Sums of Squares in Leavitt Path *-Algebras and beyond

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Overview

- Definition of ordered *-algebras and the grand unified problem
- The uniform norm and the Archimedean Positivstellensatz
- Closed ordered *-algebras and the generalized Gelfand–Naimark Theorem

Ordered *-Algebras

Definition

A **-algebra* is a *unital* associative algebra $\mathcal{A}$ over the field of complex numbers $\mathbb{C}$ endowed with an antilinear involution $\cdot^* \colon \mathcal{A} \to \mathcal{A}$ that fulfils $(ab)^* = b^* a^*$ for all $a, b \in \mathcal{A}$.

$$\mathcal{A}_\mathrm{h} \coloneqq \{\, a \in \mathcal{A} \mid a = a^* \,\}$$

is the real linear subspace of *hermitian* elements of $\mathcal{A}$.

Ordered *-Algebras

Definition

- A *quadratic module* on a *-algebra $\mathcal{A}$ is a subset $\mathcal{Q} \subseteq \mathcal{A}_{\mathrm{h}}$ fulfilling
$$q + r \in \mathcal{Q}\,, \qquad a^* q\, a \in \mathcal{Q} \qquad \text{and} \qquad \mathbb{1} \in \mathcal{Q}$$
for all $q, r \in \mathcal{Q}$, $a \in \mathcal{A}$. The *support* **-ideal* of $\mathcal{Q}$ is
$$\operatorname{supp} \mathcal{Q} := \big(\mathcal{Q} \cap (-\mathcal{Q})\big) \otimes_{\mathbb{R}} \mathbb{C} = \big(\mathcal{Q} \cap (-\mathcal{Q})\big) + \mathrm{i}\big(\mathcal{Q} \cap (-\mathcal{Q})\big)$$
- An *ordered* **-algebra* is a *-algebra $\mathcal{A}$ with a partial order $\le$ (reflexive, transitive, and antisymmetric relation) on $\mathcal{A}_{\mathrm{h}}$ such that
$$b + d \le c + d\,, \qquad a^* b\, a \le a^* c\, a \qquad \text{and} \qquad 0 \le \mathbb{1}$$
hold for all $a \in \mathcal{A}$ and $b, c, d \in \mathcal{A}_{\mathrm{h}}$ with $b \le c$. Then the *positive* hermitian elements
$$\mathcal{A}_{\mathrm{h}}^+ := \{\, a \in \mathcal{A}_{\mathrm{h}} \mid 0 \le a \,\}$$
are a quadratic module and $\operatorname{supp} \mathcal{A}_{\mathrm{h}}^+ = \{0\}$.
- Conversely, $\mathcal{A}/\operatorname{supp} Q$ with $[a] \le [b] \iff b - a \in \mathcal{Q}$ is ordered *-algebra.

Example: C^*-algebras

Proposition (unique order on C^*-algebras)

Let $\mathcal{A}$ be a C^*-algebra, then there is a unique partial order $\leq$ on $\mathcal{A}_{\mathrm{h}}$ that turns $\mathcal{A}$ into an ordered *-algebra. This order is determined by

$$\mathcal{A}_{\mathrm{h}}^+ := \{\, a \in \mathcal{A}_{\mathrm{h}} \mid \operatorname{spec}(a) \subseteq [0, \infty[\,\} = \{\, a^* a \mid a \in \mathcal{A} \,\} = \{\, a^2 \mid a \in \mathcal{A}_{\mathrm{h}} \,\}. \quad (*)$$

Proof

$\mathcal{Q}$ is q.m. means: $q + r \in \mathcal{Q}$, $a^* q\, a \in \mathcal{Q}$ for all $q, r \in \mathcal{Q}$, $a \in \mathcal{A}$, and $\mathbb{1} \in \mathcal{Q}$.

