Modular invariants of compact quantum groups

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based on a joint work with Piotr Sołtan

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Part I: introduction

Definition (Compact quantum group) [Woronowicz]

Compact quantum group $\mathbb{G}$ consists of:

- a unital C*-algebra $\mathfrak{A}$,
- a unital $\star$-homomorphism $\Delta\colon \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$ such that:

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta,$$
$$\overline{\operatorname{span}}\,\Delta(\mathfrak{A})(\mathbb{1} \otimes \mathfrak{A}) = \overline{\operatorname{span}}\,\Delta(\mathfrak{A})(\mathfrak{A} \otimes \mathbb{1}) = \mathfrak{A} \otimes \mathfrak{A}.$$

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- [Woronowicz, Van Daele] There exisits a unique Haar integral: state $h \in \mathfrak{A}^*$ which is invariant:

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- Let $(\mathrm{L}^2(\mathbb{G}), \pi_h, \Omega_h)$ be the GNS representation for h.
- We write $\mathrm{C}(\mathbb{G}) = \pi_h(\mathfrak{A})$, $\mathrm{L}^\infty(\mathbb{G}) = \pi_h(\mathfrak{A})''$, $\mathrm{L}^1(\mathbb{G}) = \mathrm{L}^\infty(\mathbb{G})_*$.

Examples

- Let G be a compact Hausdorff group with Haar measure μ. Define $\mathrm{C}(\mathbb{G}) = \mathrm{C}(G)$ and Δ via

$$\Delta(f)(x,y) = f(xy) \quad (f \in \mathrm{C}(G), x, y \in G).$$

Then $\mathrm{L}^\infty(\mathbb{G}) = \mathrm{L}^\infty(G), h = \int_G \cdot \, \mathrm{d}\mu$.

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- Let Γ be a discrete group. Define $\mathrm{C}(\mathbb{G}) = \mathrm{C}^*_r(\Gamma)$ and

$$\Delta\colon \mathrm{L}(\Gamma) \ni \lambda_\gamma \mapsto \lambda_\gamma \otimes \lambda_\gamma \in \mathrm{L}(\Gamma) \bar{\otimes} \mathrm{L}(\Gamma).$$

  Then $h(\lambda_\gamma) = \delta_{e,\gamma}$ and $\mathrm{L}^\infty(\mathbb{G}) = \mathrm{L}(\Gamma)$. We write $\mathbb{G} = \widehat{\Gamma}$.

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  Then $h(\lambda_\gamma)=\delta_{e,\gamma}$ and $\mathrm{L}^\infty(\mathbb{G})=\mathrm{L}(\Gamma)$. We write $\mathbb{G}=\widehat{\Gamma}$.
- With any quantum group $\mathbb{G}$ we can associate its dual $\widehat{\mathbb{G}}$ and $\widehat{\widehat{\mathbb{G}}}\simeq\mathbb{G}$. If $\mathbb{G}$ is compact, $\widehat{\mathbb{G}}$ is discrete.

Example: $\mathbb{G} = \mathrm{SU}_q(2)$ $(0 < q < 1)$

- $\mathrm{C}(\mathrm{SU}_q(2))$ is defined as the universal unital C^*-algebra generated by $\alpha, \gamma \in \mathrm{C}(\mathrm{SU}_q(2))$ such that

$$\begin{bmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{bmatrix} \quad \text{is unitary.}$$

- $\Delta\colon \mathrm{C}(\mathrm{SU}_q(2)) \to \mathrm{C}(\mathrm{SU}_q(2)) \otimes \mathrm{C}(\mathrm{SU}_q(2))$ acts via

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma,$$
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- [Hong-Szymański] $\mathrm{C}(\mathrm{SU}_q(2))$ is a graph C*-algebra.

Modular theory of $\mathbb{G}$

- There is a continuous group of modular automorphisms $\sigma_t^h \in \operatorname{Aut}(\mathrm{L}^\infty(\mathbb{G}))\,(t \in \mathbb{R})$

$$h(x\,y) = h(y\,\sigma^h_{-i}(x)) \qquad (x, y \in \mathrm{L}^\infty(\mathbb{G}) \text{ nice}).$$

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- There is a point-w^* continuous group of scaling automorphisms $\tau_t \in \mathrm{Aut}(\mathrm{L}^\infty(\mathbb{G}))$.

