Principal differential calculi on quantum flag manifolds

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Based on [**Aschieri-Fioresi-Latini**-TW '21]

Plan of talk: 1. Noncommutative differential geometry

(H, Δ, ϵ, S) Hopf algebra
A right H-comodule algebra

"Problem": A differential calculus $\Omega^\bullet(A)$ on A is not unique! Which one to choose?

1st main question: Given a (faithfully flat) Hopf-Galois extension $B := A^{\mathrm{co}H} \subseteq A$ can we find noncommutative differential calculi $\Omega^\bullet(A)$, $\Omega^\bullet(H)$ such that

$$\Omega^\bullet(B) = \Omega^\bullet(A)^{\mathrm{co}\Omega^\bullet(H)} \subseteq \Omega^\bullet(A)$$

is a Hopf-Galois extension of graded algebras?

We give conditions for this to holds as first order differential calculi:
$\rightsquigarrow$ Principal covariant calculi

$$0 \to \underbrace{A \otimes_B \Omega^1(B)}_{\cong \Omega^1_{\mathrm{hor}}(A)} \to \Omega^1(A) \xrightarrow{\mathrm{ver}} \underbrace{A \square_H \Omega^1(H)}_{=\Omega^1_{\mathrm{ver}}(A)} \to 0$$

Plan of talk: 2. Quantum principal bundles and sheaves

In noncommutative differential geometry there is (usually) no underlying topological space. Algebras and modules are understood as global functions or sections.

2nd main question: How to quantize a projective space, like $\mathbb{C}\mathbf{P}^1 = U_{\text{north}} \cup U_{\text{south}}$, which has only trivial global functions but non-trivial local data?

Quantum ringed space $(\mathbb{C}\mathbf{P}^1, \mathcal{O}_{\mathbb{C}\mathbf{P}^1})$ with $\mathcal{O}_{\mathbb{C}\mathbf{P}^1}(\mathbb{C}\mathbf{P}^1) = \mathbb{C}$, $\mathcal{O}_{\mathbb{C}\mathbf{P}^1}(U_1) = \mathbb{C}[x_1/x_0]$, $\mathcal{O}_{\mathbb{C}\mathbf{P}^1}(U_2) = \mathbb{C}[x_0/x_1]$, $\mathcal{O}_{\mathbb{C}\mathbf{P}^1}(U_1 \cap U_2) = \mathbb{C}[x_0/x_1, x_1/x_0]$.

- Study sheaves $\mathcal{F}$ of H-comodule algebras which are locally Hopf-Galois extensions.
- Define differential calculi on $\mathcal{F}$ as sheaves which are locally (principal covariant) calculi.
- Explicit construction based on $\mathcal{O}_q(G)$ with structure Hopf algebra $\mathcal{O}_q(P)$ for G complex semisimple Lie group and P a parabolic subgroup.
  $\rightsquigarrow$ Ore extension!
- Examples feature graph algebras, like $\mathcal{O}_q(\mathrm{SU}_2)$, $\mathcal{O}_q(\mathbb{S}^2)$, $\mathcal{O}(\mathbb{C}\mathbf{P}^1)$,...

  Open questions: Are differential calculi on graph algebras determined by graphs? Is the theory of QPBs (sometimes) better understood in graph language?

Hopf-Galois Extensions

Let $\Bbbk$ be a field.

(H, Δ, ϵ, S) Hopf algebra with coproduct $\Delta\colon H \to H \otimes H$, counit $\epsilon\colon H \to \Bbbk$ and antipode $S\colon H \to H$.
Sweedler's notation $\Delta(h) = h_1 \otimes h_2$.

(A, δ_A) right H-comodule algebra with coaction $\delta_A\colon A \to A \otimes H$.
In particular δ_A is algebra morphism $\delta_A(aa') = \delta_A(a)\delta_A(a')$, $\delta_A(1_A) = 1_A \otimes 1_H$.
Sweedler's notation $\delta_A(a) = a_0 \otimes a_1$.

$B := A^{\mathrm{co}H} := \{a \in A \mid \delta_A(a) = a \otimes 1\}$

Definition (Kreimer-Takeuchi '81)

$B \subseteq A$ is called a Hopf-Galois extension if the canonical map

$$\chi\colon A \otimes_B A \to A \otimes H, \qquad a \otimes_B a' \mapsto aa'_0 \otimes a'_1$$

is a bijection.

