

Few bilinear operators on spaces of continuous functions

by

LEANDRO CANDIDO

Abstract. Motivated by recent work exhibiting a locally compact scattered space L constructed under Ostaszewski's $\clubsuit$ -principle, which yielded a complete classification of linear operators on $C_0(L \times L)$, we extend the analysis to the bilinear setting. We show that, for this space L , every bilinear operator $G : C_0(L) \times C_0(L) \rightarrow C_0(L)$ admits a unique decomposition into the sum of trivially predictable components. This establishes a bilinear analogue of the *few operators phenomenon*.

1. Introduction. To place our work within a broader context, we briefly recall some historical background. Questions concerning the structure of bounded linear operators on infinite-dimensional Banach spaces have been present since the early days of functional analysis, already within the environment shaped by Banach and his collaborators.

Although ideas related to the scarcity of operators were already implicit in this early period, a precise formulation of the problem was made explicit by J. Lindenstrauss in 1976 [15]. Among several influential questions, he asked whether there exists an infinite-dimensional Banach space X such that every bounded linear operator $T : X \rightarrow X$ can be written as

$$T = rI + S,$$

where r is a scalar, I denotes the identity, and S is compact. This became known as the *scalar-plus-compact problem*. Since operators of this form exist on any Banach space, a space satisfying this condition can be said to have “very few operators”.

The first example of a Banach space that can be said to have “few operators” was provided by S. Shelah [21], initially under additional set-theoretic assumptions, and later in **ZFC** in joint work with J. Steprāns [22]: a non-

2020 *Mathematics Subject Classification*: Primary 46E15; Secondary 47L20, 54G12, 46B25, 03E65, 54B10.

Key words and phrases: Banach spaces of continuous functions, bilinear operators, few operators phenomenon.

Received 8 October 2025; revised 8 January 2026.

Published online 10 October 2026.

separable Banach space in which every operator is the sum of a scalar multiple of the identity plus an operator with separable range. Further contributions were made by W. T. Gowers and B. Maurey [8] (see also [16]), who obtained a Banach space exhibiting few operators in a slightly different sense, namely, a space in which every operator is a scalar multiple of the identity plus a *strictly singular* operator. Although the Gowers–Maurey construction does not solve the scalar-plus-compact problem, since strictly singular operators form a class strictly larger than compact operators (see [1]), it represents a significant advance in the study of the operator structure on exotic Banach spaces.

The scalar-plus-compact problem was completely resolved in the affirmative in 2011 by S. A. Argyros and R. G. Haydon [2], representing a major milestone in 21st-century functional analysis.

Between the work of Gowers and Maurey and the complete solution of the scalar-plus-compact problem by Argyros and Haydon, the study of few operators in spaces of continuous functions $C(K)$ began to flourish. The first notable contribution in this context was made by P. Koszmider [11], who, assuming the continuum hypothesis, constructed a connected compact Hausdorff space K such that every operator $T : C(K) \rightarrow C(K)$ is of the form

$$T = gI + S,$$

with $g \in C(K)$ and S weakly compact (equivalently, strictly singular for $C(K)$ spaces). Later, G. Plebanek [18] removed the continuum hypothesis assumption. Koszmider also extended the study of few operators to Asplund spaces of the form $C(K)$, that is, when K is a scattered compact Hausdorff space. In [12], he constructed a scattered compact Hausdorff space K such that every operator is a scalar multiple of the identity plus an operator with range contained in a copy of c_0 , initially under extra set-theoretic assumptions, later removed in joint work with N. J. Laustsen [9]. Numerous other contributions have since been made in this area; see [10] for a comprehensive survey of this topic, and in the present paper we provide further results along the same lines.

Our research continues this line of investigation, focusing on few operators in Banach spaces of the form $C(K)$ with K scattered. In a recent development [4], working under Ostaszewski’s $\clubsuit$ -principle [17], we identified a locally compact Hausdorff space L that exhibits a notion of few operators for spaces of the form $C_0(L \times L)$. The rigidity of the underlying space $C_0(L \times L)$, recalling that it is linearly isometric to the injective tensor product $C_0(L) \hat{\otimes}_\varepsilon C_0(L)$, suggests that similar rigidity phenomena might be reflected at the level of the projective tensor product $C_0(L) \hat{\otimes}_\pi C_0(L)$. This observation leads us to ask whether the few operators phenomenon can be extended

to bilinear operators

$$G : C_0(L) \times C_0(L) \rightarrow C_0(L),$$

which, by the universal property of the projective tensor product (see [19, Theorem 2.9]), is equivalent to asking whether there are few bounded linear operators from $C_0(L) \hat{\otimes}_\pi C_0(L)$ into $C_0(L)$.

Among all bilinear operators on $C_0(L)$, besides those with separable range, two elementary types arise. The first is what we refer to as *scaled multiplication*:

$$M(f, g)(t) = r f(t)g(t),$$

for a fixed scalar r . The second is what we call a *cross-interaction* operator, derived from functionals $\varphi, \psi \in C_0(L)^*$:

$$H(f, g)(t) = \varphi(f)g(t) + f(t)\psi(g).$$

In this paper, we show that for an exotic locally compact Hausdorff space L , constructed under additional set-theoretic assumptions, every bilinear operator $G : C_0(L) \times C_0(L) \rightarrow C_0(L)$ decomposes uniquely as the sum of a scaled multiplication component, a cross-interaction component, and a separable-range component. Since operators of these types exist independently of the topology of L , this provides an example of the *few operators phenomenon* in the bilinear setting. More precisely, the main result of this work is the following.

THEOREM 1.1. *Assuming Ostaszewski's $\clubsuit$ -principle, there exists a non-metrizable, locally compact, scattered Hausdorff space L such that for every bilinear operator $G : C_0(L) \times C_0(L) \rightarrow C_0(L)$ there exist a unique scalar $r \in \mathbb{R}$, unique functionals $\varphi, \psi \in C_0(L)^*$, and a unique bilinear operator $S : C_0(L) \times C_0(L) \rightarrow C_0(L)$ with separable range such that*

$$G(f, g)(t) = r f(t)g(t) + \varphi(f)g(t) + f(t)\psi(g) + S(f, g)(t)$$

for all $f, g \in C_0(L)$ and all $t \in L$.

Following the historical development of this line of research, where results are often first obtained under strong set-theoretic assumptions and later refined or reproduced within **ZFC**, it seems to be a natural and interesting direction for further investigation to determine whether a version of Theorem 1.1 can be established under different set-theoretic assumptions, or even in **ZFC**. Moreover, it would be of independent interest to develop a notion of few bilinear operators $G : X \times X \rightarrow X$ for other classes of Banach spaces X .

This paper is organized as follows. In Section 2, we establish the basic terminology and recall the notion of a *collapsing space*, along with its exotic properties that will be exploited throughout the paper. In Section 3, we present fundamental results on Fréchet measures and discuss their intrinsic connection with bilinear forms on $C_0(L)$. Section 4 contains the main

technical part of the paper, which reveals the pathologies of many bilinear operators on $C_0(L)$. Finally, in Section 5, we provide a detailed proof of Theorem 1.1.

2. Terminology and preliminaries. We begin by introducing some basic terminology that will be used throughout this work. All ordinals are equipped with the usual order topology. As is customary, we denote by ω the first infinite ordinal and by ω_1 the first uncountable ordinal.

We use the letter L to denote a locally compact Hausdorff space, and let $K = L \cup \{\infty\}$ denote its one-point (Aleksandrov) compactification.

For such spaces L and K , we denote by $C_0(L)$ the Banach space of continuous real-valued functions on L vanishing at infinity, and by $C(K)$ the Banach space of continuous real-valued functions on K , both equipped with the supremum norm. Throughout this paper, we always regard $C_0(L)$ as the closed subspace of $C(K)$ consisting of functions $f : K \rightarrow \mathbb{R}$ with $f(\infty) = 0$. All Banach spaces are taken over the real numbers for convenience and in line with previous works that inspired this research. While some technical details may require adaptation, we expect that the main results can be extended to the complex case.

For functions $f, g \in C(K)$, we denote by $f \otimes g : K \times K \rightarrow \mathbb{R}$ the function defined by $f \otimes g(x, y) = f(x)g(y)$. For any subset $A \subset K$, we denote by χ_A the indicator function of A . For a singleton set $\{x\}$, we write χ_x instead of $\chi_{\{x\}}$.

For a compact Hausdorff space K , we denote by $\mathfrak{B}(K)$ its Borel σ -algebra, and by $\wp(K)$ its power set. By the Riesz representation theorem [20, Theorem 18.4.1], the topological dual of $C(K)$ is isometrically identified with $\mathcal{M}(K)$, the space of Radon measures on K endowed with the total variation norm. If $\mu \in \mathcal{M}(K)$, we denote by $|\mu|$ its total variation. The dual of $C_0(L)$ is identified with the subspace $\mathcal{M}_0(K) = \{\mu \in \mathcal{M}(K) : \mu(\{\infty\}) = 0\}$.

At the core of this study lies a locally compact Hausdorff space L constructed in [4, Theorem 3.6] under Ostaszewski's $\clubsuit$ -principle. We refer to this space as the *collapsing space* L and recall the properties that will be relevant throughout this paper.

We recall the class $\mathcal{S}$ of topological spaces [4, 5]. A locally compact Hausdorff space L belongs to $\mathcal{S}$ if there exists a continuous finite-to-one surjection $\varphi : L \rightarrow \omega_1$, that is, $\varphi^{-1}(\{\alpha\})$ is finite for every $\alpha \in \omega_1$. Consequently, L is a first-countable scattered space [5, Proposition 3.2]. We always consider such a space L together with the collection $\{L_\alpha : \alpha \in \omega_1\}$, where $L_\alpha = \varphi^{-1}[\alpha + 1] = \{x \in L : \varphi(x) \in \alpha + 1\}$.

This collection forms a covering of L by countable clopen sets and satisfies $L_\alpha \subset L_\beta$ whenever $\alpha \leq \beta$. In particular, every function $f \in C_0(L)$ has countable support. For each $\alpha < \omega_1$, we identify $C_0(L_\alpha)$ with the subspace

of $C_0(L)$ consisting of functions that vanish outside L_α . With this, we can establish the following simple but important fact.

PROPOSITION 2.1. *Let L be a member of the class $\mathcal{S}$. If W is a separable subset of $C_0(L)$, then there exists $\eta < \omega_1$ such that $W \subset C_0(L_\eta)$.*

Proof. Let W be a separable subset of $C_0(L)$, and fix a dense sequence $\{f_n\}_{n < \omega}$ in W . Since each f_n has countable support, we can choose η_n such that f_n vanishes on $L \setminus L_{\eta_n}$. Let $\eta = \sup \{\eta_n : n < \omega\}$. Hence every $g \in W$ vanishes outside L_η . ■

Next, we recall the following terminology adapted from [4]. If $\{I_1, \dots, I_k\}$ is a partition of $\{1, \dots, m\}$, then a net $\mathcal{C} = \{(x_1^\alpha, \dots, x_m^\alpha)\}_{\alpha < \omega_1}$ in L^m is said to be $(I_1, \dots, I_k)$ -separated if

$$\{x_i^\alpha : i \in I_j\} \cap \{x_i^\beta : i \in I_r\} = \emptyset$$

whenever $j \neq r$ or $\alpha \neq \beta$.

