L'évy-Khintchine decomposition for $SU_q(N)$

Anna Wysoczańska-Kula
Uniwersytet Wrocławski

joint work with:
Uwe Franz, Martin Lindsay, Michael Skeide

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The essence of the problem

Let A be a unital $*$-algebra with a character ε.

Definition

A linear mapping $\psi : A \to \mathbb{C}$ is called **generating functional** if

- $\psi(1_A) = 0$,
- $\psi(a^*) = \overline{\psi(a)}$ for $a \in A$,
- $\psi(a^* a) \geq 0$ for $a \in \ker \varepsilon$.

A generating functional ψ is called **Gaussian** if $\psi(abc) = 0$ for any $a, b, c \in \ker \varepsilon$.

Question 1: Whan are all possible generating functionals on a given (A, ε)?

Question 2: Is it always possible to decompose a given generating functional ψ into $\psi = \psi_G + \psi_R$, where ψ_G and ψ_R are generating functionals and ψ_G is maximally Gaussian?

Motivations: Lévy process with values in $\mathbb{R}^n$

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Definition

A family $(X_t)_{t\geq 0}$ of $\mathcal{F}$-mesurable functions $X_t : \Omega \to \mathbb{R}^n$ is called **Lévy process** if

- $X_0 = 0$ P-almost everywhere,
- the increments are stationary: the law of $X_t - X_s$ depends only on $t - s$
- the increments $(X_{t_{j+1}} - X_{t_j})_{j=1,\dots,n}$ are independent whenever $0 \leq t_1 < t_2 < \dots < t_{n+1}$,
- (stochastic continuity) X_t converges in probability to X_0 when $t \searrow 0$, i.e. $\Leftrightarrow P(|X_t| > a) \to 0$ as $t \searrow 0$.

Examples: Gaussian process (Brownian motion), Poisson process, compound Poisson, etc.

Lévy process with values in $\mathbb{R}$: examples

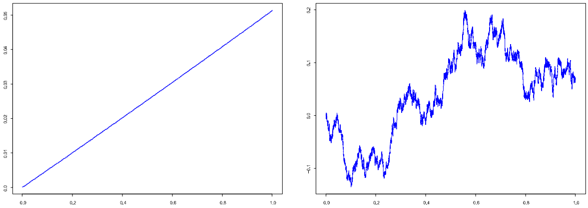

FIGURE 2.4. Examples of Lévy processes: linear drift (left) and Brownian motion.

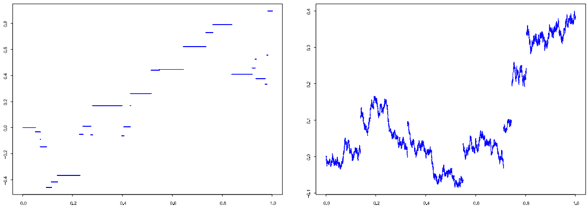

FIGURE 2.5. Examples of Lévy processes: compound Poisson process (left) and Lévy jump-diffusion.

Source: A. Papapantoleon, An Introduction to Lévy Processes with Applications in Finance.

Classification

Lévy-Khintchine Formula (1934/1937)

$X = (X_t)_t$ is a Lévy process on $\mathbb{R}^n$ iff the characteristic function

$$\phi_X(u) := \int_{\mathbb{R}^n} e^{\mathrm{i}\langle u,x\rangle} \mu_{X_1}(dx) = e^{\eta_X(u)},$$

where

$$\eta_X(u) = \mathrm{i}\langle b, u\rangle - \frac{1}{2}\langle u, \sigma u\rangle + \int_{\mathbb{R}^n} (e^{\mathrm{i}\langle u,y\rangle} - 1 - \mathrm{i}\langle u, y\rangle 1_{|y|\leq 1})\nu(dy)$$

for some $b \in \mathbb{R}^n$, $\sigma \in M(n, n)$ positive-definite and a 'Lévy measure' ν on $\mathbb{R}^n$.

Note: η_X is a sum of η_G (for a Gaussian process) and a limit of η_J (for compound Poisson processes).

Generalizations of Lévy process

In order to generalize the notion of Lévy process to processes with values in S, i.e. $(X_t : \Omega \to S)_{t \geq 0}$, we need:

- a neutral element $0 \in S$ (since $X_0 = 0$);
- a composition rule and inverse elements (for the notion of increments $X_t - X_s$ or, multiplicatively, $X_s^{-1} X_t$);

We need a **group structure**!

Classification

Let G be a Lie group, $\mathfrak{g}$ – the related Lie algebra.

