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À la recherche de l'espace perdu:
Du côté de chez Swan

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based on works joint with:
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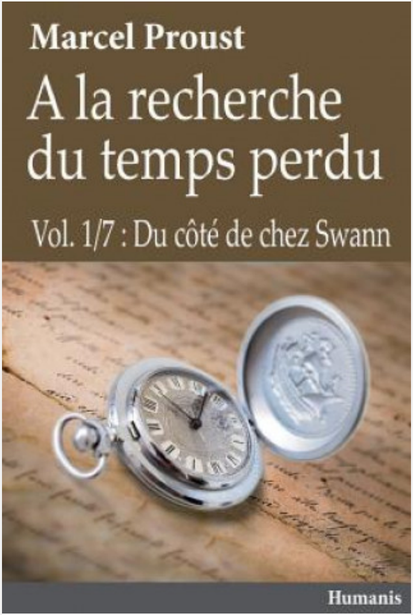
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$$X' \ni x' \longmapsto \big(x'G, f(x')\big) \in X'/G \underset{X/G}{\times} X, \tag{1}$$

- its inverse, given by means of the *translation map* $\tau : X \underset{X/G}{\times} X \to G$, $\tau(x, xg) = g$, is as follows

$$X'/G \underset{X/G}{\times} X \ni \big(x'G, x\big) \longmapsto x'\tau\big(f(x'), x\big) \in X'. \tag{2}$$

Classification of principal bundles

Theorem

- *Any G-equivariant continuous map between total spaces of two compact principal G-bundles together with their projections onto their bases related by the induced continuous map form a pullback diagram.*

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- *any compact principal G-bundle is a pullback of a standard one of the form*

$$G * \cdots * G \to (G * \cdots * G)/G,$$

*which is the Milnor compact approximation of the **universal principal G-bundle** $EG \to BG$.*

Associated vector bundles

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This is a consequence of the compatibility of associating and pulling back

$$X' \overset{G}{\times} V = \left(X'/G \underset{X/G}{\times} X\right) \overset{G}{\times} V$$

$$= X'/G \underset{X/G}{\times} \left(X \overset{G}{\times} V\right) = (f/G)^* \left(X \overset{G}{\times} V\right),$$

which is afforded by the G-equivariant homeomorphism

$$X' \ni x' \longmapsto (x'G, f(x')) \in X'/G \underset{X/G}{\times} X.$$

Naturality of the Chern character

Let $X \to Y \to S$ be a family of G-principal bundles of spaces over S. These bundles correspond to a principal G-action $X \times G \to X$ over S with the family of orbit spaces $Y = X/G$ over S.

Naturality of the Chern character

Let $X \to Y \to S$ be a family of G-principal bundles of spaces over S. These bundles correspond to a principal G-action $X \times G \to X$ over S with the family of orbit spaces $Y = X/G$ over S. Then, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{K}^0(BG) & \xrightarrow{\mathrm{K}^0(\mathrm{cl})} & \mathrm{K}^0(Y) \\ \Big\downarrow{\scriptstyle \mathrm{ch}_n(BG)} & & \Big\downarrow{\scriptstyle \mathrm{ch}_n(Y\,|\,S)} \\ \mathrm{H}^{2n}_{dR}(BG) & \xrightarrow{\mathrm{H}^{2n}(\mathrm{cl})} & \mathrm{H}^{2n}_{dR}(Y\,|\,S). \end{array}$$

where $\mathrm{cl} : Y \to BG$ is a family of classifying maps (parameterized by a space S).

Ad-invariant polynomials on $\mathfrak{g}$

- If $\mathrm{Ad}(G)$ is a linear algebraic group G regarded as a G-variety with respect to its adjoint action by conjugations, its coordinate algebra $\mathscr{O}(\mathrm{Ad}(G))$ is a ***coalgebra*** with the comultiplication equivalent to the algebraic group law.

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- Its Ad-***invariant part*** $\mathscr{O}(\mathrm{Ad}(G))^G$ (invariants with respect to the action of G on itself by conjugations, aka class functions) is related with the Chern–Weil map as follows.

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  The $\mathfrak{m}$-adic filtration of $\mathscr{O}(\mathrm{Ad}(G))$, for $\mathfrak{m} := \mathrm{k}er(\varepsilon)$, is G- invariant.

Therefore

$$\mathrm{gr}_{\mathfrak{m}} \mathscr{O}\,(\mathrm{Ad}(G))^G = \left(\bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1} \right)^G$$

$$\cong \mathrm{Sym}(\mathfrak{m}/\mathfrak{m}^2)^G = \bigoplus_{n \geq 0} (\mathrm{Sym}^n \mathfrak{g}^*)^G$$

of Ad-invariant polynomials on the Lie algebra $\mathfrak{g}$.

