Banach Center
INSTITUTE OF MATHEMATICS PAS

THE COVARIANT FUNCTORIALITY OF GRAPH ALGEBRAS

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Morphisms of graphs

Definition

A homomorphism $f\colon E \to F$ of graphs is a pair of maps

$$(f^0 : E^0 \to F^0, f^1 : E^1 \to F^1)$$

satisfying the conditions:

$$s_F \circ f^1 = f^0 \circ s_E\,, \qquad t_F \circ f^1 = f^0 \circ t_E\,.$$

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A path homomorphism of graphs is a map $f\colon FP(E) \to FP(F)$ satisfying:

1. $f(E^0) \subseteq F^0$,
2. $s_F \circ f = f \circ s_E\,, \quad t_F \circ f = f \circ t_E\,,$
3. $\forall\, p, q \in FP(E)$ such that $t(p) = s(q)$: $f(pq) = f(p)f(q)$.

Note that a path homomorphism of graphs is a homomorphism of graphs if and only if it preserves the lengths of paths.

Path algebras

Let k be a field. Consider the vector space

$kE := \{f \in \mathrm{Map}(FP(E), k) \mid f(p) \neq 0$ for finitely many $p \in FP(E)\}$,

where the addition and scalar multiplication are pointwise. Then the set of functions $\{\chi_p\}_{p \in FP(E)}$ given by

$$\chi_p(q) = \begin{cases} 1 & \text{for } p = q \\ 0 & \text{otherwise} \end{cases}$$

is a linear basis of kE. Next, we will use $\{\chi_p\}_{p \in FP(E)}$ to define a bilinear map:

$$m : kE \times kE \longrightarrow kE, \qquad m(\chi_p, \chi_q) := \begin{cases} \chi_{pq} & \text{if } t(p) = s(q) \\ 0 & \text{otherwise} \end{cases}.$$

The bilinear map $m : kE \times kE \to kE$ defines an algebra structure on kE. We call this algebra the path algebra of E.

Extended graphs and Cohn path algebras

Definition

Let $E = (E^0, E^1, s_E, t_E)$ be a graph. The extended graph $\bar{E} := (\bar{E}^0, \bar{E}^1, s_{\bar{E}}, t_{\bar{E}})$ of the graph E is given as follows:

$$\bar{E}^0 := E^0, \quad \bar{E}^1 := E^1 \sqcup (E^1)^*, \quad (E^1)^* := \{e^* \mid e \in E^1\},$$
$$\forall\, e \in E^1 : \quad s_{\bar{E}}(e) := s_E(e), \quad t_{\bar{E}}(e) := t_E(e),$$
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Definition

Let E be a graph and k be a field. The Cohn path algebra $C_k(E)$ of E is the path algebra $k\bar{E}$ of the extended graph $\bar{E}$ divided by the ideal generated by

$$\{\chi_{e^*}\chi_f - \delta_{e,f}\chi_{t(e)} \mid e, f \in E^1\}.$$

Leavitt path algebras and graph C*-algebras

Definition

Let E be a graph and k be a field. The Leavitt path algebra $L_k(E)$ of E is the path algebra $k\bar{E}$ of the extended graph $\bar{E}$ divided by the ideal generated by the union of the following sets:

1. $\{\chi_{e^*}\chi_f - \delta_{e,f}\chi_{t(e)} \mid e, f \in E^1\}$,
2. $\{\sum_{e\in s^{-1}(v)} \chi_e\chi_{e^*} - \chi_v \mid v \in \mathrm{reg}(E)\}$.

Here $\mathrm{reg}(E)$ is the set of all regular vertices of E, and a vertex is called regular iff it emits at least one edge and at most finitely many edges.

