Curvature and Noncommutative Probability

Yang Liu
July 6, 2023

SISSA, Trieste

Operator Algebras That One Can See, Będlewo

Spectral Triples

- Prototype: Connes' model for spin geometry: $(C^\infty(M), L^2(\not{S}), \not{D})$
- In general $(\mathcal{A}, \mathcal{H}, D)$
  - $\pi : \mathcal{A} \to B(\mathcal{H})$ is a $*$-representation.
  - D is an unbounded self-adjoint operator on $\mathcal{H}$;
  - $\forall a \in \mathcal{A}$, $[D, \pi(a)]$ is bounded;
- In applications, we often have more properties that allow us to establish the analytic continuation of the spectral zeta functions

$$\zeta(s) = \mathrm{Tr}(a\,|D|^{-s}), \quad a \in \mathcal{A}, \Re s \gg 1.$$

Basic notations for $C^\infty(\mathbb{T}^2_\theta)$

- Smooth structure: $C^\infty(\mathbb{T}^2)$, $\mathbb{T}^2 \cong \mathbb{R}^2/(2\pi\mathbb{Z})^2$
- Generators: $U = e^{ix}$, $V = e^{iy}$
- Fourier series:

$$\sum_{n,m\in\mathbb{Z}} a_{n,m}U^nV^m, \ a_{n,m} \in \mathcal{S}(\mathbb{Z}^2)$$

- $*$-involution: $a \mapsto \bar{a}$;
- Partial derivatives: $\delta_1 = -i\partial_x$ and $\delta_2 = -i\partial_y$

- Smooth structure: $C^\infty(\mathbb{T}^2_\theta)$,
- Generators: U, V, $U^*U = UU^* = 1$, $VV^* = V^*V = 1$ and $UV = e^{2\pi i\theta}VU$.
- Fourier series:

$$\sum_{n,m\in\mathbb{Z}} a_{n,m}U^nV^m, \ a_{n,m} \in \mathcal{S}(\mathbb{Z}^2)$$

- $*$-involution
- Basic derivations δ_1 and δ_2:

$$\delta_1(U) = U, \delta_1(V) = 0.$$

Basic notations for $C^\infty(\mathbb{T}^2_\theta)$

- (Normalized) Lebesgue measure

$$\int_{\mathbb{T}^2} \sum_{n,m\in\mathbb{Z}} a_{n,m} U^n V^m dt = a_{0,0}$$

- Inner product: $a, b \in C^\infty(\mathbb{T}^2)$:

$$\langle a, b\rangle = \int_{\mathbb{T}^2} \overline{b} a dt$$

- Hilbert space $L^2(\mathbb{T}^2, dt)$.

- Canonical trace: $\varphi_0 : C^\infty(\mathbb{T}^2_\theta) \to \mathbb{C}$:

$$\varphi_0\left(\sum_{n,m\in\mathbb{Z}} a_{n,m} U^n V^m\right) = a_{0,0}$$

- Inner product: $a, b \in C^\infty(\mathbb{T}^2_\theta)$:

$$\langle a, b\rangle = \varphi_0(b^* a)$$

- Hilbert space completion:

$$\mathcal{H}_0 = L^2(C^\infty(\mathbb{T}^2_\theta), \varphi_0)$$

Hermitian Structure on $(1,0)$-forms

- Choose $\tau \in \mathbb{C}$ with $\Im\tau > 0$, consider:

$$(x,y) \in (\mathbb{R}/2\pi\mathbb{Z})^2 \mapsto z = (2\pi)^{-1}(x + \tau y) \in \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$$

- Cauchy-Riemann operator:

$$\delta = \delta_1 + \bar{\tau}\delta_2, \quad \delta^* = \delta_1 + \tau\delta_2$$

- $(1,0)$-forms:

$$\Omega^{(1,0)} = \left\{ \sum a\partial(b) \mid a, b \in C^\infty(\mathbb{T}^2_\theta) \right\}.$$