DEFINITION 2.2. A locally compact space L is called *collapsing* if it belongs to the class $\mathcal{S}$ and, for every integer $1 \leq m < \omega$, every partition $\{I_1, \dots, I_k\}$ of $\{1, \dots, m\}$, and every $(I_1, \dots, I_k)$ -separated net $\{(x_1^\alpha, \dots, x_m^\alpha)\}_{\alpha < \omega_1}$ in L^m , it admits an accumulation point $(q_1, \dots, q_m) \in K^m$, that is, the limit of a convergent subnet, such that there exist distinct points $a_1, \dots, a_k \in K$, with $a_k = \infty$, for which

$$\{q_i : i \in I_j\} = \{a_j\} \quad \text{for every } 1 \leq j \leq k.$$

We recall the following result from [4, Theorem 3.6], which provides the collapsing space central to our study. For other related constructions, see [5, Theorem 4.2], [6, Proposition 3.1], and [13, Section 4].

THEOREM 2.3. *Under Ostaszewski's $\clubsuit$ -principle, there exists a collapsing space.*

REMARK 2.4. It should be emphasized that collapsing spaces cannot be constructed within **ZFC** alone, as their existence requires additional set-theoretic assumptions; see [4, Remark 3.8]. However, it is not excluded that a version of Theorem 1.1 might hold under other combinatorial principles, as in [12], or even within **ZFC** itself, as in [9].

The following result can be obtained by combining [4, Theorem 4.3 and Corollary 4.6]. We recall that $\nu : K \rightarrow \mathcal{M}_0(K)$, written for simplicity as $\nu(x) = \nu_x$, is said to be a *weak*-continuous function vanishing at infinity* if ν is continuous when $\mathcal{M}_0(K)$ is endowed with the weak*-topology induced by $C_0(L)$, and $\nu_\infty = 0$. Under the identification of weak*-continuous functions $\nu : K \rightarrow \mathcal{M}_0(K)$ vanishing at infinity with operators $T : C_0(L) \rightarrow C_0(L)$ (see [4, Theorem 4.1]), this result is essentially a restatement of the fact that $C_0(L)$ has few operators when L is a collapsing space.

PROPOSITION 2.5. *Let $K = L \cup \{\infty\}$, where L is a collapsing space, and let $\nu : K \rightarrow \mathcal{M}_0(K)$ be a weak*-continuous function vanishing at infinity. Then there exist $r \in \mathbb{R}$ and $\rho < \omega_1$ such that, for all $x, y \in L \setminus L_\rho$, one has*

$$\nu_x(\{y\}) = \begin{cases} r & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

To conclude this section, we introduce some additional terminology that will be used throughout the paper.

DEFINITION 2.6. Let $\{x_\alpha\}_{\alpha < \omega_1}$ be a transfinite sequence of objects. A *refinement* (or *refined sequence*) of this family is any subsequence $\{x_{\alpha_\gamma}\}_{\gamma < \omega_1}$, possibly selected to satisfy additional properties. More generally, given several transfinite sequences $\{x_\alpha^1\}_{\alpha < \omega_1}, \dots, \{x_\alpha^m\}_{\alpha < \omega_1}$, a *simultaneous refinement* is obtained by choosing a strictly increasing sequence of ordinals $\{\alpha_\gamma\}_{\gamma < \omega_1}$ and considering the corresponding subsequences $\{x_{\alpha_\gamma}^1\}_{\gamma < \omega_1}, \dots, \{x_{\alpha_\gamma}^m\}_{\gamma < \omega_1}$. If no ambiguity arises, we continue to denote these refined sequences by the original symbols $\{x_\alpha^1\}_{\alpha < \omega_1}, \dots, \{x_\alpha^m\}_{\alpha < \omega_1}$.

All remaining terminology used in this paper can be found in [4].

3. Fréchet measures and the structure of bilinear forms on $C(K)$.

In this section, we collect several results concerning the interplay between bilinear forms on $C(K)$ and the so-called Fréchet measures on $K \times K$, focusing on the particular case where K is a scattered compact Hausdorff space. Our main reference for this material is the comprehensive monograph by Ron Blei [3, Chapters IV and VI], from which we adopt much of the basic terminology and adapt several results to our setting.

DEFINITION 3.1. An F_2 -measure (or a 2-Fréchet measure) on $K \times K$ is a function

$$\mu : \mathfrak{B}(K) \times \mathfrak{B}(K) \rightarrow \mathbb{R}$$

such that μ is a measure in each coordinate separately. That is, for every fixed $A \in \mathfrak{B}(K)$, both mappings $B \mapsto \mu(A, B)$ and $B \mapsto \mu(B, A)$ are (signed) measures on $\mathfrak{B}(K)$.

We denote by $\mathcal{F}(K, K)$ the collection of all such F_2 -measures on $K \times K$. If $K = L \cup \{\infty\}$ is the one-point compactification of L , we denote by $\mathcal{F}_0(K, K)$ the subspace of $\mathcal{F}(K, K)$ consisting of all F_2 -measures μ such that the mappings $B \mapsto \mu(\{\infty\}, B)$ and $B \mapsto \mu(B, \{\infty\})$ are identically zero.

For the benefit of readers not familiar with F_2 -measures, we briefly outline the notion of integration with respect to a measure $\mu \in \mathcal{F}(K, K)$. The exposition below follows the text of Blei [3, Chapter IV, Section 5], with notation inspired by his presentation but slightly adapted to better suit our framework. First, for each pair of functions $f, g \in C_0(L)$, we denote by μ^f

and μ_g the measures on K defined by the formulae

$$\mu^f(A) = \int_K f(x) \mu(dx, A) \quad \text{and} \quad \mu_g(B) = \int_K g(y) \mu(B, dy).$$

For $f, g \in C(K)$, the iterated integral of $f \otimes g$ with respect to μ is given by

$$\int f \otimes g \, d\mu = \int f \, d\mu_g = \int_K f(x) \left(\int_K g(y) \mu(\cdot, dy) \right) (dx).$$

It is important to note (see [3, Chapter VI, Theorem 10]) that the following Fubini-type property holds:

$$(3.1) \quad \begin{aligned} \int f \, d\mu_g &= \int_K f(x) \left(\int_K g(y) \mu(\cdot, dy) \right) (dx) \\ &= \int_K g(y) \left(\int_K f(x) \mu(dx, \cdot) \right) (dy) = \int g \, d\mu^f. \end{aligned}$$

The space $\mathcal{F}(K, K)$ can be equipped with a norm under which it becomes a Banach space. This norm is given by the Fréchet variation, which we recall below; see [3, Chapter VI, Section 3].

We say that a collection $\Lambda \subset \mathfrak{B}(K) \times \mathfrak{B}(K)$ is a *grid* for $K \times K$ if $\Lambda = \Lambda_1 \times \Lambda_2$, where each Λ_i is a finite collection of pairwise disjoint Borel subsets of K . If $\Lambda = \Lambda_1 \times \Lambda_2$ is a grid for $K \times K$, we denote by $\{r_A : A \in \Lambda_i\}$ the *Rademacher system* on $\{-1, 1\}^{\Lambda_i}$, that is, $r_A : \{-1, 1\}^{\Lambda_i} \rightarrow \{-1, 1\}$ is defined by $r_A(h) = h(A)$ for $h \in \{-1, 1\}^{\Lambda_i}$, for $i = 1, 2$.

Now, if μ is a Fréchet measure in $\mathcal{F}(K, K)$, the *Fréchet variation* is given by

$$\|\mu\|_{\mathcal{F}_2} = \sup \left\{ \left\| \sum_{(A,B) \in \Lambda} \mu(A, B) r_A \otimes r_B \right\|_{\infty} : \Lambda \text{ is a grid for } K \times K \right\}.$$

According to [3, Chapter VI, Theorem 5], we have $\|\mu\|_{\mathcal{F}_2} < \infty$.

Since our focus is on the case where K is a scattered compact Hausdorff space, the next definition introduces an important object that will play a fundamental role in our study. To simplify the notation, we shall adopt the following convention: for $\mu \in \mathcal{F}(K, K)$ and $x, y \in K$, we write $\mu(x, y) = \mu(\{x\}, \{y\})$. Similarly, if $\mu \in \mathcal{M}(K)$, we write $\mu(x) = \mu(\{x\})$ for every $x \in K$.

DEFINITION 3.2. Let K be a scattered compact Hausdorff space and let $\mu \in \mathcal{F}(K, K)$. The *total variation* of μ is the function $\text{Var}(\mu) : \wp(K) \times \wp(K) \rightarrow [0, \infty)$ defined by

$$\begin{aligned} &\text{Var}(\mu)(U, V) \\ &= \sup \left\{ \sum_{x \in S} \sum_{y \in T} \mu(x, y) u_x v_y : S \times T \subset U \times V \text{ finite, } u_x, v_y \in [-1, 1] \right\}. \end{aligned}$$

We first note that $\text{Var}(\mu)$ is well-defined, since by an argument similar to that of [3, Chapter I, Lemma 8] one has

$$\text{Var}(\mu)(U, V) \leq \|\mu\|_{\mathcal{F}_2} \quad \text{for every } (U, V) \in \wp(K) \times \wp(K).$$

Although $\text{Var}(\mu)$ does not, in general, belong to $\mathcal{F}(K, K)$, it enjoys several properties that will be fundamental in our analysis. For instance, it is subadditive: if $U, V_1, \dots, V_n \in \wp(K)$ with $V_i \cap V_j = \emptyset$ for $i \neq j$, then

$$\begin{aligned} \text{Var}(\mu)\left(U, \bigcup_{i=1}^n V_i\right) &\leq \sum_{i=1}^n \text{Var}(\mu)(U, V_i), \\ \text{Var}(\mu)\left(\bigcup_{i=1}^n V_i, U\right) &\leq \sum_{i=1}^n \text{Var}(\mu)(V_i, U). \end{aligned}$$

It is also monotone: if $U_1 \subset U_2$ and $V_1 \subset V_2$, then

$$\text{Var}(\mu)(U_1, V_1) \leq \text{Var}(\mu)(U_2, V_2).$$

A key fact for our research is the close relationship between F_2 -measures on $K \times K$ and bilinear forms on $C(K)$. The following result provides a Riesz-type representation theorem for bilinear forms on $C(K)$ when K is a scattered compact Hausdorff space.

THEOREM 3.3. *Let K be a scattered compact Hausdorff space. Then, for every bilinear form G on $C(K) \times C(K)$, there exists a unique $\mu \in \mathcal{F}(K, K)$ such that, for all $f, g \in C(K)$,*

$$(3.2) \quad G(f, g) = \sum_{x \in K} \left(\sum_{y \in K} \mu(x, y) g(y) \right) f(x) = \sum_{y \in K} \left(\sum_{x \in K} \mu(x, y) f(x) \right) g(y).$$

Moreover, the expression in (3.2) is well-defined for all $f, g \in \ell_\infty(K)$, and therefore G can be naturally extended to a bilinear form on $\ell_\infty(K) \times \ell_\infty(K)$ with the same norm. In addition, $\|G\| = \text{Var}(\mu)(K, K)$.