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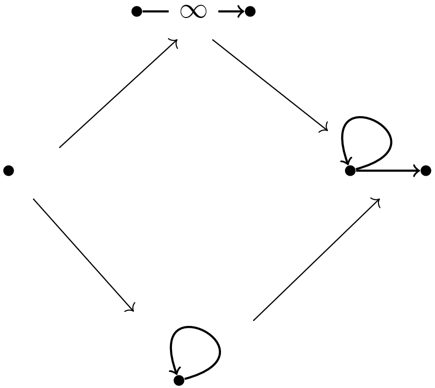
Definition

Let E be a graph and $k = \mathbb{C}$ be a field. Then the formulas

$$\forall\, v \in E^0\colon (\chi_v)^* := \chi_v\,,\ \forall\, e \in E^1\colon (\chi_e)^* := \chi_{e^*}\,,\ (\chi_{e^*})^* := \chi_e\,,$$

define involutions rendering $\mathbb{C}\bar{E}$, $C_{\mathbb{C}}(E)$, and $L_{\mathbb{C}}(E)$ $*$-algebras. The universal C*-algebra of $L_{\mathbb{C}}(E)$ is called the graph C*-algebra of E, and denoted by $C^*(E)$.

Quantum $\mathbb{C}P^1$



A pullback theorem

Theorem (A. Chirvasitu, P.M.H., M. Tobolski)

Let $F_i \subseteq E_i$, $i = 1, 2$, be admissible inclusions of graphs such that

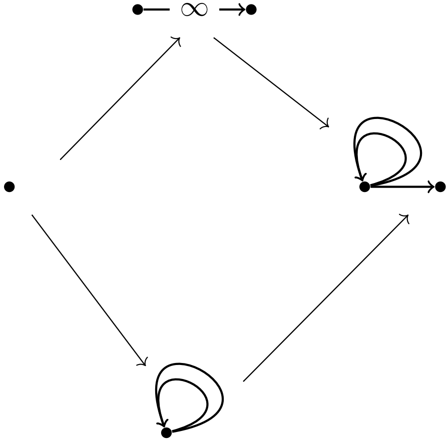
1. E_1 *has no loops,* E_2 *has no egde loops at vertices in* $E_2^0 \setminus F_2^0$*,* $E_1^0 = E_2^0$ *is finite, and* $F_1^0 = F_2^0$*;*
2. *there is a functor* $f\colon E_1 \to E_2$ *such that: it is* id *on objects, its image is the set of all pointed paths (paths whose last edge is not a loop), and it maps any two different edges to two paths such that none of them is a prolongation of the other.*

*Then the induced *-homomorphisms exist and render the diagram*

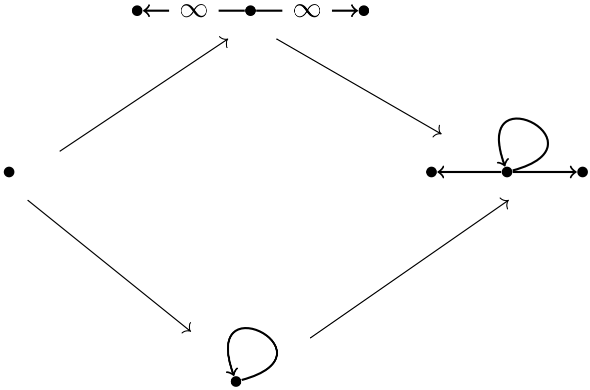
$$\begin{array}{ccccc} & & C^*(E_1) & & \\ & \swarrow & & \searrow^{f_*} & \\ C^*(F_1) & & & & C^*(E_2) \\ & {}_{\bar{f}_*}\searrow & & \swarrow & \\ & & C^*(F_2) & & \end{array}$$