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- The latter is the infinitesimal counterpart of $\mathscr{O}(\mathrm{Ad}(G))^G$ and the domain of the ***classical Chern–Weil map***.
- Replacing the Lie algebra $\mathfrak{g}$ by the G-space $\mathrm{Ad}(G)$ plays a fundamental role in the construction of ***G-equivariant cyclic homology*** of Block–Getzler.

Example

For $G = SU(2)$ (when $BG = \mathbb{HP}^\infty$), the restriction of the tautological quaternionic line bundle $\tau_{\mathbb{HP}^n}$ from $\mathbb{HP}^n$ to $\mathbb{HP}^1 \cong S^4$ is the tautological quaternionic line bundle $\tau_{\mathbb{HP}^1}$ over $\mathbb{HP}^1$, so that the Chern character computation proving the nontriviality of $\tau_{\mathbb{HP}^1}$ proves also the nontriviality of $\tau_{\mathbb{HP}^n}$, and hence the nontriviality of all the principal bundles

$$S^{4n+3} \to S^{4n+3}/SU(2) = \mathbb{HP}^n.$$

Another standard classical example

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Theorem (Atiyah–Todd)

The standard filtration by skeleta

$$\mathbb{CP}^0 \hookrightarrow \mathbb{CP}^1 \hookrightarrow \cdots \hookrightarrow \mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$$

induces a tower of standard nilpotent ring extensions in K-theory.

$$\begin{array}{ccccccccc} 0 \twoheadleftarrow & K^*(\mathbb{CP}^0) & \twoheadleftarrow & K^*(\mathbb{CP}^1) & \twoheadleftarrow \cdots \twoheadleftarrow & K^*(\mathbb{CP}^{n-1}) & \twoheadleftarrow & K^*(\mathbb{CP}^n) \\ & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 \twoheadleftarrow & \mathbb{Z} & \twoheadleftarrow & \mathbb{Z}[x]/(x^2) & \twoheadleftarrow \cdots \twoheadleftarrow & \mathbb{Z}[x]/(x^n) & \twoheadleftarrow & \mathbb{Z}[x]/(x^{n+1}) \end{array}.$$

From the classical to noncommutative

The aim: to generalize this reasoning to the noncommutative setting.

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- ***Serre–Swan***: vector bundles as finitely generated projective modules.

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- ***Serre–Swan***: vector bundles as finitely generated projective modules.
- ***Baum–Hajac–Matthes–Szymański***: associated vector bundles as associated finitely generated projective modules using the ***Milnor-Moore*** cotensor product.

Problems with tenets of NCG

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- For some noncommutative algebras it is impossible to define even a rank of a free module, so triviality of a fgp module doesn't make sense.
- Despite succesful noncommutative extension of some aspects of classical topology, many others do not survive plain forgetting commutativity.

NC principal and associated vector bundles

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Ingredients:

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- If $\mathcal{A}'$ is an algebra with a principal coaction of $\mathcal{C}$, and B' is its coaction-invariant subalgebra, then any equivariant (colinear) algebra homomorphism $\mathcal{A} \to \mathcal{A}'$ induces an algebra homomorphism $B \to B'$ making B' a (B', B)-bimodule.

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Then $\mathcal{A}\Box^{\mathcal{C}}V$ is an ***associated finitely generated projective module*** over B.
The module $\mathcal{A}\Box^{\mathcal{C}}V$ is the ***section module of the associated noncommutative vector bundle***.

Association commutes with pullbacks

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Theorem (H–M)

- *The canonical morphism*

$$B' \otimes_B (\mathcal{A}\square^{\mathcal{C}} V) \to \mathcal{A}'\square^{\mathcal{C}} V$$

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$$f_*\colon K_0(B) \to K_0(B'),$$

where B and B' are the respective fixed-point subalgebras,satisfies

$$f_*([A \square^{\mathcal{C}} V]) = [A' \square^{\mathcal{C}} V].$$

The noncommutative Chern-Weil map

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Next, we construct a factorization of this restriction through a ***noncommutative Chern-Weil homomorphism*** chw_n as follows

$$\begin{array}{ccc} \mathrm{Corep}(C) & \xrightarrow{[A\Box^C(-)]} & \mathrm{K}_0(B) \\ \downarrow \chi & & \downarrow \mathrm{ch}_n \\ C^{\mathrm{tr}} & \xrightarrow{\mathrm{chw}_n} & \mathrm{HC}_{2n}(B\,|\,R) \end{array}$$

where

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- the ***map*** $[A\Box^C(-)]$ associating a finitely generated projective module with a given corepresentation should be understood as the ***map induced by the classifying map on K-theory***,

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- the character χ of a corepresentation should be understood as the ***Chern character for the classifying space***,
- the cyclic-homology Chern–Weil homomorphism chw_n should be understood as the map induced by the classifying map on cyclic homology.

The diagonal composite in this diagram is the ***Chern–Galois character*** chg_n of Hajac–Brzeziński.