- Hermitian structure: $\forall a, a', b, b' \in C^\infty(\mathbb{T}^2_\theta)$:

$$\langle a\partial(b), a'\partial(b')\rangle := \psi\left((a')^* a, b, (b')^*\right)$$

  where ψ is the positive Hochschild 2-cocycle (cf. [Con94, §VI. 2]) on $C^\infty(\mathbb{T}^2_\theta)$

$$\psi(a,b,c) = -\varphi_0\left(a\delta(b)\delta^*(c)\right)$$

Hermitian Structure on $(1,0)$-forms

$$\psi(a,b,c) = -\varphi_0\left(a\delta(b)\delta^*(c)\right)$$

- ψ comes from the Lelong's positive current on Riemann surfaces.
- On $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, let $z = u + iv$,

$$\begin{aligned}\psi(f_0,f_1,f_2) &= \frac{i}{\pi}\int_{\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})} f_0\partial(f_1)\wedge\overline{\partial}(f_2)\\ &= \frac{i}{\pi}\int_{\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})} f_0\partial_z(f_1)\partial_{\bar{z}}(f_2)dz\wedge d\bar{z}\\ &= \frac{2}{\pi}\int_{\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})} f_0\partial_z(f_1)\partial_{\bar{z}}(f_2)du\wedge dv.\end{aligned}$$

- Denote by $\mathcal{H}^{(1,0)}$ the Hilbert space completion of $\Omega^{(1,0)}$.
- The map

$$\partial : C^\infty(\mathbb{T}_\theta^2) \to \Omega^{(1,0)} : a \to \partial(a)$$

extends to a closed unbounded operator

$$d : \mathcal{H}_0 \to \mathcal{H}^{(1,0)}$$

- Flat spectral triple

$$\left(C^\infty(\mathbb{T}_\theta^2), \mathcal{H}_0 \oplus \mathcal{H}^{(1,0)}, D\right), \quad D = \begin{bmatrix} 0 & d^* \\ d & 0 \end{bmatrix}.$$

Conformal Change of Metrics $g = e^{-h} g_0$

- Choose $h = h^* \in C^\infty(\mathbb{T}^2_\theta)$ and rescale the volume functional: $\varphi(a) = \varphi_0(ae^{-h})$. Inner product and Hilbert space

$$\langle a, b \rangle_\varphi := \varphi(b^* a), \quad \mathcal{H}_\varphi = L^2(C^\infty(\mathbb{T}^2_\theta), \varphi).$$

- KMS-property for φ:

$$\varphi(ab) = \varphi\left(b e^{-h} a e^{h}\right).$$

- Right $C^\infty(\mathbb{T}^2_\theta)$-module structure of $(C^\infty(\mathbb{T}^2_\theta), \langle \cdot, \cdot \rangle_\varphi)$:

$$a^{\mathrm{op}} := J_\varphi a^* J_\varphi \in B(\mathcal{H}_\varphi)$$

where $J_\varphi = e^{-h/2}(\cdot)^* e^{h/2}$ is the Tomita antilinear unitary of the GNS representation associated to φ.

- One has $\forall \xi, a \in C^\infty(\mathbb{T}^2_\theta)$:

$$\xi \triangleleft a = \xi e^{-h/2} a e^{h/2}.$$

Modular Spectral Triples

- Modular spectral triple

$$\left(C^\infty(\mathbb{T}^2_\theta)^{\mathrm{op}}, \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)}, D_\varphi\right), \quad D_\varphi = \begin{bmatrix} 0 & d_\varphi^* \\ d & 0 \end{bmatrix},$$

  where only the twisted commutator is bounded:

$$D_\varphi a^{\mathrm{op}} - (e^{-h/2} a e^h)^{\mathrm{op}} D_\varphi, \quad \forall a \in C^\infty(\mathbb{T}^2_\theta).$$

- Transposed modular spectral triple:

$$\left(C^\infty(\mathbb{T}^2_\theta), \mathcal{H}_0 \oplus \mathcal{H}_0, \overline{D}_\varphi\right).$$

- The Laplacians:

$$\overline{D}_\varphi^2 = \begin{bmatrix} \Delta_\varphi & 0 \\ 0 & \Delta_\varphi^{(1,0)} \end{bmatrix} = \begin{bmatrix} e^{h/2} \Delta e^{h/2} & 0 \\ 0 & \delta^* e^h \delta \end{bmatrix}$$

  where $\Delta = \delta^* \delta$ is the flat Dolbeault Laplacian.