Proof. By [3, Chapter VI, Theorem 12], there exists a unique $\mu \in \mathcal{F}(K, K)$ such that

$$G(f, g) = \int f \otimes g d\mu = \int f d\mu_g$$

with $\|\mu\|_{\mathcal{F}_2} \leq \|G\|$. Since K is a scattered compact Hausdorff space, μ_g is an atomic measure (see [20, Theorem 19.7.6]), and therefore

$$G(f, g) = \sum_{x \in K} f(x) \mu_g(x),$$

where this summation is well-defined because $(\mu_g(x))_{x \in K}$ belongs to $\ell_1(K)$ (see [20, Corollary 19.7.7]). Next, for each $x \in K$, since the marginal measure $E \mapsto \mu(\{x\}, E)$ is atomic, we have

$$\mu_g(x) = \int_K g(y) \mu(\{x\}, dy) = \sum_{y \in K} g(y) \mu(x, y),$$

and again the summation is well-defined since $(\mu(x, y))_{y \in K} \in \ell_1(K)$. Therefore,

$$G(f, g) = \sum_{x \in K} \left(\sum_{y \in K} \mu(x, y) g(y) \right) f(x).$$

The second equality in (3.2) follows from the Fubini-type property (3.1).

Now let f and g be elements of the unit ball $B_{\ell_\infty(K)}$. Since $f(x), g(y) \in [-1, 1]$ for all $x, y \in K$, it follows that for all finite sets $S, T \subset K$,

$$\left| \sum_{x \in S} \sum_{y \in T} \mu(x, y) f(x) g(y) \right| \leq \text{Var}(\mu)(K, K) \leq \|\mu\|_{\mathcal{F}_2}.$$

This shows that the map (3.2) is well-defined and defines a bilinear form $\tilde{G}$ on $\ell_\infty(K) \times \ell_\infty(K)$, extending G . Moreover, $\|\tilde{G}\| \leq \text{Var}(\mu)(K, K) \leq \|G\|$, and hence $\|G\| = \|\tilde{G}\| = \text{Var}(\mu)(K, K)$. ■

REMARK 3.4. In our analysis, we focus on bilinear forms on $C_0(L)$ rather than on $C(K)$. Recall that $C_0(L) = \{f \in C(K) : f(\infty) = 0\}$; any bilinear form $G : C_0(L) \times C_0(L) \rightarrow \mathbb{R}$ admits a natural extension $\tilde{G} : C(K) \times C(K) \rightarrow \mathbb{R}$ defined by $\tilde{G}(f, g) = G(f - f(\infty), g - g(\infty))$. Applying Theorem 3.3 to $\tilde{G}$ yields a unique $\mu \in \mathcal{F}(K, K)$ such that (3.2) holds when restricted to $C_0(L)$, and all remaining statements, including the extension to $\ell_\infty(K) \times \ell_\infty(K)$ and the norm equality, remain valid. Since $\mu(\{\infty\}, U) = \mu(U, \{\infty\}) = 0$ for every $U \subset K$, it follows that $\mu \in \mathcal{F}_0(K, K)$.

REMARK 3.5. From Theorem 3.3 we deduce that if K is a scattered compact Hausdorff space and $\mu \in \mathcal{F}(K, K)$, then for all $U, V \subset K$ we have

$$\mu(U, V) = \sum_{x \in U} \left(\sum_{y \in V} \mu(x, y) \right).$$

This, in turn, yields the important fact that for all $U, V \in \wp(K)$ one has $|\mu(U, V)| \leq \text{Var}(\mu)(U, V)$. Indeed, given $\epsilon > 0$, there exist finite sets $S \subset U$ and $T \subset V$ such that

$$|\mu(U, V)| - \epsilon < \left| \sum_{x \in S} \sum_{y \in T} \mu(x, y) \right| \leq \text{Var}(\mu)(S, T) \leq \text{Var}(\mu)(U, V).$$

Finally, for every $\mu \in \mathcal{F}(K, K)$ we can establish a finite-support regularity property associated with $\text{Var}(\mu)$ (Theorem 3.9), which will play a central role in the next section. To this end, we require the following auxiliary result, adapted from [3, Chapter IV, Lemmas 3 and 4].

LEMMA 3.6. *Let K be a scattered compact Hausdorff space and let $\mu \in \mathcal{F}(K, K)$. Let $\{A_n\}_{n < \omega}$ and $\{B_n\}_{n < \omega}$ be sequences of pairwise disjoint sets*

such that $A_n \cap B_m = \emptyset$ whenever $m \neq n$. Then

$$\sum_{n=0}^{\infty} \text{Var}(\mu)(A_n, B_n) \leq \text{Var}(\mu)(K, K).$$

Proof. Let $\Omega = \{-1, 1\}^\omega$ be endowed with the usual product σ -algebra, and let $\mathbb{P}$ be the Bernoulli product measure on Ω (so that each coordinate is $\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$). Denote by $\{r_n : n < \omega\}$ the Rademacher system on Ω .

For $h \in \Omega$ and $x \in K$, consider the functions $f_h, g_h \in \ell_\infty(K)$ given by

$$f_h(x) = \sum_{n=0}^{\infty} r_n(h) \chi_{A_n}(x), \quad g_h(x) = \sum_{n=0}^{\infty} r_n(h) \chi_{B_n}(x).$$

Note that f_h and g_h are well-defined by the disjointness hypothesis on the collections $\{A_n\}_{n < \omega}$ and $\{B_n\}_{n < \omega}$. Moreover, if $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$, then for all $x, y \in K$,

$$\mathbb{E}[f_h(x)g_h(y)] = \sum_{n=0}^{\infty} \chi_{A_n \times B_n}(x, y).$$

Let $\epsilon > 0$ be arbitrary. For each $i < \omega$ fix finite sets S_i and T_i , together with coefficients $\{u_x : x \in S_i\} \subset [-1, 1]$ and $\{v_y : y \in T_i\} \subset [-1, 1]$, such that

$$\text{Var}(\mu)(A_i, B_i) - \frac{\epsilon}{2^{i+1}} < \sum_{x \in S_i} \sum_{y \in T_i} \mu(x, y) u_x v_y.$$

Next, given $n < \omega$, let $S = \bigcup_{i=0}^n S_i$ and $T = \bigcup_{i=0}^n T_i$. For each $h \in \Omega$, it is clear that $f_h, g_h \in B_{\ell_\infty(K)}$, hence by the definition of $\text{Var}(\mu)(K, K)$,

$$\sum_{x \in S} \sum_{y \in T} \mu(x, y) (f_h(x) u_x) (g_h(y) v_y) \leq \text{Var}(\mu)(K, K).$$

Taking expectation with respect to $\mathbb{P}$ yields

$$\begin{aligned} \sum_{i=0}^n \left(\sum_{x \in S_i} \sum_{y \in T_i} \mu(x, y) u_x v_y \right) &= \sum_{x \in S} \sum_{y \in T} \mu(x, y) \mathbb{E}[f_h(x)g_h(y)] u_x v_y \\ &\leq \text{Var}(\mu)(K, K). \end{aligned}$$

Therefore, for each $n < \omega$ we have

$$\sum_{i=0}^n \text{Var}(\mu)(A_i, B_i) \leq \text{Var}(\mu)(K, K) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the conclusion follows. ■

From Lemma 3.6 we can also deduce the following fact, which will be relevant at a later stage of this work.

PROPOSITION 3.7. *Let K be a scattered compact Hausdorff space. If $\mu \in \mathcal{F}(K, K)$, then μ has countable support, that is, $\text{Supp}(\mu) = \{(x, y) \in K \times K : \mu(x, y) \neq 0\}$ is countable.*

Proof. Suppose, towards a contradiction, that $\text{Supp}(\mu)$ is uncountable. Then there exist $\epsilon > 0$ and a sequence $\{(x_n, y_n)\}_{n < \omega}$ in $\text{Supp}(\mu)$ such that $|\mu(x_n, y_n)| \geq \epsilon$ for all $n < \omega$, and the points are pairwise distinct, that is, $(x_n, y_n) \neq (x_m, y_m)$ whenever $n \neq m$.

We define a coloring of the pairs $\{m, n\}$, $m < n$, with five colors as follows, assigning to each pair the first applicable color in the list:

- Color 1: $x_m = x_n$,
- Color 2: $y_m = y_n$,
- Color 3: $x_m = y_n$,
- Color 4: $y_m = x_n$,
- Color 5: none of the above hold.

By Ramsey's Theorem (infinite version), there exists an infinite set $H \subset \omega$ such that all pairs in H have the same color. Write $H = \{n_k : k < \omega\}$ with $n_0 < n_1 < \dots$.

If the color were 1, then all x_{n_k} would coincide, say $x_{n_k} = x$ for every k , contradicting the fact that the marginal measure $E \mapsto \mu(\{x\}, E)$ is finite. Similarly, if the color were 2, all y_{n_k} would coincide, leading to the same contradiction for the marginal measure $E \mapsto \mu(E, \{y\})$. The color cannot be 3: if $m < n < p$ belong to H , then $x_m = y_p = x_n$, so the pair $\{m, n\}$ would have color 1. Similarly, the color cannot be 4, since $y_m = x_p = y_n$ would give color 2 to $\{m, n\}$.

Therefore, the only possibility for H is color 5. Thus, the subsequence $\{(x_{n_k}, y_{n_k})\}_{k < \omega}$ satisfies, for $k \neq j$,

$$x_{n_k} \neq x_{n_j}, \quad y_{n_k} \neq y_{n_j}, \quad x_{n_k} \neq y_{n_j}.$$

By Lemma 3.6, we then obtain

$$\sum_{k=1}^{\infty} \text{Var}(\mu)(\{x_{n_k}\}, \{y_{n_k}\}) \leq \text{Var}(\mu)(K, K) < \infty,$$

which is a contradiction since $\text{Var}(\mu)(\{x_{n_k}\}, \{y_{n_k}\}) = |\mu(x_{n_k}, y_{n_k})| \geq \epsilon$ for all k . ■

LEMMA 3.8. *Let K be a scattered compact Hausdorff space and let $\mu \in \mathcal{F}(K, K)$. For every $\epsilon > 0$, there exists a finite set $I_0 \subset K$ such that $\text{Var}(\mu)(K \setminus I, K \setminus I) \leq \epsilon$ for every finite set $I \supset I_0$.*

Proof. Suppose the assertion is false. Then there exists $\mu \in \mathcal{F}(K, K)$ and $\epsilon > 0$ such that, for every finite set $I \subset K$, we have $\text{Var}(\mu)(K \setminus I, K \setminus I) > \epsilon$.

We construct recursively two sequences $\{A_n\}_{n < \omega}$ and $\{B_n\}_{n < \omega}$ of finite sets as follows. Let $A_0 = B_0 = \emptyset$, and suppose that we have already defined $\{A_0, \dots, A_n\}$ and $\{B_0, \dots, B_n\}$ as pairwise disjoint finite sets with $A_i \cap B_j = \emptyset$ whenever $i \neq j$. Set

$$I = \bigcup_{i=0}^n (A_i \cup B_i).$$

Since $\text{Var}(\mu)(K \setminus I, K \setminus I) > \epsilon$, there exist finite sets $S, T \subset K \setminus I$ and coefficients $\{u_x : x \in S\} \subset [-1, 1]$, $\{v_y : y \in T\} \subset [-1, 1]$ such that

$$\epsilon < \sum_{x \in S} \sum_{y \in T} \mu(x, y) u_x v_y \leq \text{Var}(\mu)(K \setminus I, K \setminus I).$$

We then set $A_{n+1} = S$ and $B_{n+1} = T$.

