C^*-bundles containing the Effros-Shen algebras

Jack Spielberg, Arizona State University
(joint work with Ian Mitscher,
Colorado School of Mines)

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Effros-Shen algebras: let $t \in (0,1) \setminus \mathbb{Q}$.
Elliott: there is a unique simple unital AF algebra A_t with

$K_0(A_t) = \mathbb{Z} + t\mathbb{Z} \cong \mathbb{Z}^2$
$K_0(A_t)_+ = (\mathbb{Z} + t\mathbb{Z}) \cap [0, \infty)$
$[1]_0 = 1$

To realize the algebra, Effros and Shen used the continued fraction for t: every $t \in (0,1) \setminus \mathbb{Q}$ has a unique infinite simple continued fraction expansion

$$t = [0, a_1, a_2, \ldots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}, \quad a_i \in \mathbb{N}_+.$$

The nth convergent $\dfrac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}$ converges to t as $n \to \infty$,

$$p_0 = 0,\ \ p_1 = 1,\ \ p_n = a_n p_{n-1} + p_{n-2}$$
$$q_0 = 1,\ \ q_1 = a_1,\ \ q_n = a_n q_{n-1} + q_{n-2}.$$

Then can embed $M_{q_{n-1}} \oplus M_{q_{n-2}} \hookrightarrow M_{q_n} \oplus M_{q_{n-1}}$ with multiplicities given by $\left(\begin{smallmatrix} a_n & 1 \\ 1 & 0 \end{smallmatrix}\right)$:

$$x \oplus y \mapsto (\underbrace{x \oplus \cdots \oplus x}_{a_n \text{ times}} \oplus y) \oplus x.$$

$A_t = \overline{\bigcup_{n=1}^{\infty} M_{q_n} \oplus M_{q_{n-1}}}$.

The map

$$(a_1, a_2, \dots) \in (\mathbb{N}_+)^\infty \mapsto [0, a_1, a_2, \dots] \in (0,1) \setminus \mathbb{Q}$$

is a homeomorphism. Then

$$\rho_0 : \bigsqcup_{t \in (0,1) \setminus \mathbb{Q}} A_t \to (0,1) \setminus \mathbb{Q},\ \rho_0(A_t) = t,$$

can be made into a continuous C^*-bundle - but the base space is not locally compact. What can be done at the rational points?

Each $t \in (0,1) \cap \mathbb{Q}$ has two distinct finite simple continued fraction expansions:

$$t = [0, a_1, \ldots, a_n] = [0, a_1, \ldots, a_{n-1}, a_n - 1, 1].$$

These correspond to two finite dimensional C^*-algebras:

$$M_{q_n} \oplus M_{q_{n-1}} \text{ and } M_{q'_{n+1}} \oplus M_{q'_n},$$

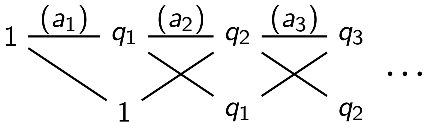
defined by the last two convergents of each expansion. (Of course $q_n = q'_{n+1}$ is the denominator of t.)

In fact, the even-length expansion corresponds to approximation of t from the right, and the odd-length expansion from the left.

Write A_{t+} and A_{t-} for these two C^*-algebras, and let $X = \big((0,1) \setminus \mathbb{Q}\big) \cup \{t_+, t_- : t \in (0,1) \cap \mathbb{Q}\}$ be the usual disconnection of $(0,1)$ at the rational points. Then X is a locally compact Cantor set, and we can make a continuous C^*-bundle

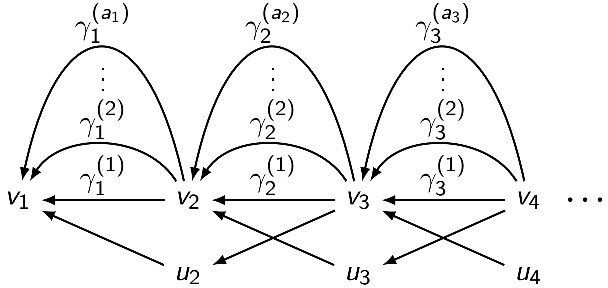
$$\rho : \bigsqcup_{x \in X} A_x \to X, \ \rho(A_x) = x.$$

