The K-distribution of random graph C^*-algebras

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Graph algebras in Będlewo, 03.07.2023

Motivation

If a finite graph $\mathbb{E}_n$ is created by randomly adding edges to a fixed large vertex set $\{1, \ldots, n\}$, then what is

$$\mathbb{P}\left(C^*(\mathbb{E}_n) \cong_{\otimes \mathbb{K}} \mathcal{O}_m \text{ for some } m \right) ?$$

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Each possible edge (allowing loops but not multiple edges) occurs independently with probability q.

$\Leftrightarrow$ The entries $A(n)_{ij}$ of the adjacency matrix are independent Bernoulli random variables with law $\mu(\{1\}) = q$, $\mu(\{0\}) = 1-q$.

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- loops are not allowed (but this does not affect connectivity!)
- a different (but closely related!) distribution is used
  $\mathbb{E}_{n,m}$ — uniform distribution on the family of graphs with $m \in [0, \binom{n}{2}]$ edges on n labelled vertices.

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Theorem (Erdős - Rényi)

If $q = (\log n + \omega)/n$ for some function $\omega = \omega(n)$,

then

$$\lim_{n \to \infty} \mathbb{P}\big((\mathbb{E}_{n,q}) \text{ is connected}\big) = \begin{cases} 0 & \omega \to -\infty \\ e^{-e^{-c}} & \text{if } \omega \to c \\ 1 & \omega \to +\infty. \end{cases}$$

Corollary With q as above,

$$\lim_{n\to\infty} \mathbb{P}\big((\mathbb{D}_{n,q}) \text{ is strongly connected}\big) = \begin{cases} 0 & \omega \to -\infty \\ e^{-2e^{-c}} & \text{if } \omega \to c \\ 1 & \omega \to +\infty. \end{cases}$$

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Corollary If q is kept constant as n varies, and $\mathbb{E}_n = \mathbb{D}_{n,q}$ or $\mathbb{E}_{n,q}$, then $C^*(\mathbb{E}_n)$ is asymptotically almost surely (= with high probability) a simple Cuntz-Krieger algebra, i.e.

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Recall : In the absence of sinks,

$K_0(C^*(\mathbb{E}_n)) \cong \operatorname{coker} M(n)$

$K_1(C^*(\mathbb{E}_n)) \cong \ker M(n) \cong$ torsion-free part of K_0

where $M(n) = A(n)^t - I$.

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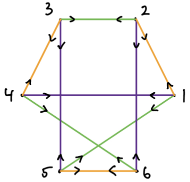

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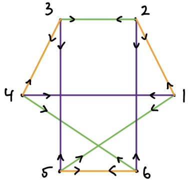

$A(n)$ is symmetric, upper triangular entries are <u>not</u> independent.

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Corollary If $\mathbb{E}_n = \mathbb{D}_{n,q}$ or $\mathbb{E}_{n,q}$ or $\mathbb{G}_{n,r}$, then whp $C^*(\mathbb{E}_n)$ is a simple Cuntz-Krieger algebra with trivial K_1.

Definition (Cuntz polygons)

For $\bar{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$, let $E_{\bar{m}}$ be the graph with vertex set $\{1, \ldots, n\}$ and

edge set $\{l_1, \ldots, l_n\} \cup \{e_{ij}\}_{\substack{1 \le i \le n \\ 1 \le j \le m_i}}$

with $r(l_i) = s(l_i) = r(e_{ij}) = i$

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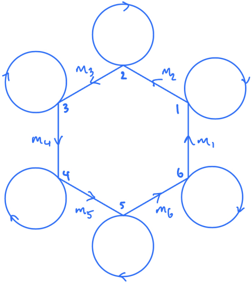

$$A_{\bar{m}} = I + \begin{pmatrix} & m_2 & & & & \\ & & m_3 & & & \\ & & & m_4 & & \\ & & & & m_5 & \\ & & & & & m_6 \\ m_1 & & & & & \end{pmatrix}$$

$$A_{\bar{m}}^t - I = \operatorname{diag}(m_1, \ldots, m_n) \cdot P$$

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e.g. $G = \underbrace{\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3^4\mathbb{Z}}_{G_3} \oplus \underbrace{\mathbb{Z}/7\mathbb{Z}}_{G_7}$ noncyclic