- $(X_1, \dots, X_n)$ basis in $\mathfrak{g}$
- $(e_1, \dots, e^n)$ are canonical coordinates in a neighborhood of e,
- $(X_1^L, \dots, X_n^L)$ derivations in the direction related to X_i

Hunt's Formula (1956)

Lévy process on G are in one-to-one correspondence with the generating functionals L of the form

$$Lf(x) = \sum_i b_i X_i^L f(x) + \sum_{i,j} a_{ij} X_i^L X_j^L f(x) + \int_{G\setminus\{e\}} \left[f(xy) - f(x) - \sum_i e^i(x) X_i^L(y) \right] \nu(dy)$$

for some $b \in \mathbb{R}^n$, $a = (a_{ij})_{i,j} \in M_n(\mathbb{R})$ positive definite, symmetric and a Lévy measure ν on $G \setminus \{e\}$. The domain of L contains $C_c^\infty(G)$-functions.

A noncommutative generalization of Lévy processes

Let $(\mathcal{A}, \Delta, \varepsilon)$ be a $*$-bialgebra, $(\mathcal{B}, \Phi)$ – a nc probability space.

Definition (Accardi, Schürmann, von Waldenfels'88)

A **Lévy process on $\mathcal{A}$** is a quantum stochastic process, that is a family of a $*$-homomorphisms $(j_{st} : \mathcal{A} \to \mathcal{B})_{t \geq s \geq 0}$, which satisfies:

- $j_{tt} = \varepsilon 1_{\mathcal{A}}$;
- the increments (j_{st}) are stationary, i.e. $\varphi_{st} = \Phi \circ j_{st} \sim t - s$,
- the increments (j_{st}) are (tensor) independent: for nonoverlapping intervals $I_k = [s_k, t_k]$ (with $j_k := j_{s_k, t_k}$) and all $b_j \in \mathcal{A}$ we have
  (i) $[j_k(b_1), j_l(b_2)] = 0$ for $k \neq l$;
  (ii) $\Phi\big(j_1(b_1) \dots j_n(b_n)\big) = \Phi\big(j_1(b_1)\big) \dots \Phi\big(j_n(b_n)\big)$;
- (increment property) for all $0 \leq r \leq s \leq t$
  $$j_{rs} \star j_{st} = j_{rt},$$
  where $j_1 \star j_2 := m_{\mathcal{B}} \circ (j_1 \otimes j_2) \circ \Delta$;
- (weak continuity) j_{st} converges to j_{ss} in distribution for $t \searrow s$.

Classification

???

Lévy processes and equivalent objects

- Given a Lévy process $(j_{st} : \mathcal{A} \to \mathcal{B})_{0 \leq s < t}$, define

$$\varphi_t = \Phi \circ j_{0t} = \Phi \circ j_{s,s+t}.$$

Then $(\varphi_t)_{t \geq 0}$ is a semigroup of states on $\mathcal{A}$, i.e. for $a \in \mathcal{A}$, $s, t \geq 0$

  - $\varphi_t(a^* a) \geq 0$, $\varphi_t(1_{\mathcal{A}}) = 1$,
  - $\varphi_s \star \varphi_t = \varphi_{s+t}$, $\lim_{t \searrow 0} \varphi_t(a) = \varphi_0(a) = \varepsilon(a)$.

- For the semigroup of states there exists an infinitesimal generator

$$\psi = \frac{d}{dt}\Big|_{t=0} \varphi_t \ .$$

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Theorem

A linear mapping $\psi : \mathcal{A} \to \mathbb{C}$ is an infinitesimal generator of a semigroup of states iff it is a generating fuctional:

$$\bullet\ \psi(1) = 0, \quad \bullet\ \psi(a^*) = \overline{\psi(a)} \quad \bullet\ \psi(a^* a) \ge 0 \ (a \in \ker \varepsilon).$$

For the proof set $\varphi_t = \exp^\star(t\psi) = \sum_{n=0}^{\infty} \frac{t^n \psi^{\star n}}{n!}$.

Lévy processes and equivalent objects

For a generating functional ψ we can get via the GNS-type construction:

Schürmann triples (π, η, ψ)

- $\pi : \mathcal{A} \to L^{\#}(H)$ is a **unital $*$-representation** of $\mathcal{A}$ on some pre-Hilbert space H;
- $\eta : \mathcal{A} \to H$ is a linear mapping satisfying

$$\eta(ab) = \pi(a)\eta(b) + \eta(a)\varepsilon(b)$$

  (**1-π-ε-cocycle**);
- $\psi : \mathcal{A} \to \mathbb{C}$ is a linear hermitian functional satisfying

$$\psi(ab) = \varepsilon(a)\psi(b) + \langle \eta(a^*), \eta(b) \rangle + \psi(a)\varepsilon(b)$$

  $(-\partial\psi = \eta \cup \eta)$.