- $(*)$ defines quadratic module $\mathcal{A}_{\mathrm{h}}^+$ and $\operatorname{supp} \mathcal{A}_{\mathrm{h}}^+ = \{0\}$: standard.
- If $\mathcal{Q} \subseteq \mathcal{A}_{\mathrm{h}}$ is a quadratic module and $\operatorname{supp} \mathcal{Q} = \{0\}$, then $\mathcal{Q} = \mathcal{A}_{\mathrm{h}}^+$:

"$\supseteq$": Given $a \in \mathcal{A}_{\mathrm{h}}^+$, then $a = \sqrt{a}\, \mathbb{1} \sqrt{a} \in \mathcal{Q}$.

"$\subseteq$": Given $a \in \mathcal{Q}$, then $a = a_+ - a_-$ with $a_+, a_- \in \mathcal{A}_{\mathrm{h}}^+$, $a_+ a_- = a_- a_+ = 0$. From $-(a_-)^3 = a_- a\, a_- \in \mathcal{Q}$ and $(a_-)^3 \in \mathcal{A}_{\mathrm{h}}^+ \subseteq \mathcal{Q}$ it follows that $(a_-)^3 = 0$, therefore $a_- = 0$ and $a = a_+ \in \mathcal{A}_{\mathrm{h}}^+$.

Corollary

Every C^*-norm on a *-algebra $\mathcal{A}$ turns $\mathcal{A}$ into an ordered *-algebra.

Constructing quadratic modules – order from *-representations

Ordered *-algebras of functions

X a set, $\mathbb{C}^X$ the *-algebra of complex-valued functions on X with pointwise operations and pointwise order.
Then $(\mathbb{C}^X)_{\mathrm{h}}$ are the $\mathbb{R}$-valued functions, $(\mathbb{C}^X)_{\mathrm{h}}^+$ the $[0,\infty[$-valued functions.

Ordered *-algebras of operators (O^*-algebras)

$\mathcal{D}$ a pre-Hilbert space with inner product $\langle\,\cdot\,|\,\cdot\,\rangle\colon \mathcal{D}\times\mathcal{D}\to\mathbb{C}$,
$\mathcal{L}^*(\mathcal{D})$ the *-algebra of *adjointable endomorphisms* of $\mathcal{D}$, i.e. of linear maps $a\colon\mathcal{D}\to\mathcal{D}$ such that there exists linear $a^*\colon\mathcal{D}\to\mathcal{D}$ fulfilling

$$\langle\phi\,|\,a(\psi)\rangle = \langle a^*(\phi)\,|\,\psi\rangle \qquad \text{for all } \phi,\psi\in\mathcal{D}.$$

Then

$$\mathcal{L}^*(\mathcal{D})_{\mathrm{h}} = \big\{\, a\in\mathcal{L}^*(\mathcal{D}) \;\big|\; \langle\psi\,|\,a(\psi)\rangle\in\mathbb{R} \text{ for all } \psi\in\mathcal{D} \,\big\}$$

and $\mathcal{L}^*(\mathcal{D})$ becomes an ordered *-algebra with the operator order on $\mathcal{L}^*(\mathcal{D})_{\mathrm{h}}$,

$$\mathcal{L}^*(\mathcal{D})_{\mathrm{h}}^+ = \big\{\, a\in\mathcal{L}^*(\mathcal{D}) \;\big|\; \langle\psi\,|\,a(\psi)\rangle\in[0,\infty[\text{ for all } \psi\in\mathcal{D} \,\big\}.$$

Constructing quadratic modules – order from generators

Definition

Let $\mathcal{A}$ be a *-algebra, then

$$\mathcal{A}_{\mathrm{h}}^{++} := \Big\{ \sum\nolimits_{n=1}^{N} a_n^* a_n \,\Big|\, N \in \mathbb{N}_0;\ a_1, \dots, a_N \in \mathcal{A} \Big\}$$

are the *sums of hermitian squares*.

$\mathcal{A}_{\mathrm{h}}^{++}$ is the smallest quadratic module of a *-algebra $\mathcal{A}$.

Definition

Let $\mathcal{A}$ be a *-algebra and $G \subseteq \mathcal{A}_{\mathrm{h}}$, then

$$\langle\!\langle\, G \,\rangle\!\rangle := \Big\{ \sum\nolimits_{n=1}^{N} a_n^* g_n a_n \,\Big|\, N \in \mathbb{N}_0;\ a_1, \dots, a_N \in \mathcal{A};\ g_1, \dots, g_N \in G \cup \{\mathbb{1}\} \Big\}$$

is the *quadratic module generated by G*.