$G_p = 0$ for all other p

$H = \mathbb{Z}/2^4\mathbb{Z} \oplus \mathbb{Z}/5^2\mathbb{Z} \cong \mathbb{Z}/400\mathbb{Z}$ cyclic

(4) If $\mathbb{E}_n = \mathbb{D}_{n,q}$ or $\mathbb{E}_{n,q}$ or $\mathbb{G}_{n,r}$, then

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(c) $$c(\mathbb{D}) = \prod_{p \text{ prime}} \left(1 + \frac{1}{p^2-p}\right) \prod_{k=2}^{\infty} \zeta(k)^{-1} \approx 0.85 .$$

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(a) $\lim_{n\to\infty} \mathbb{P}\left(K_0(C^*(\mathbb{E}_n))_p \cong G\right) = N(G) \prod_{k=1}^{\infty} (1-p^{-2k+1})$

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$$N(G) = \frac{|\{\text{perfect pairings } G\times G \to \mathbb{C}^*\}|}{|G|\cdot|\mathrm{Aut}(G)|}$$

$$= p^{-N} \text{ if } G = \mathbb{Z}/p^N\mathbb{Z}$$

Aside From (a) we can deduce $\lim_{n\to\infty} \mathbb{P}(K_1 \neq 0) = 0$:

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What about $\mathbb{G}_{n,r}$?

To start with, what is c_p when $p \mid 2(r-1)$?

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The typical sample size was $m = 10^5$.

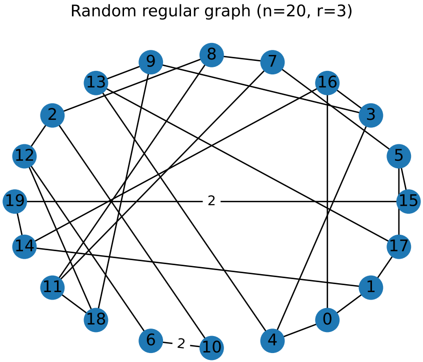

Figure: Sample generated graph $\mathbb{G}_{20,3}$

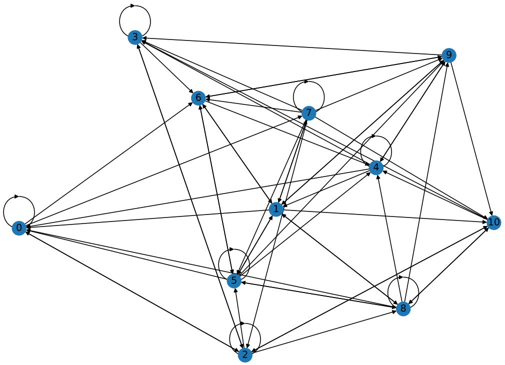

Figure: Sample generated graph $\mathbb{D}_{11,1/2}$

(n,q)	$\mathbb{D}_{n,q}$ connected	$K_1 \neq 0$	$(\mathrm{Tor}(K_0))_p$ cyclic				
			all p	$p=2$	$p=3$	$p=5$	$p=7$
$(50,1/2)$	100000	0	84881	86769	98104	99788	99954
$(100,1/2)$	100000	0	85098	86928	98086	99819	99961
$(50,1/3)$	100000	0	84597	86598	97975	99784	99950
$(100,1/3)$	100000	0	84727	86676	98003	99801	99952
$(50,1/4)$	99994	0	84756	86679	98057	99793	99955
$(100,1/4)$	100000	0	84586	86570	97982	99805	99958
$n \to \infty$	$10^5 - O(1/n)$	$10^5(1/\sqrt{2}+o(1))^n$	84694	86636	98022	99794	99951

Table: $\mathrm{C}^*(\mathbb{D}_{n,1/k}),\ n=50,100,\ k=2,3,4$

(n, q)	$\mathbb{D}_{n,q}$ connected	$K_1 \neq 0$	$(\mathrm{Tor}(K_0))_p$ cyclic				
			all p	$p = 2$	$p = 3$	$p = 5$	$p = 7$
$(50, 1/2)$	100000	0	84881	86769	98104	99788	99954
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Table: $\mathrm{C}^*(\mathbb{D}_{n,1/k})$, $n = 50, 100$, $k = 2, 3, 4$

q $(n = 100)$	$\mathbb{D}_{100,q}$ connected	$K_1 \neq 0$	$(\mathrm{Tor}(K_0))_p$ cyclic				
			all p	$p = 2$	$p = 3$	$p = 5$	$p = 7$
$3 \log n/n$	99993	3	84617	86586	97990	99781	99958
$2 \log n/n$	98606	114	84880	86786	98062	99805	99952
$\log n/n$	15623	8829	85713	87481	98183	99801	99941
$\log n/2n$	0	51702	87968	89172	98776	99866	99978