a pullback diagram of unital C-algebras.*

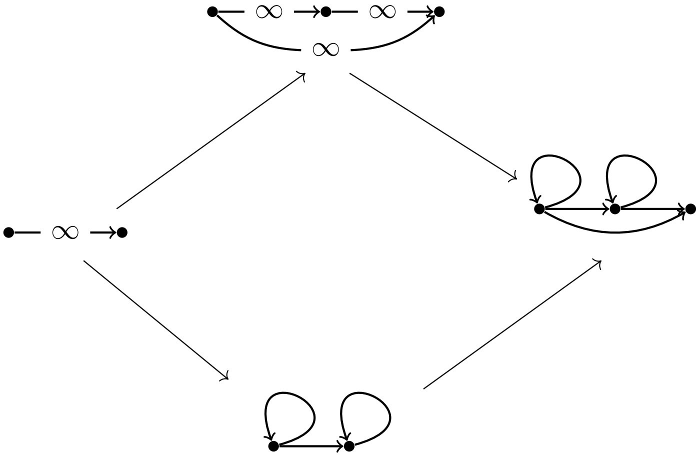
A quantum bonus



Quantum weighted complex projective line



Quantum $\mathbb{C}P^2$



Covariant conditions

C1 IPG is the category of graphs and path homomorphisms of graphs that are injective when restricted to vertices.

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when restricted to the sets of edges.

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when restricted to the sets of edges.

C3 $RMIPG$ is the subcategory of $MIPG$ whose morphisms satisfy the regularity conditions:

(A) For any $v \in \operatorname{reg}(E)$, the vertex $f(v)$ emits $|s_E^{-1}(v)|$-many positive-length paths $p_1, \ldots, p_{n_v}$, $n_v := |s_E^{-1}(v)|$, whose all edges begin at regular vertices. Also, we require that the set $FP_{f(v)} := \{p_1, \ldots, p_{n_v}\}$ is constructed in the following way: we take $x \in s_F^{-1}(f(v))$ and either set it aside as a length-one element of $FP_{f(v)}$, or extend it by all edges emitted from $t_F(x)$. Any thus obtained path of length two, we either set aside as an element of $FP_{f(v)}$, or extend it by all edges emitted from its end. Then we iterate this procedure until we obtain the set $FP_{f(v)}$.

(B) For any $v \in \operatorname{reg}(E)$, the map f when restricted to $s_E^{-1}(v)$ is a bijection onto $FP_{f(v)}$.

RMIPG

Covariant functors

Let k be a field. The following assignments define covariant functors to the category of k-algebras:

1. $\forall\,(f\colon E \to F) \in \mathrm{Mor}(IPG)$, $p \in FP(E)$:
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2. $\forall\,(f\colon E \to F) \in \mathrm{Mor}(MIPG)$, $p \in FP(\bar{E})$:
$$C_k(E) \ni [\chi_p] \xmapsto{f_*^C} [\chi_{\bar{f}(p)}] \in C_k(F),$$

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3. $\forall\,(f\colon E \to F) \in \mathrm{Mor}(RMIPG)$, $p \in FP(\bar{E})$:
$$L_k(E) \ni [\chi_p] \xrightarrow{f_*^L} [\chi_{\bar{f}(p)}] \in L_k(F).$$

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Here $\bar{f}\colon FP(\bar{E}) \to FP(\bar{F})$ is the obvious extension of $f\colon FP(E) \to FP(F)$.

Main result

Theorem

The directed graphs together with path homomorphisms satisfying the three covariant conditions form a subcategory in the category of graphs and path homomorphisms. We call this subcategory RMIPG. Moreover, the assignments

$$\forall\, E \in \mathrm{Obj}(RMIPG)\colon E \stackrel{f_*^L}{\longmapsto} C^*(E),$$

$$\forall\, (f\colon E \to F) \in \mathrm{Mor}(RMIPG),\, p \in FP(\bar{E})\colon$$

$$C^*(E) \ni [\chi_p] \stackrel{f_*^L}{\longmapsto} [\chi_{\bar{f}(p)}] \in C^*(F),$$

define a covariant functor into the category of C-algebras and *-homomorphisms.*

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define a covariant functor into the category of C-algebras and *-homomorphisms. Finally, restricting the functor to the subcategory $RMBPG$ (graphs have finitely many vertices and path homomorphisms are bijective when restricted to the sets of vertices) gives a functor into the subcategory of unital C*-algebras and unital *-homomorphisms.*