To get a bundle over $(0,1)$ we use a different construction. A_t is usually presented by a Bratteli diagram:

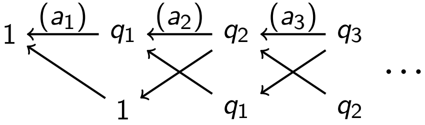


It can equally well be described as (the compression of) a graph algebra: $A_t = P_{v_1} C^*(E) P_{v_1}$, where

E:



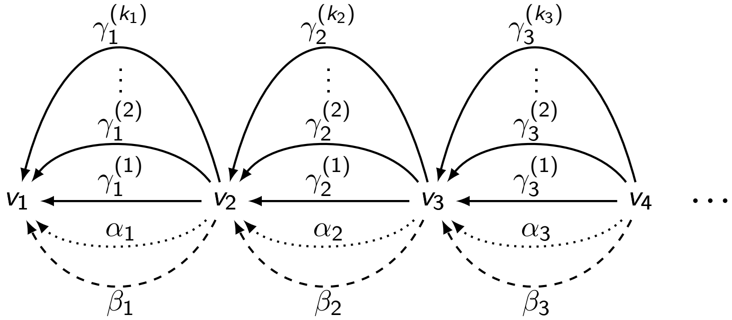
How does this go? The directed graph E has no singular vertices, so $v_1 \partial E = v_1 E^\infty$. For $\mu \in v_1 E^*$ we let $Z(\mu) = \{\mu x : x \in s(\mu) E^\infty\}$, the set of all infinite paths that begin with μ. The typical generator $S_\mu S_\nu^*$ (where $\mu, \nu \in v_1 E^*$ and $s(\mu) = s(\nu)$) can be thought of as a (partial) homeomorphism: $Z(\nu) \to Z(\mu)$. Recall the graph E



$$q_n = |v_1 E^* v_{n+1}|$$
$$q_{n-1} = |v_1 E^* u_{n+1}|$$

Therefore $\{\mu \in v_1 E^* : |\mu| = n\}$ define the minimal diagonal projections in $M_{q_n} \oplus M_{q_{n-1}}$, and $S_\mu S_\nu^*$ is the matrix unit $e_{\mu,\nu}$. The action given by $\{S_\mu S_\nu^*\}$ is *free*.

Now we replace E with a different structure, a *category of paths* Λ. Let $k = (k_i)_{i=1}^{\infty} \in \mathbb{N}^{\infty}$, and suppose that $k_i > 0$ infinitely often.

$\Lambda(k)$: 

where we make the identifications $\alpha_i\beta_{i+1} = \beta_i\alpha_{i+1}$ for all i. The subgraph formed by $\{\alpha_i, \beta_i\}$ is a 2-graph, but Λ is not a higher rank graph. It is a small category satisfying cancellation and having no nontrivial inverses.

Categories of paths define C^*-algebras in much the same way that directed graphs, higher rank graphs, and submonoids of groups do - for example, using an étale groupoid. We briefly analyze the above example.

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An element composed of α_i's and β_j's can be written with the edges permuted - it is determined only by how many edges of each type occur. A typical element of $v_1\Lambda$ looks like

$$v_1 \alpha^{i_1} \beta^{j_1} \gamma^{(\ell_1)}_{i_1+j_1+1} \alpha^{i_2} \beta^{j_2} \gamma^{(\ell_2)}_{..} \dots \gamma^{(\ell_m)}_{..} \alpha^{i_{m+1}} \beta^{j_{m+1}}$$

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The boundary $v_1\partial\Lambda = v_1\Lambda^\infty$ has two kinds of elements:

- $v_1\alpha^{i_1}\beta^{j_1}\gamma^{(\ell_1)}_{i_1+j_1+1}\alpha^{i_2}\beta^{j_2}\gamma^{(\ell_2)}_{..}\dots\gamma^{(\ell_m)}_{..}\alpha^{i_{m+1}}\beta^{j_{m+1}}\dots$
- $v_1\alpha^{i_1}\beta^{j_1}\gamma^{(\ell_1)}_{i_1+j_1+1}\alpha^{i_2}\beta^{j_2}\gamma^{(\ell_2)}_{..}\dots\gamma^{(\ell_m)}_{..}\alpha^{p}\beta^{q}$, $p+q=\infty$.

Again, for $\mu, \nu \in v_1\Lambda$, $S_\mu S_\nu^*$ is a typical element of a total set for $P_{v_1} C^*(\Lambda) P_{v_1}$. But they no longer act like matrix units. For example,

$$S_{\beta_1} S_{\alpha_1}^* : Z(\alpha_1) \to Z(\beta_1)$$

does not act freely on $Z(\alpha_1) \subseteq v_1\partial\Lambda$, since

$$S_{\beta_1} S_{\alpha_1}^* (v_1 \alpha^\infty \beta^\infty) = \alpha^\infty \beta^\infty.$$

For these examples of categories of paths we have the following

Theorem. (Mitscher-S) Let $(k_i)_1^\infty \in \mathbb{N}^\infty$ with $k_i > 0$ infinitely often.

1. The (in general) nonsimple continued fraction $[0, 1, k_1, 1, k_2, 1, \ldots]$ converges to an irrational point of $(0, 1)$.
2. Each $t \in (0, 1) \setminus \mathbb{Q}$ has a unique expansion of this form.
3. $P_{v_1} C^*(\Lambda) P_{v_1} \cong A_t$.

This is very different from the usual construction. The 2-graph inside Λ produces nontrivial isotropy in the groupoid underlying the C^*-algebra - it is not an AF groupoid.

The proof has three steps:

- calculate the Elliott invariant
- show that the algebra is classifiable
- use the classification theorem (Tikuisis, White, Winter, ...).

The third step means that the proof is nonconstructive - we cannot exhibit the dense union of finite dimensional subalgebras.

What happens at rational numbers? Suppose that $(k_i) \in \mathbb{N}^\infty$ is finitely nonzero. Say $k_m > 0$ and $k_i = 0$ for $i > m$. We still have convergence of the continued fraction, but to a rational number:

$$\begin{aligned}[0, 1, k_1, 1, k_2, 1, \ldots] &= [0, 1, k_1, 1, \ldots, 1, k_m, 1, 0, 1, 0, 1, 0, \ldots] \\ &= [0, 1, k_1, 1, \ldots, 1, k_m].\end{aligned}$$

Each rational number in $(0, 1)$ has a *unique* finite continued fraction in the above form. This alternate continued fraction expansion *chooses* one of the two expansions of a rational number.

Let $\pi : \mathbb{N}^\infty \to [0, 1)$ be given by $\pi(k) = [0, 1, k_1, 1, k_2, 1, \ldots]$. Then π is bijective and continuous, and π^{-1} is continuous from the right (but not from the left).

The category of paths $\Lambda(k)$ has only finitely many edges that are not part of the sub-2-graph, but the construction of the C^*-algebra goes through without difficulties. The algebra is type I:

$$0 \to \mathcal{K} \oplus \mathcal{K} \to P_{v_1} C^*(\Lambda(k)^\infty) P_{v_1} \to M_q \otimes C(\mathbb{T}) \to 0.$$

(The superscript $^\infty$ is a technicality present in the finitely nonzero case.)

