Compact quantum surfaces

joint with Arley Sierra

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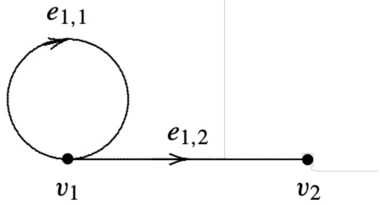
1. Overview

- Definition of closed quantum surfaces of any genus.
- Noncommutative CW complexes of dimension 2.
- Isomorphism clases: Quantization reduces degeneracy.
- BDF-Theory: Essentially normal generators.
- K-groups: 6-term exact sequence.
- K-Theory: Spectral sequences.

- **Graph C*-algebra:**



⇒ C*-algebra generated by $P_1,\ P_2,\ S_1,\ S_2$ with relations
$P_i^2 = P_i = P_i^*,\ i = 1,2,\quad P_1P_2 = P_2P_1,\quad S_1^*S_2 = 0,$
$S_1^*S_1 = P_1,\quad S_2^*S_2 = P_2,\quad S_1S_1^* + S_2S_2^* = P_1$

- **Irreducible Hilbert space representation:**
  $\mathcal{H} = \mathbb{C}e_0 \oplus \ell_2(\mathbb{N}) \cong \ell_2(\mathbb{N}_0) = \overline{\mathrm{span}}\{e_0, e_1, \dots\}$
  $\mathrm{Id} = P_2 \oplus P_1 : \mathbb{C}e_0 \oplus \ell_2(\mathbb{N}) \longrightarrow \mathbb{C}e_0 \oplus \ell_2(\mathbb{N}),$
  $S_1e_0 = 0,\ S_1e_i = e_{i+1},\ i > 0,\quad S_2e_0 = e_1,\ S_2e_i = 0,\ i > 0$
- $S := S_1 + S_2 \ \Rightarrow\ Se_i = e_{i+1},\ \forall i \in \mathbb{N}_0$ (single generator)

⇒ Toeplitz algebra

3. Quantization of the Unit Disk

- **Notations:**
  - open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with closure $\bar{\mathbb{D}}$
  - $L_2(\mathbb{D})$ with respect to the Lebesgue measure
  - $A_2(\mathbb{D}) := \{f \in L_2(\mathbb{D}) : \text{analytic in } \mathbb{D}\}$ (Bergman space)
  - $B_{\mathbb{D}} : L_2(\mathbb{D}) \twoheadrightarrow A_2(\mathbb{D})$, $B_{\mathbb{D}}^2 = B_{\mathbb{D}} = B_{\mathbb{D}}^*$ (Bergman projection)
- **Toeplitz operators:**

  $$T_f : A_2(\mathbb{D}) \to A_2(\mathbb{D}), \quad T_f(\psi) := B_{\mathbb{D}}(f\,\psi), \quad \forall f \in C(\bar{\mathbb{D}})$$
- **Toeplitz algebra:** $\mathcal{T} :=$ C*-alg$\{\, T_f : f \in C(\bar{\mathbb{D}}) \,\} \subset \mathcal{B}(A_2(\mathbb{D}))$
- **Generators:** $\{\mathrm{Id},\ T_z,\ T_{\bar{z}} = T_z^*\} \cup \mathcal{K}(A_2(\mathbb{D}))$, $\quad z(x + \mathrm{i}y) := x + \mathrm{i}y$
- **Commutators:** $[T_f, T_g] \in \mathcal{K}(A_2(\mathbb{D})) =: \mathcal{K}$

4. C*-algebra extension

- **Toeplitz extension:** $[T_f, T_g] \bmod \mathcal{K} \; = \; 0$

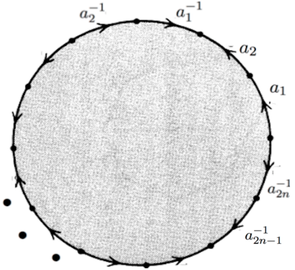
$\Rightarrow$ $$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(\mathbb{S}^1) \longrightarrow 0$$