Note: If $+g, -g \in S$, then $g \in \operatorname{supp}\langle\!\langle\, G \,\rangle\!\rangle$.

Examples (commutative)

Polynomials

- $\mathbb{C}[x_1, \dots, x_n]$ *-algebra of polynomials in hermitian variables $x_1, \dots, x_n$.
- Consider $G \subseteq \mathbb{C}[x_1, \dots, x_n]_{\mathrm{h}} = \mathbb{R}[x_1, \dots, x_n]$.
- Set $\mathcal{P}(G) := \{\, \xi \in \mathbb{R}^n \mid g(\xi) \geq 0 \text{ for all } g \in G \,\}$.
- How is $\langle\!\langle\, G \,\rangle\!\rangle$ related to polynomials pointwise positive on $\mathcal{P}(G)$?

Polynomials on $\mathbb{CP}^n$ via symmetry reduction

- $\mathbb{C}[z_0, \dots, z_n, \overline{z}_0, \dots, \overline{z}_n]$ *-algebra of polynomials and $z_i^* := \overline{z}_i$.
- $\mathbb{C}[z_0, \dots, z_n, \overline{z}_0, \dots, \overline{z}_n]^{U(1)}$ *-subalgebra of $U(1)$-invariant functions.
- Momentum map $\mathcal{J} := z_0\overline{z}_0 + \dots + z_n\overline{z}_n \in \mathbb{C}[z_1, \dots, z_n, \overline{z}_1, \dots, \overline{z}_n]^{U(1)}$, and $\mathcal{J}^{-1}(\{\mu\}) \cong \mathbb{S}^{2n+1}$ for $\mu > 0$.
- How is $\langle\!\langle\, \{\mathcal{J} - \mu, \mu - \mathcal{J}\} \,\rangle\!\rangle$ related to $U(1)$-invariant polynomials pointwise positive on $\mathcal{J}^{-1}(\{\mu\})$, $\mu > 0$?

Example (non-commutative)

Berezin quantization of $\mathbb{CP}^n$, but via symmetry reduction

- Weyl *-algebra $\mathcal{W}(n) := \left\langle a_0, \dots, a_n \;\middle|\; a_i a_j = a_j a_i,\ a_i a_j^* - a_j^* a_i = \delta_{ij}\hbar \right\rangle$, $\hbar > 0$.
- $\mathcal{W}(n)^{U(1)}$ the $U(1)$-invariant elements (#creators = #annihilators).
- Momentum map $\mathcal{J} := a_0 a_0^* + \dots + a_n a_n^* \in \mathcal{W}(n)^{U(1)}$.
- How is $\langle\!\langle \{\mathcal{J} - \mu, \mu - \mathcal{J}\} \rangle\!\rangle$, $\mu \geq 0$, related to $U(1)$-invariant elements positive in representations $\pi_\mu \colon \mathcal{W}(n)^{U(1)} \to \mathcal{L}^*(\mathcal{D}_\mu)$ of the Berezin quantization of $\mathbb{CP}^n$?
  - $\mu = \hbar k$, $k \in \mathbb{N}_0$:
    $\mathcal{D}_\mu$ are holomorphic sections of a complex line bundle over $\mathbb{CP}^n$.
  - Otherwise: $\mathcal{D}_\mu = \{0\}$.

Another Example (non-commutative)

Leavitt path algebras

- Consider a directed graph $\mathrm{G} := (E_0, E_1, r\colon E_1 \to E_0, s\colon E_1 \to E_0)$, E_0 finite.
- Let $\mathcal{A}$ be the *-algebra freely generated by: hermitian elements $\{\, p_v \mid v \in E_0 \,\}$ and arbitrary elements $\{\, s_e \mid e \in E_1 \,\}$.
- Let $\mathcal{Q}$ be the quadratic module implementing the Cuntz–Krieger relations, namely

$$\left\langle\!\!\left\langle \begin{array}{l} \{\, \pm(p_v p_w - \delta_{v,w} p_v) \mid v, w \in E_0 \,\} \cup \{\, \pm(s_e^* s_f - \delta_{e,f} p_{r(e)}) \mid e, f \in E_1 \,\} \\ \cup \{\, \pm\big(p_v - \sum_{s(e)=v} s_e s_e^*\big) \mid v \in E_0 \text{ regular} \,\} \cup \{\, p_{s(e)} - s_e s_e^* \mid e \in E_1 \,\} \end{array} \right\rangle\!\!\right\rangle$$