Set $\langle a, b \rangle_\psi := \psi\big((a - \varepsilon(a)1)^*(b - \varepsilon(b)1)\big)$ and $H := A/\{a : \langle a, a \rangle_\psi = 0\}$. Moreover take $\eta(a) := [a]$ and $\pi(a)\eta(b) := \eta(a(b - \varepsilon(b)))$.

Lévy processes and equivalent objects

We have one-to-one correspondences between the following objects:

Lévy processes $(j_{st})_{t>s\geq 0}$ on $\mathcal{A}$

$\updownarrow$

(Weak-*) cont. convolutions semigroups of states $(\varphi_t)_{t\geq 0}$ on $\mathcal{A}$

$\updownarrow$

Generating functionals $\psi : \mathcal{A} \to \mathbb{C}$

$\updownarrow$

Surjective[1] **Schürmann triples** (π, η, ψ)

[1] i.e. surjective: $\eta(\mathcal{B}) = H$ or $\overline{\eta(\mathcal{B})} = H$

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$$\updownarrow$$

Surjective[1] **Schürmann triples** (π, η, ψ)

Remarks

1. If we want to describe all Lévy processes on a given $\mathcal{A}$, it is enough to describe Schürmann triples on it.
2. For considering generating functionals and Schürmann triples, augmented algebra $(\mathcal{A}, \varepsilon)$ is enough. To recover the Lévy process, the comultiplication is necessary.

[1] i.e. surjective: $\eta(\mathcal{B}) = H$ or $\overline{\eta(\mathcal{B})} = H$

Schürmann triples and classification of GFs

Lévy processes $\leftrightarrow$ **Schürmann triple** (π, η, ψ)

Procedure for describing all LPs

1. Find all $*$-representations π of $\mathcal{A}$ (on some H).
2. Describe all linear mappings η satisfying $\eta(ab) = \pi(a)\eta(b) + \eta(a)\varepsilon(b)$.
3. Check whether for (π, η) there exists ψ such that (π, η, ψ) is a Schürmann triple. If so, we say that η is **completable**.
4. Find all ψ's associated to a given (π, η): if there exists one, then all others are of the form ψ+drift term(*)

(*) A **drift** is a generating functional vanishing on $(\ker \varepsilon)^2$, or equivalently having $\eta = 0$.

Apart from (1), the most difficult part is (3). What can go wrong?

What can go wrong?

- Given a representation π and a cocycle η it is necessary to define

$$\psi(1_{\mathcal{A}}) = 0, \quad \psi(ab) = \langle \eta(a^*), \eta(b) \rangle, \; a, b \in \ker \varepsilon.$$

  Then it it normalized, cond. positive and hermitian on $(\ker \varepsilon)^2$.
- Potential conflicts: $ab = cd \Rightarrow \langle \eta(a^*), \eta(b) \rangle = \langle \eta(c^*), \eta(d) \rangle$

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Example of a pair (π, η) without generating functional

- $\mathcal{A}$: free unital commutative $*$-algebra generated by x,
- counit : $\varepsilon(1) = 1$, $\varepsilon(x) = 0$.

For $z, w \in \mathbb{C}$ set

$$\eta(x) = z, \quad \eta(x^*) = w$$

and extend to a ε-ε-cocyle.
If $|z| \neq |w|$, then (ε, η) does not admit generating functional, since

$$|z|^2 = \psi(x^* x) = \psi(x x^*) = |w|^2.$$

When it can't go wrong?

Inner derivations are always completable

Let π be a representation of $\mathcal{A}$ on $B(H)$ and $h \in H$ a vector. Define

$$\eta_{\pi,h}(a) = \pi(a)h - \varepsilon(a)h, \;\; a \in \mathcal{A}.$$

Then $\eta_{\pi,h}$ is a π-ε-cocycle, which is called a **coboundary** or an **inner derivation**. The associated generating functional is

$$\psi_{\pi,h}(a) = \langle h, (\pi(a) - \varepsilon(a)I)h\rangle.$$

Note that for $a, b \in \ker\varepsilon$ we have $\psi_{\pi,h}(ab) = \langle \eta(a^*), \eta(b)\rangle = \langle h, \pi(ab)h\rangle$.

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Approximately inner cocycles are always completable

A π-ε-cocycle η is called **approximately inner** if it is a a pointwise limit of coboundaries $\eta_{\pi,h(\lambda)}$ for some net $(h(\lambda))_\lambda$. Then any approximately inner cocycle is completable (by the poitwise limit of $\psi_{\pi,h(\lambda)}$).