- **Symbol map:**

$$\sigma : \mathcal{T} \longrightarrow C(\mathbb{S}^1), \;\; \sigma(T_f) = f\!\upharpoonright_{\mathbb{S}^1}, \quad f \in C(\bar{\mathbb{D}})$$

- **Classical picture:**

$$0 \longrightarrow C_0(\mathbb{D}) \longrightarrow C(\bar{\mathbb{D}}) \xrightarrow{\upharpoonright_{\mathbb{S}^1}} C(\mathbb{S}^1) \longrightarrow 0$$



- **Classical closed orientable surface of genus g:**

$\Rightarrow$ $C(\mathbb{T}^g) \cong \{f \in C(\bar{\mathbb{D}}) : f(a_k(t)) = f(a_k^{-1}(t)),\ t \in [0,1],\ k = 1, \ldots, 2g\}$

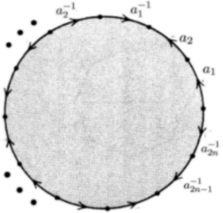
Isomorphism by identify arcs a_k and a_k^{-1}.

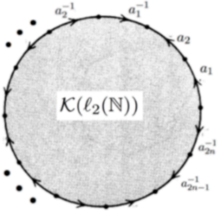
- **Closed orientable quantum surface of genus g:** $\sigma : \mathcal{T} \longrightarrow C(\mathbb{S}^1)$

$$C(\mathbb{T}_q^g) := \{f \in \mathcal{T} : \sigma(f)(a_k(t)) = \sigma(f)(a_k^{-1}(t)),\ t \in [0,1],\ k = 1, \ldots, 2g\}$$

Definition! (Family of C*-algebras.)

6. C*-algebra extension





- **Classical C*-algebra extension:** $C(\mathbb{T}^g) \subset C(\bar{\mathbb{D}})$

$$0 \longrightarrow C_0(\mathbb{D}) \longrightarrow C(\mathbb{T}^g) \xrightarrow{\restriction \mathbb{S}^1} C(\mathbb{S}^1 \vee \ldots \vee \mathbb{S}^1) \longrightarrow 0$$

- **Quantum case:** $C(\mathbb{T}^g_q) \subset \mathcal{T}$

$$0 \longrightarrow \mathcal{K}(\ell_2(\mathbb{N})) \longrightarrow C(\mathbb{T}^g_q) \xrightarrow{\sigma} C(\mathbb{S}^1 \vee \ldots \vee \mathbb{S}^1) \longrightarrow 0$$

- **Topological motivation (K-groups):**

$$K_*(C_0(\mathbb{D})) \cong K_*(C_0(\mathbb{R}^2)) \cong K_*(\Sigma^2\mathbb{C}) \cong K_*(\mathbb{C}) \cong K_*(\mathcal{K}(\ell_2(\mathbb{N})))$$

7. Closed Non-Orientable Quantum Surfaces

- **Classical closed non-orientable surface of genus n:**

$$\mathbb{P}^n = \bar{\mathbb{D}}/\!\sim\; \cong \underbrace{\mathbb{P}^1_1 \# \cdots \# \mathbb{P}^1_1}_{k \text{ times}} \# \underbrace{\mathbb{T}^1 \# \cdots \# \mathbb{T}^1}_{(n-k)/2 \text{ times}} \cong \mathbb{P}^k_k \# \mathbb{T}^{(n-k)/2} =: \mathbb{P}^n_k$$

- **Closed non-orientable quantum surface of genus n:**

Divide $\partial\bar{\mathbb{D}}$ into $2n$ arcs $a_1, \dots, a_k, b_1, \dots, b_k, a_{k+1}, \dots, a_n, a^{-1}_{k+1}, \dots, a^{-1}_n$ with $n+k$ times the same orientation such that $\bar{\mathbb{D}}/\!\sim\; \cong \bigvee_{k=1}^{n} \mathbb{S}^1$.

$$C(\mathbb{P}^n_{k,q}) := \{ f \in \mathcal{T} : \sigma(f)(a_j(t)) = \sigma(f)(b_j(t)),\ j \le k,$$
$$\sigma(f)(a_j(t)) = \sigma(f)(a^{-1}_j(t)),\ j > k,\ t \in [0,1] \}$$

- **Isomorphism classes:** $C_0(\mathbb{D}_q) := \mathcal{K}$ simple C*-algebra

$\Rightarrow$ No cut-and-paste technique.