- There is a canonical map $\Phi\colon \mathcal{A} \to C^*(\mathrm{G})$ in the corresponding graph C^*-algebra $C^*(\mathrm{G})$.
- How is $\mathcal{Q}$ related to $\Phi^{-1}\big(C^*(\mathrm{G})^+_{\mathrm{h}}\big)$?

The grand unified problem

Definition

- A *positive *-representation* of an ordered *-algebra $\mathcal{A}$ is a unital *-homomorphism $\pi\colon \mathcal{A} \to \mathcal{L}^*(\mathcal{D})$ to the *-algebra of adjointable endomorphisms on a pre-Hilbert space $\mathcal{D}$ such that $\langle \phi \,|\, \pi(a)(\phi) \rangle \geq 0$ for all $a \in \mathcal{A}_{\mathrm{h}}^+$, $\phi \in \mathcal{D}$.
- Such a positive *-representation is called *bounded* if $\mathcal{D}$ is complete, i.e. $\mathcal{D} = \mathfrak{H}$ a Hilbert space (cf. Hellinger–Toeplitz theorem).

The Problem

Let $\mathcal{A}$ be an ordered *-algebra (typically $\mathcal{A}_{\mathrm{h}}^+ = \langle\!\langle\, G \,\rangle\!\rangle$ for some $G \subseteq \mathcal{A}_{\mathrm{h}}$). Define

$$\mathcal{Q} := \big\{\, a \in \mathcal{A}_{\mathrm{h}} \;\big|\; \langle \phi \,|\, \pi(a)(\phi) \rangle \geq 0 \text{ for all } \; \pi\colon \mathcal{A} \to \mathcal{L}^*(\mathfrak{H}), \phi \in \mathfrak{H} \,\big\},$$

π bounded positive *-representations. Clearly $\mathcal{Q}$ is a quadratic module of $\mathcal{A}$.

$\rightsquigarrow$ How are $\mathcal{A}_{\mathrm{h}}^+$ and $\mathcal{Q}$ related? Certainly $\mathcal{A}_{\mathrm{h}}^+ \subseteq \mathcal{Q}$, but conversely?

The uniform norm

Definition

Let $\mathcal{A}$ be an ordered *-algebra, then define the map $\|\cdot\|_\infty \colon \mathcal{A} \to [0,\infty]$,

$$a \mapsto \|a\|_\infty := \inf\{\, \lambda \in [0,\infty] \;\big|\; a^* a \leq \lambda^2 \,\}.$$

The set of *infinitesimal elements* of $\mathcal{A}$ is defined as

$$\mathcal{I}_{\mathrm{bd}} := \{\, a \in \mathcal{A} \;\big|\; \|a\|_\infty = 0 \,\},$$

and the set of *uniformly bounded elements* of $\mathcal{A}$ as

$$\mathcal{A}_{\mathrm{bd}} := \{\, a \in \mathcal{A} \;\big|\; \|a\|_\infty < \infty \,\}.$$

The ordered *-algebra $\mathcal{A}$ is called *uniformly bounded* if $\mathcal{A} = \mathcal{A}_{\mathrm{bd}}$.

Cimprič [1]; Schmüdgen [10], ...; part I

Let $\mathcal{A}$ be an ordered *-algebra.

- The uniformly bounded elements $\mathcal{A}_{\mathrm{bd}}$ form a unital *-subalgebra of $\mathcal{A}$.
- The infinitesimal elements $\mathcal{I}_{\mathrm{bd}}$ form a *-ideal of $\mathcal{A}_{\mathrm{bd}}$.
- The map $\|\cdot\|_\infty$ descends to a C^*-norm on $\mathcal{A}_{\mathrm{bd}}/\mathcal{I}_{\mathrm{bd}}$.