Example

i.) If $A = H$ then $\Bbbk = A^{\mathrm{co}H} \subseteq A$ is Hopf-Galois extension with $\chi^{-1}(h \otimes h') = hS(h'_1) \otimes h'_2$.

ii.) $\pi\colon P \to M$ principal G-bundle, $A = \mathcal{C}^\infty(P)$, $H = \mathcal{C}^\infty(G)$.
Right G-action $r\colon P \times G \to P$ induces right coaction $\delta_A := r^*\colon A \to A \otimes H$.
$B := A^{\mathrm{co}H} = \mathcal{C}^\infty(P/G) = \mathcal{C}^\infty(M)$
$\phi\colon P \times G \to P \times_M P$, $(p, g) \mapsto (p, r(p, g))$ induces $\chi := \phi^*$ and χ is bijection if r is free and transitive on fibers.

iii.) $A = \mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ free algebra generated by $\alpha, \beta, \gamma, \delta$ modulo

$$\begin{aligned} \alpha\beta =& q^{-1}\beta\alpha, \quad \alpha\gamma = q^{-1}\gamma\alpha, \quad \beta\delta = q^{-1}\delta\beta, \quad \gamma\delta = q^{-1}\delta\gamma, \\ \beta\gamma =& \gamma\beta, \quad \alpha\delta - \delta\alpha = (q^{-1} - q)\beta\gamma, \quad \alpha\delta - q^{-1}\beta\gamma = 1 \end{aligned}$$

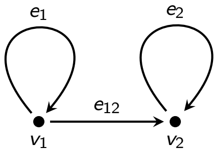
is Hopf algebra with $\Delta \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.
Hopf algebra quotient $\pi\colon A \to H = \mathcal{O}(U(1))$, $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$.
Then A is a right H-comodule algebra with $\delta_A = (\mathrm{id} \otimes \pi) \circ \Delta\colon A \to A \otimes H$ and $B = A^{\mathrm{co}H} = \mathcal{O}_q(\mathbb{S}^2)$ is the Podleś sphere.

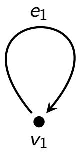
One can show that $B \subseteq A$ is faithfully flat Hopf-Galois extension.

A real form of $\mathrm{SL}_q(2,\mathbb{C})$ is $\mathrm{SU}_q(2)$ and the QPB

$$\mathcal{O}_q(\mathbb{S}^2) = \mathcal{O}_q(\mathrm{SU}(2))^{\mathrm{co}\,\mathcal{O}(U(1))}$$

corresponds to the graph algebras





Principal comodule algebras

Definition

(A, δ_A) is called a principal comodule algebra if

i.) $B := A^{\mathrm{co}H} \subseteq A$ is a Hopf-Galois extension and

ii.) A is a faithfully flat right (or left) B-module, i.e. ${}_B\mathcal{M} \to {}_A\mathcal{M}^H$, $N \mapsto A \otimes_B N$ is an exact functor which reflects exactness.

Functor of coinvariants ${}_A\mathcal{M}^H \to {}_B\mathcal{M}$, $M \mapsto M^{\mathrm{co}H}$.

Theorem (Schneider '90)

The following are equivalent.

i.) (A, δ_A) *is a principal comodule algebra.*

ii.) ${}_B\mathcal{M} \cong {}_A\mathcal{M}^H$ *are equivalent categories, namely*

$$(A \otimes_B N)^{\mathrm{co}H} \cong N, \qquad A \otimes_B M^{\mathrm{co}H} \cong M.$$

Example

i.) $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ with Hopf algebra $H = \mathcal{O}(U(1))$ as before.

ii.) Every crossed product algebra $B\#_\sigma H$, where $\sigma \colon H \otimes H \to B$ is a 2-cocycle with values in B.

First Order Differential Calculi

(A, δ_A) right H-comodule algebra.

Definition (Woronowicz '89)

We call (Γ, d) a first order differential calculus (FODC) on A, if

1. Γ is A-bimodule.
2. $\mathrm{d}\colon A \to \Gamma$ is $\Bbbk$-linear s.t.
   $$\mathrm{d}(ab) = \mathrm{d}(a)b + a\mathrm{d}(b) \qquad \text{(Leibniz rule)}$$
   holds for all $a, b \in A$.
3. $\Gamma = A\mathrm{d}A := \mathrm{span}_{\Bbbk}\{a\mathrm{d}(b) \mid a, b \in A\}$. (Surjectivity)

We call a FODC (Γ, d) on A right H-covariant if $\Gamma \in {}_A\mathcal{M}_A^H$ and d is right H-colinear.

Example

i.) $A = \mathscr{C}^\infty(M)$, $\Gamma = \Gamma^\infty(T^*M)$, $\mathrm{d}\colon A \to \Gamma$ de Rham differential
$\mathrm{d}f|_U = \frac{\partial f}{\partial x^i}\mathrm{d}x^i$ in local chart (U, x). Coaction dual to Lie group action $G \circlearrowright M$.

ii.) The universal FODC $\Gamma_u = \ker \mu_A$, $\mathrm{d}_u(a) = 1 \otimes a - a \otimes 1$.
Every FODC on A is a quotient of (Γ_u, d_u).