Now we have a single algebra to offer at the rational points of the interval: for $t \in [0,1) \cap \mathbb{Q}$ let $k = \pi^{-1}(t) \in \mathbb{N}^\infty$. We set

$$A_t := P_{v_1} C^*(\Lambda(k)^\infty) P_{v_1}.$$

We now define a bundle of C^*-algebras over $[0,1)$,

$$\mathscr{A} = \bigsqcup_{t \in [0,1)} A_t, \text{ by } p : \mathscr{A} \to [0,1),\ p(A_t) = t.$$

Let's see how we might topologise this bundle. For this we consider a typical generating element independent of the choice of k. Let $\mu = \mu_1\mu_2\cdots\mu_n, \nu = \nu_1\nu_2\cdots\nu_n$ with $\mu_i, \nu_i \in \{\alpha_i, \beta_i, \gamma_i^{(j)} : j \geq 1\}$ for each i. We ask:

for which k does $S_\mu S_\nu^*$ belong to $A_{\pi(k)}$?

The answer is: those k for which μ and ν belong to $\Lambda(k)$. Put

$$\ell_i = \max\{j : \mu_i = \gamma_i^{(j)} \text{ or } \nu_i = \gamma_i^{(j)}\}$$
$$\ell = (\ell_1, \ell_2, \ldots) \in \mathbb{N}^\infty.$$

Then $S_\mu S_\nu^* \in A_{\pi(k)}$ if and only if $k_i \geq \ell_i$, all i.

We need to identify $D(\ell) := \{\pi(k) : k_i \geq \ell_i \text{ for all } i\}$. Let $m = \max\{i : \ell_i > 0\}$, so $\ell = (\ell_1, \ldots, \ell_m, 0, 0, 0, \ldots)$. We build the answer step by step.

$$[0,1) = [0,\tfrac{1}{2}) \sqcup [\tfrac{1}{2},\tfrac{2}{3}) \sqcup [\tfrac{2}{3},\tfrac{3}{4}) \sqcup \cdots = \bigsqcup_{h\geq 0}\left[\tfrac{h}{h+1},\tfrac{h+1}{h+2}\right)$$

Note that $\dfrac{h}{h+1} = \dfrac{1}{1+\frac{1}{h}} = [0,1,h]$. Then

$$[0,1) = \bigsqcup_{h\geq 0}\Big[[0,1,h],[0,1,h+1]\Big)$$

$$\{\pi(k) : k_1 \geq \ell_1\} = \bigsqcup_{h\geq \ell_1}\Big[[0,1,h],[0,1,h+1]\Big) = \Big[[0,1,\ell_1],1\Big).$$

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We apply the same decomposition to these subintervals:

$$\left[\frac{h}{h+1},\frac{h+1}{h+2}\right) = \bigsqcup_{\ell\geq 0}\left[\frac{h+\frac{\ell}{\ell+1}}{h+1+\frac{\ell}{\ell+1}},\frac{h+\frac{\ell+1}{\ell+2}}{h+1+\frac{\ell+1}{\ell+2}}\right).$$

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$$= [0, 1, h, 1, \ell].$$

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Then we have $\left[\frac{h}{h+1}, \frac{h+1}{h+2}\right) = \bigsqcup_{\ell\geq 0}\Big[[0,1,h,1,\ell],[0,1,h,1,\ell+1]\Big)$.

Thus

$$\{\pi(k) : k_1 = h, k_2 \geq \ell_2\} = \bigsqcup_{\ell \geq \ell_2} \Big[[0,1,h,1,\ell],[0,1,h,1,\ell+1]\Big)$$
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In general we find that

$$D(\ell) := \{\pi(k) : k_i \geq \ell_i \text{ for } 1 \leq i \leq m\}$$

$$= \bigsqcup_{\substack{k_i \geq \ell_i, \\ 1 \leq i \leq m-1}} \Big[[0,1,k_1,1,\cdots,k_{m-1},1,\ell_m],$$

$$[0,1,k_1,1,\cdots,k_{m-2},1,k_{m-1}+1]\Big).$$

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$$[0,1,k_1,1,\cdots,k_{m-2},1,k_{m-1}+1]\Big).$$

Incidentally, we see that the particular nonsimple continued fraction expansions we are forced to use are, in fact, quite natural.