Which ordered *-algebras are uniformly bounded?

If $\mathcal{A}$ is a uniformly bounded ordered *-algebras, then all its positive *-representations are uniformly bounded! But conversely...?

A pathological example

Set $g_1 := 2x_1 - 1, g_2 := 2x_2 - 1, g_3 := 1 - x_1 x_2 \in \mathbb{C}[x_1, x_2]_{\mathrm{h}}$ and consider the set $\mathcal{P}(\{g_1, g_2, g_3\}) = \{\, \xi \in \mathbb{R}^2 \mid g_i(\xi) \geq 0 \text{ for all } i \in \{1, 2, 3\} \,\}$.
Then $\mathcal{P}(G)$ is compact but $\mathbb{C}[x_1, x_2]$ with $\mathbb{C}[x_1, x_2]^+_{\mathrm{h}} := \langle\!\langle \{g_1, g_2, g_3\} \rangle\!\rangle$ is not uniformly bounded (see [2, p. 146]).

Schmüdgen's Positivstellensatz, part I

Consider any finite set $g_1, \dots, g_k \in \mathbb{C}[x_1, \dots, x_n]_{\mathrm{h}}$ and let G be the set of all finite products of $g_1, \dots, g_k$.
If $\mathcal{P}(G)$ is compact, then $\mathbb{C}[x_1, \dots, x_n]$ with $\mathbb{C}[x_1, \dots, x_n]^+_{\mathrm{h}} := \langle\!\langle G \rangle\!\rangle$ is uniformly bounded.

Schmüdgen, S. (2023)

Consider any set of real polynomials of degree 1 and let G be the set of all their finite products.
If $\mathcal{P}(G)$ is compact and non-empty, then $\mathbb{C}[x_1, \dots, x_n]$ with $\mathbb{C}[x_1, \dots, x_n]^+_{\mathrm{h}} := \langle\!\langle G \rangle\!\rangle$ is uniformly bounded.

Which ordered *-algebras are uniformly bounded?

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Schmüdgen, S. [9]

Consider any set of real polynomials of degree 1 and let G be the set of all their finite products.
If $\mathcal{P}(G)$ is compact and non-empty, then $\mathbb{C}[x_1, \dots, x_n]$ with $\mathbb{C}[x_1, \dots, x_n]^+_{\mathrm{h}} := \langle\!\langle\, G \,\rangle\!\rangle$ is uniformly bounded.

Another pathological example

Set $G := \{\, x - n \mid n \in \mathbb{N} \,\} \subseteq \mathbb{C}[x]_{\mathrm{h}}$.
Then $\mathcal{P}(G) = \emptyset$, but $\langle\!\langle\, G \,\rangle\!\rangle = \{\, p \in \mathbb{C}[x]_{\mathrm{h}} \mid p(+\infty) \geq 0 \,\}$, and $\mathbb{C}[x]$ with $\mathbb{C}[x]^+_{\mathrm{h}} := \langle\!\langle\, G \,\rangle\!\rangle$ is not uniformly bounded.

Which ordered *-algebras are uniformly bounded?

As $\mathcal{A}_{\mathrm{bd}}$ is a unital *-subalgebra of $\mathcal{A}$, uniform boundedness of a generating subset is sufficient!

- $\mathbb{CP}^n$ via symmetry reduction:
  $\mathbb{C}[z_0, \dots, z_n, \overline{z}_0, \dots, \overline{z}_n]^{U(1)}$ is generated by $z_i\overline{z}_j$, $i, j \in \{0, \dots, n\}$.
  Recall: $\mathcal{J} := z_0\overline{z}_0 + \dots + z_n\overline{z}_n$ and we consider $\langle\!\langle \{\mathcal{J} - \mu, \mu - \mathcal{J}\} \rangle\!\rangle$, $\mu > 0$.
  In $\mathbb{C}[z_0, \dots, z_n, \overline{z}_0, \dots, \overline{z}_n]^{U(1)} / \operatorname{supp}\langle\!\langle \{\mathcal{J} - \mu, \mu - \mathcal{J}\} \rangle\!\rangle$:

  $$[z_i\overline{z}_j]^*[z_i\overline{z}_j] = [z_j\overline{z}_i z_i\overline{z}_j] \leq [z_j\mathcal{J}\overline{z}_j] = \mu[z_j\overline{z}_j] \leq \mu\mathcal{J} = \mu^2$$