Base forms and horizontal forms

(A, δ_A) principal comodule algebra, i.e. faithfully flat H-Galois extension of $B = A^{\mathrm{co}H}$.
(Γ, d) right H-covariant FODC on A, in particular, for $a, a' \in A$, $\omega \in \Gamma$

$$\Delta_\Gamma(a \cdot \omega \cdot a') = \delta_A(a)\Delta_\Gamma(\omega)\delta_A(a').$$

Base forms: $\Gamma_B := B\mathrm{d}B \subseteq \Gamma$ with differential $\mathrm{d}_B := \mathrm{d}|_B \colon B \to \Gamma_B$

Horizontal forms: $\Gamma^{\mathrm{hor}} := A\Gamma_B$

Proposition

$\Gamma_B = \Gamma^{\mathrm{hor}} \cap \Gamma^{\mathrm{co}H}$

Proof.

Clearly $\Gamma_B \subseteq \Gamma^{\mathrm{co}H}$ and thus, by the flatness of A,

$$A \otimes_B \Gamma_B \to A \otimes_B \Gamma^{\mathrm{co}H} \cong \Gamma$$

is an injection $\implies$ $A \otimes_B \Gamma_B \cong A\Gamma_B$. Then

$$\Gamma_B \cong (A \otimes_B \Gamma_B)^{\mathrm{co}H} \cong (A\Gamma_B)^{\mathrm{co}H} = \Gamma^{\mathrm{hor}} \cap \Gamma^{\mathrm{co}H}.$$

□

Principal calculi and vertical forms

(A, δ_A) principal comodule algebra.

Definition

A right H-covariant FODC (Γ, d) on A and a bicovariant FODC (Γ_H, d_H) on H are called a principal covariant calculus if ver is well-defined and we have exact sequence

$$0 \to A \otimes_B \Gamma_B \to \Gamma \xrightarrow{\mathrm{ver}} A \square_H \Gamma_H \to 0.$$

Above $A \square_H \Gamma_H := \mathrm{span}_{\Bbbk}\{a \otimes \omega_H \in A \otimes \Gamma_H \mid \delta_A(a) \otimes \omega_H = a \otimes {}_{\Gamma_H}\Delta(\omega_H)\}$ is the cotensor product over H and

$$\mathrm{ver} \colon \Gamma \to A \square_H \Gamma_H, \qquad a\mathrm{d}a' \mapsto a_0 a'_0 \otimes a_1 \mathrm{d}_H a'_1$$

the vertical map. Warning: ver might not be well-defined!

Example

Consider the principal comodule algebra $B := \mathcal{O}_q(\mathbb{S}^2) \subseteq A := \mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ with structure Hopf algebra $H = \mathcal{O}(U(1))$.

- The 3-dim. right covariant FODC on A is a principal covariant calculus.
- The 4-dim. bicovariant FODC on A is not a principal covariant calculus.

Example (The q-monopole bundle)

Consider the principal comodule algebra $B := \mathcal{O}_q(\mathbb{S}^2) \subseteq A := \mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ with structure Hopf algebra $H = U(1)$. We define Γ as the free left A-module with basis

$$e^- = \delta \mathrm{d}\beta - q^{-1}\beta \mathrm{d}\delta, \quad e^+ = q\alpha \mathrm{d}\gamma - q^2\gamma \mathrm{d}\alpha, \quad e^0 = \delta \mathrm{d}\alpha - q^{-1}\beta \mathrm{d}\gamma$$

with commutation relations

$$e^{\pm} f = q^{|f|} f e^{\pm}, \qquad e^0 f = q^{2|f|} f e^0,$$

where $f \in \{\alpha, \beta, \gamma, \delta\}$ and $|\alpha| = |\gamma| = -1$, $|\beta| = |\delta| = 1$ and differential

$$\mathrm{d}\alpha = \alpha e^0 + q^{-1}\beta e^+, \quad \mathrm{d}\beta = \alpha e^- - q^2\beta e^0,$$
$$\mathrm{d}\gamma = \gamma e^0 + q^{-1}\delta e^+, \quad \mathrm{d}\delta = \gamma e^- - q^2\delta e^0.$$

(Γ, d) is right H-covariant. On B we induce a FODC (Γ_B, d_B) via the injection $\iota \colon B \to A$.

On H we induce a bicovariant calculus (Γ_H, d_H) via the projection $\pi \colon A \to H$. Then, the vertical map

$$\mathrm{ver} \colon \Gamma \to A \square \Gamma_H, \qquad \mathrm{ver}(\omega) = \omega_{-1} \otimes [\omega_0]$$

is well-defined and $0 \to A \otimes_B \Gamma_B \to \Gamma \xrightarrow{\mathrm{ver}} A \square_H \Gamma_H \to 0$ is exact. Thus, (Γ, d) is a principal covariant calculus.