By recursion, this yields sequences $\{A_n\}_{n < \omega}$ and $\{B_n\}_{n < \omega}$ of pairwise disjoint finite sets with $A_i \cap B_j = \emptyset$ for $i \neq j$, and such that $\text{Var}(\mu)(A_n, B_n) > \epsilon$ for all $0 < n < \omega$.

This contradicts Lemma 3.6, proving the current lemma. ■

THEOREM 3.9. *Let K be a scattered compact Hausdorff space and let $\mu \in \mathcal{F}(K, K)$. For every $\epsilon > 0$ and all $a, b \in K$, there exists a finite set $I_0 \subset K$ such that, for every $I \subset K$ containing I_0 ,*

$$\text{Var}(\mu)(\{a\}, K \setminus I) + \text{Var}(\mu)(K \setminus I, \{b\}) + \text{Var}(\mu)(K \setminus I, K \setminus I) < \epsilon.$$

Proof. Let $\epsilon > 0$ be arbitrary. By Lemma 3.8, we may fix a finite set $J_0 \subset K$ such that, for every $J \supset J_0$,

$$\text{Var}(\mu)(K \setminus J, K \setminus J) \leq \epsilon/3.$$

Given $a, b \in K$, let μ_a and μ^b denote the marginal measures in $\mathcal{M}(K)$ defined by $\mu_a(E) = \mu(\{a\}, E)$ and $\mu^b(E) = \mu(E, \{b\})$ for every $E \in \mathfrak{B}(K)$, with total variation (in the usual sense of Radon measures) denoted by $|\mu_a|$ and $|\mu^b|$ respectively. Since μ_a and μ^b are atomic measures, there exist finite sets $J_1, J_2 \subset K$ such that, for every $J \supset J_1 \cup J_2$,

$$\text{Var}(\{a\}, K \setminus J) = |\mu_a|(K \setminus J) < \epsilon/3, \quad \text{Var}(K \setminus J, \{b\}) = |\mu^b|(K \setminus J) < \epsilon/3.$$

The claim follows by taking $I_0 = J_0 \cup J_1 \cup J_2$. ■

4. Dismantling of bilinear operators. In this section, we always consider L to be a collapsing space (see Definition 2.2), as introduced in Section 2, and let $K = L \cup \{\infty\}$ denote its one-point compactification. Our aim is to establish, via a suitable characterization of bilinear operators $G : C_0(L) \times C_0(L) \rightarrow C_0(L)$, a series of results that will allow us to dismantle these operators by exploiting the exotic properties of L . We begin by introducing the main object of this section.

DEFINITION 4.1. We say that a mapping $\nu : K \rightarrow \mathcal{F}_0(K, K)$, written for simplicity as $\nu(t) = \nu_t$, is a *weak*-continuous function vanishing at infinity* if, for all $f, g \in C_0(L)$, the formula

$$t \mapsto \int f \otimes g d\nu_t = \sum_{x \in K} \left(\sum_{y \in K} \nu_t(x, y) g(y) \right) f(x)$$

defines an element of $C_0(L)$.

This terminology is due to the fact that $\mathcal{F}_0(K, K)$, which is isometrically isomorphic to the space of bilinear forms on $C_0(L) \times C_0(L)$ by Theorem 3.3, is the topological dual of the projective tensor product $C_0(L) \hat{\otimes}_\pi C_0(L)$ (see [19, Theorem 2.9]).

The relevance of such functions in our research stems from the following characterization:

THEOREM 4.2. *Every weak*-continuous function $\nu : K \rightarrow \mathcal{F}_0(K, K)$ vanishing at infinity defines a bilinear operator $G : C_0(L) \times C_0(L) \rightarrow C_0(L)$ through the formula*

(4.1)

$$G(f, g)(t) = \sum_{x \in K} \left(\sum_{y \in K} \nu_t(x, y) g(y) \right) f(x) \quad \text{for all } f, g \in C_0(L), t \in K.$$

Conversely, every bilinear operator $G : C_0(L) \times C_0(L) \rightarrow C_0(L)$ admits a unique weak-continuous function $\nu : K \rightarrow \mathcal{F}_0(K, K)$ vanishing at infinity and satisfying the relation above. In either case, the operator norm is given by*

$$\|G\| = \sup_{t \in K} \text{Var}(\nu_t)(K, K).$$

Proof. If $\nu : K \rightarrow \mathcal{F}_0(K, K)$ is a weak*-continuous function vanishing at infinity, then (4.1) defines a bilinear mapping $G : C_0(L) \times C_0(L) \rightarrow C_0(L)$. According to the Banach–Steinhaus theorem (applied twice), we have $\|G\| < \infty$. For each $t \in K$, the mapping $G_t : C_0(L) \times C_0(L) \rightarrow \mathbb{R}$ given by $G_t(f, g) = G(f, g)(t)$ is a bilinear form on $C_0(L)$. By the uniqueness of the representation in Theorem 3.3, we obtain $\|G_t\| = \text{Var}(\nu_t)(K, K)$ and therefore

$$\|G\| = \sup_{t \in K} \|G_t\| = \sup_{t \in K} \text{Var}(\nu_t)(K, K).$$

Conversely, if $G : C_0(L) \times C_0(L) \rightarrow C_0(L)$ is a bilinear operator, then $G_t(f, g) = G(f, g)(t)$ defines a bilinear form on $C_0(L)$. By Theorem 3.3 (see also Remark 3.4), there exists a unique $\nu_t \in \mathcal{F}_0(K, K)$ such that

$$G(f, g)(t) = \sum_{x \in K} \left(\sum_{y \in K} \nu_t(x, y) g(y) \right) f(x).$$

Since $G(f, g) \in C_0(L)$ for all $f, g \in C_0(L)$, it follows that $\nu : K \rightarrow \mathcal{F}_0(K, K)$ is a weak*-continuous function vanishing at infinity, as in Definition 4.1. Moreover, by the second part of Theorem 3.3 we obtain

$$\|G\| = \sup_{t \in K} \|G_t\| = \sup_{t \in K} \text{Var}(\nu_t)(K, K). \blacksquare$$

REMARK 4.3. The attentive reader will note that both the characterization given in the previous theorem and the one discussed in Section 2, namely, the identification of weak*-continuous functions $\nu : K \rightarrow \mathcal{M}_0(K)$ vanishing at infinity with bounded linear operators $T : C_0(L) \rightarrow C_0(L)$ (see [4, Theorem 4.1]), can be viewed as instances of a more general principle: weak*-continuous functions $\nu : K \rightarrow X^*$ correspond to bounded linear operators $T : X \rightarrow C(K)$; see [7, Chapter VI, Section 7, Theorem 1]. Keeping in mind that bilinear mappings $G : X \times X \rightarrow C(K)$ correspond, via the universal property of the projective tensor product (see [19, Theorem 2.9]), to linear operators $\tilde{G} : X \hat{\otimes}_\pi X \rightarrow C(K)$, one could alternatively approach our results from this broader perspective. We nevertheless adopt the presentation above, as we believe it is clearer and better suits the purposes of this article.

The characterization given in Theorem 4.2 allows us to replace a bilinear operator $G : C_0(L) \times C_0(L) \rightarrow C_0(L)$ by a function that associates to each $t \in K$ an infinite matrix $\nu_t \in \mathcal{F}_0(K, K)$. The weak*-continuity of ν , together with the exotic topological properties of L , forces a certain regularity in the behavior of the coefficients $\nu_t(y, z)$, which can be exploited to decompose the operator step by step.

The results established below make this strategy precise. At a conceptual level, Theorem 4.6 isolates the cross-interaction component, while Theorem 4.11 identifies the scaled multiplication component. After these components have been removed, the remaining Theorems 4.10 and 4.13 ensure that the residual bilinear operator has separable range.

We now begin the technical implementation of this strategy. The following proposition was inspired by [5, Lemma 4.4].

PROPOSITION 4.4. *Let $A, B \subset L$ be countable subsets of L . If $\nu : K \rightarrow \mathcal{F}_0(K, K)$ denotes a weak*-continuous function vanishing at infinity, then there exists $\rho < \omega_1$ such that $\text{Var}(\nu_t)(A, B) = 0$ for every $t \in K \setminus L_\rho$.*

Proof. Suppose the proposition is false. Then, since A and B are countable and L is uncountable, there exist $\epsilon > 0$ and $(a, b) \in A \times B$ such that $|\nu_t(a, b)| > \epsilon$ for each t in an uncountable set $\Lambda \subset L$. Since L is a first-countable, locally compact, scattered space, we may fix countable local bases $\mathcal{U}_a = \{U_n : n < \omega\}$ and $\mathcal{V}_b = \{V_m : m < \omega\}$ consisting of compact clopen neighborhoods of a and b , respectively.

For each $t \in \Lambda$, from the regularity of the measure $E \mapsto \nu_t(\{a\}, E)$, we may select $V_t \in \mathcal{V}_b$ such that $|\nu_t(\{a\}, V_t \setminus \{b\})| < \epsilon/4$. Then by the regularity of the measure $E \mapsto \nu_t(E, V_t)$ we may select $U_t \in \mathcal{U}_a$ such that $|\nu_t(U_t \setminus \{a\}, V_t)| < \epsilon/4$. Since $\mathcal{U}_a$ and $\mathcal{V}_b$ are countable, there exists an uncountable subset $\Lambda' \subset \Lambda$ such that $U_t = U$ and $V_t = V$ for all $t \in \Lambda'$.

For each $t \in \Lambda'$ we have

$$\begin{aligned} |\nu_t(U, V)| &= |\nu_t(a, b) + \nu_t(U \setminus \{a\}, V) + \nu_t(\{a\}, V \setminus \{b\})| \\ &\geq |\nu_t(a, b)| - |\nu_t(U \setminus \{a\}, V)| - |\nu_t(\{a\}, V \setminus \{b\})| \geq \epsilon - \epsilon/2 = \epsilon/2. \end{aligned}$$

This is a contradiction, since $t \mapsto \nu_t(U, V) = \int \chi_U \otimes \chi_V d\nu_t$ belongs to $C_0(L)$ and therefore must have countable support. ■

PROPOSITION 4.5. *Let $(a_w)_{w \in L}$ be an element of $\ell_\infty(L)$ with the following property: for every $\rho < \omega_1$, $(a_w)_{w \in L_\rho}$ is an element of $\ell_1(L_\rho)$. Then $(a_w)_{w \in L}$ belongs to $\ell_1(L)$.*

Proof. Assume, towards a contradiction, that $(a_w)_{w \in L} \notin \ell_1(L)$. Then, for each $n < \omega$, there exists a finite set $F_n \subset L$ such that $\sum_{w \in F_n} |a_w| \geq n$. Let $\rho < \omega_1$ be such that $\bigcup_{n < \omega} F_n \subset L_\rho$. It follows that $\sup\{\sum_{w \in F} |a_w| : F \subset L_\rho \text{ finite}\} = \infty$, which contradicts the assumption that $(a_w)_{w \in L_\rho} \in \ell_1(L_\rho)$. ■

The next result was inspired by [4, Lemma 5.9] and [5, Theorem 4.8].