- Just as to express K-theory in terms of matrix idempotents one introduces the functor of forming an ***H-unital algebra of locally finite matrices*** $M_\infty(-)$, to embrace the symmetry of a principal bundle in terms of representations of the symmetry, we introduce another ***H-unital algebra*** M ***which is the Ehresmann–Schauenburg quantum groupoid*** with a non-standard multiplication.

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- we introduce an abstract ***cyclic-homology Chern character*** unifying both construction

which can be subsumed in the following commutative diagram:

$$
\begin{array}{ccccc}
\mathrm{Corep}(C) & & \xrightarrow{\quad [A\square^{C}(-)] \quad} & & \mathrm{K}_0(B) \\
\Big\downarrow{\scriptstyle \widetilde{\mathrm{chg}}_n} & \searrow^{\mathrm{chg}_n} & & \swarrow^{\mathrm{ch}_n} & \Big\downarrow{\scriptstyle \widetilde{\mathrm{ch}}_n} \\
\mathrm{HC}_{2n}(M\,|\,R) & \underset{\cong}{\xrightarrow{\mathrm{HC}_{2n}(\varepsilon_M)}} & \mathrm{HC}_{2n}(B\,|\,R) & \underset{\cong}{\xleftarrow{[\mathrm{tr}_{2n}]}} & \mathrm{HC}_{2n}(\mathrm{M}_\infty(B)\,|\,R).
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Here the bottom horizontal arrows are isomorphisms of ***H-unital models of cyclic homology of B***, and the vertical arrows are tautological constructions. Thus the left-hand-side ***factorization of the Chern–Galois character*** becomes analogous to the well-known right-hand-side ***factorization of the Chern character***.

KK-theory

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The present state of the art of Noncommutative Topology: bivariant Kasparov's ***KK-theory***.

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It restricts as a bivariant theory to ***metrizable locally compact Hausdorff spaces***:

$$KK(X,Y) := KK(C_0(Y), C_0(X)).$$

KK-equivalence of different homotopy types

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Similarly as ***stable homotopy equivalences***, KK-equivalences do not recognize the ***plain homotopy type***.

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Note: $\mathbb{CP}^n$ connected, P_{n+1} disconnected, but their K-groups isomorphic.

Multiplicative K-equivalences

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Definition

A map of compact Hausdorff spaces is called ***K-equivalence*** if it induces an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded K-groups.

But then any K-equivalence automatically induces an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded K-rings.

Therefore, a K-equivalence of (metrizable compact Hausdorff) spaces can be promoted to a ***multiplicative K-equivalence***.

Cup products and the diagonal map

Cup products and the diagonal map

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- On the way of constructing ***category of motives*** in Algebraic Geometry all the cohomology theories of interest are equipped with a ***cycle class map*** which sends ***intersection products*** to ***cup products***.

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- On the way of constructing ***category of motives*** in Algebraic Geometry all the cohomology theories of interest are equipped with a ***cycle class map*** which sends ***intersection products*** to ***cup products***.
- Under the diagonal map and the constant map into a one point space, every space is a ***comonoid*** in the monoidal category of spaces, the ***category of spectra*** is monoidal and the ***suspension spectrum functor*** from spaces to spectra is strong monoidal what makes a suspension spectrum a comonoid as well.

Cup product helps

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Example

$K^\bullet(\mathbb{CP}^n)$ and $K^\bullet(P_{n+1})$ are isomorphic as $\mathbb{Z}/2\mathbb{Z}$-graded K-groups but not as $\mathbb{Z}/2\mathbb{Z}$-graded K-rings:

$$K^\bullet(\mathbb{CP}^n) \cong \mathbb{Z}[x]/(x^{n+1}) \text{ for } x \text{ even},$$

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Example

For $n > 1$, the cup product in K-theory can be used to show that there is no a retraction of $\mathbb{CP}^n$ onto its complex projective hyperplane $\mathbb{CP}^{n-1}$.

Mapping K-groups $\not\Rightarrow$ mapping K-rings

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Example

Since K-theory is an (extraordinary) cohomology theory, the ***Mayer–Vietoris principle*** applies, but not all maps in the corresponding long (six term, in fact) exact sequence respect the cup product.

Augmented rings instead of rings

The last example suggests the following replacement of the $\mathbb{Z}/2\mathbb{Z}$-graded ring structure of the K-theory.

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A ***morphism*** $(S^\bullet, N^\bullet, \mathbb{1}_N) \to (R^\bullet, M^\bullet, \mathbb{1}_M)$ of $\mathbb{Z}/2\mathbb{Z}$-graded augmented rings consists of

- a unital $\mathbb{Z}/2\mathbb{Z}$-graded ring map $S^\bullet \to R^\bullet$,
- a unitary $\mathbb{Z}/2\mathbb{Z}$-graded $S^\bullet$-module map $N^\bullet \to M^\bullet$,
- $\mathbb{1}_N \mapsto \mathbb{1}_M$.