Modular theory of $\mathbb{G}$

- Compact quantum groups have representation theory resembling the classical one.
- $\mathrm{Irr}(\mathbb{G})$ – set of (classes of) irreducible representations

$$\alpha \in \mathrm{Irr}(\mathbb{G}) \quad \rightsquigarrow \quad U^{\alpha} \in \mathrm{C}(\mathbb{G}) \otimes \mathrm{B}(\mathsf{H}_{\alpha}), \quad \dim \mathsf{H}_{\alpha} < +\infty.$$

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$$\{\xi_i^\alpha\}_{i=1}^{\dim(\alpha)} \text{ orthonormal basis of } \mathsf{H}_\alpha \rightsquigarrow U^\alpha_{i,j} = (\mathrm{id} \otimes \omega_{\xi_i^\alpha, \xi_j^\alpha}) U^\alpha \in \mathrm{C}(\mathbb{G}).$$

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- There is a family of positive, invertible operators $\rho_\alpha \in \mathrm{B}(\mathsf{H}_\alpha)$.
- Automorphisms σ_t^h, τ_t can be expressed using ρ_α.

$$\sigma_t^h(U_{i,j}^\alpha) = (\rho_{\alpha,i}\rho_{\alpha,j})^{it} U_{i,j}^\alpha \quad \tau_t(U_{i,j}^\alpha) = (\tfrac{\rho_{\alpha,i}}{\rho_{\alpha,j}})^{it} U_{i,j}^\alpha,$$

where $\rho_\alpha = \mathrm{diag}(\rho_{\alpha,1}, \dots, \rho_{\alpha,\dim(\alpha)})$.

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  where $\rho_\alpha = \mathrm{diag}(\rho_{\alpha,1}, \dots, \rho_{\alpha,\dim(\alpha)})$.
- h is tracial $\Leftrightarrow \forall_t\, \sigma_t^h = \mathrm{id} \Leftrightarrow \forall_\alpha\, \rho_\alpha = \mathbb{1} \Leftrightarrow \forall_t\, \tau_t = \mathrm{id}$.
  In this case $\mathbb{G}$ is of Kac type.

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- If $\mathbb{G} = \mathrm{SU}_q(2)$ then the Haar integral h is not tracial and

$$\sigma_t^h(\alpha) = q^{-2it}\alpha, \quad \sigma_t^h(\gamma) = \gamma,$$
$$\tau_t(\alpha) = \alpha, \quad \tau_t(\gamma) = q^{2it}\gamma.$$

Modular invariants – motivation

- Fix $0 < \lambda < 1$. With Piotr we've constructed a family of CQGs $\{\mathbb{K}_j\}_{j\in\mathbb{J}}$ such that $\mathrm{L}^\infty(\mathbb{K}_j)$ is the injective type III_λ factor.
- How can we show that that $\mathbb{K}_j \neq \mathbb{K}_{j'}$?

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- $\{1, \alpha_j\}_{j\in\mathbb{J}}$ – basis of $\mathbb{R}$ over $\mathbb{Q}$, $\Gamma_j = \alpha_j \frac{2\pi}{\log(\lambda)}\mathbb{Z}$,

$$\mathbb{K}_j = \Gamma_j \bowtie \mathbb{G} \;\text{ via } \Gamma_j \times \mathrm{L}^\infty(\mathbb{G}) \ni (\gamma, x) \mapsto \tau_\gamma(x) \in \mathrm{L}^\infty(\mathbb{G}).$$

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- $\tau_t \in \mathrm{Inn}(\mathrm{L}^{\infty}(\mathbb{K}_j))$ if and only if

$$t \in \Gamma_j + \tfrac{2\pi}{\log(\lambda)}\mathbb{Z} \ \Rightarrow\ \mathbb{K}_j \neq \mathbb{K}_{j'} \ (j \neq j').$$

Let $\mathbb{G}$ be a compact quantum group.

Modular invariants

Define subgroups of $\mathbb{R}$:

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And similarly $T^{\sigma}(\mathbb{G}) = \{t \in \mathbb{R} \,|\, \sigma^h_t = \mathrm{id}\}$, $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}), T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G})$.

- $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) = T(\mathrm{L}^{\infty}(\mathbb{G}))$ is the Connes' T-invariant.

Let $\mathbb{G}$ be a locally compact quantum group.

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$$\mathrm{Mod}(\mathbb{G}) = \{t \in \mathbb{R} \,|\, \delta^{it} = \mathbb{1}\} \text{ where } \delta \eta \, \mathrm{L}^{\infty}(\mathbb{G}) \text{ is the modular element.}$$

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- All these sets depends only on the isomorphism class of $\mathbb{G}$.