Remark

Exactness of $0 \to A \otimes_B \Gamma_B \to \Gamma \xrightarrow{\mathrm{ver}} A \square_H \Gamma_H \to 0$ *is equivalent to the exactness of* $0 \to A\Gamma_B \to \Gamma \xrightarrow{\mathrm{ver}} A \otimes {}^{\mathrm{co}H}\Gamma_H \to 0$ *(= strong quantum principal bundle à la Brzeziński, Majid, Hajac).*

Theorem (Aschieri-Fioresi-Latini-TW '21)

For any principal covariant calculus (Γ, d) *on* A *with bicovariant FODC* (Γ_H, d_H) *on* H *we have a faithfully flat Hopf-Galois extension*

$$\Omega_B^{\leqslant 1} = \left(\Omega_A^{\leqslant 1}\right)^{\mathrm{co}\Omega_H^{\leqslant 1}} \subseteq \Omega_A^{\leqslant 1}$$

such that δ_A *is differentiable, i.e. such that the diagram commutes*

$$\begin{array}{ccc} \Gamma & \xrightarrow{\delta_A^1} & \Gamma \otimes H \oplus A \otimes \Gamma_H \\ \uparrow{\scriptstyle \mathrm{d}} & & \uparrow{\scriptstyle \mathrm{d}\otimes\mathrm{id}_H + \mathrm{id}_A \otimes \mathrm{d}_H} \\ A & \xrightarrow{\delta_A} & A \otimes H \end{array}$$

On the other hand, if the above is a faithfully flat Hopf-Galois extension and the diagram commutes (Γ, d) *is a principal covariant calculus.*

For the proof we use this lemma.

Lemma

Let (Γ, d) *principal covariant calculus on* (A, δ_A)
(Γ_H, d_H) *the corresponding bicovariant FODC on* H. *Then*

i.) $\Omega_H^{\leqslant 1} = H \oplus \Gamma_H$ *is a graded Hopf algebra with*

$$\Delta^1 = \Delta_{\Gamma_H} + {}_{\Gamma_H}\Delta \colon \Gamma_H \to \Gamma_H \otimes H \oplus H \otimes \Gamma_H$$

and $S^1 \colon \Gamma_H \to \Gamma_H$, $\omega \mapsto -S(\omega_{-1})\omega_0 S(\omega_1)$.

ii.) $\Omega_A^{\leqslant 1} = A \oplus \Gamma$ *is a graded right* $\Omega_H^{\leqslant 1}$*-comodule algebra with*

$$\delta_A^1 = \Delta_\Gamma + \mathrm{ver} \colon \Gamma \to \Gamma \otimes H \oplus A \otimes \Gamma_H.$$

iii.) $\Omega_B^{\leqslant 1} = (\Omega_A^{\leqslant 1})^{\mathrm{co}\Omega_H^{\leqslant 1}}$.

Proof.

For part iii.) we note that $\omega \otimes 1 = \delta_A^1(\omega) := \Delta_\Gamma(\omega) \oplus \mathrm{ver}(\omega)$ if and only if

$$\omega \in \underbrace{\Gamma^{\mathrm{hor}}}_{=\ker \mathrm{ver}} \cap \Gamma^{\mathrm{co}H} = \Gamma_B.$$

□

Quantum principal bundles

M topological space, H Hopf algebra.
$\mathcal{F}$ sheaf of right H-comodule algebras, i.e.

- $U \mapsto \mathcal{F}(U)$ gives a right H-comodule algebra $\forall$ opens U of M, $\mathcal{F}(\emptyset) = \{0\}$
- for $U \subseteq V$ there is a morphism $r_{UV} \colon \mathcal{F}(V) \to \mathcal{F}(U)$ of right H-com. algebras
- compatibility $r_{UV} \circ r_{VW} = r_{UW}$ if $U \subseteq V \subseteq W$ and $r_{UU} = \mathrm{id}$

Moreover, for any U and open cover $\{U_i\}$ of U we have

- if $a \in \mathcal{F}(U)$ s.t. $a|_{U_i} = 0$ for all i then $a = 0$
- if $\exists\ a_i \in \mathcal{F}(U_i)\ \forall\ i$ such that $a_i|_{U_i \cap U_j} = a_j|_{U_i \cap U_j}$ then $\exists\ a \in \mathcal{F}(U)$ s.t. $a|_{U_i} = a_i$

$(M, \mathcal{O}_M)$ quantum ringed space, i.e. sheaf of (noncommutative) algebras.