THEOREM 4.6. *Let $\nu : K \rightarrow \mathcal{F}_0(K, K)$ be a weak*-continuous function vanishing at infinity. Then there exist $(a_z)_{z \in L}, (b_z)_{z \in L} \in \ell_1(L)$ such that, for each $z \in L$, there exists $\gamma_z < \omega_1$ with the property that, for all $x, y \in L \setminus L_{\gamma_z}$,*

$$\nu_x(z, y) = \begin{cases} a_z & \text{if } x = y, \\ 0 & \text{if } x \neq y, \end{cases} \quad \nu_x(y, z) = \begin{cases} b_z & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Proof. Let $z \in L$ be arbitrary, and fix a sequence of compact clopen neighborhoods $V_1 \supset V_2 \supset \dots$ of z in L such that $\bigcap_{n < \omega} V_n = \{z\}$. For each $n < \omega$, define the functions $\nu^{(1,n)}, \nu^{(2,n)} : K \rightarrow \mathcal{M}_0(K)$ by

$$\nu_x^{(1,n)}(U) = \nu_x(V_n, U), \quad \nu_x^{(2,n)}(U) = \nu_x(U, V_n).$$

These functions are weak*-continuous and vanish at infinity (in the sense presented in Section 2). We analyze the sequence $\{\nu^{(1,n)}\}_{n < \omega}$ and the arguments for $\{\nu^{(2,n)}\}_{n < \omega}$ are analogous. Since L is a collapsing space, for each $n < \omega$ Proposition 2.5 yields $s_n \in \mathbb{R}$ and $\beta_n < \omega_1$ such that

$$\nu_x^{(1,n)}(y) = \begin{cases} s_n & \text{if } x = y, \\ 0 & \text{if } x \neq y, \end{cases}$$

for all $x, y \in L \setminus L_{\beta_n}$. Define $\xi_1(z) = \sup\{\beta_n : n < \omega\}$. Let $m, n < \omega$, pick $x \in L \setminus L_{\xi_1(z)}$ and let $\nu_x^x \in \mathcal{M}_0(K)$ be the marginal measure $\nu_x^x(U) =$

$\nu_x(U, \{x\})$. If $n \leq m$, then

$$\begin{aligned} |s_n - s_m| &= |\nu_x^{(1,n)}(x) - \nu_x^{(1,m)}(x)| = |\nu_x(V_n, \{x\}) - \nu_x(V_m, \{x\})| \\ &= |\nu_x(V_n \setminus V_m, \{x\})| = |\nu_x^x(V_n \setminus V_m)| \leq |\nu_x^x(V_n \setminus \{z\})|. \end{aligned}$$

Since $\bigcap_{n < \omega} (V_n \setminus \{z\}) = \emptyset$, it follows that $\lim_{n \rightarrow \infty} |\nu_x^x(V_n \setminus \{z\})| = 0$, so the sequence $\{s_n\}_{n < \omega}$ converges to some limit $a_z \in \mathbb{R}$. Hence, for each $x \in L \setminus L_{\xi_1(z)}$ and $n < \omega$,

$$\begin{aligned} |\nu_x(z, x) - a_z| &\leq |\nu_x(z, x) - s_n| + |s_n - a_z| \leq |\nu_x(V_n \setminus \{z\}, \{x\})| + |s_n - a_z| \\ &\leq |\nu_x^x(V_n \setminus \{z\})| + |s_n - a_z|, \end{aligned}$$

which implies $\nu_x(z, x) = a_z$. Now let $x, y \in L \setminus L_{\xi_1(z)}$ with $x \neq y$, and fix $n < \omega$. If $\nu_x^y \in \mathcal{M}_0(K)$ denotes the marginal measure $\nu_x^y(U) = \nu_x(U, \{y\})$, we have

$$\begin{aligned} |\nu_x(z, y)| &= |\nu_x^{(1,n)}(y)| + |\nu_x^{(1,n)}(y) - \nu_x(z, y)| \\ &= |\nu_x(V_n \setminus \{z\}, \{y\})| \leq |\nu_x^y(V_n \setminus \{z\})|. \end{aligned}$$

Since $\bigcap_{n < \omega} (V_n \setminus \{z\}) = \emptyset$, it follows that $\nu_x(z, y) = 0$.

Applying the same reasoning to $\{\nu^{(2,n)}\}_n$, we obtain an ordinal $\xi_2(z)$ and a real number b_z such that, whenever $x, y \in L \setminus L_{\xi_2(z)}$, if $x = y$ then $\nu_x(y, z) = b_z$, and $\nu_x(y, z) = 0$ if $x \neq y$. We set $\gamma_z = \max\{\xi_1(z), \xi_2(z)\}$.

Finally, we prove that $(a_z)_{z \in L}$ and $(b_z)_{z \in L}$ belong to $\ell_1(L)$. Let $G : C_0(L) \times C_0(L) \rightarrow C_0(L)$ be the bilinear operator induced by ν as in Theorem 4.2. Let $\beta < \omega_1$ be arbitrary and set $\rho = \sup\{\gamma_z : z \in L_\beta\}$. Pick $t \in L \setminus L_\rho$ and consider the bilinear form $G_t : C_0(L) \times C_0(L) \rightarrow \mathbb{R}$ given by $G_t(f, g) = G(f, g)(t)$. If $\tilde{G}_t$ denotes the extension of G_t to $\ell_\infty(L) \times \ell_\infty(L)$ as in Theorem 3.3, then the formula

$$\begin{aligned} \varphi(f) &= \tilde{G}_t(f, \chi_t) = \sum_{z \in L_\beta} \left(\sum_{y \in K} \nu_t(z, y) \chi_t(y) \right) f(z) \\ &= \sum_{z \in L_\beta} \nu_t(z, t) f(z) = \sum_{z \in L_\beta} a_z f(z) \end{aligned}$$

defines a bounded linear functional $\varphi : C_0(L_\rho) \rightarrow \mathbb{R}$. Therefore $(a_z)_{z \in L_\beta}$ belongs to $\ell_1(L_\rho)$ (see [20, Corollary 19.7.7]). From Proposition 4.5 we deduce that $(a_z)_{z \in L}$ lies in $\ell_1(L)$, and the same reasoning applies to $(b_z)_{z \in L}$. ■

For the next two lemmas, recalling that $L_\alpha = \varphi^{-1}[\alpha + 1]$ as presented in Section 2, we denote $L_\alpha^* = \varphi^{-1}[\alpha] = \{x \in L : \varphi(x) \in \alpha\}$.

LEMMA 4.7. *Let $\nu : K \rightarrow \mathcal{F}_0(K, K)$ be a weak*-continuous function vanishing at infinity. Suppose that, for each $z \in L$, there exists $\gamma_z < \omega_1$ such that, for all $x, y \in L \setminus L_{\gamma_z}$, we have $\nu_x(y, z) = \nu_x(z, y) = 0$. Then, for every $\xi < \omega_1$, there exists $\xi \leq \rho < \omega_1$ such that, for every $t \in L \setminus L_\rho$ and every $(u, v) \in (L_\rho^* \times K \setminus L_\rho^*) \cup (K \setminus L_\rho^* \times L_\rho^*) \cup (L_\rho^* \times L_\rho^*)$, one has $\nu_t(u, v) = 0$.*

Proof. Fix $\xi < \omega_1$ and set $\rho_0 = \xi$. Given $n < \omega$, suppose that ρ_n has been defined. By Proposition 4.4, there exists $\rho_n \leq \theta < \omega_1$ such that $\text{Var}(\nu_y)(L_{\rho_n}, L_{\rho_n}) = 0$ for every $y \in L \setminus L_\theta$. Moreover, by hypothesis, for each $z \in L_{\rho_n}$ there exists $\theta \leq \gamma_z < \omega_1$ such that $\nu_x(y, z) = \nu_x(z, y) = 0$ whenever $x, y \in L \setminus L_{\gamma_z}$. Define

$$\rho_{n+1} = \sup \{\gamma_z : z \in L_{\rho_n}\}.$$

By recursion, this yields a strictly increasing sequence $\rho_0 < \rho_1 < \rho_2 < \dots$ such that, for every $z \in L_{\rho_n}$ and all $x, y \in K \setminus L_{\rho_{n+1}}$, one has $\nu_x(y, z) = \nu_x(z, y) = 0$ and $\nu_x(z, z) = 0$.

Let $\rho = \sup \{\rho_n : n < \omega\}$. If $(u, v) \in L_\rho^* \times (K \setminus L_\rho^*)$ and $t \in L \setminus L_\rho$, then some $\rho_n < \rho$ satisfies $u \in L_{\rho_n}$, while $t, v \in K \setminus L_\rho^* \subset K \setminus L_{\rho_{n+1}}$, giving $\nu_t(u, v) = 0$. A symmetric argument covers the case $(u, v) \in (K \setminus L_\rho^*) \times L_\rho^*$.

Finally, if $(u, v) \in L_\rho^* \times L_\rho^*$, then $(u, v) \in L_{\rho_n} \times L_{\rho_n}$ for some n , and for all $t \in L \setminus L_\rho \subset L \setminus L_{\rho_{n+1}}$ we obtain $\nu_t(u, v) = 0$. ■

LEMMA 4.8. *Let $\nu : K \rightarrow \mathcal{F}_0(K, K)$ be a weak*-continuous function vanishing at infinity. Suppose that, for each $z \in L$, there exists $\gamma_z < \omega_1$ such that, for all $x, y \in L \setminus L_{\gamma_z}$, we have $\nu_x(y, z) = \nu_x(z, y) = 0$. Let $\{x_\alpha\}_{\alpha < \omega_1}$, $\{y_\alpha\}_{\alpha < \omega_1}$, $\{z_\alpha\}_{\alpha < \omega_1}$ and $\{w_\alpha\}_{\alpha < \omega_1}$ be sequences of pairwise distinct points of L . Then, for each $\epsilon > 0$, there is a simultaneous refinement $\{x_{\alpha_\gamma}\}_{\gamma < \omega_1}$, $\{y_{\alpha_\gamma}\}_{\gamma < \omega_1}$, $\{z_{\alpha_\gamma}\}_{\gamma < \omega_1}$ and $\{w_{\alpha_\gamma}\}_{\gamma < \omega_1}$ of these sequences (see Definition 2.6), together with a sequence $\{I_\gamma\}_{\gamma < \omega_1}$ of finite subsets of L , such that*

- (Ba) $\{x_{\alpha_\gamma}, y_{\alpha_\gamma}, z_{\alpha_\gamma}, w_{\alpha_\gamma}\} \subset I_\gamma$ for all $\gamma < \omega_1$;
- (Bb) $I_\gamma \cap I_\xi = \emptyset$ whenever $\gamma \neq \xi$;
- (Bc) $\text{Var}(\nu_{x_{\alpha_\gamma}})(\{y_{\alpha_\gamma}\}, K \setminus I_\gamma) + \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus I_\gamma, \{z_{\alpha_\gamma}\})$
 $+ \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus I_\gamma, K \setminus I_\gamma) < \epsilon$;
- (Bd) $\text{Var}(\nu_{x_{\alpha_\gamma}})(\{w_{\alpha_\gamma}\}, K \setminus I_\gamma) + \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus I_\gamma, \{w_{\alpha_\gamma}\}) < \epsilon$.

Additionally, if $n < \omega$ is a fixed natural number, we may assume $|I_\gamma| \geq n$ for each $\gamma < \omega_1$.