Rings are coreflective in augmented rings

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Theorem (easy)

The functor

$$\mathcal{R}ing^{\mathbb{Z}/2\mathbb{Z}} \longrightarrow \mathcal{A}ug\mathcal{R}ing^{\mathbb{Z}/2\mathbb{Z}}, \quad R^\bullet \rightsquigarrow (R^\bullet, R^\bullet, 1_R)$$

is a ***full embedding****, admitting a right adjoint (a* ***coreflector****)*

$$\mathcal{A}ug\mathcal{R}ing^{\mathbb{Z}/2\mathbb{Z}} \longrightarrow \mathcal{R}ing^{\mathbb{Z}/2\mathbb{Z}}, \quad (R^\bullet, M^\bullet, \mathbb{1}_M) \rightsquigarrow R^\bullet.$$

Recall that ***compact spaces*** are coreflective in ***compact quantum spaces*** (the coreflector is the ***space of classical points***).

Augmented rings vs abelian groups

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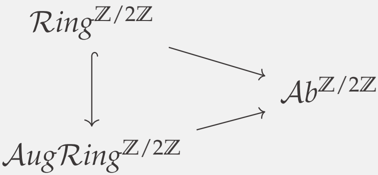
We will need another functor.

Definition

$$\mathcal{A}ug\mathcal{R}ing^{\mathbb{Z}/2\mathbb{Z}} \longrightarrow \mathcal{A}b^{\mathbb{Z}/2\mathbb{Z}}, \quad (R^\bullet, M^\bullet, \mathbb{1}_M) \rightsquigarrow M^\bullet.$$

It is related to the previous full embedding and forgetting the unital ring structure as follows



Multiplicative K-theory of spaces

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Theorem (a truism)

The association

$$X \rightsquigarrow (R^\bullet, M^\bullet, \mathbb{1}_M) := (K^\bullet(X), K^\bullet(X), [\mathbb{1}_X])$$

extends by functoriality of the K-ring uniquely to a contravariant functor from spaces to augmented rings.

Note that this is entirely equivalent to the K-ring functor, so ...

why anyone might care?

The justification of the new approach

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Theorem (D'A–M)

There exists a full embedding ***Q*** *of* ***cw-Waldhausen categories***

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- *the following diagram*

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  commutes,
- *the* **cofibrations** *of both Waldhausen structures are monomorphisms and* **weak equivalences** *are K-equivalences,*
- *the composition of* **K** *with* $\mathcal{AugRing}^{\mathbb{Z}/2\mathbb{Z}} \to \mathcal{Ab}^{\mathbb{Z}/2\mathbb{Z}}$ *factors through K-theory of unital C*-algebras.*

cw-Waldhausen categories

Definition

An ***unpointed Waldhausen category*** $\mathscr{C}$ is a category with

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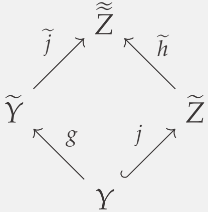
Cofibration weakening

Definition (A–H–M–S–Z)

We call an unpointed Waldhausen category ***cw-Waldhausen*** (***cofibration-weakening***-Waldhausen category) iff for every pushout diagram



with j being a cofibration, $\widetilde{h}$ is a weak equivalence if and only if so is g.

Weak cofibrations

Theorem (A–H–M–S–Z)

Any cw-Waldhausen category admits a calculus of left fractions of the form $Weq^{-1} \circ Cof$ *in the homotopy category* $Ho(\mathscr{C}) := \mathscr{C}[Weq^{-1}]$.

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depict as $X \rightarrowtail Y$, and compose in $Ho(\mathscr{C})$ as follows

Composition of weak cofibrations

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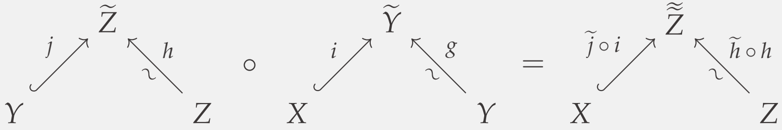
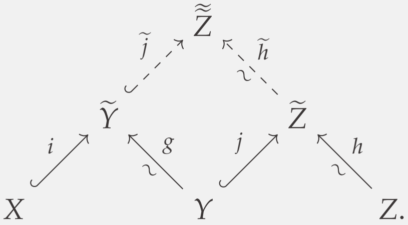
$$\left(Y \xhookrightarrow{j} \widetilde{Z} \xleftarrow[\sim]{h} Z\right) \circ \left(X \xhookrightarrow{i} \widetilde{Y} \xleftarrow[\sim]{g} Y\right) = \left(X \xhookrightarrow{\widetilde{j} \circ i} \widetilde{\widetilde{Z}} \xleftarrow[\sim]{\widetilde{h} \circ h} Z\right)$$

Composition of weak cofibrations



where $\widetilde{j}$ and $\widetilde{h}$ are the arrows completing the pushout square in the diagram below



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Compact Hausdorff spaces

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Essentially, it is an enhancement of the Gelfand–Naimark duality.

k-topology

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A ***k-topology*** on X is a collection of k-covering families. When equipped with a k-topology X will be called ***k-compact quantum space***.