$\Rightarrow$ Brown-Douglas-Fillmore theory

8. Busby Invariant

- $\mathbb{S}^1/\!\sim \;\cong\; \bigvee_{k=1}^{N} \mathbb{S}^1\,, \quad a_j(t) \sim a_j^{-1}(t), \quad a_j(t) \sim b_j(t), \quad t \in [0,1]$

$\Rightarrow \; \rho_N : \mathbb{S}^1 \longrightarrow \mathbb{S}^1/\!\sim, \quad \rho_N^* : C(\bigvee_{k=1}^{N} \mathbb{S}^1) \longrightarrow \mathbb{S}^1$

- **C*-algebra extensions:** $C_0(\mathbb{D}_q) := \mathcal{K}, \;\; \partial\bar{\mathbb{D}} = \mathbb{S}^1$

$\Rightarrow \; 0 \longrightarrow C_0(\mathbb{D}_q) \xrightarrow{\;\iota\;} C(\mathbb{T}_q^g) \xrightarrow{\;\sigma\;} C(\bigvee_{k=1}^{2g} \mathbb{S}^1) \longrightarrow 0\,,$

$\Rightarrow \; 0 \longrightarrow C_0(\mathbb{D}_q) \xrightarrow{\;\iota\;} C(\mathbb{P}_{k,q}^n) \xrightarrow{\;\sigma\;} C(\bigvee_{k=1}^{n} \mathbb{S}^1) \longrightarrow 0\,,$

- **Busby invariant:** $\tau_N : C(\bigvee_{k=1}^{N} \mathbb{S}^1) \longrightarrow \mathcal{M}(\mathcal{K})/\mathcal{K} = \mathcal{B}/\mathcal{K}$

$\tau_N(f) := \sigma(T_{\widehat{\rho_N^*(f)}}) \in \mathcal{T}/\mathcal{K} \subset \mathcal{B}/\mathcal{K}$

$\widehat{\rho_N^*(f)}(r\mathrm{e}^{\mathrm{i}\theta}) := r\, f(\mathrm{e}^{\mathrm{i}\theta})$

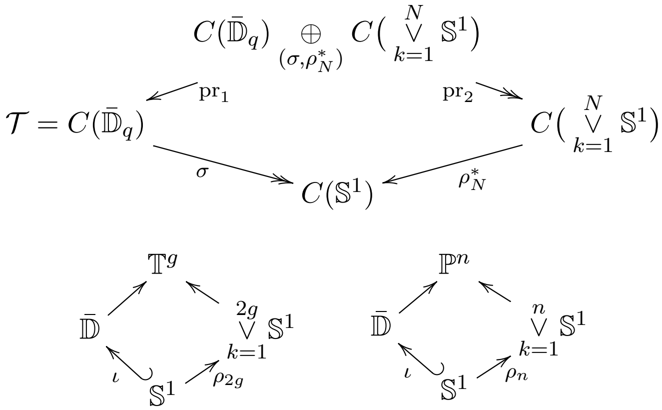
9. Noncommutative CW complexes

- **Pullback C*-algebras:**

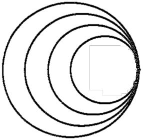
$$
\begin{array}{ccccc}
 & & \mathcal{B} \underset{(\sigma,\tau_N)}{\oplus} C\Big(\bigvee_{k=1}^{N} \mathbb{S}^1\Big) & & \\
 & \swarrow^{\mathrm{pr}_1} & & \searrow^{\mathrm{pr}_2} & \\
\mathcal{B} & & & & C\Big(\bigvee_{k=1}^{N} \mathbb{S}^1\Big) \\
 & \searrow_{\sigma} & & \swarrow_{\tau_N} & \\
 & & \mathcal{B}/\mathcal{K} & &
\end{array}
$$