Lévy-Khintchine decomposition for $*$-bialgebras

Problem (Schürmann 1990)

Does there exists an analogue of Lévy-Khintchine formula, i.e. a decomposition of any generating functional on a Hopf $*$-algebra into "maximal Gaussian" and "purely non-Gaussian" part?

Definition (Schürmann 1990)

A generating functional $\psi : \mathcal{A} \to \mathbb{C}$ is called **Gaussian** if $\psi(abc) = 0$ whenever $a, b, c \in \ker \varepsilon$.

Let ψ have the Schürmann triple (π, η, ψ). Then TFCAE:

- ψ is Gaussian, i.e vanishes on $(\ker \varepsilon)^3$,
- $\eta(ab) = \varepsilon(a)\eta(b) + \eta(a)\varepsilon(b)$.
- $\pi(a) = \varepsilon(a)\mathrm{id}_H$ for all $a \in \mathcal{A}$

Lévy-Khintchine decomposition for $*$-bialgebras

Can we always extract Gaussian part from a generating functional?

Reformulation in terms of Schürmann triple

- If $((H, \pi), \eta, \psi)$ is a Schürmann triple, then

$$H_{\mathrm{G}} := \bigcap_{a \in \ker \varepsilon} \ker \pi(a) = \{u \in H : \pi(a)u = \varepsilon(a)u, a \in \mathcal{A}\}$$

  is the maximal Gaussian subspace[a] of H which is reducing for π. Hence $\pi = \pi_{\mathrm{G}} \oplus \pi_{\mathrm{R}}$.

- Let P_{G} be the orthogonal projection onto H_{G}. Then

$$\eta_{\mathrm{G}} := P_{\mathrm{G}} \circ \eta$$

  is a Gaussian cocycle (with values in H_G)

- Then $\eta_{\mathrm{R}} = (\mathrm{id} - P_{\mathrm{G}}) \circ \eta$ is a cocycle (purely non-Gaussian) and $\eta = \eta_{\mathrm{G}} \oplus \eta_{\mathrm{R}}$.

[a] maximal subspace of H such that $\pi|_{H_G}(a) = \varepsilon(a)\mathrm{id}_{H_G}$

Lévy-Khintchine decomposition for CQG-algebras

Let ψ be a generating functional with the Schürmann triple (π, η, ψ), and let

$$(\pi, \eta) = (\pi_{\mathrm{G}}, \eta_{\mathrm{G}}) \oplus (\pi_{\mathrm{R}}, \eta_{\mathrm{R}}).$$

Definition (Schürmann'90; see also Franz, Gerhold, Thom'15)

We say that $\psi : \mathcal{A} \to \mathbb{C}$ **admits a Lévy-Khintchine decomposition** if there exist generating functionals $\psi_{\mathrm{G}}, \psi_{\mathrm{R}} : \mathcal{A} \to \mathbb{C}$ such that

- $\psi = \psi_{\mathrm{G}} + \psi_{\mathrm{R}}$,
- $(\pi_{\mathrm{G}}, \eta_{\mathrm{G}}, \psi_{\mathrm{G}})$ and $(\pi_{\mathrm{R}}, \eta_{\mathrm{R}}, \psi_{\mathrm{R}})$ are Schürmann triples.

We say that $\mathcal{A}$ **has the property (LK)** if all generating functionals on $\mathcal{A}$ admit Lévy-Khintchine decomposition.

- If exists, ψ_{G} (the maximal Gaussian part) is unique up to a drift.

Lévy-Khintchine decomposition for Hopf $*$-algebras/CQGs

$\mathcal{A}$ has the property (LK) if for any generating functional

$$(\pi, \eta, \psi) = (\pi_{\mathrm{G}}, \eta_{\mathrm{G}}, \psi_{\mathrm{G}}) \oplus (\pi_{\mathrm{R}}, \eta_{\mathrm{R}}, \psi_{\mathrm{R}}).$$

Remark

If we show that one of ψ_x ($x \in \{\mathrm{G}, \mathrm{R}\}$) exists, then the other one can be defined $\psi_y = \psi - \psi_x$.