Definition

We call a sheaf of right H-comodule algebras $\mathcal{F}$ a quantum principal bundle (QPB) over $(M, \mathcal{O}_M)$ if there is an open cover $\{U_i\}$ of M s.t.

- $\mathcal{F}(U_i)^{\mathrm{co}H} = \mathcal{O}_M(U_i)$
- $\mathcal{F}(U_i)$ is a principal comodule algebra, i.e. $\mathcal{O}_M(U_i) \subseteq \mathcal{F}(U_i)$ is faithfully flat Hopf-Galois extension

Example $\mathrm{SL}_q(2)$ over $\mathbb{C}\mathbf{P}^1$

Consider $A := \mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ and $H := \mathcal{O}_q(P) := \mathbb{C}_q[t, t^{-1}, p]/(tp - q^{-1}pt)$ on parabolic subgroup P with Hopf algebra quotient

$$\pi\colon A \to H, \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} t & p \\ 0 & t^{-1} \end{pmatrix}.$$

Consider the topology $\{\emptyset, U_1, U_2, U_{12}, \mathbb{C}\mathbf{P}^1\}$ on $\mathbb{C}\mathbf{P}^1$.
We define the sheaves

$$\mathcal{F}(\emptyset) := \{0\},\ \mathcal{F}(U_1) := A[\alpha^{-1}],\ \mathcal{F}(U_2) := A[\gamma^{-1}],$$
$$\mathcal{F}(U_{12}) := (A[\alpha^{-1}])[\gamma^{-1}],\ \mathcal{F}(\mathbb{C}\mathbf{P}^1) := A$$

of right H-comodule algebras and

$$\mathcal{O}_{\mathbb{C}\mathbf{P}^1}(\emptyset) := \{0\},\ \mathcal{O}_{\mathbb{C}\mathbf{P}^1}(U_1) := \mathbb{C}_q[\alpha^{-1}\gamma] = \mathbb{C}_q[u],$$
$$\mathcal{O}_{\mathbb{C}\mathbf{P}^1}(U_2) := A[\gamma^{-1}\alpha] = \mathbb{C}_q[v],$$
$$\mathcal{O}_{\mathbb{C}\mathbf{P}^1}(U_{12}) := \mathbb{C}_q[u, u^{-1}],\ \mathcal{O}_{\mathbb{C}\mathbf{P}^1}(\mathbb{C}\mathbf{P}^1) := \mathbb{C}_q$$

of algebras with restriction morphism $r_{U_{12}U_2}\colon v \mapsto u^{-1}$.

$\Rightarrow \mathcal{F}$ is QPB over $\mathcal{O}_{\mathbb{C}\mathbf{P}^1}$ with cleaving maps $j_1\colon t^{\pm} \mapsto \alpha^{\pm}$, $p \mapsto \beta$ and $j_2\colon t^{\pm} \mapsto \gamma^{\pm}$, $p \mapsto \delta$.

Consider the following data

- G complex semisimple algebraic group
- P closed algebraic subgroup of G (usually parabolic subgroup)
- $\chi\colon P \to \mathbb{C}^\times$ character of P
- $\mathcal{L}$ line bundle on G/P associated with χ with global sections
$$\mathcal{O}(G/P)_1 = \{f\colon G \to \mathbb{C} \mid f(gh) = \chi^{-1}(h)f(g)\}$$
- $\mathcal{L}$ is ample and gives projective embedding of G/P.

Then $\mathcal{O}(G/P) := \sum_n \mathcal{O}(G/P)_n$ with $\mathcal{O}(G/P)_n = \{f\colon G \to \mathbb{C} \mid f(gh) = \chi^{-n}(h)f(g)\}$.

In Hopf algebra language: χ determines an element $s \in \mathcal{O}(G)$ such that

- $(\mathrm{id} \otimes \pi)\Delta(s) = s \otimes \pi(s)$
- $\pi(s^m) \neq \pi(s^n)$ for all $m \neq n$

where $\pi\colon \mathcal{O}(G) \to \mathcal{O}(P)$ and we obtain

$$\mathcal{O}(G/P)_n = \{f \in \mathcal{O}(G) \mid (\mathrm{id} \otimes \pi)\Delta(f) = f \otimes \pi(s^n)\}.$$

We sometimes call s a classical section of the line bundle $\mathcal{L}$ on G/P.