Table: $\mathrm{C}^*(\mathbb{D}_{n,k \log n/n})$, $n = 100$, $k = 3, 2, 1, 0.5$

(n, q)	$\mathbb{E}_{n,q}$ connected	$K_1 \neq 0$	$(\mathrm{Tor}(K_0))_p$ cyclic				
			all p	$p = 2$	$p = 3$	$p = 5$	$p = 7$
$(50, 1/2)$	100000	0	79251	83796	95908	99129	99689
$(100, 1/2)$	100000	0	79463	83990	95806	99182	99713
$(50, 1/3)$	100000	0	79355	83960	95792	99137	99698
$(100, 1/3)$	100000	0	79488	84056	95890	99172	99675
$(50, 1/4)$	99999	0	79349	83804	95919	99215	99688
$(100, 1/4)$	100000	0	79279	83774	95890	99141	99690
$n \to \infty$	$10^5 - O(1/n)$	$o(1)$	79352	83884	95851	99167	99702

Table: $\mathrm{C}^*(\mathbb{E}_{n,1/k})$, $n = 50, 100$, $k = 2, 3, 4$

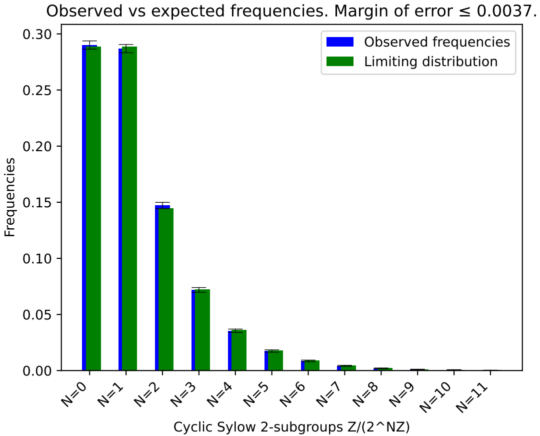

Figure: Frequency distribution for $K_0(\mathrm{C}^*(\mathbb{D}_{100,1/4}))_2$

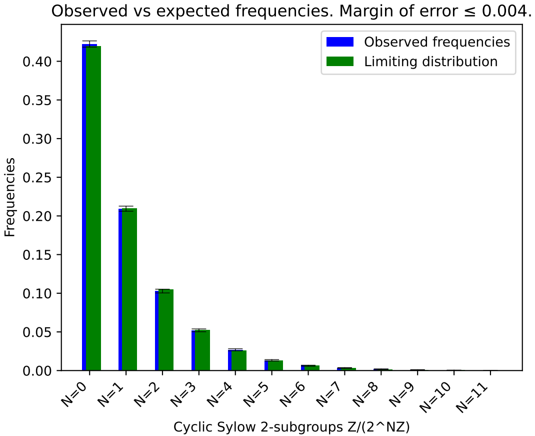

Figure: Frequency distribution for $K_0(\mathrm{C}^*(\mathbb{E}_{100,1/2}))_2$

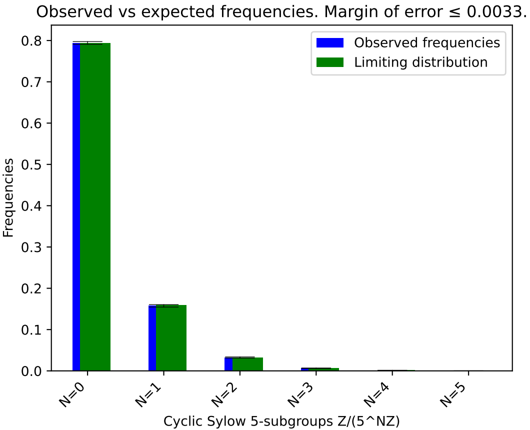

Figure: Frequency distribution for $K_0(\mathrm{C}^*(\mathbb{G}_{200,3}))_5$

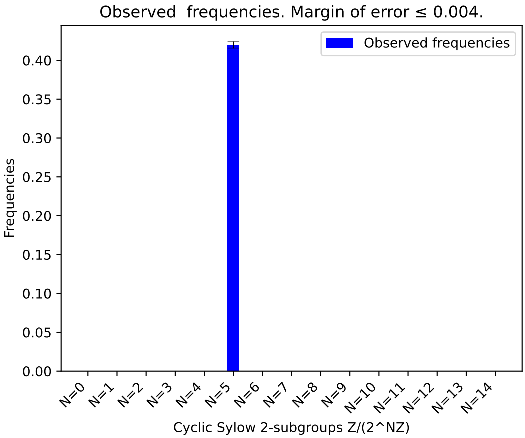

Figure: Frequency distribution for $K_0(\mathrm{C}^*(\mathbb{G}_{100,9}))_2$

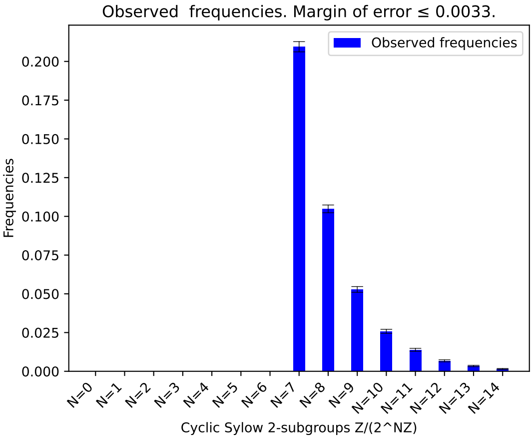

Figure: Frequency distribution for $K_0(\mathrm{C}^*(\mathbb{G}_{200,9}))_2$

r	4	5	6	7	8	9	10	11	12	17
$\widehat{\pi}_{2,r}$	**0.416**	**0.418**	**0.419**	**0.415**	**0.416**	**0.420**	**0.420**	**0.420**	**0.418**	**0.419**