We say that $\mathcal{A}$ has the property:

- **(AC)** if any cocycle is completable;
- **(GC)** if any Gaussian cocycle is completable;
- **(NC)** if any purely non-Gaussian cocycle is completable.

$$(\mathrm{AC}) \Rightarrow (\mathrm{GC}) \text{ or } (\mathrm{NC}) \Rightarrow (\mathrm{LK})$$

$\mathcal{SU}_q(2)$, $q \in (0,1)$

$SU_q(2)$: the compact quantum groupwith $C(SU_q(2))$ the universal unital C*-algebra generated by α, γ satisfying

$$\alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma,$$
$$\alpha\alpha^* + q^2\gamma\gamma^* = 1, \quad \alpha^*\alpha + \gamma\gamma^* = 1$$

and $\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma$, $\Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$.
(Note that $C(SU_q(2))$ a graph C*-algebra! [Hong, Szymański 2002])

$\mathcal{SU}_q(2) := \mathrm{Pol}(SU_q(2))$ is a $*$-bialgebra, $\varepsilon(\alpha) = 1$ and $\varepsilon(\gamma) = 0$.

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$\mathcal{SU}_q(2) := \mathrm{Pol}(SU_q(2))$ is a $*$-bialgebra, $\varepsilon(\alpha) = 1$ and $\varepsilon(\gamma) = 0$.

Remark

There is a one-parameter family of 1-dim (irreducible) $*$-representations of $\mathcal{SU}_q(2)$ given by

$$\varepsilon_\theta(\alpha) = e^{i\theta}, \quad \varepsilon_\theta(\gamma) = 0, \qquad \theta \in [0, 2\pi).$$

Note that $\varepsilon_0 = \varepsilon$. Moreover, this family is pointwise C^∞ in θ (i.e. the function $\theta \mapsto \varepsilon_\theta(a)$ is C^∞ for any fixed $a \in \mathcal{A}$).

LK-decomposition on $\mathcal{SU}_q(2)$: Gaussian part

Remarks

- The mapping
$$\varepsilon' : \mathcal{A} \ni a \mapsto \frac{d}{d\theta}\varepsilon_\theta(a)\Big|_{\theta=0} \in \mathbb{C}$$
is a ε-ε-cocycle (i.e Gaussian cocycle).
- The mapping
$$\frac{\varepsilon''}{2} : \mathcal{A} \ni a \mapsto \frac{1}{2}\frac{d^2}{d\theta^2}\varepsilon_\theta(a)\Big|_{\theta=0} \in \mathbb{C}$$
is a generating functional associated to ε'.

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Theorem (Schürmann, Skeide'98)

Any gaussian cocyle on $\mathcal{SU}_q(2)$ is of the form $\eta(a) = \varepsilon'(a)h$ with $h \in H$. It admits a generating functional with
$$\psi = r\varepsilon' + b\varepsilon''$$
with $r \in \mathbb{R}$ and $b \geq 0$. Since any Gaussian cocycle is completable, $\mathcal{SU}_q(2)$ has the property (GC), and hence (LK).

LK-decomposition on $\mathcal{SU}_q(2)$: non-Gaussian part

Theorem (Schürmann, Skeide'98)

- Let π be the representation of $\mathcal{SU}_q(2)$ such that $H_{\mathrm{G}} = \{0\}$ and let $h \in H$. Set $h_p = (1 - p\pi(\alpha^*))^{-1} h$. Then for any $a \in \mathcal{A}$ there exists the limit $\lim_{p \to 1} \eta_{\pi, h_p}(a)$, and thus

$$\eta(a) = \lim_{p \to 1} \eta_{\pi, h_p}(a) = \lim_{p \to 1} (\pi(a) - \varepsilon(a) I)(1 - p\pi(\alpha^*))^{-1} h$$

  is an approximately inner π-ε cocycle, which is purely non-gaussian.
- Any purely non-gaussian cocycle appears this way with $h = \eta(\alpha^*)$.
- $\mathcal{SU}_q(2)$ has (NC): any purely non-gaussian cocycle η on $\mathcal{SU}_q(2)$ is completable. The associated generating functional is

$$\psi(a) = \lim_{p \to 1} \langle h_p, (\pi(a) - \varepsilon(a) I) h_p \rangle.$$

LK-decomposition on $\mathcal{SU}_q(2)$

Hunt formula for $\mathcal{SU}_q(2)$ (Schürmann, Skeide'98)

Any generating functional is of the form

$$\psi(a) = r\varepsilon'(a) + b\varepsilon''(a) + \lim_{p\to 1}\langle h_p, (\pi(a) - \varepsilon(a)I)h_p\rangle$$

with $r \in \mathbb{R}$, $b \geq 0$, π a purely non-gaussian representation of $\mathcal{SU}_q(2)$ on H and $h \in H$.

