal{L}_X \setminus \{\omega\}$;
- $s_\beta s_\beta^* = p_{C(\omega,\beta)}$ for all $\beta \in \mathcal{L}_X \setminus \{\omega\}$.

Proposition

For any subshift X, either $\widetilde{\mathcal{O}}_X = \mathcal{O}_X$, or $\widetilde{\mathcal{O}}_X$ is the minimal unitisation of $\mathcal{O}_X$.

Examples

- If $\mathcal{A}$ is finite, then $\widetilde{\mathcal{O}}_X = \mathcal{O}_X$ and it coincides with Carlsen's algebra associated with a one-sided subshift.
- Let I be a set of indices and $A = (A_{ij})_{i,j\in I}$ a 0-1 matrix with no identically zero rows. We let $X_A = \{(x_n) \in I^{\mathbb{N}} : A_{x_n x_{n+1}} = 1 \text{ for all } n \in \mathbb{N}\}$. Then we obtain the two versions of Exel-Laca algebras: $\widetilde{\mathcal{O}}_A \cong \widetilde{\mathcal{O}}_{X_A}$ and $\mathcal{O}_A \cong \mathcal{O}_{X_A}$.
- Let $\mathcal{E}$ be a graph with no sinks and with no vertex that is simultaneously a source and an infinite emitter. Let $X_{\mathcal{E}} = \mathcal{E}^\infty$ be the associated one-sided edge subshift of $\mathcal{E}$. Then, $\mathcal{O}_{X_{\mathcal{E}}} \cong C^*(\mathcal{E})$.

Two interesting commutative subalgebras

We now focus on the unital C*-algebra $\widetilde{\mathcal{O}}_X$. We consider the diagonal subalgebra

$$D(X) := \overline{\text{span}}\{s_\alpha p_A s_\alpha^* : \alpha \in \mathcal{L}_X, A \in \mathcal{U}\} = \overline{\text{span}}\{p_A : A \in \mathcal{U}\} \cong C(\widehat{\mathcal{U}}),$$

where $\widehat{\mathcal{U}}$ is the Stone dual of $\mathcal{U}$. We also consider

$$\overline{\text{span}}\{s_\alpha s_\alpha^* : \alpha \in \mathcal{L}_X\} \cong C(X^{OTW}).$$

For the groupoid model, we use $\widehat{\mathcal{U}}$ as the unit space either via a local homeomorphism or via a partial action by the free group generated by $\mathcal{A}$.

Topological conjugacy for OTW-subshifts

Definition

Let X_1^{OTW} and X_2^{OTW} be OTW-subshifts over alphabets $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. A map $h : X_1^{OTW} \to X_2^{OTW}$ is a conjugacy if it is a homeomorphism, commutes with the shift and is length-preserving. If there is a conjugacy, we say that X_1^{OTW} and X_2^{OTW} are topologically conjugate.

Theorem (Boava, de C., Gonçalves, van Wyk - WIP)

Let X_1 and X_2 be two subshifts over alphabets $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. Let X_1^{OTW} and X_2^{OTW} be the corresponding OTW-subshifts.

- *If X_1^{OTW} and X_2^{OTW} are topologically conjugate, then there exists a diagonal-preserving gauge-invariant isomorphism between $\widetilde{\mathcal{O}}_{X_1}$ and $\widetilde{\mathcal{O}}_{X_2}$.*
- *Reciprocally, if there exists a diagonal-preserving isomorphism between $\widetilde{\mathcal{O}}_{X_1}$ and $\widetilde{\mathcal{O}}_{X_2}$ with some extra conditions, then X_1^{OTW} and X_2^{OTW} are topologically conjugate.*

Remark

The purely algebraic result is done in arXiv:2211.02148 [Boava, de C., Gonçalves, van Wyk].

A few words about the extra conditions

Let X be a subshift over an alphabet $\mathcal{A}$. For each $M \subseteq \mathcal{A}$, define

$$\begin{array}{rccc} \tau_M : & \widetilde{\mathcal{O}}_X & \longrightarrow & \widetilde{\mathcal{O}}_X \\ & y & \longmapsto & \sum_{a,b\in M} s_a y s_b^* \end{array}$$

and

$$e_M = \sum_{a\in M} s_a s_a^*.$$

The converse of the previous theorem uses the maps τ_M and the projections e_M. Moreover, the isomorphism must preserve the corresponding subalgebras $C(X^{OTW})$.

K-theory

We say that a set $A \in \mathcal{U}$ is regular if the set $\{a \in \mathcal{A} : Z_a \cap A \neq \emptyset\}$ is finite and non-empty. The set of regular sets in $\mathcal{U}$ is denoted by $\mathcal{U}_{reg}$.

For $A \in \mathcal{U}$ and $a \in \mathcal{A}$, we define the set $r(A, a) = \{x \in X : ax \in A\}$.

Theorem (Bates, Carlsen, Pask - 2017 - taken from the context of labelled graphs C*-algebras)

Let X be a one-sided subshift over an arbitrary alphabet $\mathcal{A}$. Let $(1 - \Phi) : \mathrm{span}_{\mathbb{Z}} \{\chi_A : A \in \mathcal{U}_{reg}\} \to \mathrm{span}_{\mathbb{Z}} \{\chi_A : A \in \mathcal{U}\}$ be the linear map given by

$$(1 - \Phi)(\chi_A) = \chi_A - \sum_{a : Z_a \cap A \neq \emptyset} \chi_{r(A,a)}, \quad A \in \mathcal{U}_{reg}.$$

Then $K_1(\widetilde{\mathcal{O}}_X)$ is isomorphic to $\ker(1 - \Phi)$ and there exists an isomorphism from $K_0(\widetilde{\mathcal{O}}_X)$ to $\mathrm{coker}(1 - \Phi)$ which maps $[S_\alpha^* S_\alpha]_0$ to $\chi_{F_\alpha} + \mathrm{Im}(1 - \Phi)$ for each $\alpha \in \mathcal{L}_X$.

Thank you!