Spectral Geometry of Riemannian Manifolds

Let (M, g) be a closed Riemannian manifold and P_g be a Laplacian type operator. Examples: $P_g = \not{D}_g^2$ or $P_g = \Delta$.

- Small time asymptotic:

$$\mathrm{Tr}(f e^{-tP_g}) \backsim_{t \searrow 0} \sum_{j=0}^{\infty} V_j(f, P_g) t^{(j-m)/2}, \quad \forall f \in C^\infty(M),$$

  where $m = \dim M$.
- Local invariants:

$$V_j(f, P) = \varphi_0 \left(f v_j(P) \right),$$

  where $\varphi_0 : C^\infty(M) \to \mathbb{C}$ is the volume functional:

$$\varphi_0(f) := \int_M f dg.$$

- Examples: for scalar Laplacian Δ, upto some constant:

$$v_0(\Delta) = 1, \quad v_2(\Delta) = \frac{1}{6} S_g.$$

Modular Geometry

- Spectral zeta function

$$\zeta_\varphi(s) = \mathrm{Tr}(\Delta_\varphi^{-s}), \quad \Re s \gg 1.$$

  It admits a meromorphic extension to $\mathbb{C}$.

- Gauss-Bonnet Theorem [FK12, CT11]: the functional

$$h = h^* \in C^\infty(\mathbb{T}_\theta^2) \mapsto \zeta_{\Delta_\varphi}(0)$$

  is constant in h.

- OPS (Osgood-Phillips-Sarnak) functional

$$F_{\mathrm{OPS}}(h) = \zeta'{}_{\Delta_\varphi}(0) + \log \varphi_0(e^{-h})$$

Modular Gaussian Curvature

- For any functional $F(h)$ on self-adjoint elements, we consider the variational problem:

$$h \mapsto h + \varepsilon a, \quad \boldsymbol{\delta}_a = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0},$$

  that is $\boldsymbol{\delta}_a$ is the directional derivative along some self-adjoint $a \in C^\infty(\mathbb{T}^2_\theta)$.
- Define the functional gradient at h w.r.t the inner product $\langle \cdot, \cdot \rangle_{\varphi_0}$ to be the unique element in $C^\infty(\mathbb{T}^2_\theta)$:

$$\boldsymbol{\delta}_a F(h) = \varphi_0 \left(a \operatorname{grad}_h F \right), \ \forall a = a^* \in C^\infty(\mathbb{T}^2_\theta).$$

Modular Gaussian Curvature [CM14]

We define the analog of the Gaussian curvature of the curvature metric Δ_φ to be

$$K_\varphi := \operatorname{grad}_h F_{\mathrm{OPS}}.$$

Why called "modular"

- Modular operator and modular derivation

$$\not\Delta := e^{-h}(\cdot)e^{h} = e^{\not\nabla}, \quad \not\nabla := [\cdot, h]$$

with respect to the weight $\varphi(\cdot) = \varphi_0((\cdot)e^{-h})$.

- Set $\not\nabla^{(1)} = \not\nabla \otimes 1$ and $\not\nabla^{(2)} = 1 \otimes \not\nabla$,

$$K_\varphi = \mathsf{K}(\not\nabla)\,(\Delta h) + \mathsf{H}\left(\not\nabla^{(1)}, \not\nabla^{(2)}\right)(\delta h \otimes \delta h)$$

where

$$K(s) = 8\sum_{1}^{\infty} \frac{B_{2n}}{(2n)!} s^{2n-2}.$$

and there is a simple functional relation for being an Euler-Langrange equation

$$-H(s_1, s_2) = \frac{K(s_2) - K(-s_1)}{s_1 + s_2} + \frac{K(s_1 + s_2) - K(s_2)}{s_1} - \frac{K(s_1 + s_2) - K(s_1)}{s_2}$$