$\mathcal{SU}_q(N)$, $q \in (0,1)$

$SU_q(N)$, $N \in \mathbb{N}$: the compact quantum group with $C(SU_q(N))$ being the universal unital C*-algebra generated by $u = (u_{jk})_{j,k=1}^N$ with the relations

a) **unitarity condition**: $\sum_{s=1}^{N} u_{js} u_{ks}^* = \delta_{jk} 1 = \sum_{s=1}^{N} u_{sj}^* u_{sk}$

b) **twisted determinant condition**:

$$\sum_{\sigma \in S_N} (-q)^{i(\sigma)} u_{\sigma(1),\tau(1)} u_{\sigma(2),\tau(2)} \cdots u_{\sigma(N),\tau(N)} = (-q)^{i(\tau)} 1$$

equipped with $\Delta(u_{jk}) = \sum_{p=1}^{N} u_{jp} \otimes u_{pk}$.

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equipped with $\Delta(u_{jk}) = \sum_{p=1}^N u_{jp} \otimes u_{pk}$.

Examples of relations:

$$u_{ij} u_{kj} = q u_{kj} u_{ij} \ (i < k), \quad u_{ij} u_{kl} = u_{kl} u_{ij} - (q^{-1} - q) u_{il} u_{kj} \ (i < k, j < l)$$

$$u_{ij} u_{kl}^* = u_{kl}^* u_{ij} \ (i \neq k, j \neq l), \quad u_{NN}^* u_{NN} = q^2 u_{NN} u_{NN}^* + (1 - q^2) 1$$

$\mathcal{SU}_q(N) := \mathrm{Pol}(SU_q(N))$ is a $*$-bialgebra with $\varepsilon(u_{jk}) = \delta_{jk}$.

Decomposition on $\mathcal{SU}_q(N)$: Gaussian part

Remark

There is a (N-1)-parameter family of 1-dim (irreducible) $*$-representations of $\mathcal{SU}_q(N)$ given by

$$\varepsilon_{\theta_2,\dots,\theta_N}(u_{kl}) := e^{i\theta_k}\delta_{k,l},$$

where $\theta_2,\dots,\theta_N \in [0,2\pi)$ and $\theta_1 = (-\sum_{k=2}^{N}\theta_k)_{\mathrm{mod}\,2\pi}$.
This family is pointwise C^∞ in θ_j for $j = 2,\dots,N$, and $\varepsilon_{0,\dots,0} = \varepsilon$.

- For any $j = 2,\dots,N$, the mapping

$$\varepsilon'_j : \mathcal{A} \ni a \mapsto \frac{\partial\varepsilon_{\theta_2,\dots,\theta_N}(a)}{\partial\theta_j}\Big|_{\theta_2=\dots=\theta_N=0} \in \mathbb{C}$$

  is a Gaussian cocycle. And so it any linear combination of ε'_j's.
- For any j,k the mapping

$$\frac{\varepsilon''_{jk}}{2} : \mathcal{A} \ni a \mapsto \frac{1}{2}\frac{\partial^2\varepsilon_{\theta_2,\dots,\theta_N}(a)}{\partial\theta_j\theta_k}\Big|_{\theta_2=\dots=\theta_N=0} \in \mathbb{C}$$

  is a generating functional.

Decomposition on $\mathcal{SU}_q(N)$: Gaussian part

Theorem (FKLS)

- Gaussian cocycles $\mathcal{SU}_q(N)$ are precisely of the form

$$\eta(a) = \sum_{j=2}^{N} \varepsilon_j'(a) h_j$$

for some $h_2, \ldots, h_N \in H$.

- Any Gaussian generating functional on $\mathcal{SU}_q(N)$ will be of the form

$$\psi(a) = \sum_{j=2}^{N} \varepsilon_j'(a) r_j + \sum_{j=2}^{N} B_{jk} \varepsilon_{jk}''(a), \quad r_j \in \mathbb{R}, B \in M_n(\mathbb{R}), B \geq 0.$$

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.
- A Gaussian η admits a generating functional iff it is hermitian. i.e.
$$\langle \eta(a), \eta(b) \rangle = \langle \eta(b^*), \eta(a^*) \rangle.$$
- For $N \geq 3$, $\mathcal{SU}_q(N)$ does not have (GC), since there exist non-hermitian Gaussian cocycles on $\mathcal{SU}_q(N)$ ($\langle h_j, h_k \rangle \neq \langle h_k, h_j \rangle$).