$\prod_{\substack{p \text{ prime} \\ p \leq 37}} \widehat{\pi}_{p,r}$	0.264	0.395	0.317	0.262	0.338	0.396	0.265	0.318	0.359	0.397
$\widehat{\gamma}_{100,r}$	0.265	0.395	0.316	0.261	0.338	0.396	0.264	0.317	0.360	0.397

Table: Cyclicity frequencies for $K_0(\mathrm{C}^*(\mathbb{G}_{100,r}))$

r	4	5	6	7	8	9	10	11	12	17
$\widehat{\pi}_{2,r}$	**0.416**	**0.418**	**0.419**	**0.415**	**0.416**	**0.420**	**0.420**	**0.420**	**0.418**	**0.419**
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Table: Cyclicity frequencies for $K_0(\mathrm{C}^*(\mathbb{G}_{100,r}))$

r	6	7	8	10	11	12	13	14	20
$p \mid r-1$	5	3	7	3	5	11	3	13	19
$\widehat{\pi}_{p,r}$	**0.794**	**0.639**	**0.856**	**0.639**	**0.794**	**0.908**	**0.638**	**0.922**	**0.947**
$\prod_{k=1}^{\infty} \left(1 - p^{-2k+1}\right)$	0.793	0.639	0.855	0.639	0.793	0.908	0.639	0.923	0.947

Table: Cyclicity frequencies for $K_0(\mathrm{C}^*(\mathbb{G}_{100,r}))_p$, $p \mid r-1$

Conjecture If $\mathbb{E}_n = \mathbb{G}_{n,r}$, then :

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(1) $$c_p = \prod_{k=1}^{\infty}\left(1 - p^{-2k+1}\right) \text{ if } p \mid 2(r-1)$$ ≈ 0.42 if $p = 2$

Conjecture If $\mathbb{E}_n = \mathbb{G}_{n,r}$, then :

(1) $c_p = \prod_{k=1}^{\infty}\left(1-p^{-2k+1}\right)$ if $p \mid 2(r-1)$ ≈ 0.42 if $p=2$

(2) $c(\mathbb{G}) = \prod_{\substack{p \text{ prime} \\ p \mid 2(r-1)}} (1-p)^{-1} \prod_{p \text{ prime}} \prod_{k=2}^{\infty}\left(1-p^{-2k+1}\right)$ ≈ 0.40 if $r = 2^j+1$

Děkuji!