Proof. We proceed by transfinite induction. Let $\gamma < \omega_1$ be arbitrary and assume that we have obtained sequences $\{x_{\alpha_\xi}\}_{\xi < \gamma}$, $\{y_{\alpha_\xi}\}_{\xi < \gamma}$, $\{z_{\alpha_\xi}\}_{\xi < \gamma}$, $\{w_{\alpha_\xi}\}_{\xi < \gamma}$, and a sequence $\{I_\xi\}_{\xi < \gamma}$ of finite subsets of L satisfying conditions (Ba)–(Bd), together with $|I_\xi| \geq n$ for each $\xi < \gamma$, where n is a fixed natural number. Choose $\theta < \omega_1$ such that

$$\bigcup_{\xi < \gamma} I_\xi \times I_\xi \subset L_\theta \times L_\theta.$$

By Lemma 4.7, there exists $\theta \leq \rho < \omega_1$ such that, for each $t \in L \setminus L_\rho$ and all

$$(4.2) \quad (u, v) \in (L_\rho^* \times (K \setminus L_\rho^*)) \cup ((K \setminus L_\rho^*) \times L_\rho^*) \cup (L_\rho^* \times L_\rho^*),$$

we have $\nu_t(u, v) = 0$.

We now choose $\alpha_\gamma < \omega_1$, strictly greater than $\sup\{\alpha_\xi : \xi < \gamma\}$, such that $x_{\alpha_\gamma}, y_{\alpha_\gamma}, z_{\alpha_\gamma}, w_{\alpha_\gamma} \in L \setminus L_\rho$. According to Theorem 3.9, we may obtain a finite set $J_0 \subset L$ containing the points $x_{\alpha_\gamma}, y_{\alpha_\gamma}, z_{\alpha_\gamma}, w_{\alpha_\gamma}$ such that for every $J \subset L$ containing J_0 we have

$$\begin{aligned} \text{Var}(\nu_{x_{\alpha_\gamma}})(\{y_{\alpha_\gamma}\}, K \setminus J) + \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus J, \{z_{\alpha_\gamma}\}) \\ + \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus J, K \setminus J) < \epsilon. \end{aligned}$$

Next, we let μ_1 and μ_2 denote the marginal measures defined by $\mu_1(E) = \nu_{x_{\alpha_\gamma}}(\{w_{\alpha_\gamma}\}, E)$ and $\mu_2(E) = \nu_{x_{\alpha_\gamma}}(E, \{w_{\alpha_\gamma}\})$ for every $E \in \mathfrak{B}(K)$, with total variation (in the usual sense of Radon measures) denoted $|\mu_1|$ and $|\mu_2|$, respectively. Since μ_1 and μ_2 are atomic measures, there exist finite sets $J_1, J_2 \subset K$ such that, for every $J \supset (J_1 \cup J_2)$,

$$\begin{aligned} \text{Var}(\nu_{x_{\alpha_\gamma}})(\{w_{\alpha_\gamma}\}, K \setminus J) &= |\mu_1|(K \setminus J) < \epsilon/2, \\ \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus J, \{w_{\alpha_\gamma}\}) &= |\mu_2|(K \setminus J) < \epsilon/2. \end{aligned}$$

We then fix a finite set J containing $J_0 \cup J_1 \cup J_2$ such that $|J \setminus L_\rho^*| \geq n$ and let $I_\gamma = J \setminus L_\rho^*$. Since $K \setminus I_\gamma = (K \setminus J) \cup (J \cap L_\rho^*)$, by the choice of ρ and by condition (4.2), we have

$$\begin{aligned} \text{Var}(\nu_{x_{\alpha_\gamma}})(\{y_{\alpha_\gamma}\}, K \setminus I_\gamma) \\ = \text{Var}(\nu_{x_{\alpha_\gamma}})(\{y_{\alpha_\gamma}\}, K \setminus J) + \text{Var}(\nu_{x_{\alpha_\gamma}})(\{y_{\alpha_\gamma}\}, J \cap L_\rho^*) \\ = \text{Var}(\nu_{x_{\alpha_\gamma}})(\{y_{\alpha_\gamma}\}, K \setminus J). \end{aligned}$$

With a similar argument, we may deduce that

$$\begin{aligned} \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus I_\gamma, \{z_{\alpha_\gamma}\}) &= \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus J, \{z_{\alpha_\gamma}\}), \\ \text{Var}(\nu_{x_{\alpha_\gamma}})(\{w_{\alpha_\gamma}\}, K \setminus I_\gamma) &= \text{Var}(\nu_{x_{\alpha_\gamma}})(\{w_{\alpha_\gamma}\}, K \setminus J), \\ \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus I_\gamma, \{w_{\alpha_\gamma}\}) &= \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus J, \{w_{\alpha_\gamma}\}). \end{aligned}$$

Next, by the subadditivity of $\text{Var}(\nu_{x_{\alpha_\gamma}})$ we can write

$$\begin{aligned} \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus I_\gamma, K \setminus I_\gamma) \\ \leq \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus J, K \setminus J) + \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus J, J \cap L_\rho^*) \\ + \text{Var}(\nu_{x_{\alpha_\gamma}})(J \cap L_\rho^*, K \setminus J) + \text{Var}(\nu_{x_{\alpha_\gamma}})(J \cap L_\rho^*, J \cap L_\rho^*). \end{aligned}$$

By monotonicity of $\text{Var}(\nu_{x_{\alpha_\gamma}})$ and by the choice of ρ , using (4.2), we deduce that

$$\text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus I_\gamma, K \setminus I_\gamma) = \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus J, K \setminus J).$$

Therefore, conditions (Ba)–(Bd) are satisfied for $\{x_{\alpha_\xi}\}_{\xi \leq \gamma}$, $\{y_{\alpha_\xi}\}_{\xi \leq \gamma}$, $\{z_{\alpha_\xi}\}_{\xi \leq \gamma}$, $\{w_{\alpha_\xi}\}_{\xi \leq \gamma}$, and $\{I_\xi\}_{\xi \leq \gamma}$. The result then follows by transfinite induction. ■

REMARK 4.9. In the previous lemma, the sequences $\{x_\alpha\}_{\alpha < \omega_1}$, $\{y_\alpha\}_{\alpha < \omega_1}$, $\{z_\alpha\}_{\alpha < \omega_1}$, and $\{w_\alpha\}_{\alpha < \omega_1}$ need not be pairwise distinct; in fact, two or more of them may coincide. We also emphasize that condition (Bd) of the lemma will only play a role in Theorem 4.13.

THEOREM 4.10. *Let $\nu : K \rightarrow \mathcal{F}_0(K, K)$ be weak*-continuous vanishing at infinity. Suppose that, for each $z \in L$, there exists $\gamma_z < \omega_1$ such that, for all $x, y \in L \setminus L_{\gamma_z}$, we have $\nu_x(y, z) = \nu_x(z, y) = 0$. Then there exists $\rho < \omega_1$ such that, for all $x, y, z \in L \setminus L_\rho$, if $x \notin \{y, z\}$ then $\nu_x(y, z) = 0$.*

Proof. If the theorem were false, for some $\epsilon > 0$ we could obtain transfinite sequences of pairwise distinct points $\{x_\alpha\}_{\alpha < \omega_1}$, $\{y_\alpha\}_{\alpha < \omega_1}$, and $\{z_\alpha\}_{\alpha < \omega_1}$ such that, for each $\alpha < \omega_1$, we have $x_\alpha \notin \{y_\alpha, z_\alpha\}$ and $|\nu_{x_\alpha}(y_\alpha, z_\alpha)| \geq \epsilon$.

By employing Lemma 4.8, we may obtain a simultaneous refinement of these sequences, together with a sequence $\{I_\alpha\}_{\alpha < \omega_1}$ of finite subsets of L of cardinality strictly greater than 3, such that

- (Ca) $\{x_\alpha, y_\alpha, z_\alpha\} \subset I_\alpha$ for all $\alpha < \omega_1$;
- (Cb) $I_\alpha \cap I_\beta = \emptyset$ whenever $\alpha \neq \beta$;
- (Cc) $\text{Var}(\nu_{x_\alpha})(\{y_\alpha\}, K \setminus I_\alpha) + \text{Var}(\nu_{x_\alpha})(K \setminus I_\alpha, \{z_\alpha\})$
 $+ \text{Var}(\nu_{x_\alpha})(K \setminus I_\alpha, K \setminus I_\alpha) \leq \epsilon/2$.

We may take another simultaneous refinement so that either $y_\alpha \neq z_\alpha$ for all $\alpha < \omega_1$, or $y_\alpha = z_\alpha$ for all $\alpha < \omega_1$. We proceed with the proof for the former case, and the latter can be obtained with simple adjustments. By taking a further simultaneous refinement, we may assume that $|I_\alpha| = m + 3$ (for some $m > 0$) for each $\alpha < \omega_1$. Denoting $I_\alpha = \{y_\alpha, z_\alpha, x_\alpha, w_1^\alpha, \dots, w_m^\alpha\}$, we may consider the net $\mathcal{C} = \{(y_\alpha, z_\alpha, x_\alpha, w_1^\alpha, \dots, w_m^\alpha)\}_{\alpha < \omega_1}$ in L^{m+3} , which is $(\{1\}, \{2\}, \{3, 4, \dots, m+3\})$ -separated.

Since L is a collapsing space (see Definition 2.2) there exist $a, b \in L$, with $a \neq b$, such that $\ell = (a, b, \infty, \dots, \infty)$ is an accumulation point of $\mathcal{C}$. We then fix a subnet $\{(y_{\alpha_\gamma}, z_{\alpha_\gamma}, x_{\alpha_\gamma}, w_1^{\alpha_\gamma}, \dots, w_m^{\alpha_\gamma})\}_{\gamma \in \Gamma}$ of $\mathcal{C}$ converging to ℓ .

Next, we select disjoint clopen compact neighborhoods U of a and V of b , and, by passing to a subnet if necessary, we can assume that $U \cap I_{\alpha_\gamma} = \{y_{\alpha_\gamma}\}$ and $V \cap I_{\alpha_\gamma} = \{z_{\alpha_\gamma}\}$ for each $\gamma \in \Gamma$. Therefore, we have $U \setminus \{y_{\alpha_\gamma}\} \subset K \setminus I_{\alpha_\gamma}$ and $V \setminus \{z_{\alpha_\gamma}\} \subset K \setminus I_{\alpha_\gamma}$. Hence,

$$\begin{aligned}
 & |\nu_{x_{\alpha_\gamma}}(U, V)| \\
 & \geq |\nu_{x_{\alpha_\gamma}}(y_{\alpha_\gamma}, z_{\alpha_\gamma})| - (|\nu_{x_{\alpha_\gamma}}(\{y_{\alpha_\gamma}\}, V \setminus \{z_{\alpha_\gamma}\})| \\
 & \quad + |\nu_{x_{\alpha_\gamma}}(U \setminus \{y_{\alpha_\gamma}\}, \{z_{\alpha_\gamma}\})| + |\nu_{x_{\alpha_\gamma}}(U \setminus \{y_{\alpha_\gamma}\}, V \setminus \{z_{\alpha_\gamma}\})|) \\
 & \geq \epsilon - (\text{Var}(\nu_{x_{\alpha_\gamma}})(\{y_{\alpha_\gamma}\}, V \setminus \{z_{\alpha_\gamma}\}) \\
 & \quad + \text{Var}(\nu_{x_{\alpha_\gamma}})(U \setminus \{y_{\alpha_\gamma}\}, \{z_{\alpha_\gamma}\}) + \text{Var}(\nu_{x_{\alpha_\gamma}})(U \setminus \{y_{\alpha_\gamma}\}, V \setminus \{z_{\alpha_\gamma}\}))
 \end{aligned}$$

$$\begin{aligned}
&\geq \epsilon - (\text{Var}(\nu_{x_{\alpha_\gamma}})(\{y_{\alpha_\gamma}\}, K \setminus I_{\alpha_\gamma}) \\
&\quad + \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus I_{\alpha_\gamma}, \{z_{\alpha_\gamma}\}) + \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus I_{\alpha_\gamma}, K \setminus I_{\alpha_\gamma})) \\
&> \epsilon - \epsilon/2 = \epsilon/2.
\end{aligned}$$