  So $[z_i\overline{z}_j] \in \big(\mathbb{C}[z_0, \dots, z_n, \overline{z}_0, \dots, \overline{z}_n]^{U(1)} / \operatorname{supp}\langle\!\langle \{\mathcal{J} - \mu, \mu - \mathcal{J}\} \rangle\!\rangle\big)_{\mathrm{bd}}$ for all $i, j \in \{0, \dots, n\}$.
- Berezin quantization of $\mathbb{CP}^n$: completely analogous.
- Leavitt path *-algebras:
  $p_v^2 = p_v$ and $s_e^* s_e = p_{r(e)}$, $v \in E_0$, $e \in E_1$, enforce uniform boundedness.
  $\rightsquigarrow$ The norm of the graph C^*-algebra is the uniform norm $\|\cdot\|_\infty$.

The Archimedean Positivstellensatz

Schmüdgen [11]

Let $\mathcal{A}$ be a uniformly bounded ordered *-algebra and $a \in \mathcal{A}_{\mathrm{h}}$.
If $\langle \phi \,|\, \pi(a)(\phi) \rangle > 0$ for all bounded positive *-representations $\pi \colon \mathcal{A} \to \mathcal{L}^*(\mathfrak{H})$ and $\phi \in \mathfrak{H} \setminus \{0\}$, then $a \in \mathcal{A}_{\mathrm{h}}^+$.

Proof (idea)

If $a \in A_{\mathrm{h}} / \mathcal{A}_{\mathrm{h}}^+$, construct positive real linear functional $\omega \colon \mathcal{A}_{\mathrm{h}} \to \mathbb{R}$ with $\omega(a) \leq 0$ (Hahn–Banach). Extend $\mathbb{C}$-linearly and apply GNS-construction.

The Archimedean Positivstellensatz

Schmüdgen [11]

Let $\mathcal{A}$ be a uniformly bounded ordered *-algebra and $a \in \mathcal{A}_{\mathrm{h}}$.
If $\langle \phi \,|\, \pi(a)(\phi) \rangle > 0$ for all bounded positive *-representations $\pi\colon \mathcal{A} \to \mathcal{L}^*(\mathfrak{H})$ and $\phi \in \mathfrak{H} \setminus \{0\}$, then $a \in \mathcal{A}_{\mathrm{h}}^+$.

Corollary (Schmüdgen's Positivstellensatz, part II)

Consider any finite set $g_1, \dots, g_k \in \mathbb{C}[x_1, \dots, x_n]_{\mathrm{h}}$ and assume $\mathcal{P}(G)$ is compact. Let G be the set of all finite products of $g_1, \dots, g_k$.
If $p \in \mathbb{C}[x_1, \dots, x_n]_{\mathrm{h}}$ fulfils $p(\xi) > 0$ for all $\xi \in \mathcal{P}(G)$, then $p \in \langle\!\langle\, G \,\rangle\!\rangle$.

Corollary

Let G be a directed graph with finitely many vertices. Write $L^*(\mathrm{G})$ for the Leavitt path *-algebra with its natural quadratic module $\mathcal{Q}$ and let $\iota\colon L^*(\mathrm{G}) \to C^*(\mathrm{G})$ be the embedding in its graph C^*-algebra.
Consider $a \in L^*(\mathrm{G})_{\mathrm{h}}$. If $\mathrm{spec}(\iota(a)) \in [\epsilon, \infty[$ for some $\epsilon > 0$, then $a \in \mathcal{Q}$.

Closed ordered *-algebras

Grand unified problem is almost solved...