$$
\begin{array}{ccccc}
 & & C(\bar{\mathbb{D}}_q) \underset{(\sigma,\rho_N^*)}{\oplus} C\Big(\bigvee_{k=1}^{N} \mathbb{S}^1\Big) & & \\
 & \swarrow^{\mathrm{pr}_1} & & \searrow^{\mathrm{pr}_2} & \\
\mathcal{T} = C(\bar{\mathbb{D}}_q) & & & & C\Big(\bigvee_{k=1}^{N} \mathbb{S}^1\Big) \\
 & \searrow_{\sigma} & & \swarrow_{\rho_N^*} & \\
 & & C(\mathbb{S}^1) & &
\end{array}
$$

- **Classical interpretation:**



11. Isomorphism classes: BDF theory

- **Hawaiian Earing:**  $=: X_N \subset \mathbb{C}, \quad z(x+\mathrm{i}y) = x+\mathrm{i}y$

$$\varphi_N : \bigvee_{k=1}^{N} \mathbb{S}^1 \xrightarrow{\cong} X_N := \bigcup_{k=1}^{N} \mathbb{S}^1_{\frac{k+1}{k}}(-\tfrac{1}{k}) = \bigcup_{k=1}^{N} \{x \in \mathbb{C} : |x + \tfrac{1}{k}| = \tfrac{k+1}{k}\}.$$

$\Rightarrow$ $\zeta_N := (\varphi_N \circ \rho_N)^* z : \mathbb{S}^1 \to X_N \subset \mathbb{C}$ separates the points of $\bigvee_{k=1}^{N} \mathbb{S}^1$

$\Rightarrow$ $T_{\widehat{\zeta_N}} \cup \mathcal{K}$ generate $C(\mathbb{T}_q^{N/2})$ resp. $C(\mathbb{P}^N_{k,q})$, $\widehat{\zeta_N}(r\mathrm{e}^{\mathrm{i}\theta}) := r\,\zeta_N(\mathrm{e}^{\mathrm{i}\theta})$

- $[T_{\widehat{\zeta_N}}, T^*_{\widehat{\zeta_N}}] \in \mathcal{K}$

$\Rightarrow$ $T_{\widehat{\zeta_N}}$ is essentially normal operator with $\operatorname{ess\,spec}(T_{\widehat{\zeta_N}}) = X_N$

$\Rightarrow$ BDF-Theory: Group of equivalence classes of C*-algebra extensions

$\Rightarrow$ Classification by $KK_1(C(X_N), \mathbb{C}) \cong K^1(C(X_N)) \cong K_1(C(X_N))$

Ismorphism: $\alpha : C(\mathbb{M}_q) \to C(\mathbb{M}'_q)$, $\mathbb{M}_q, \mathbb{M}'_q \in \{\mathbb{T}^g_q, \mathbb{P}^n_{k,q} : g, n, k \in \mathbb{N}\}$

$\Rightarrow$ $\alpha : \mathcal{K} \longrightarrow \mathcal{K}$ isomorphism

$\Rightarrow$ $\exists\, U_\alpha \in \mathcal{B},\ U^*_\alpha U_\alpha = \mathrm{Id} = U_\alpha U^*_\alpha$ such that $\alpha(t) = U_\alpha t U^*_\alpha$

$\Rightarrow$ $T_{\widehat{\zeta_N}}$ and $U_\alpha T_{\widehat{\zeta_N}} U^*_\alpha$ have the same essential spectrum $0 \notin X_n \subset \mathbb{C}$

$\Rightarrow$ Classification by $K_1(X_N) = K^1(C(X_N)) \cong \mathbb{Z}^N$ (BDF Theory)

$\Rightarrow$ Classification by $\mathrm{ind}(T_{\widehat{\zeta_N}} - \lambda \mathrm{Id}) = \mathrm{wind}(\zeta_N - \lambda)$

