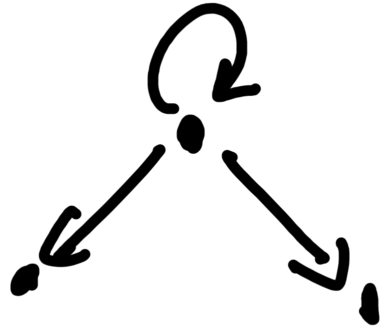
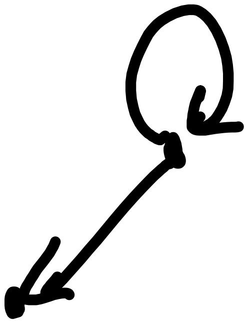
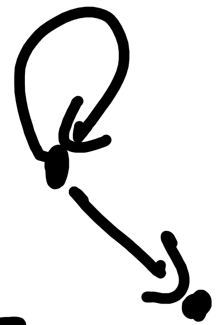
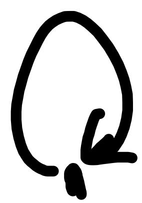




CONTRAVARIANT FUNCTORIALITY OF GRAPH ALGEBRAS





j/w P.M. Hajac

Motivation : NC topology

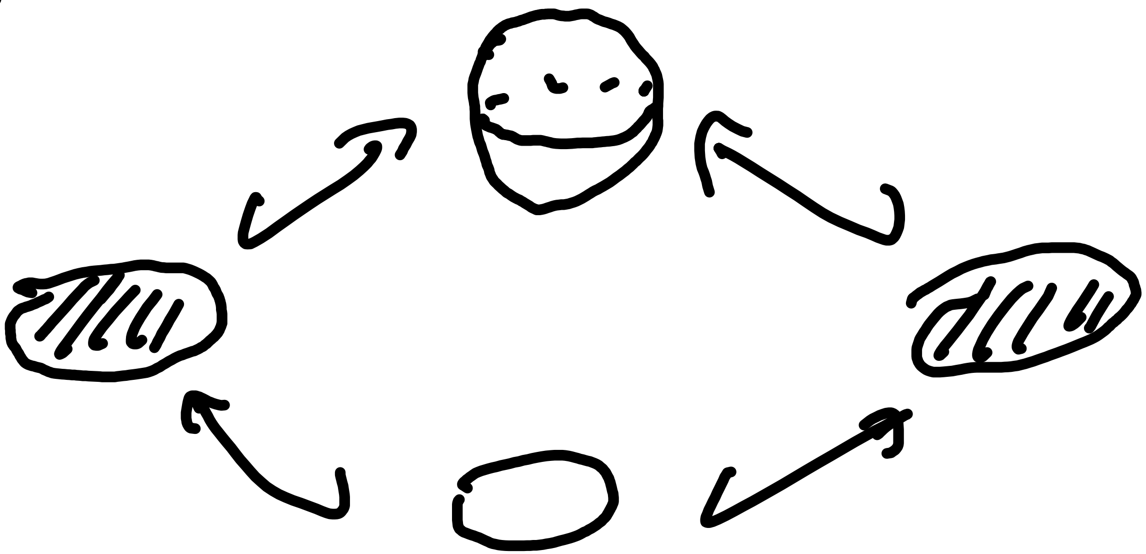
$$\begin{tikzcd} & C(S^2_q) \arrow[dl] \arrow[dr] & \\ \mathcal{T} \arrow[dr, "\sigma"'] \arrow[rr, phantom, "\text{pull}"] & & \mathcal{T} \arrow[dl, "\sigma"] \\ & C(S^1) & \end{tikzcd}$$

Toeplitz alg $\to \mathcal{T}$

symbol map $\to \sigma$

equatorial Podleś sphere

(Podleś, Sheu)



Hong, Szymański (2002)

Question: When do pushouts of graphs induce pullbacks of graph C^*-algebras?

Hong, Szymański (2008): $C(S_q^{2n})$

Robertson, Szymański (2011): Cuntz-Pimsner algs

Kaygun, Hajac, T.; Hajac, Reznikoff, T. (2018)

$\rightsquigarrow$ arbitrary graphs without breaking vertices

Brooker, Spielberg (2022): Toeplitz algebras

+ breaking vertices

All results are for two-injective pushouts.

What we learned so far?

- This problem "has nothing to do with analysis"!

- The most general approach is to find a suitable subcategory of graphs

→ In this talk we deal with one-surjective pullbacks.