Modular Curvature for Toric Noncommutative Manifolds

- Connes-Landi deformation: M admits a $\mathbb{T}^n$-action as diffeomorphisms, and θ is a $n \times n$ skew-symmetric matrix, choose a metric which is $\mathbb{T}^n$-invariant:

$$\left(C^\infty(M)_\theta, L^2(\not{S}), \not{D}\right)$$

- Conformal change of metric: choose e^h with $h = h^* \in C^\infty(M)_\theta$,

$$\not{D} \mapsto D_g = e^h \not{D} e^h$$

- [Liu18]: construction of a pseudodifferential calculus for M_θ
- [Liu17]: computation of the V_2-term

$$\operatorname{Tr}(f e^{-tD_g^2}) \backsim_{t \searrow 0} \sum_{j=0}^{\infty} V_j(f, D_g) t^{(j-m)/2}$$

What's Next?

- What is the next class of noncommutative manifolds on which one can set up similar problems?
- Potential applications of such notion of curvature?
- More examples involving both geometry and modular theory?

Outline of the Construction by Cipriani, Franz, Wysoczańska-Kula [CFK14]

Ingredients:

- Compact quantum group: (A, Δ), where $\mathcal{A} \subset \mathsf{A}$ is a Hopf $*$-subalgebra.
- For any linear functional $\phi : \mathcal{A} \to \mathbb{C}$, denote by $L_\phi : \mathcal{A} \to \mathcal{A}$ the (left) convolution operator:

$$L_\phi(a) = (\phi \otimes 1)(\Delta(a)) = \sum \phi\left(a_{(1)}\right) a_{(2)}.$$

- $h : \mathsf{A} \to \mathbb{C}$ Haar measure, $\mathcal{H}_0 = L^2(\mathcal{A}, h)$, the GNS-representation.
- $(\mathcal{H}_\pi, \eta, \phi)$: a Schürmann triple, where $\pi : \mathcal{A} \to B(\mathcal{H}_\pi)$ is a representation of $\mathcal{A}$

The spectral triple $(\mathcal{A}, \mathcal{H}_\phi, D)$

- the Hilbert space: $\mathcal{H}_\phi = \mathcal{H}_0 \oplus \mathcal{H}_1$, with representation

$$\pi_L : \lambda_L \otimes \rho_L : \mathcal{A} \to B(\mathcal{H}_\phi).$$

  where λ_L is the left multiplication representation and

$$\rho_L(a)\,(b \otimes v) = (\lambda_L \otimes \pi)\Delta(a)(b \otimes v) = \sum a_{(1)} b \otimes \pi(a_{(2)})(v).$$

- Dirac operator

$$D = \begin{bmatrix} 0 & d^* \\ d & 0 \end{bmatrix} : \mathcal{H}_\phi \to \mathcal{H}_\phi$$

Theorem (Cipriani, Franz, Wysoczańska-Kula)

We have bounded commutators for any $a \in \mathcal{A}$:

$$\|[D, \pi_L(a)]\| \leq \|\partial a\|_{\mathcal{H}_1}$$

Schürmann Triples

For a pre-Hilbert space $(\mathcal{E}, \langle\cdot,\cdot\rangle)$, we denote by $\mathcal{L}(\mathcal{E})$ the set of all linear operators with a well-defined adjoint.

Definition

A Schürmann triple (π, η, ϕ) on a unital $*$-bialgebra $(\mathcal{A}, \Delta, \epsilon)$ consists of:

- a $*$-representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{E})$ on a pre-Hilbert space $\mathcal{E}$.
- a π-ϵ 1-cocycle $\eta : \mathcal{A} \to \mathcal{E}$

$$\eta(ab) = \pi(a)\eta(b) + \eta(a)\epsilon(b).$$

- The bilinear form $a \otimes b \to -\langle \eta(a^*), \eta(b)\rangle$ is a ϵ-ϵ 2-coboundary

$$-\langle \eta(a^*), \eta(b)\rangle = \partial\phi(a,b) := \epsilon(a)\phi(b) - \phi(ab) + \phi(a)\epsilon(b)$$

A Remark on the 2-coboundary Condition

Consider the scalar Laplacian $\triangle = -\nabla^2 : C^\infty(M) \to C^\infty(M)$ on some manifold M, the Carré du Champ operator $\Gamma : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ looks like

$$\Gamma(a, b) := \triangle(ab) - a\triangle(b) - \triangle(a)b = (\nabla a)(\nabla b)$$

At the level of Dirichlet form

$$\langle a, \triangle b \rangle = \int_M a(-\nabla^2 b) = \int_M (\nabla a)(\nabla b) = \langle \nabla a, \nabla b \rangle \,,$$

provided that M has no boundary.