Example

The unital subalgebra

$$\mathcal{A} := \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{C} \right\}$$

of $\mathbb{C}^{2\times 2}$ with elementwise complex conjugation is a commutative *-algebra and

$$\mathcal{A}_{\mathrm{h}}^{++} = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \text{ with } a > 0 \text{ or } a = b = 0 \right\}.$$

So $\mathcal{A}$ becomes a uniformly bounded ordered *-algebra with $\mathcal{A}_{\mathrm{h}}^{+} := \mathcal{A}_{\mathrm{h}}^{++}$.

- There are $M \in \mathcal{A}_{\mathrm{h}} \setminus \{0\}$ with $M^2 = 0$ (namely if $a = 0$, $b \neq 0$).
- Consequently $||M||_\infty = 0$, i.e. $M \in \mathcal{I}_{\mathrm{bd}}$.
- Note also: $\left(\begin{smallmatrix} \epsilon & b \\ 0 & \epsilon \end{smallmatrix}\right)$ with $\epsilon > 0$ and $b \neq 0$ is in $\mathcal{A}_{\mathrm{h}}^{++}$, unlike its limit $\epsilon \to 0$.

Closed ordered *-algebras

Definition

An ordered *-algebra $\mathcal{A}$ is *(integrally) closed* if the following holds: Whenever $a, b \in \mathcal{A}_\mathrm{h}$ fulfil $a \le \epsilon b$ for all $\epsilon \in \,]0, \infty[$, then $a \le 0$.

Cimprič [1], Schmüdgen [10], ...; part II

If $\mathcal{A}$ is a closed ordered *-algebra, then $\|\cdot\|_\infty$ is a C^*-norm on $\mathcal{A}_\mathrm{bd}$, $\mathcal{I}_\mathrm{bd} = \{0\}$.

Corollary

The category of closed and uniformly bounded ordered *-algebras with positive unital *-homomorphisms between them is equivalent to the category of pre-C^*-algebras (*-algebras with C^*-norm) and continuous unital *-homomorphisms between them.

Corollary (Archimedean Positivstellensatz revisited)

Let $\mathcal{A}$ be a closed and uniformly bounded ordered *-algebra and $a \in \mathcal{A}_\mathrm{h}$. Then $a \in \mathcal{A}_\mathrm{h}^+$ if and only if $\langle \phi \,|\, \pi(a)(\phi) \rangle \ge 0$ for all bounded positive *-representations $\pi \colon \mathcal{A} \to \mathcal{L}^*(\mathfrak{H})$ and $\phi \in \mathfrak{H}$.

But how to choose generators of $\mathcal{A}_\mathrm{h}^+$ so that $\mathcal{A}$ is closed?

σ-bounded ordered *-algebras

Definition

An ordered *-algebra $\mathcal{A}$ is called *σ-bounded* if there exists an increasing sequence $(\hat{a}_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_\mathrm{h}^+$ that is cofinal, i.e. for all $b \in \mathcal{A}_\mathrm{h}$ there is some $n \in \mathbb{N}$ such that $b \leq \hat{a}_n$.

Examples

- Every uniformly bounded ordered *-algebra is σ-bounded, choose $\hat{a}_n := n\mathbb{1}$ for all $n \in \mathbb{N}$.
- Every countably generated ordered *-algebra is σ-bounded, choose

$$\hat{a}_n := n \sum_{j=1}^{n} \frac{\mathbb{1} + b_j^2}{2}$$

with $b_1, b_2, \ldots \in \mathcal{A}_\mathrm{h}$ a vector space basis of $\mathcal{A}_\mathrm{h}$; use $\pm b_j \leq (\mathbb{1} + b_j^2)/2$.

An unbounded Gelfand–Naimark theorem

S. [12]

Let $\mathcal{A}$ be a σ-bounded closed ordered *-algebra, then $\mathcal{A}$ has a faithful positive *-representation.