From the weak*-continuity of the function ν and since $x_{\alpha_\gamma} \rightarrow \infty$, we obtain $|\nu_\infty(U, V)| \geq \epsilon/2$, but this is a contradiction since $\nu_\infty = 0$. ■

THEOREM 4.11. *Let $\nu : K \rightarrow \mathcal{F}_0(K, K)$ be weak*-continuous vanishing at infinity. Suppose that, for each $z \in L$, there exists $\gamma_z < \omega_1$ such that, for all $x, y \in L \setminus L_{\gamma_z}$, we have $\nu_x(y, z) = \nu_z(x, y) = 0$. Then there are $r \in \mathbb{R}$ and $\rho < \omega_1$ such that $\nu_y(y, y) = r$ for all $y \in L \setminus L_\rho$.*

Proof. Towards a contradiction, assume that for every $r \in \mathbb{R}$ and every $\beta < \omega_1$, there exists $y \in L \setminus L_\beta$ such that $\nu_y(y, y) \neq r$. By [5, Lemma 4.6], this yields rational numbers $p < q$ and uncountable sets $A, B \subset L$ such that

$$\nu_y(y, y) < p < q < \nu_z(z, z) \quad \text{for all } y \in A, z \in B.$$

As A and B are uncountable, we may construct sequences $\{y_\alpha\}_{\alpha < \omega_1} \subset A$ and $\{z_\alpha\}_{\alpha < \omega_1} \subset B$ of pairwise distinct points. Then, applying Lemma 4.8 individually to each of these sequences (see Remark 4.9), we obtain subsequences $\{y_\alpha\}_{\alpha < \omega_1} \subset A$ and $\{z_\alpha\}_{\alpha < \omega_1} \subset B$ of pairwise distinct points, together with sequences $\{I_\alpha\}_{\alpha < \omega_1}$ and $\{J_\alpha\}_{\alpha < \omega_1}$ of finite subsets of L with cardinality strictly greater than 2, such that

- (Da) $y_\alpha \in I_\alpha$ and $z_\alpha \in J_\alpha$ for all $\alpha < \omega_1$;
- (Db) $I_\alpha \cap I_\beta = \emptyset$ and $J_\alpha \cap J_\beta = \emptyset$ whenever $\alpha \neq \beta$;
- (Dc) for all $\alpha < \omega_1$,

$$\begin{aligned}
&\text{Var}(\nu_{y_\alpha})(\{y_\alpha\}, K \setminus I_\alpha) + \text{Var}(\nu_{y_\alpha})(K \setminus I_\alpha, \{y_\alpha\}) \\
&\quad + \text{Var}(\nu_{y_\alpha})(K \setminus I_\alpha, K \setminus I_\alpha) \leq (q - p)/3,
\end{aligned}$$

and

$$\begin{aligned}
&\text{Var}(\nu_{z_\alpha})(\{z_\alpha\}, K \setminus J_\alpha) + \text{Var}(\nu_{z_\alpha})(K \setminus J_\alpha, \{z_\alpha\}) \\
&\quad + \text{Var}(\nu_{z_\alpha})(K \setminus J_\alpha, K \setminus J_\alpha) \leq (q - p)/3.
\end{aligned}$$

Moreover, after another simultaneous refinement, we may assume that $\{I_\alpha \cup J_\alpha\}_{\alpha < \omega_1}$ is a sequence of pairwise disjoint sets with $|I_\alpha \cup J_\alpha| = n + 2$ (for some $n > 0$) for every $\alpha < \omega_1$. Indeed, by applying the Δ -System Lemma [14, Chapter II, Theorem 1.5], we may assume that $\{I_\alpha \cup J_\alpha\}_{\alpha < \omega_1}$ forms a Δ -system with root R , and that all members of the sequence have the same cardinality, strictly greater than 2. Then condition (Db) forces $R = \emptyset$.

Let us assume that $I_\alpha \cup J_\alpha = \{y_\alpha, z_\alpha, w_1^\alpha, \dots, w_n^\alpha\}$ for all $\alpha < \omega_1$. This allows us to consider the net $\mathcal{C} = \{(y_\alpha, z_\alpha, w_1^\alpha, \dots, w_n^\alpha)\}_{\alpha < \omega_1}$ in L^{n+2} , which is $(\{1, 2\}, \{3, 4, \dots, n + 2\})$ -separated.

Since L has the collapsing property (see Definition 2.2), there exists a point $a \in L$ such that $\ell = (a, a, \infty, \dots, \infty)$ is an accumulation point of $\mathcal{C}$. We then fix a subnet $\{(y_{\alpha_\gamma}, z_{\alpha_\gamma}, w_1^{\alpha_\gamma}, \dots, w_n^{\alpha_\gamma})\}_{\gamma \in \Gamma}$ of $\mathcal{C}$ converging to ℓ .

Finally, we fix a clopen neighborhood V of a and, if necessary, by passing to a subnet we may assume that $V \cap I_{\alpha_\gamma} = \{y_{\alpha_\gamma}\}$ and $V \cap J_{\alpha_\gamma} = \{z_{\alpha_\gamma}\}$ for all $\gamma \in \Gamma$. It then follows that

$$V \setminus \{y_{\alpha_\gamma}\} \subset K \setminus I_{\alpha_\gamma} \quad \text{and} \quad V \setminus \{z_{\alpha_\gamma}\} \subset K \setminus J_{\alpha_\gamma},$$

which in turn allows us to write

$$\begin{aligned} \nu_{y_{\alpha_\gamma}}(V, V) &\leq \nu_{y_{\alpha_\gamma}}(y_{\alpha_\gamma}, y_{\alpha_\gamma}) + (|\nu_{y_{\alpha_\gamma}}(\{y_{\alpha_\gamma}\}, V \setminus \{y_{\alpha_\gamma}\})| \\ &\quad + |\nu_{y_{\alpha_\gamma}}(V \setminus \{y_{\alpha_\gamma}\}, \{y_{\alpha_\gamma}\})| + |\nu_{y_{\alpha_\gamma}}(V \setminus \{y_{\alpha_\gamma}\}, V \setminus \{y_{\alpha_\gamma}\})|) \\ &\leq p + (\text{Var}(\nu_{y_{\alpha_\gamma}})(\{y_{\alpha_\gamma}\}, V \setminus \{y_{\alpha_\gamma}\}) \\ &\quad + \text{Var}(\nu_{y_{\alpha_\gamma}})(V \setminus \{y_{\alpha_\gamma}\}, \{y_{\alpha_\gamma}\}) + \text{Var}(\nu_{y_{\alpha_\gamma}})(V \setminus \{y_{\alpha_\gamma}\}, V \setminus \{y_{\alpha_\gamma}\})) \\ &\leq p + (\text{Var}(\nu_{y_{\alpha_\gamma}})(\{y_{\alpha_\gamma}\}, K \setminus I_{\alpha_\gamma}) \\ &\quad + \text{Var}(\nu_{y_{\alpha_\gamma}})(K \setminus I_{\alpha_\gamma}, \{y_{\alpha_\gamma}\}) + \text{Var}(\nu_{y_{\alpha_\gamma}})(K \setminus I_{\alpha_\gamma}, K \setminus I_{\alpha_\gamma})) \\ &< p + \frac{q-p}{3} = \frac{2p+q}{3}, \end{aligned}$$

and similar arguments also show that

$$\begin{aligned} \nu_{z_{\alpha_\gamma}}(V, V) &\geq \nu_{z_{\alpha_\gamma}}(z_{\alpha_\gamma}, z_{\alpha_\gamma}) - (|\nu_{z_{\alpha_\gamma}}(\{z_{\alpha_\gamma}\}, V \setminus \{z_{\alpha_\gamma}\})| \\ &\quad + |\nu_{z_{\alpha_\gamma}}(V \setminus \{z_{\alpha_\gamma}\}, \{z_{\alpha_\gamma}\})| + |\nu_{z_{\alpha_\gamma}}(V \setminus \{z_{\alpha_\gamma}\}, V \setminus \{z_{\alpha_\gamma}\})|) \\ &\geq q - (\text{Var}(\nu_{z_{\alpha_\gamma}})(\{z_{\alpha_\gamma}\}, V \setminus \{z_{\alpha_\gamma}\}) \\ &\quad + \text{Var}(\nu_{z_{\alpha_\gamma}})(V \setminus \{z_{\alpha_\gamma}\}, \{z_{\alpha_\gamma}\}) + \text{Var}(\nu_{z_{\alpha_\gamma}})(V \setminus \{z_{\alpha_\gamma}\}, V \setminus \{z_{\alpha_\gamma}\})) \\ &\geq q - (\text{Var}(\nu_{z_{\alpha_\gamma}})(\{z_{\alpha_\gamma}\}, K \setminus J_{\alpha_\gamma}) \\ &\quad + \text{Var}(\nu_{z_{\alpha_\gamma}})(K \setminus J_{\alpha_\gamma}, \{z_{\alpha_\gamma}\}) + \text{Var}(\nu_{z_{\alpha_\gamma}})(K \setminus J_{\alpha_\gamma}, K \setminus J_{\alpha_\gamma})) \\ &> q - \frac{q-p}{3} = \frac{2q+p}{3}. \end{aligned}$$

Recalling that ν is weak*-continuous and that V is a clopen compact subset of L , and since both nets $\{y_{\alpha_\gamma}\}_{\gamma \in \Gamma}$ and $\{z_{\alpha_\gamma}\}_{\gamma \in \Gamma}$ converge to a , the preceding inequalities yield

$$\begin{aligned} \nu_a(V, V) &= \lim_{\gamma \rightarrow \infty} \nu_{y_{\alpha_\gamma}}(V, V) \leq \frac{2p+q}{3} \\ &< \frac{2q+p}{3} \leq \lim_{\gamma \rightarrow \infty} \nu_{z_{\alpha_\gamma}}(V, V) = \nu_a(V, V), \end{aligned}$$

which is a contradiction. ■

LEMMA 4.12. *Let $\nu : K \rightarrow \mathcal{F}_0(K, K)$ be a weak*-continuous function vanishing at infinity. Suppose that, for each $z \in L$, there exists $\gamma_z < \omega_1$ such that, for all $x, y \in L \setminus L_{\gamma_z}$, we have $\nu_x(y, z) = \nu_x(z, y) = 0$. Let $\{y_\alpha\}_{\alpha < \omega_1}$ and $\{z_\alpha\}_{\alpha < \omega_1}$ be sequences of pairwise distinct points of L with $y_\alpha \neq z_\alpha$ for each $\alpha < \omega_1$. Then there is a simultaneous refinement $\{y_{\alpha_\gamma}\}_{\gamma < \omega_1}$ and $\{z_{\alpha_\gamma}\}_{\gamma < \omega_1}$ together with a new sequence $\{x_\gamma\}_{\gamma < \omega_1}$ of pairwise distinct points of L such that, for all $\gamma, \xi, \eta < \omega_1$, the points $x_\gamma, y_{\alpha_\xi}, z_{\alpha_\eta}$ are pairwise distinct and*

- (Ea) $\nu_{y_{\alpha_\gamma}}(x_\gamma, z_{\alpha_\gamma}) = \nu_{y_{\alpha_\gamma}}(z_{\alpha_\gamma}, x_\gamma) = \nu_{z_{\alpha_\gamma}}(x_\gamma, y_{\alpha_\gamma}) = \nu_{z_{\alpha_\gamma}}(y_{\alpha_\gamma}, x_\gamma) = 0$;
- (Eb) $\nu_{x_\gamma}(y_{\alpha_\gamma}, x_\gamma) = \nu_{x_\gamma}(x_\gamma, y_{\alpha_\gamma}) = \nu_{x_\gamma}(z_{\alpha_\gamma}, x_\gamma) = \nu_{x_\gamma}(x_\gamma, z_{\alpha_\gamma}) = 0$.