Category of graphs :

Obj : graph $E = (E_0, E_1, t_E, s_E)$

Mor. graph homomorphisms

$$f : E \longrightarrow F$$

$$f_0 : E_0 \to F_0 \qquad f_1 : E_1 \to F_1$$

$$t_F \circ f_1 = f_0 \circ t_E$$

$$s_F \circ f_1 = f_0 \circ s_E$$

Morphisms: $f: E \to F$ graph homomorphisms

- f is <u>proper</u> f_0, f_1 are finite to one

$$0 < |f_0^{-1}(v)| < \infty \quad \forall v \in E_0 \qquad 0 \leq |f_1^{-1}(e)| < \infty \quad \forall e \in E_1$$

- f satisfies <u>target bijectivity condition</u> (TBC)

$\iff \forall x \in F_1 : f_1^{-1}(x) \ni e \longmapsto t_E(e) \in f_0^{-1}(t_F(x))$ is bijective

- f is <u>regular</u> iff

$$f_0(E_0 \setminus \mathrm{reg}(E)) \subseteq F_0 \setminus \mathrm{reg}(F)$$

Admissible category of graphs.

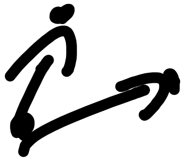
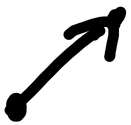
Obj: graphs

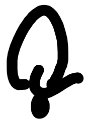
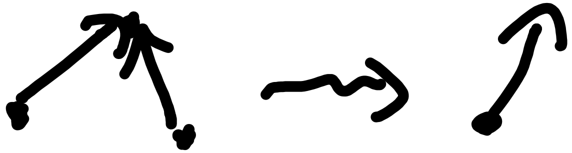
Mor: proper regular graph homomorphisms satisfying TBC

<u>Ex</u> $f: E \hookrightarrow F$ inj $\Rightarrow$ proper

bij $\Leftrightarrow$ TBC

"quotient graph" $\subseteq$ graph admissible

 $\longrightarrow$  folding admissible

$\bullet \hookrightarrow$  not regular  does not sat TBC

Leavitt path algebras

Thm Let $f : E \to F$ be a morphism in the admissible category of graphs then the assignments

$$L_k(F) \ni w \overset{f^*}{\longmapsto} \sum_{v \in f_0^{-1}(w)} v \in L_k(E)$$

$$L_k(F) \ni x \longmapsto \sum_{e \in f_1^{-1}(x)} e \in L_k(E)$$

$$L_k(F) \ni x^* \longmapsto \sum_{e^* \in f_1^{-1}(x^*)} e^* \in L_k(E)$$

give rise to a $\mathbb{Z}$-graded alg homomorphism.

Question: pushouts $\xmapsto{Lk(\cdot)}$ pullbacks ?

$$E \sqcup_G F = (E_0 \sqcup_{G_0} F_0 \ , E_1 \sqcup_{G_1} F_1, t_\sqcup, s_\sqcup)$$

$\rightsquigarrow$ it is a pushout in the category of graphs

Warning: this construction takes you outside of admissible category

Breaking vertices: E arbitrary graph

$$H \subseteq E_0$$

$v \in E_0$ is a breaking vertex wrt H

$$\Leftrightarrow \left\{ v \in E_0 \setminus H, \ |s_E^{-1}(v)| = \infty, \ 0 < |s_E^{-1}(v) \cap t_E^{-1}(E_0 \setminus H)| < \infty \right\}$$

$B_H = \{$ breaking vertices wrt $H \}$

Thm Let

$$\begin{tikzcd} & P & \\ E \arrow[ur, "\iota_E"] & & F \arrow[ul, "\iota_F"'] \\ & G \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

be a pushout in the admissible category of graphs

Assume. 1) f is injective

2) $g_0\big|_{f_0^{-1}(B_{E_0}\setminus f_0(G_0))}$ is injective

3) $g_0\left(f_0^{-1}(B_{E_0}\setminus f_0(G_0))\right) \subseteq B_{F_0}\setminus g_0(G_0)$

Then

$$\begin{tikzcd}
 & L_K(P) \arrow[dl, "q_E^*"'] \arrow[dr, "q_F^*"] & \\
L_K(E) \arrow[dr, "f^*"'] & & L_K(F) \arrow[dl, "g^*"] \\
 & L_K(G) &
\end{tikzcd}$$

a pullback of $\mathbb{Z}$-graded algebras.