Proof

- GNS-construction yields *-representations from positive functionals.
- Hahn–Banach theorem yields positive functionals.
- How to construct the l.c. topology? Use σ-boundedness:

Given a cofinal sequence $(\hat{a}_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_\mathrm{h}^+$ and a sequence $(\delta_n)_{n\in\mathbb{N}}$ in $]0,\infty[$. The union of order intervals

$$U_\delta := \bigcup_{n\in\mathbb{N}} \Big[-\sum\nolimits_{j=1}^n \delta_j \hat{a}_j \, , \, \sum\nolimits_{j=1}^n \delta_j \hat{a}_j \Big]$$

is an absorbing, balanced, and convex subset of $\mathcal{A}_\mathrm{h}$, hence a 0-neighbourhood.

For any $a \in \mathcal{A}_\mathrm{h} \setminus \mathcal{A}_\mathrm{h}^+$ there is U_δ such that $(a + U_\delta) \cap \mathcal{A}_\mathrm{h}^+ = \emptyset$ (construct $(\delta_n)_{n\in\mathbb{N}}$ recursively using that $\mathcal{A}$ is closed).

$\rightsquigarrow$ Commutative version also available, but more tricky...

So which examples are closed ordered *-algebras?

- Consider a finite set $g_1, \dots, g_k \in \mathbb{C}[x_1, \dots, x_n]_{\mathrm{h}}$, let G be the set of all finite products of $g_1, \dots, g_k$ and assume that $\mathcal{P}(G)$ is compact. There are many examples in which $\mathbb{C}[x_1, \dots, x_n] / \operatorname{supp}\langle\!\langle\, G \,\rangle\!\rangle$ is a closed uniformly bounded ordered *-algebra with $\mathcal{P}(G)$ having dimension 1 or 2, but not in higher dimensions (see Scheiderer [5], [6], [7]).
- In higher dimensions: Krivine–Stengle Positivstellensatz.
- Especially for $\mathbb{CP}^n$ via symmetry reduction: $\mathbb{C}[z_0, \dots, z_n, \overline{z}_0, \dots, \overline{z}_n]^{U(1)} / \operatorname{supp}\langle\!\langle\, \{\mathcal{J} - \mu, \mu - \mathcal{J}\} \,\rangle\!\rangle$, $\mu > 0$ is a closed uniformly bounded ordered *-algebra if and only if $n = 1$.
- But for the quantization of $\mathbb{CP}^n$ we find (Schmitt, S. [8]): $\mathcal{W}(n)^{U(1)} / \operatorname{supp}\langle\!\langle\, \{\mathcal{J} - \mu, \mu - \mathcal{J}\} \,\rangle\!\rangle$, $\mu \geq 0$ is a closed uniformly bounded ordered *-algebra for all $n \in \mathbb{N}$. So $\mathcal{W}(n)^{U(1)} / \operatorname{supp}\langle\!\langle\, \{\mathcal{J} - \mu, \mu - \mathcal{J}\} \,\rangle\!\rangle \cong \mathbb{C}^{d \times d}$, $d \in \mathbb{N}$ for $\mu/\hbar \in \mathbb{N}_0$ and $\mathcal{W}(n)^{U(1)} / \operatorname{supp}\langle\!\langle\, \{\mathcal{J} - \mu, \mu - \mathcal{J}\} \,\rangle\!\rangle \cong \{0\}$ otherwise.
- And for Leavitt path *-algebras?

So which Leavitt path *-algebras give closed ordered *-algebras?

- Of course, all complex matrix algebras.
- Fejér-Riesz theorem:
  Pointwise positive complex polynomials on the circle are sums of squares.
- Matrix-valued Fejér-Riesz theorem:
  Pointwise positive matrix-valued polynomials on the circle are sums of squares (Rosenblum [3]).
- Non-commutative Fejer-Riesz theorem (Savchuk, Schmüdgen [4]):
  Let $\mathcal{A} := \langle\, s, s^* \mid s^* s = 1 \,\rangle$ and let $\pi \colon \mathcal{A} \to \mathcal{L}^*\big(\ell^2(\mathbb{N}_0)\big)$ be the *-representation given by the right shift $\pi(s)$. Consider $a \in \mathcal{A}_\mathrm{h}$ such that $\pi(a)$ is positive semi-definite. Then there is $b \in \mathcal{A}$ such that $a = b^* b$.
- $\rightsquigarrow$ Some examples are understood, but no general theory on par with the commutative case.

References — Thank you for your attention!

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