Let $\mathbb{G}$ be a locally compact quantum group.

Modular invariants

- A priori we obtain 14 subgroups of $\mathbb{R}$:

$$T^{\tau}, T^{\tau}_{\mathrm{Inn}}, T^{\tau}_{\overline{\mathrm{Inn}}}, T^{\sigma}, T^{\sigma}_{\mathrm{Inn}}, T^{\sigma}_{\overline{\mathrm{Inn}}} \text{ and } \operatorname{Mod} \text{ for } \mathbb{G}, \widehat{\mathbb{G}}.$$

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- There are easy reductions:

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so we have 11 subgroups.

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  - $(\tau_t \otimes \widehat{\tau}_t)\mathrm{W}^{\mathbb{G}} = \mathrm{W}^{\mathbb{G}}$,
  - $\nabla_{\psi}^{it} = \widehat{\delta}^{-it} P^{-it}$ (P^{it} implements τ_t and $\widehat{\tau}_t$).

Modular invariants

- If $\mathbb{G}$ is compact then $\delta = \mathbb{1}, \ell^\infty(\widehat{\mathbb{G}}) = \prod_{\alpha \in \mathrm{Irr}(\mathbb{G})} \mathrm{B}(\mathsf{H}_\alpha)$ so we are left with 6 invariants

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- If additionally $\mathrm{L}^\infty(\mathbb{G})$ is semifinite, then $T^\sigma_{\mathrm{Inn}}(\mathbb{G}) = T^\sigma_{\overline{\mathrm{Inn}}}(\mathbb{G}) = \mathbb{R}$ and there are 4 possibly non-trivial invariants. This is the case for G_q.

Question

Let $\mathbb{G}$ be a second countable compact quantum group. Assume $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = \mathbb{R}$. Is $\mathbb{G}$ of Kac type?

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- Equivalently: $\mathbb{G}$ second countable, not of Kac type. Do we have $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) \neq \mathbb{R}$?
- [K., Sołtan] The answer is affirmative in special cases:
    - there is a unitary representation U with $2 = \dim(U) < \dim_q(U)$,
    - $\mathrm{C}^u(\mathbb{G})$ is type I, in particular $\mathbb{G} = G_q$,
    - $\mathbb{G} = U_F^+$,
    - $\widehat{\mathbb{G}}$ satisfies an ICC-type condition.

Part II: q-deformations

Lie group G and companions

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  - Weyl group W and W-invariant inner product on $\mathfrak{h}$ such that $\langle \alpha | \alpha \rangle = 2$ for short roots α.

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For $SU(n+1)$: $r = n, W = S_{n+1}$,

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Fix $0 < q < 1$.

q-DEFORMED ENVELOPING ALGEBRA OF $\mathfrak{g}$

- $U_q\mathfrak{g}$ is the unital algebra generated by E_i, F_i, K_i, K_i^{-1} $(1 \leqslant i \leqslant r)$ satisfying certain relations.
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- $\mathrm{Pol}(G_q) = \{\text{matrix coefficients of } \pi_\varpi \text{ as above}\} \subseteq (U_q\mathfrak{g})^*$.

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- [Giselsson] $\mathrm{C}(\mathrm{SU}_q(3))$ is a higher-rank graph C*-algebra.

Automorphisms for G_q

- $\sigma^h_t(U^\varpi(\xi,\eta)) = q^{\langle 2\rho |\, \mathrm{wt}(\xi)+\mathrm{wt}(\eta)\rangle it}\, U^\varpi(\xi,\eta)$,
- $\tau_t(U^\varpi(\xi,\eta)) = q^{\langle 2\rho |\, \mathrm{wt}(\xi)-\mathrm{wt}(\eta)\rangle it}\, U^\varpi(\xi,\eta)$.

where $\rho \in \mathbf{P}^+$ is the Weyl vector and $\langle\cdot|\cdot\rangle$ is W-invariant scalar product on $\mathfrak{h}^*$.

C*-ALGEBRA $C(G_q)$

- [Soibelman] Irreducible representations of $C(G_q)$ are (up to equivalence) precisely

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- $\mathrm{C}(G_q)$ is type I.