- $\bar{\mathbb{D}}/\!\sim\ \cong\ \mathbb{T}^g\ \Rightarrow\ \mathrm{wind}(\zeta_N - \lambda) = (0, \dots, 0)\ \in\ \mathbb{Z}^{2g}$
- $\bar{\mathbb{D}}/\!\sim\ \cong\ \mathbb{P}^n_k\ \Rightarrow\ \mathrm{wind}(\zeta_N - \lambda) = (2, \dots, 2, 0, \dots, 0)\ \in\ \mathbb{Z}^k \oplus \mathbb{Z}^{n-k}$

Theorem: Let $a_1, \ldots, a_{2g}, a_1^{-1}, \ldots, a_{2g}^{-1}$ be an assignment of arcs such that $\bar{\mathbb{D}}/\!\sim\; \cong\; \mathbb{T}^g$. Then $C(\mathbb{T}_q^g)$ is isomorphic to the C*-algebra generated by

$$T_g := \bigoplus_{j=1}^{2g} \left(\tfrac{j+1}{j} U - \tfrac{1}{j}\right) \text{ on } \mathcal{H} := \bigoplus_{j=1}^{2g} \ell_2(\mathbb{Z}), \quad U e_k = e_{k+1}$$

Let $a_1, \ldots, a_k, b_1, \ldots, b_k, a_{k+1}, \ldots, a_n, a_{k+1}^{-1}, \ldots, a_n^{-1}$ be an assignment of arcs such that $\bar{\mathbb{D}}/\!\sim\; \cong\; \mathbb{P}_k^n$. Then $C(\mathbb{P}_{k,q}^n)$ is isomorphic to the C*-algebra generated by

$$T_{n,k} := \bigoplus_{j=1}^{k} \left(\tfrac{j+1}{j} S^2 - \tfrac{1}{j}\right) \oplus \bigoplus_{j=k+1}^{n} \left(\tfrac{j+1}{j} U - \tfrac{1}{j}\right) \text{ on } \mathcal{H} := \bigoplus_{j=1}^{k} \ell_2(\mathbb{N}) \oplus \bigoplus_{j=k+1}^{n} \ell_2(\mathbb{Z}).$$

Proof: $\bar{\mathbb{D}}/\!\sim\; \cong\; \mathbb{T}^g \;\Rightarrow\; \mathrm{wind}(\zeta_{2g}) = (0, \ldots, 0) \in \mathbb{Z}^{2g}$

$\bar{\mathbb{D}}/\!\sim\; \cong\; \mathbb{P}_k^n \;\Rightarrow\; \mathrm{wind}(\zeta_n) = (2, \ldots, 2, 0, \ldots, 0) \in \mathbb{Z}^k \oplus \mathbb{Z}^{n-k}$

up to orden and orientation of circles. $\square$

- **C*-algebra extensions:**

$$0 \longrightarrow \mathcal{K}(\ell_2(\mathbb{N})) \longrightarrow C(\mathbb{T}_q^g) \xrightarrow{\sigma} C(\mathbb{S}^1 \vee \ldots \vee \mathbb{S}^1) \longrightarrow 0$$

- **Six-term exact sequence:**

$$\begin{array}{ccccc} K_0(\mathcal{K}(\ell_2(\mathbb{N}))) & \longrightarrow & K_0(C(\mathbb{T}_q^g)) & \longrightarrow & K_0(C(\bigvee_{k=1}^{2g} \mathbb{S}^1)) \\ \text{ind} \uparrow & & & & \downarrow \exp \\ K_1(C(\bigvee_{k=1}^{2g} \mathbb{S}^1)) & \longleftarrow & K_1(C(\mathbb{T}_q^g)) & \longleftarrow & K_1(\mathcal{K}(\ell_2(\mathbb{N}))) \end{array}$$

$$K_*(C(\mathbb{S}^1 \vee \ldots \vee \mathbb{S}^1)) = K_*(C_0((0,1)) \oplus \ldots \oplus C_0((0,1)) \dotplus \mathbb{C}\mathbf{1})$$

$$\Rightarrow \quad K_0(C(\bigvee_{k=1}^{2g} \mathbb{S}^1)) = \mathbb{Z}, \qquad K_1(C(\bigvee_{k=1}^{2g} \mathbb{S}^1)) = \mathbb{Z}^{2g}$$