Theorem (Ciccoli-Fioresi-Gavarini '08)

Given a quantum section of the line bundle $\mathcal{L}$ on G/P, i.e. $s_q \in \mathcal{O}_q(G)$ such that

- $(\mathrm{id} \otimes \pi)\Delta(s_q) = s_q \otimes \pi(s_q)$
- $\lim_{q \to 1} s_q = s$, *the classical section*

then $\mathcal{O}_q(G/P) := \sum_n \mathcal{O}_q(G/P)_n$ *with*

$$\mathcal{O}_q(G/P)_n := \{f \in \mathcal{O}_q(G) \mid (\mathrm{id} \otimes \pi)\Delta(f) = f \otimes \pi(s_q^n)\}$$

is a projective homogeneous quantum variety. If we write $\Delta(s) = s^i \otimes s_i$, *the set* $\{s_i\}$ *determines an open cover* $\{U_i\}$ *of* $M = G/P$.

Define $U_I := U_{i_1} \cap \ldots \cap U_{i_r}$ for $I = (i_1, \ldots, i_r)$.

Theorem (Aschieri-Fioresi-Latini '21)

1. $U_I \mapsto \mathcal{O}_M(U_I) := \mathbb{C}_q[s_{k_1} s_{i_1}^{-1}, \ldots, s_{k_r} s_{i_r}^{-1}]$ *for* $1 \leq k_j \leq n$ *defines a sheaf* $\mathcal{O}_M$ *of algebras on* $M = G/P$.
2. $U_I \mapsto \mathcal{F}_G(U_I) := \mathcal{O}_q(G)\{s_I^r \mid r \leq 0\}$ *defines a sheaf* $\mathcal{F}_G$ *of right H-comodule algebras.*
3. $\mathcal{F}_G^{\mathrm{co}\mathcal{O}_q(P)} \cong \mathcal{O}_M$.

For $G = \mathrm{SL}_{n+1}(\mathbb{C})$ the above gives a QPB over $(\mathbb{C}\mathbf{P}^n, \mathcal{O}_{\mathbb{C}\mathbf{P}^n})$.

Ore Extension of Calculi

Let (A, δ_A) be a right H-comodule algebra and $\alpha \in A$ be an Ore element such that $\delta_A(\alpha) \in A \otimes H$ is invertible.

Then $A[\alpha^{-1}]$ is a right H-comodule algebra with $\delta_{A[\alpha^{-1}]}(\alpha^{-1}) = \delta_A(\alpha)^{-1}$.

Lemma

Consider a right H-covariant FODC (Γ, d) on A and let $\alpha \in A$ be as before. We define the $A[\alpha^{-1}]$-bimodule

$$\Gamma_\alpha := A[\alpha^{-1}]\,\Gamma\,A[\alpha^{-1}] := A[\alpha^{-1}] \otimes_A \Gamma \otimes_A A[\alpha^{-1}]$$

and the $\Bbbk$-linear map

$$\mathrm{d}_\alpha \colon A[\alpha^{-1}] \to \Gamma_\alpha, \qquad \mathrm{d}_\alpha(a) = \begin{cases} \mathrm{d}a & a \in A \\ -\alpha^{-1}\,\mathrm{d}\alpha\,\alpha^{-1} & a = \alpha^{-1} \end{cases}$$

where we extend d_α to $A[\alpha^{-1}]$ by the Leibniz rule.

Then $(\Gamma_\alpha, \mathrm{d}_\alpha)$ is a right H-covariant FODC on $A[\alpha^{-1}]$.

Calculi on Sheaves of Comodule Algebras

Stalk of a sheaf: for $x \in M$

$$\mathcal{F}_x = \{(U, a) \mid x \in U \text{ open and } a \in \mathcal{F}(U)\}/\sim$$

where $(U, a) \sim (V, a')$ iff $\exists W \subseteq U \cap V$ s.t. $a|_W = a'|_W$.

Definition

A right H-covariant FODC on sheaf $\mathcal{F}$ of right H-comodule algebras is a sheaf Υ of right H-covariant $\mathcal{F}$-bimodules together with a morphism $\mathrm{d}\colon \mathcal{F} \to \Upsilon$ of sheaves of right H-comodules, such that on the stalks

1. $\mathrm{d}_x(aa') = \mathrm{d}_x(a)a' + a\mathrm{d}_x(a')$ for all $a, a' \in \mathcal{F}_x$
2. $\Upsilon_x = \mathcal{F}_x \mathrm{d}_x \mathcal{F}_x$

hold for all $x \in M$, where $\mathrm{d}_x \colon \mathcal{F}_x \to \Upsilon_x$ is the induced map on the stalks.

Example

M algebraic variety, G algebraic group acting on M. Then

- the structure sheaf $\mathcal{O}_M$ carries an $H = \mathcal{O}(G)$-action.
- the sheaf Ω of Kähler differentials is a right H-covariant FODC.