Haar integral on G_q

- [Reshetikhin-Yakimov] Haar integral on G_q can be calculated as

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- $b_\rho = U^\rho(\xi_\rho, \eta_{w_\circ\rho}) \in \mathrm{Pol}(G_q)$
- Desmedt's theorem: we obtain:
  - unitary $\mathcal{Q}_L \colon \mathrm{L}^2(G_q) \to \int_T^\oplus \mathrm{HS}(\mathscr{H}_\lambda)\,\mathrm{d}\lambda$ such that
  - $\mathcal{Q}_L\,\mathrm{L}^\infty(G_q)\mathcal{Q}_L^* = \int_T^\oplus \mathrm{B}(\mathscr{H}_\lambda) \otimes \mathbb{1}_{\overline{\mathscr{H}_\lambda}}\,\mathrm{d}\lambda.$

Scaling group

[K., Sołtan] For $t \in \mathbb{R}, x = \int_T^{\oplus} x_\lambda \otimes \mathbb{1}_{\overline{\mathscr{H}_\lambda}} \mathrm{d}\lambda \in \mathrm{L}^\infty(G_q)$ we have

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- Corollary: $T^\tau_{\mathrm{Inn}}(G_q) = T^\tau_{\overline{\mathrm{Inn}}}(G_q)$.

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- $\{\langle 2\rho|\varpi\rangle \,|\, \varpi \in \mathbf{P}\} = \Upsilon_w\mathbb{Z}$ is a nontrivial subgroup of $\mathbb{Z}$ ($\Upsilon_w \in \mathbb{N}$).

Theorem [K., Sołtan]

Modular invariants for G_q are given by

$$T^{\tau}(G_q) = \tfrac{\pi}{\log(q)}\mathbb{Z}, \quad T^{\tau}_{\mathrm{Inn}}(G_q) = T^{\tau}_{\overline{\mathrm{Inn}}}(G_q) = \mathrm{Mod}(\widehat{G_q}) = \tfrac{\pi}{\Upsilon_w \log(q)}\mathbb{Z}.$$

Irreducible root systems

- Every root system Φ decomposes as $\Phi_1 \oplus \cdots \oplus \Phi_l$ for Φ_i irreducible.

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- $\Upsilon_w = \gcd(\Upsilon_w^{(1)}, \ldots, \Upsilon_w^{(l)})$.
- Irreducible root systems are classifed:
  - type $A_n \, (n \geqslant 1), G = \mathrm{SU}(n+1)$,
  - type $B_n \, (n \geqslant 2), G = \mathrm{Spin}(2n+1)$,
  - type $C_n \, (n \geqslant 3), G = \mathrm{Sp}(2n)$,
  - type $D_n \, (n \geqslant 4), G = \mathrm{Spin}(2n)$,
  - exceptional: types E_6, E_7, E_8, F_4, G_2.

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[K., Sołtan] Invariants in simple case

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If $\Upsilon_w \geqslant 2$, then G_q has non-trivial, inner scaling automorphisms not implemented by a group-like unitary.

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- $D_n (n \geqslant 4)$: $\Upsilon_w = 2\,(n \in 4\mathbb{N} + \{0,1\}), \Upsilon_w = 1\,(n \in 4\mathbb{N} + \{2,3\})$.
- For E_6, E_7, E_8, F_4, G_2 number Υ_w is equal to $2, 1, 2, 2, 2$.

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- For $\mathrm{SU}_q(3)$ we obtain $t_0 = \frac{\pi}{2\log(q)}$ such that $\tau_{t_0} \in \mathrm{Aut}(\mathrm{L}^\infty(\mathrm{SU}_q(3)))$ is inner, non-trivial, not implemented by any group-like unitary.

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- Is it the case that $\tau_{t_0} \neq \mathrm{Ad}(u)$ for any $u \in \mathrm{C}(\mathrm{SU}_q(3))$?
  - Grasp on $\pi_{\lambda,w_\circ}(b_\rho) \in \mathrm{B}(\ell^2(\mathbb{Z}_+)^{\ell(w_\circ)})$ – work in progress.

Theorem [K., Sołtan]

Let $\mathbb{G}$ be a second countable compact quantum group, assume:

- there is a finite dimensional unitary representation U with $2 = \dim(U) < \dim_q(U)$.

Then $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) \neq \mathbb{R}$.

- Assume by contradiction $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = \mathbb{R}$. Then $\tau_t = \mathrm{Ad}(a^{it})$ for strictly positive $a \,\eta\, \mathrm{L}^{\infty}(\mathbb{G})$.

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$$\varepsilon_t \xrightarrow[t \to 0^+]{} 0, \ \Gamma(U^n) \xrightarrow[n \to \infty]{} +\infty \ \rightsquigarrow \ \text{contradiction.}$$