- **Index map:** $\text{ind} : K_1(C(\bigvee_{k=1}^{2g} \mathbb{S}^1)) \subset \frac{\mathcal{B}}{\mathcal{K}} \longrightarrow \mathbb{Z} \cong K_0(\mathcal{K})$

$$K_1(C(\bigvee_{k=1}^{2g} \mathbb{S}^1)) \ni [v] \mapsto T_{rv} \in C(\mathbb{T}_q^g) \mapsto \text{ind}(T_{rv}) = -\text{wind}(v) = 0 \in \mathbb{Z}$$

15. K-groups of closed orientable quantum surfaces

- **Six-term exact sequence:**

$$
\begin{array}{ccccc}
\mathbb{Z} & \hookrightarrow & K_0(C(\mathbb{T}^g_q)) & \xrightarrow{[1]\mapsto[1]} & \mathbb{Z} \\
\uparrow {\scriptstyle 0} & & & & \downarrow {\scriptstyle \mathbf{0}} \\
\mathbb{Z}^{2g} & \xleftarrow{\cong} & K_1(C(\mathbb{T}^g_q)) & \longleftarrow & 0
\end{array}
$$

$$\Rightarrow K_0(C(\mathbb{T}^g_q)) \cong \mathbb{Z}^2 \cong K_0(C(\mathbb{T}^g)), \quad K_1(C(\mathbb{T}^g_q)) \cong \mathbb{Z}^{2g} \cong K_1(C(\mathbb{T}^g))$$

- **Generalized Bott projections:** $u \in C(\mathbb{S}^1)$ unitary generator, $n \in \mathbb{Z}$

$$P_n := \begin{pmatrix} |T^*_{ru^n}|^2 & T_{ru^n}\sqrt{1-|T_{ru^n}|^2} \\ \sqrt{1-|T_{ru^n}|^2}\,T^*_{ru^n} & 1-|T_{ru^n}|^2 \end{pmatrix}$$

$$\Rightarrow \ [P_n] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \in K_0(\mathcal{K}) \ \hookrightarrow \ K_0(C(\mathbb{T}^g_q))$$

16. K-groups of closed non-orientable quantum surfaces

- **Six-term exact sequence:**

$$\begin{array}{ccccc}
2j_1 + \cdots + 2j_k \in \mathbb{Z} & \longrightarrow & K_0(C(\mathbb{T}_q^g)) & \xrightarrow{[1]\mapsto[1]} & \mathbb{Z} \\
\uparrow {\scriptstyle \mathrm{ind} \neq 0} & & & & \downarrow {\scriptstyle \mathbf{0}} \\
(j_1, \ldots, j_k, j_{k+1}, \ldots, j_n) \in \mathbb{Z}^n & \hookleftarrow & K_1(C(\mathbb{T}_q^g)) & \longleftarrow & 0
\end{array}$$

$\Rightarrow K_0(C(\mathbb{P}^n_{k,q})) \cong \mathbb{Z}_2 \oplus \mathbb{Z} \cong K_0(C(\mathbb{P}^n)),\ \ K_1(C(\mathbb{P}^n_{k,q})) \cong \mathbb{Z}^{n-1} \cong K_1(C(\mathbb{P}^n))$

- **Generalized Bott projections:**

$$P_2 := \begin{pmatrix} |T^*_{ru^2}|^2 & T_{ru^2}\sqrt{1-|T_{ru^2}|^2} \\ \sqrt{1-|T_{ru^2}|^2}\,T^*_{ru^2} & 1-|T_{ru^2}|^2 \end{pmatrix}$$

$$\Rightarrow\ [P_2] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = 2\Big([P_1] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]\Big) = \mathbf{0} \in K_0(\mathcal{K})$$