Quantum subgroup chain

We have the chain of quantum subgroups

$$\{e\} = \mathcal{SU}_q(1) \leqslant \mathcal{SU}_q(2) \leqslant \cdots \leqslant \mathcal{SU}_q(N) \leqslant \cdots$$

with the epimorphisms $s_n : \mathcal{SU}_q(n) \to \mathcal{SU}_q(n-1)$, which is determined by

$$s_n : \begin{bmatrix} u_{11} & \cdots & u_{1,n-1} & u_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ u_{n-1,1} & \cdots & u_{n-1,n-1} & u_{n-1,n} \\ u_{n1} & \cdots & u_{n,n-1} & u_{nn} \end{bmatrix} \longmapsto \begin{bmatrix} u_{11} & \cdots & u_{1,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ u_{n-1,1} & \cdots & u_{n-1,n-1} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

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Definition

We say that a linear map $T : \mathcal{SU}_q(n) \to V$ **lives on $\mathcal{SU}_q(n-1)$** if it factors through s_n, i.e. $T = \tilde{T} \circ s_n$ for some $\tilde{T} : \mathcal{SU}_q(n-1) \to V$.

For example, a rep π on $\mathcal{SU}_q(N)$ lives on $\mathcal{SU}_q(N-1)$ iff $\pi(u_{NN}) = I$. Then necesarily, by the unitarity condition, $\pi(u_{jN}) = 0 = \pi(u_{Nj})$ for $j < N$.

LK-decomposition on $SU_q(N)$: non-Gaussian part

Take a Schürmann triple (π, η, ψ) on $\mathcal{SU}_q(N)$ with $\pi : \mathcal{SU}_q(N) \to B(H)$.

1. Space decomposition
   The space $L_N := \ker(\mathrm{id} - \pi(u_{NN}))^\perp$ is invariant, so $H = L_N \oplus R_N$.
2. Decomposition of the representation
   Accordingly, $\pi = \lambda \oplus \rho$.
   - $\lambda(1 - u_{NN})$ has trivial kernel, so it is injective;
   - $\rho(u_{NN}) = I$, since ρ acts on $R_N = L_N^\perp = \ker(\mathrm{id} - \pi(u_{NN}))$
3. Decomposition of the cocycle
   Let $\eta = \eta^\lambda \oplus \eta^\rho$ with $\eta^\lambda := P_{L_N} \circ \eta$, $\eta^\rho := P_{R_N} \circ \eta$.
   - $\eta^\rho(u_{NN}) = 0$
4. Ready for induction
   We have the decomposition $(\pi, \eta) = (\lambda, \eta^\lambda) \oplus (\rho, \eta^\rho)$ and
   - $\rho(u_{NN}) = I$ implies ρ lives on $\mathcal{SU}_q(N-1)$.
   - $\eta^\rho(u_{NN}) = 0$ implies η^ρ lives on $\mathcal{SU}_q(N-1)$.

   If we complete (λ, η^λ), the induction may start.

4. η^λ is approximately inner

   We know that $\lambda(1 - u_{NN})$ is injective. In this case, the value $h = \eta^\lambda(u_{NN})$ determines η^λ uniquely.

   One shows that the limits on the RHS exist and it equals

$$\eta^\lambda(a) = \lim_{p\to 1} \left(\lambda(a) - \varepsilon(a)\mathrm{id}\right) \underbrace{\left(\mathrm{id} - p\lambda(u_{NN})\right)^{-1} h}_{h_p}$$

5. Finding generating functional

   Approximately inner cocylces are completable with

$$\psi^\lambda(a) := \lim_{p\to 1} \langle h_p, [\pi(a) - \varepsilon(a)I]h_p\rangle, \quad a \in \mathcal{SU}_q(N).$$

   So $(\lambda, \eta^\lambda, \psi^\lambda)$ is a Schürmann triple.

LK-decomposition on $SU_q(N)$: non-Gaussian part

Theorem

Any purely non-Gaussian pair (π, η) on $\mathcal{SU}_q(N)$ can be completed to a Schürmann triple (ρ, η, ψ) with

$$(\pi, \eta, \psi) = (\pi|_{H_N}, \eta_N, \psi_N) \oplus \cdots \oplus (\pi|_{H_2}, \eta_2, \psi_2),$$

where $\pi|_{H_j}(1 - u_{jj})$ is injective and $\pi|_{H_j}$ as well as η_j lives on $\mathcal{SU}_q(j)$.

Consequently, $\mathcal{SU}_q(N)$ has (NC), hence admits a Lévy-Kchintchine type decomposition.

LK-decomposition on $SU_q(N)$

Hunt formula for $\mathcal{SU}_q(N)$ (FKLS)

For $q \in (0,1)$ and $N \geq 3$ $\mathcal{SU}_q(N)$ has the property (LK). Moreover, every generating functional of a Lévy process on $\mathcal{SU}_q(N)$ is of the form

$$\psi(a) = \sum_{j=2}^{N} r_j \varepsilon_j'(a) + \sum_{j,k=2}^{N} B_{jk} \varepsilon_{jk}''(a) + \lim_{p \to 1} \sum_{j=2}^{N} \langle h_{j,p}, [\pi_j(a) - \varepsilon(a)] h_{j,p} \rangle,$$

where $r_j \in \mathbb{R}$, $B \in M_n(\mathbb{R})$ is positive definite and some net $(h_{j,p})_p$ in H.