Given (Υ, d) right H-covariant FODC on sheaf $\mathcal{F}$ we induce the following sheaves of $\mathcal{F}^{\mathrm{co}H}$-bimodules

- Base forms $\Upsilon_M \colon U \mapsto \Upsilon_M(U) := \mathcal{F}^{\mathrm{co}H}(U)\mathrm{d}_U\mathcal{F}^{\mathrm{co}H}(U) = \mathcal{O}_M(U)\mathrm{d}_U\mathcal{O}_M(U)$
- Horizontal forms $\Upsilon^{\mathrm{hor}} \colon U \mapsto \mathcal{F}(U)\Upsilon_M(U)$
- Coinvariant forms $\Upsilon^{\mathrm{co}H} = \ker \Delta_\Upsilon$, with sheaf morphism $\Delta_\Upsilon \colon \Upsilon \to \Upsilon \otimes H$

Lemma

$(\Upsilon_M, \mathrm{d}_M := \mathrm{d}|_{\mathcal{O}_M})$ *is a FODC on the sheaf* $\mathcal{O}_M$.

As in the affine case one proves...

Theorem (Aschieri-Fioresi-Latini-TW '21)

For any right H-covariant FODC (Υ, d) on a QPB $\mathcal{F}$ we have an isomorphism of sheaves of $\mathcal{O}_M$-bimodules $\Upsilon_M \cong \Upsilon^{\mathrm{hor}} \cap \Upsilon^{\mathrm{co}H}$.

Proof.

$\mathcal{F}(U_i)$ principal comodule algebra for open cover $\{U_i\}$ implies $\mathcal{F}_x$ principal comodule algebra $\forall\ x \in M$. Then use the affine results on stalks! □

We construct a class of examples of calculi on quantum flag manifolds given the following data

- G complex semisimple algebraic group, P parabolic subgroup
- $\mathcal{O}_q(G), \mathcal{O}_q(P)$ Hopf algebra quantizations
- $s_q \in \mathcal{O}_q(G)$ quantum section with corresponding sheaves $\mathcal{F}_G$ and $\mathcal{O}_M$

Theorem (Aschieri-Fioresi-Latini-TW '21)

Let (Γ, d) be a right $\mathcal{O}_q(P)$-covariant FODC on the Hopf algebra $\mathcal{O}_q(G)$. Then

i.) *there is a right $\mathcal{O}_q(P)$-covariant FODC $(\Upsilon_G, \mathrm{d}_G)$ on the sheaf $\mathcal{F}_G$.*

ii) *$(\Upsilon_G, \mathrm{d}_G)$ induces a FODC $(\Upsilon_M, \mathrm{d}_M)$ on the sheaf $\mathcal{O}_M$.*

iii.) *if $\mathcal{F}_G$ is a QPB we have $\Upsilon_M \cong \Upsilon_G^{\mathrm{hor}} \cap \Upsilon_G^{\mathrm{co}\mathcal{O}_q(P)}$.*

Proof.

Recall that the topology of M is generated by a finite open cover. Consider $x \in M$ and the smallest open $U_x := \bigcap_{U_i \ni x} U_i$ containing x. Then $(\mathcal{F}_G)_x = \mathcal{F}(U_x)$. Apply the Ore extension of calculi... □

Definition

Let $\mathcal{F}$ be a QPB over $(M, \mathcal{O}_M)$. We say that a right H-covariant FODC (Υ, d) on $\mathcal{F}$ and a bicovariant FODC (Γ_H, d_H) on H form a principal covariant calculus on $\mathcal{F}$, if there are exact sequences on all stalks, $x \in M$,

$$0 \to \mathcal{F}_x \otimes_{(\mathcal{O}_M)_x} (\Upsilon_M)_x \to \Upsilon_x \xrightarrow{\mathrm{ver}_x} \mathcal{F}_x \square_H \Gamma_H \to 0.$$

Example ($A = \mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$, $H = \mathcal{O}_q(P)$ parabolic subgroup P)

Let (Γ_A, d_A) be the 3-dimensional left covariant FODC on A, consider the quotient calculus (Γ_H, d_H) on H and the left H-covariant FODC $(\Upsilon_{\mathrm{SL}_2}, \mathrm{d}_{\mathrm{SL}_2})$ on $\mathcal{F}_{\mathrm{SL}_2}$. Then

i.) $\Upsilon_{\mathrm{SL}_2}(U_I) = \Gamma_{A_I}$ is a free left $\mathcal{F}_{\mathrm{SL}_2}(U_I) = A_I$-module generated by $\{\omega^0, \omega^1, \omega^2\}$.