Conditionally Positive Linear Functionals

A linear functional $\phi : \mathcal{A} \to \mathbb{C}$ is called

- Hermitian: $\phi(a^*) = \overline{\phi(a)}$ for $a \in \mathcal{A}$
- Conditionally positive:

$$\phi\left(a^* a\right) \geq 0, \quad \forall a \in K_1 = \ker \epsilon,$$

  where $\epsilon : \mathcal{A} \to \mathbb{C}$ is the counit.

To recover a Schürmann triple (π, η, ϕ):

- Define a positive sequilinear form on $\mathcal{A}$

$$\langle a, b \rangle_\phi = \phi\left((a - \epsilon(a))^* (b - \epsilon(b))\right)$$

- $\mathcal{E} := \mathcal{A}/\mathcal{N}_\phi$, where $\mathcal{N}_\phi$ is the null space of $\langle \cdot, \cdot \rangle_\phi$, $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{E})$ is induced by the left multiplication.
- $\eta : \mathcal{A} \to \mathcal{E}$ is induced by the quotient map $a \mapsto [a]$.

The Dirac Operator

- Assume we have a Schürmann triple (H_π, η, ϕ) such that the convolution operator

$$L_\phi : \mathcal{H}_0 \to \mathcal{H}_0, \; H_0 = L^2(\mathcal{A}, h)$$

  is symmetric.
- Recall that $\mathcal{H}_1 = \mathcal{H}_0 \otimes \mathcal{H}_\phi$

$$\partial : \mathcal{A} \to \mathcal{H}_1 : a \mapsto \partial(a) = (1 \otimes \eta)(\Delta(a)) = a_{(1)} \otimes \eta(a_{(2)})$$

- One can check the derivation property

$$\partial(ab) = (\partial a) \cdot b + a \cdot (\partial b)$$

- $d : \mathcal{H}_0 \to \mathcal{H}_1$ is the closed extension of ∂.

The derivation property above requires the following bimodule structure of $\mathcal{H}_1 = \mathcal{H}_0 \otimes \mathcal{H}_\pi$:

$$\rho_L(a)\,(b \otimes v) = (\lambda_L \otimes \pi)\Delta(a)(b \otimes v) = \sum a_{(1)} b \otimes \pi(a_{(2)})(v)$$

where $b \otimes v \in \mathcal{H}_0 \otimes \mathcal{H}_\pi$ and

$$\rho_R(a)\,(b \otimes v) = (\lambda_R \otimes \pi)(b \otimes v) = ba \otimes v,$$

where λ_L and λ_R are the left and right GNS-representations.

References i

Fabio Cipriani, Uwe Franz, and Anna Kula.
Symmetries of lévy processes on compact quantum groups, their markov semigroups and potential theory.
Journal of Functional Analysis, 266(5):2789–2844, March 2014.

Alain Connes and Henri Moscovici.
Modular curvature for noncommutative two-tori.
J. Amer. Math. Soc., 27(3):639–684, 2014.

Alain Connes.
***Noncommutative geometry*.**
Academic Press, Inc., San Diego, CA, 1994.

References ii

Alain Connes and Paula Tretkoff.
The Gauss-Bonnet theorem for the noncommutative two torus.
In *Noncommutative geometry, arithmetic, and related topics*, pages 141–158. Johns Hopkins Univ. Press, Baltimore, MD, 2011.

Farzad Fathizadeh and Masoud Khalkhali.
The Gauss-Bonnet theorem for noncommutative two tori with a general conformal structure.
J. Noncommut. Geom., 6(3):457–480, 2012.

Yang Liu.
Scalar curvature in conformal geometry of connes–landi noncommutative manifolds.
Journal of Geometry and Physics, 121:138 – 165, 2017.

Yang Liu.
Modular curvature for toric noncommutative manifolds.
J. Noncommut. Geom., 12(2):511–575, 2018.