Recall the bundle from earlier:

$\mathscr{A} = \bigsqcup_{t\in[0,1)} A_t$, by $p : \mathscr{A} \to [0,1)$, $p(A_t) = t$.

We will use a space of sections to define a continuous field; this will then give a topology on the bundle. Let μ, ν, ℓ be as before. Define $f : [0,1) \to \mathscr{A}$ by

$$f(\pi(k)) = \begin{cases} S_\mu S_\nu^*, & \text{if } k_i \geq \ell_i, \text{ all } i \\ 0, & \text{otherwise.} \end{cases}$$

Then $D(\ell) = \{f \neq 0\}$. We have to manage the discontinuities of f at the right endpoints of the intervals in $D(\ell)$. Since the intervals making up $D(\ell)$ do not accumulate at any of their left endpoints, $D(\ell)$ is a locally compact subset of $[0,1)$. We will use sections $\phi \cdot f$ where $\phi \in C_0(D(\ell))$.

Theorem. $\text{span}\{\phi \cdot f : \phi \in C_0(D(\ell)), f \text{ as above}\}$ defines an upper semicontinuous field of C^*-algebras over $[0,1)$.

There is another way to fill in the bundle at rational points. For μ, ν, ℓ as above, let

$$D_0(\ell) := \mathrm{int}(D(\ell)),$$

the (disjoint) union of the interiors of the half-open intervals making up $D(\ell)$. We consider the continuous field $\mathcal{F}$ defined by

$$\mathrm{span}\{\phi \cdot f : \phi \in C_0(D_0(\ell)), f \text{ as above}\}.$$

The sections in $\mathcal{F}$ are continuous at all endpoints of the intervals in $D(\ell)$. However, if we let $s = \pi(\ell)$, then $S_\mu S_\nu^*$ is no longer the value at s of a section in $\mathcal{F}$.

Let $B_s = \{x \in A_s : x = g(s) \text{ for some } g \in \mathcal{F}\}$.

Then $B_s = A_{\widetilde{s}}$, where $\widetilde{s} = [0, 1, \ell_1, 1, \ldots, 1, \ell_{m-1}, 1, \ell_m - 1]$. (Thus B_s is a proper subalgebra of A_s - we have lost some of the elements.)

For $t \in (0,1) \setminus \mathbb{Q}$ we set $B_t = A_t$.

Let $\mathscr{B} = \bigsqcup_{t \in [0,1)} B_t$, $q : \mathscr{B} \to [0,1)$ by $q(B_t) = t$.

Theorem. $\mathcal{F}$ is a continuous field of C^*-algebras, and it topologises $\mathscr{B}$ as a continuous C^*-bundle.

It turns out that the usual Effros-Shen algebras may be completed to a continuous C^*-bundle over $(0,1)$ by a similar device. Recall that a rational number $t \in (0,1)$ has two simple continued fraction expansions:

$$t = [0, a_1, \dots, a_n] = [0, a_1, \dots, a_{n-1}, a_n - 1, 1],$$

with corresponding algebras A_{t+} and A_{t-} (not nec. in this order)

$$M_{q_n} \oplus M_{q_{n-1}} \text{ and } M_{q'_{n+1}} \oplus M_{q'_n}.$$

Let $\widetilde{t} = [0, a_1, \dots, a_{n-1}, a_n - 1]$. Let C_t be the finite dimensional algebra corresponding to this expansion of $\widetilde{t}$. Then

$$C_t \subseteq A_{t+} \cap A_{t-}.$$

Letting $C_t = A_t$ for $t \in (0,1) \setminus \mathbb{Q}$, we obtain a bundle $\mathscr{C} = \bigsqcup_{t \in (0,1)} C_t$ It is possible to use the same kind of elements $S_\mu S_\nu^*$ with coefficient functions to define continuous sections of $\mathscr{C}$.

Theorem. $\mathscr{C}$ can be topologised as a continuous C^*-bundle.

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What does it all mean???