ii.) The base forms $(\Upsilon_{\mathbb{C}\mathbf{P}^1}, \mathrm{d}_{\mathbb{C}\mathbf{P}^1})$ are determined by $\Gamma_{B_1} = \mathrm{span}_{B_1}\{\alpha^{-2}\omega^2\}$ and $\Gamma_{B_2} = \mathrm{span}_{B_2}\{\gamma^{-2}\omega^2\}$ as free left modules with commutation relations
$$(\mathrm{d}_1 u)u = q^2 u \mathrm{d}_1 u, \qquad (\mathrm{d}_2 v)v = q^{-2} v \mathrm{d}_2 v,$$
where $u = \gamma\alpha^{-1} \in B_1$ and $v = \alpha\gamma^{-1} \in B_2$.

iii.) $0 \to A_I \otimes_{B_I} \Gamma_{B_I} \to \Gamma_{A_I} \xrightarrow{\mathrm{ver}_I} A_I \square_H \Gamma_H \to 0$ is exact for $I \in \{1, 2, 12\}$.

iv.) $(\Upsilon_{\mathrm{SL}_2}, \mathrm{d}_{\mathrm{SL}_2})$ is not principal covariant calculus, since (Γ_H, d_H) not bicovariant.

Example $\mathcal{O}_q(\mathrm{GL}_2(\mathbb{C}))$ over $\mathbb{C}\mathbf{P}^1$

The Ore extensions of $A = \mathcal{O}_q(\mathrm{GL}_2(\mathbb{C}))$ give rise to a QPB $\mathcal{F}_{\mathrm{GL}_2}$ on $(\mathbb{C}\mathbf{P}^1, \mathcal{O}_{\mathbb{C}\mathbf{P}^1})$:

$$\mathcal{F}_{\mathrm{GL}_2}(\emptyset) = \{0\}, \quad \mathcal{F}_{\mathrm{GL}_2}(U_1) = A[\alpha^{-1}], \quad \mathcal{F}_{\mathrm{GL}_2}(U_1) = A[\gamma^{-1}],$$
$$\mathcal{F}_{\mathrm{GL}_2}(U_1 \cap U_2) = A[\alpha^{-1}, \gamma^{-1}], \quad \mathcal{F}_{\mathrm{GL}_2}(\mathbb{C}\mathbf{P}^1) = A.$$

Example

The Ore extension of the bicovariant FODC $(\Gamma_{\mathrm{GL}_2}, \mathrm{d}_{\mathrm{GL}_2})$ on A is a principal covariant calculus $(\Upsilon_{\mathrm{GL}_2}, \mathrm{d}_{\mathrm{GL}_2})$ on $\mathcal{F}_{\mathrm{GL}_2}$.

Proof.

$(\Gamma_{\mathrm{GL}_2}, \mathrm{d}_{\mathrm{GL}_2})$ is 4-dim. free A-module with basis $\omega^1, \omega^2, \omega^3, \omega^4$.
The quotient calculus (Γ_H, d_H) on $H = A/\langle\gamma\rangle$ is 3-dimensional $[\omega^1], [\omega^3], [\omega^4]$.
$B_1 = \mathcal{F}(U_1)^{\mathrm{co}H} = \mathbb{C}_q[\alpha^{-1}\gamma]$ with 1-dim. calculus generated by

$$\mathrm{d}_1(u) = \mathrm{d}_1(\alpha^{-1}\gamma) = -\alpha^{-2}\omega^2.$$

$\mathrm{ver}_1(\sum_{i=1}^4 a^i\omega^i) = \sum_{i=1}^4 a_0^i \otimes a_1^i[\omega^i]$.
So $0 \to A_I \otimes_{B_I} \Gamma_{B_I} \to \Gamma_{A_I} \xrightarrow{\mathrm{ver}_I} A_I \Box_H \Gamma_H \to 0$ is exact. □

Future directions

- In the affine setting show under which conditions a (faithfully flat) Hopf-Galois extension $B = A^{\mathrm{co}H} \subseteq A$ extends to a (faithfully flat) Hopf-Galois extension of degree $n > 1$

$$\Omega^{\leqslant n}(B) = \Omega^{\leqslant n}(A)^{\mathrm{co}\Omega^{\leqslant n}(H)} \subseteq \Omega^{\leqslant n}(A).$$

  The expectation is that no higher order exact sequences are needed since $\Omega^\bullet$ is "determined" in degree 1.
- Consider the QPB $\mathcal{F}_{\mathrm{SL}_{n+1}}$ over $(\mathbb{C}\mathbf{P}^n, \mathcal{O}_{\mathbb{C}\mathbf{P}^n})$ based on $\mathcal{O}_q(\mathrm{SL}_{n+1}(\mathbb{C}))$.
- ...and higher Grassmannians

Can we extend some of the technology to graph algebras?

- Differential calculi on graph algebras (in terms of graphs)?
- Ore extension of graph algebras?
- Sheaves of graph algebras?

...or maybe we can get some insights on QPBs from the theory of graph algebras.

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Thank you for your attention!