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$$\begin{array}{ccc} A' & \xleftarrow{\ \alpha\ } & A \\ \uparrow & & \uparrow \\ B' & \xleftarrow[\ \beta\ ]{} & B \end{array}$$

commute

k-continuous maps

Definition (A–M)

We call a morphism of compact quantum spaces $X' \to X$ equipped with k-topology ***k-continuous map*** if for any compact quantum principal bundle $(G, E \to X)$ belonging to the k-topology on X, there exists a compact quantum principal bundle $(G', E' \to X')$ belonging to the k-topology on X', a morphism of compact quantum groups $G' \to G$, and a morphism of compact quantum G'-spaces $E' \to E$ making a corresponding Peter-Weyl diagram

$$\begin{array}{ccc} A' & \xleftarrow{\ \alpha\ } & A \\ \uparrow & & \uparrow \\ B' & \xleftarrow[\ \beta\ ]{} & B \end{array}$$

commute and satisfying the condition that

Definition (continued)

the *canonical map*

$$B' \otimes_B A \to A' \square^{H'} H, \quad b' \otimes_B a \mapsto b' \alpha(a_{(0)}) \otimes a_{(1)}$$

is bijective.

Definition (continued)

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Definition (A–M)

Forgetting k-topology defines a functor from the category *k-CpctQuant* of k-topological compact quantum spaces with k-continuous morphisms to *CpctQuant*.

L’espace retrouvé

L'espace retrouvé

Remark. Classically

- any G-equivariant continuous map of G-principal bundles over the same base is an isomorphism,
- any continuous map between bases pulls back principal bundles and lifts to an equivariant map of principal bundles,
- every continuous map is k-continuous with respect to k-topology consisting of all classical compact principal bundles.

Why k-topology is a kind of topology

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There is a weakened, but sufficient for speaking about sheaves, version of (unsaturated) Grothendieck topology, introduced by Peter Johnstone as ***coverage***.

Why k-topology is a kind of topology

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There is a weakened, but sufficient for speaking about sheaves, version of (unsaturated) Grothendieck topology, introduced by Peter Johnstone as ***coverage***.

Theorem

Families of compact quantum principal bundles together with k-continuous maps form a ***coverage in the sense of Johnstone****.*

From k-topology to multiplicative K-theory

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Associated vector bundles as section bimodules act on vector bundles as module sections over X and over its unreduced suspension ΣX.

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In particular, since $K^0(X)$ contains a class $\mathbb{1}_{M(X)} := [\mathbb{1}_X]$ of a rank one trivial vector bundle $\mathbb{1}_X$ we can define, using Bott periodicity, NC join construction and reduced K-theory the desired functor.

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Definition (A–M)

$$\text{k-}\mathcal{C}pct\mathcal{Q}uant \longrightarrow \mathcal{A}ug\mathcal{M}od^{\mathbb{Z}/2\mathbb{Z}},$$

$$X \leadsto (R^\bullet(X),\, M^\bullet(X),\, \mathbb{1}_{M(X)}),$$

$$R^0(X) := \lim_{(G,\, E\to X)} R(G)/Ann(\mathbb{1}_{K(X)}), \quad R^1(X) := R^0(\Sigma X).$$

Quantum spheres and balls

Quantum spheres and balls

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Definition (A–H–M–S–Z)

By ***boundary map*** from a ***K-sphere*** to a ***K-ball*** we mean a cofibration $\partial : \mathbb{S}^{d-1} \hookrightarrow \mathbb{B}^d$ in the cw-Waldhausen category $\mathcal{C}pct\mathcal{Q}uant$

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$$0 \longrightarrow \underset{\mathbb{Z}}{\overset{K^0(\mathbb{B}^d)}{\cong}} \longrightarrow \underset{\mathbb{Z}}{\overset{K^0(\mathbb{S}^{d-1})}{\cong}} \longrightarrow 0 \longrightarrow 0,$$

$$0 \longrightarrow \underset{0}{\overset{K^1(\mathbb{B}^d)}{\parallel}} \longrightarrow \underset{\mathbb{Z}}{\overset{K^1(\mathbb{S}^{d-1})}{\cong}} \longrightarrow \mathbb{Z} \longrightarrow 0,$$

for d even,

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$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^0(\mathbb{B}^d) & \longrightarrow & K^0(\mathbb{S}^{d-1}) & \longrightarrow & 0 & \longrightarrow & 0, \\ & & \cong & & \cong & & & & \\ & & \mathbb{Z} & & \mathbb{Z} & & & & \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^1(\mathbb{B}^d) & \longrightarrow & K^1(\mathbb{S}^{d-1}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0, \\ & & \| & & \cong & & & & \\ & & 0 & & \mathbb{Z} & & & & \end{array}$$