17. Spectral Sequences: Glueing maps

$$C(X^1) \cong \bigoplus_{k=1}^{N_1} C([0,1]) \underset{(\sigma_1,\rho_1)}{\oplus} \mathbb{C}(X^0)$$

$$\begin{array}{ccccc}
\bigoplus_{k=1}^{N_1} C([0,1]) & \xleftarrow{\mathrm{pr}_1^1} & & \xrightarrow{\mathrm{pr}_2^1} & C(X^0) \cong \mathbb{C}(\{\mathrm{pt}\}) \\
& \searrow^{\sigma_1(f)=f\restriction_{\partial[0,1]}} & & \swarrow^{\rho_1(\alpha)=(\alpha,\ldots,\alpha)} & \\
& & \bigoplus_{k=1}^{N_1} \big(\mathbb{C}\{0\} \oplus \mathbb{C}\{1\}\big) & &
\end{array}$$

$$C(X_q^2) \cong \bigoplus_{k=1}^{N_2} C(\bar{\mathbb{D}}_q) \underset{(\sigma_2,\rho_2)}{\oplus} C(X^1)$$

$$\begin{array}{ccccc}
\bigoplus_{k=1}^{N_2} C(\bar{\mathbb{D}}_q) & \xleftarrow{\mathrm{pr}_1^2} & & \xrightarrow{\mathrm{pr}_2^2} & C(X^1) \cong C\big(\bigvee_{k=1}^{N_2} \mathbb{S}^1\big) \\
& \searrow^{\sigma_2=\sigma} & & \swarrow^{\rho_2} & \\
& & \bigoplus_{k=1}^{N_2} C(\partial\bar{\mathbb{D}}_q) \cong C(\mathbb{S}^1) & &
\end{array}$$

$$C(X_q^2) \overset{\mathrm{pr}_2^2}{\twoheadrightarrow} C(X_q^1) \overset{\mathrm{pr}_2^1}{\twoheadrightarrow} C(X_q^0) \overset{\mathrm{pr}_2^0}{\twoheadrightarrow} \mathbf{0}$$

18. Spectral Sequences: Filtration

$$C(X_q^2) \overset{\mathrm{pr}_2^2}{\twoheadrightarrow} C(X_q^1) \overset{\mathrm{pr}_2^1}{\twoheadrightarrow} C(X_q^0) \overset{\mathrm{pr}_2^0}{\twoheadrightarrow} \mathbf{0}$$

- **Filtration:** $A_{n-k} := \ker(\mathrm{pr}_2^k \circ \ldots \circ \mathrm{pr}_2^n), \quad A_{-1} := \{0\}$

$\Rightarrow \{0\} = A_{-1} \subset A_0 \subset A_1 \subset A_2 = C(X^2).$

$\Rightarrow \{0\} = C_0(X_q^2 \setminus X_q^2) \subset C_0(X_q^2 \setminus X_q^1) \subset C_0(X_q^2 \setminus X_q^0) \subset C(X_q^2)$

$\Rightarrow \frac{A_0}{A_{-1}} = \bigoplus_{j=1}^{n_2} C_0(\mathbb{D}^2), \ \frac{A_1}{A_0} = \bigoplus_{j=1}^{n_1} C_0((0,1)), \ \frac{A_2}{A_1} = \bigoplus_{j=1}^{n_0} \mathbb{C}(\{\mathrm{pt}_j\})$

$\Rightarrow K_1(\frac{A_0}{A_{-1}}) = 0, \qquad K_0(\frac{A_1}{A_0}) = 0, \qquad K_1(\frac{A_2}{A_1}) = 0$

$K_0(\frac{A_0}{A_{-1}}) = \mathbb{Z}^{n_2}, \ K_1(\frac{A_1}{A_0}) = \mathbb{Z}^{n_1}, \ K_0(\frac{A_2}{A_1}) = \mathbb{Z}^{n_0}$

$\Rightarrow$ Schochet Spectral Sequence with E^2_{pq} stationary and $d_1 = \pi \circ \partial \cong \rho_{2*}$