Note that $h_{j,p} = (\mathrm{id} - p\pi_j(u_{jj}))^{-1} \eta_j(u_{jj})$.

LK-decomposition on $SU_q(N)$

Let π be a rep of $\mathcal{SU}_q(N)$ on H such that $\pi(1 - u_{NN})$ is injective.
Which vectors in H may occur as values $\eta(u_{NN})$ for a cocycle η?

LK-decomposition on $SU_q(N)$

Let π be a rep of $\mathcal{SU}_q(N)$ on H such that $\pi(1 - u_{NN})$ is injective. Which vectors in H may occur as values $\eta(u_{NN})$ for a cocycle η?

- Not every vector in H may occur as $\eta(u_{NN})$ for a cocycle! Take the ∞-dim irrep ρ of $\mathcal{SU}_q(2)$ and define $\rho_{12} \star \rho_{23}$ on $\ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$. Then there is no cocycle with the value $e_0 \otimes e_0$ on u_{33}.
- There are many vectors that do give rise to cocycles. More precisely, any element $h \in H_0 := \pi(1 - u_{NN})H$ determines a cocycle (which is also a coboundary). H_0 is a dense subspace of H.
- The map

$$\|.\|_\pi : h \mapsto \left(\sum_{j=1}^{N} \|\pi(1 - u_{jj})h\|^2 \right)^{1/2}$$

is a norm on H. Cauchy sequences w.r.t. $\|.\|_\pi$ defines pointwise converging coboundaries.

LK-decomposition: $\mathcal{SU}_q(2)$ vs. $\mathcal{SU}_q(N)$ $(N > 2)$

- $\mathcal{SU}_q(2)$ has (GC), which is no longer true for $N > 2$
- Any purely non-gaussian cocycle is a direct sum of $N - 1$ approximately inner cocylces living on smaller quantum subgroups.
- Both algebras have (NC).
- The parametrization space of purely non-gaussian cocycles (and the related generating functionals) differs.
- The Lévy-Khintchine decomposition exsits for both algebras.

Does every $*$-bialgebra admit LK decomposition?

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[YES] commutative $*$-bialgebras, $\mathrm{Pol}(G)$ for a compact G (Schürmann'90)

[YES] the Brown algebra $\mathcal{U}\langle d\rangle$ (Schürmann'90)

[YES] $\mathcal{SU}_q(2)$ (Schürmann, Skeide'98)

[YES] the free permutation group $\mathcal{S}_n^+$ (Franz, AK, Skalski'16)

[YES] $\mathcal{S}_D^+ := \mathcal{S}_n^+/\langle uD = Du\rangle$ (include: quantum automorphism group of graphs, quantum reflection groups) (Bichon, Franz, Gerhold'17)

[YES] universal quantum groups $\mathcal{U}_F^+$ and $\mathcal{O}_F^+$ provided F^*F has eigenvalues of multiplicity 1 (Das, Franz, AK, Skalski'18)

[YES] $\mathcal{SU}_q(N)$ and $\mathcal{U}_q(N)$, $N \geq 3$, $q \in (0,1)$, have (LK) (Franz, AK, Lindsay, Skeide'24?)

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[NO] the group algebra of the fundamental group of an oriented surface of genus $k \geq 2$ (Franz, Gerhold, Thom'15)

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Final remarks

General Problem: when a $*$-bialgebra admits a Levy-Khintchine decomposition?

Observation: Neither the property (LK) nor its negations transfer to quantum subgroups (quotients of algebras):

$$\mathcal{O}^+_{2\,\mathrm{(LK)}} \subset \mathcal{O}^+_{3\,\mathrm{no\,(LK)}} \subset \mathcal{U}\langle 3\rangle_{\mathrm{(LK)}}.$$

Final remarks

References

1. M.Schürmann, Gaussian states on bialgebras. In: LMM 1442 (1990).
2. M.Schürmann, M.Skeide, Infinitesimal generators on the quantum group $SU_q(2)$, *IDAQP* 1 (1998).
3. U. Franz, M. Gerhold and A. Thom, On the Lévy-Khintchine decomposition of generating functionals. *Comm. Stoch. Anal.* 9 (2015), no. 4 529-544.
4. U.Franz, AK, M.Lindsay, M.Skeide, Hunt's Formula for $SU_q(N)$ and $U_q(N)$. To appear in *Indiana Univ. Math. J.* `arXiv:2009.06323`.

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Thank you!