for d even, and

Definition (continued)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K^0(\mathbb{B}^d) & \longrightarrow & K^0(\mathbb{S}^{d-1}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0, \\
 & & \wr\| & & \wr\| & & & & \\
 & & \mathbb{Z} & & \mathbb{Z}\oplus\mathbb{Z} & & & & \\
 & & & & & & & & \\
0 & \longrightarrow & K^1(\mathbb{B}^d) & \longrightarrow & K^1(\mathbb{S}^{d-1}) & \longrightarrow & 0 & \longrightarrow & 0, \\
 & & \| & & \| & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

for d odd,

Definition (continued)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K^0(\mathbb{B}^d) & \longrightarrow & K^0(\mathbb{S}^{d-1}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0, \\
 & & \cong & & \cong & & & & \\
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 & & & & & & & & \\
0 & \longrightarrow & K^1(\mathbb{B}^d) & \longrightarrow & K^1(\mathbb{S}^{d-1}) & \longrightarrow & 0 & \longrightarrow & 0, \\
 & & \| & & \| & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

for d odd, as classical boundary maps from spheres to balls do.

K-weak quantum CW-complexes

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Definition (A–H–M–S–Z)

A *finite quantum K-weak CW-complex* is an object X of the category $\mathcal{H}o(\mathcal{C}pct\mathcal{Q}uant)$

K-weak quantum CW-complexes

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Definition (A–H–M–S–Z)

A *finite quantum K-weak CW-complex* is an object X of the category $\mathcal{H}o(\mathcal{C}pct\mathcal{Q}uant)$ admitting a finite sequence of *weak cofibrations*

$$X^{d_0} \rightarrowtail X^{d_1} \rightarrowtail \cdots \rightarrowtail X^{d_{n-1}} \rightarrowtail X^{d_n} = X$$

of the form

$$X^{d_{k-1}} \hookrightarrow \widetilde{X}^{d_k} \xleftarrow{\ \sim\ } X^{d_k}.$$

Definition (continued)

Here X^{d_0} is finite, and the above sequence (referred to as *weak filtration by skeleta*) is compatible with pushouts in *CpctQuant* (called *attaching cells*)

$$
\begin{array}{ccccc}
 & & \widetilde{X}^{d_k} & & \\
 & \nearrow & & \nwarrow & \\
X^{d_{k-1}} & & & & \coprod_{i=1}^{i_k} \mathbb{B}_i^{d_k} \\
 & \overset{a_k}{\nwarrow} & & \overset{\partial}{\nearrow} & \\
 & & \coprod_{i=1}^{i_k} \mathbb{S}_i^{d_k-1}. & &
\end{array}
$$

Strict quantum CW-complexes

Definition (A–H–M–S–Z)

If in all presentations of the weak cofibrations the K-equivalences are identities, then we suppress the adjective "K-weak" and regard the resulting objects as objects of $\mathcal{C}pct\mathcal{Q}uant$.

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Example

The Vaksman–Soibelman quantum odd spheres and their quotient quantum complex projective spaces are strict quantum CW-complexes.

Strict quantum CW-complexes

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If in all presentations of the weak cofibrations the K-equivalences are identities, then we suppress the adjective "K-weak" and regard the resulting objects as objects of $\mathcal{C}pct\mathcal{Q}uant$.

Example

The Vaksman–Soibelman quantum odd spheres and their quotient quantum complex projective spaces are strict quantum CW-complexes.

Example

The multipullback odd spheres and their quotient quantum complex projective spaces are K-weak quantum CW-complexes.

A quantum k-topological strict version

A quantum k-topological strict version

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Theorem (A–M)

The Vaksman–Soibelman quotient quantum complex projective spaces are strict quantum k-CW-complexes with k-topology given by the compact $U(1)$-principal bundle being the Vaksman-Soibelman sphere and admits a strict k-topological filtration by skeleta

$$\mathbb{CP}^0_q \hookrightarrow \mathbb{CP}^1_q \hookrightarrow \cdots \hookrightarrow \mathbb{CP}^{n-1}_q \hookrightarrow \mathbb{CP}^n_q$$

inducing a tower of standard nilpotent ring extensions in multiplicative K-theory with $M^\bullet(\mathbb{CP}^n_q) \cong R^\bullet(\mathbb{CP}^n_q)$

$$\begin{array}{ccccccccccc} 0 & \twoheadleftarrow & R^\bullet(\mathbb{CP}^0_q) & \twoheadleftarrow & R^\bullet(\mathbb{CP}^1_q) & \twoheadleftarrow \cdots \twoheadleftarrow & R^\bullet(\mathbb{CP}^{n-1}_q) & \twoheadleftarrow & R^\bullet(\mathbb{CP}^n_q) \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ 0 & \twoheadleftarrow & \mathbb{Z} & \twoheadleftarrow & \mathbb{Z}[x]/(x^2) & \twoheadleftarrow \cdots \twoheadleftarrow & \mathbb{Z}[x]/(x^n) & \twoheadleftarrow & \mathbb{Z}[x]/(x^{n+1}). \end{array}$$

A quantum k-topological K-weak version

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Theorem (A–M)

The Heegaard quotient quantum complex projective spaces are K-weak quantum k-CW-complexes with k-topology given by the compact $U(1)$*-principal bundle being the Heegaard quantum sphere and admits a K-weak k-topological filtration by skeleta*

$$\mathbb{CP}^0_{\mathcal{T}} \rightarrowtail \mathbb{CP}^1_{\mathcal{T}} \rightarrowtail \cdots \rightarrowtail \mathbb{CP}^{n-1}_{\mathcal{T}} \rightarrowtail \mathbb{CP}^n_{\mathcal{T}}$$

inducing a tower of standard nilpotent ring extensions in multiplicative K-theory with $M^\bullet(\mathbb{CP}^n_{\mathcal{T}}) \cong R^\bullet(\mathbb{CP}^n_{\mathcal{T}})$

$$\begin{array}{ccccccccc}
0 \twoheadleftarrow & R^\bullet(\mathbb{CP}^0_{\mathcal{T}}) & \twoheadleftarrow & R^\bullet(\mathbb{CP}^1_{\mathcal{T}}) & \twoheadleftarrow \cdots \twoheadleftarrow & R^\bullet(\mathbb{CP}^{n-1}_{\mathcal{T}}) & \twoheadleftarrow & R^\bullet(\mathbb{CP}^n_{\mathcal{T}}) \\
 & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
0 \twoheadleftarrow & \mathbb{Z} & \twoheadleftarrow & \mathbb{Z}[x]/(x^2) & \twoheadleftarrow \cdots \twoheadleftarrow & \mathbb{Z}[x]/(x^n) & \twoheadleftarrow & \mathbb{Z}[x]/(x^{n+1}).
\end{array}$$

A multiplicative K-equivalence

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Theorem (A–M)

There is a multiplicative K-equivalence from the weak hyperplane filtration of $\mathbb{CP}^n_\mathcal{T}$ to the strict hyperplane filtration of $\mathbb{CP}^n_q$.

Remark. Although these two quantizations of $\mathbb{CP}^n$ are not isomorphic, they define the same ***multiplicative K-theory type*** which should be understood as a quantization of the classical K-theory type of $\mathbb{CP}^n$.

A higher-categorical perspective

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The 2-category $\mathfrak{G}$

- 0-cells $\mathbf{X}$ are Grothendieck categories.
- 1-cells $\mathbf{f} : \mathbf{X}' \to \mathbf{X}$ are adjunctions $\mathbf{f} = (f^* \dashv f_*)$, where $f_* : \mathbf{X}' \to \mathbf{X}$ is an additive functor.
- 2-cells $\mathbf{f} \Longrightarrow \mathbf{g}$ are natural transformations $f_* \Longrightarrow g_*$.

Weakly Cartesian squares

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Let $\mathfrak{S}$ be a sub-2-category of $\mathfrak{G}$.

- A weakly commutative square in $\mathfrak{S}$ is a diagram in $\mathfrak{S}$:

$$\begin{array}{ccc} \mathbf{U}' & \xrightarrow{\widetilde{\mathbf{f}}} & \mathbf{U} \\ {\scriptstyle \mathbf{q}'}\downarrow & & \downarrow{\scriptstyle \mathbf{q}} \\ \mathbf{X}' & \xrightarrow{\mathbf{f}} & \mathbf{X} \end{array} \tag{4}$$

  such that there is an invertible 2-cell $\mathbf{q}\widetilde{\mathbf{f}} \Longrightarrow \mathbf{f}\mathbf{q}'$. The latter means that there is a natural isomorphism of functors $q_*\widetilde{f}_* \Longrightarrow f_*q'_*$.
- A weakly commutative square in $\mathfrak{S}$ is ***weakly Cartesian*** if the ***Beck-Chevalley condition*** is satisfied, i.e. the natural transformation of functors $q^*f_* \Longrightarrow \widetilde{f}_*q'^*$ is an isomorphism.

Weakly Cartesian coverage

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A ***weakly Cartesian coverage*** of $\mathfrak{S}$

- is a function T assigning to every Grothendieck category $\mathbf{X}$ in $\mathfrak{S}$ a collection $T(\mathbf{X})$ of families of adjunctions $\left\{\mathbf{q}_i : \mathbf{U}_i \to \mathbf{X} \,\middle|\, i \in I\right\}$ (called ***T-covering families***)
- such that, for every weakly Cartesian square as above if $\mathbf{q} : \mathbf{U} \to \mathbf{X}$ is a member of a T-covering family over $\mathbf{X}$, then $\mathbf{q}' : \mathbf{U}' \to \mathbf{X}'$ is a member of a T-covering family over $\mathbf{X}'$.

Classical case

By a ***family of group actions*** we mean a pair $(G, U \to X)$ of a group G and a G-equivariant map $U \to X$ from a G-space U to a space X with trivial G-action. Let us consider the category whose objects are families of group actions and whose morphisms

$$(G', U' \to X') \longrightarrow (G, U \to X) \tag{5}$$

are pairs consisting of a morphism $\gamma : G' \to G$ of groups and a commutative diagram

$$\begin{array}{ccc} U' & \xrightarrow{\widetilde{f}} & U \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ X' & \xrightarrow{f} & X \end{array} \tag{6}$$

where $\widetilde{f}$ is G'-equivariant.

Then the map

$$U' \times^{G'} G \longrightarrow X' \times_X U\,, \qquad [(u',g)] \mapsto \left(q'(u'),\widetilde{f}(u')g\right)\,,$$

is well-defined and it is a morphism in the category of right G-spaces equipped with a continuous map to X' and a G-equivariant continuous map to U.
A morphism of families of group actions is called ***Cartesian*** if the above map is an isomorphism.

Several important notions of equivariant topology can be rewritten in terms of Cartesian morphisms of families of group actions.

Example 1. Orbit spaces

$$G \longrightarrow \{e\}$$

$$\begin{array}{ccc} U & \longrightarrow & \{*\} \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle =} \\ X & \longrightarrow & \{*\} \end{array}$$

Then, the map

$$U \times^G \{*\} \longrightarrow X \times_{\{*\}} \{*\} , \qquad [(u,e)] \mapsto (q(u),*)$$

reads as

$$U/G \longrightarrow X , \qquad [u] \mapsto q(u) ,$$

which means that it is an isomorphism if and only if q is a quotient map onto the ***space of orbits***.

Example 2. ***Slices***

$$G \xrightarrow{=} G$$

$$\begin{array}{ccc} U & \xrightarrow{s} & G \\ {\scriptstyle q}\downarrow & & \downarrow \\ X & \longrightarrow & \{*\} \end{array}$$

Then, the map

$$U \times^G G \longrightarrow X \times_{\{*\}} G\,, \qquad [(u,g)] \mapsto \big(q(u), s(u)g\big)$$

reads as

$$U \longrightarrow X \times G\,, \qquad [u] \mapsto \big(q(u), s(u)\big)\,,$$

which means that it is an isomorphism if and only if s is a ***slice map***, where the slice is the pre-image $s^{-1}(e) \subset U$, q is a quotient map onto the space of orbits and admits a section whose the image is the slice.

Example 3. *Principal bundles*

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$$G \xrightarrow{\;=\;} G$$

$$\begin{array}{ccc} U \times G & \xrightarrow{\;\alpha\;} & U \\ {\scriptstyle \widetilde{q}}\downarrow & & \downarrow{\scriptstyle q} \\ U & \xrightarrow{\;q\;} & X \end{array}$$

Here $\widetilde{q}$ is the projection onto the first Cartesian factor and $\alpha(u,g) := ug$ the group action. The right G-action on $U \times G$ is on the second factor. Then, our map

$$(U \times G) \times^G G \longrightarrow U \times_X U\,, \qquad [((u,g_1),g_2)] \mapsto (u, ug_1g_2)$$

reads as the graph of the G-action

$$U \times G \longrightarrow U \times_X U\,, \qquad (u,g) \mapsto (u,ug),$$

which is an isomorphism if and only if $q : U \to X$ is a ***principal G-bundle***.

Example 4. *Change of the structure group*

$$G' \longrightarrow G$$

$$\begin{array}{ccc} U' & \xrightarrow{\widetilde{f}} & U \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[=]{} & X \end{array}$$

For two principal bundles over the same base, the map

$$U' \times^{G'} G \longrightarrow X \times_X U\,, \qquad [(u',g)] \mapsto \big(q'(u'),\widetilde{f}(u')g\big)$$

reads as

$$U' \times^{G'} G \longrightarrow U\,, \qquad [(u',g)] \mapsto \widetilde{f}(u')g,$$

which is an isomorphism if and only if $\widetilde{f}$ is a ***change of structure group***.

Example 5. *Locally trivial principal bundles*

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Every morphism between locally trivial principal bundles is ***Cartesian***. Indeed, notice that the triangle

$$\begin{array}{ccc} U' \times^{G'} G & \longrightarrow & X' \times_X U \\ & \searrow \quad \swarrow & \\ & X' & \end{array}$$

is a morphism of principal locally trivial G-bundles over the same base space X', and hence it must be an isomorphism.