Proof. Fix $\gamma < \omega_1$ and assume that we have obtained subsequences $\{y_{\alpha_\xi}\}_{\xi < \gamma}$ and $\{z_{\alpha_\xi}\}_{\xi < \gamma}$, and also a sequence $\{x_\xi\}_{\xi < \gamma}$, satisfying conditions (Ea) and (Eb) for each $\xi < \gamma$.

Choose α_γ strictly greater than $\sup\{\alpha_\xi : \xi < \gamma\}$ so that y_{α_γ} and z_{α_γ} avoid $\{x_\xi, y_{\alpha_\xi}, z_{\alpha_\xi} : \xi < \gamma\}$ and let $\rho_1 < \omega_1$ be such that

$$\{y_{\alpha_\xi} : \xi \leq \gamma\} \cup \{z_{\alpha_\xi} : \xi \leq \gamma\} \cup \{x_\xi : \xi < \gamma\} \subset L_{\rho_1}.$$

According to Proposition 3.7, we may fix $\rho_2 < \omega_1$ such that

$$\bigcup_{\xi \leq \gamma} (\text{Supp}(\nu_{y_{\alpha_\xi}}) \cup \text{Supp}(\nu_{z_{\alpha_\xi}})) \subset L_{\rho_2} \times L_{\rho_2}.$$

It follows that, for any $x \in L \setminus L_{\rho_2}$,

$$\nu_{y_{\alpha_\xi}}(x, z_{\alpha_\xi}) = \nu_{y_{\alpha_\xi}}(z_{\alpha_\xi}, x) = \nu_{z_{\alpha_\xi}}(x, y_{\alpha_\xi}) = \nu_{z_{\alpha_\xi}}(y_{\alpha_\xi}, x) = 0$$

for each $\xi \leq \gamma$. Moreover, by our hypothesis, there exists $\rho_3 < \omega_1$ such that, for every $x \in L \setminus L_{\rho_3}$,

$$\nu_x(y_{\alpha_\gamma}, x) = \nu_x(x, y_{\alpha_\gamma}) = \nu_x(z_{\alpha_\gamma}, x) = \nu_x(x, z_{\alpha_\gamma}) = 0.$$

We then choose $x_\gamma \in L \setminus (L_{\rho_1} \cup L_{\rho_2} \cup L_{\rho_3})$. By construction, $x_\gamma, y_{\alpha_\gamma}, z_{\alpha_\gamma}$ are pairwise distinct, and (Ea) and (Eb) are satisfied. This completes the inductive step.

Hence, by transfinite recursion, the sequences $\{y_{\alpha_\gamma}\}_{\gamma < \omega_1}$, $\{z_{\alpha_\gamma}\}_{\gamma < \omega_1}$, and $\{x_\gamma\}_{\gamma < \omega_1}$ can be constructed as required. ■

THEOREM 4.13. *Let $\nu : K \rightarrow \mathcal{F}_0(K, K)$ be a weak*-continuous function vanishing at infinity. Suppose that, for each $z \in L$, there exists $\gamma_z < \omega_1$ such that, for all $x, y \in L \setminus L_{\gamma_z}$, we have $\nu_x(y, z) = \nu_x(z, y) = 0$. Let $\{y_\alpha\}_{\alpha < \omega_1}$ and $\{z_\alpha\}_{\alpha < \omega_1}$ be sequences of pairwise distinct points in L such that $y_\alpha \neq z_\alpha$ for each $\alpha < \omega_1$. Then there exists $\rho < \omega_1$ such that, for each $\alpha \geq \rho$,*

$$\nu_{y_\alpha}(y_\alpha, z_\alpha) = \nu_{y_\alpha}(z_\alpha, y_\alpha) = 0.$$

Proof. If the theorem were false, by taking a simultaneous refinement of the sequences, we could find $\epsilon > 0$ such that either $|\nu_{y_\alpha}(y_\alpha, z_\alpha)| \geq \epsilon$ for all $\alpha < \omega_1$, or $|\nu_{y_\alpha}(z_\alpha, y_\alpha)| \geq \epsilon$ for all $\alpha < \omega_1$. We assume, without loss

of generality, the former situation; the latter can be treated with a similar argument with only minor adjustments.

By employing Lemma 4.12, we can find another simultaneous refinement, together with a sequence of pairwise distinct points $\{x_\alpha\}_{\alpha < \omega_1}$, such that $x_\alpha, y_\beta, z_\gamma$ are pairwise distinct for all $\alpha, \beta, \gamma < \omega_1$, and

$$\begin{aligned}\nu_{y_\alpha}(x_\alpha, z_\alpha) &= \nu_{y_\alpha}(z_\alpha, x_\alpha) = \nu_{z_\alpha}(x_\alpha, y_\alpha) = \nu_{z_\alpha}(y_\alpha, x_\alpha) = 0, \\ \nu_{x_\alpha}(y_\alpha, x_\alpha) &= \nu_{x_\alpha}(x_\alpha, y_\alpha) = \nu_{x_\alpha}(z_\alpha, x_\alpha) = \nu_{x_\alpha}(x_\alpha, z_\alpha) = 0\end{aligned}$$

for each $\alpha < \omega_1$.

Next, by applying Lemma 4.8 twice, we may obtain a simultaneous refinement of the above sequences, along with sequences $\{I_\alpha\}_{\alpha < \omega_1}$ and $\{J_\alpha\}_{\alpha < \omega_1}$ of finite subsets of L with cardinality strictly greater than 3, satisfying

- (Fa) $\{x_\alpha, y_\alpha, z_\alpha\} \subset I_\alpha \cap J_\alpha$ for all $\alpha < \omega_1$;
- (Fb) $I_\alpha \cap I_\beta = \emptyset$ and $J_\alpha \cap J_\beta = \emptyset$ whenever $\alpha \neq \beta$;
- (Fc) for all $\alpha < \omega_1$,

$$\begin{aligned}[\text{Var}(\nu_{x_\alpha})(\{y_\alpha\}, K \setminus I_\alpha) + \text{Var}(\nu_{x_\alpha})(K \setminus I_\alpha, \{z_\alpha\}) \\ + \text{Var}(\nu_{x_\alpha})(K \setminus I_\alpha, K \setminus I_\alpha)] \leq \epsilon/6,\end{aligned}$$

and

$$\begin{aligned}[\text{Var}(\nu_{y_\alpha})(\{y_\alpha\}, K \setminus J_\alpha) + \text{Var}(\nu_{y_\alpha})(K \setminus J_\alpha, \{z_\alpha\}) \\ + \text{Var}(\nu_{y_\alpha})(K \setminus J_\alpha, K \setminus J_\alpha)] \leq \epsilon/6;\end{aligned}$$

- (Fd) for all $\alpha < \omega_1$,

$$\max \{\text{Var}(\nu_{x_\alpha})(\{x_\alpha\}, K \setminus I_\alpha), \text{Var}(\nu_{y_\alpha})(\{x_\alpha\}, K \setminus J_\alpha)\} < \epsilon/6.$$

Note that we may assume that $\{I_\alpha \cup J_\alpha\}_{\alpha < \omega_1}$ is a sequence of pairwise disjoint sets with $|I_\alpha \cup J_\alpha| = n + 3$ (for some $n > 0$) for every $\alpha < \omega_1$. Indeed, by refining the sequence $\{I_\alpha \cup J_\alpha\}_{\alpha < \omega_1}$ via the Δ -System Lemma [14, Chapter II, Theorem 1.5], we may suppose that $\{I_\alpha \cup J_\alpha\}_{\alpha < \omega_1}$ forms a Δ -system with root R , and that all members of the sequence have the same cardinality strictly greater than 3. However, condition (Fb) forces $R = \emptyset$.

We suppose that $I_\alpha \cup J_\alpha = \{x_\alpha, y_\alpha, z_\alpha, w_1^\alpha, \dots, w_n^\alpha\}$, and consider the net

$$\mathcal{C} = \{(x_\alpha, y_\alpha, z_\alpha, w_1^\alpha, \dots, w_n^\alpha)\}_{\alpha < \omega_1}$$

in L^{n+3} , which is $(\{1, 2\}, \{3\}, \{4, \dots, n+3\})$ -separated.

Since L is a collapsing space (see Definition 2.2), there exist distinct $a, b \in L$ such that $\ell = (a, a, b, \infty, \dots, \infty)$ is an accumulation point of $\mathcal{C}$. We then fix a subnet $\{(x_{\alpha_\gamma}, y_{\alpha_\gamma}, z_{\alpha_\gamma}, w_1^{\alpha_\gamma}, \dots, w_n^{\alpha_\gamma})\}_{\gamma \in \Gamma}$ of $\mathcal{C}$ converging to ℓ .

Next, we choose disjoint clopen neighborhoods U of a and V of b , and, by passing to a subnet if necessary, we may assume that

$$U \cap I_{\alpha_\gamma} = U \cap J_{\alpha_\gamma} = \{x_{\alpha_\gamma}, y_{\alpha_\gamma}\} \quad \text{and} \quad V \cap I_{\alpha_\gamma} = V \cap J_{\alpha_\gamma} = \{z_{\alpha_\gamma}\}$$

for each $\gamma \in \Gamma$.

From condition (Fd), we have, for each $\gamma \in \Gamma$,

$$\begin{aligned}
& |\nu_{x_{\alpha_\gamma}}(U \setminus \{y_{\alpha_\gamma}\}, V \setminus \{z_{\alpha_\gamma}\})| \\
& \leq \text{Var}(\nu_{x_{\alpha_\gamma}})(U \setminus \{y_{\alpha_\gamma}\}, V \setminus \{z_{\alpha_\gamma}\}) \\
& \leq \text{Var}(\nu_{x_{\alpha_\gamma}})((K \setminus I_{\alpha_\gamma}) \cup \{x_{\alpha_\gamma}\}, K \setminus I_{\alpha_\gamma}) \\
& \leq \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus I_{\alpha_\gamma}, K \setminus I_{\alpha_\gamma}) + \text{Var}(\nu_{x_{\alpha_\gamma}})(\{x_{\alpha_\gamma}\}, K \setminus I_{\alpha_\gamma}) \\
& \leq \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus I_{\alpha_\gamma}, K \setminus I_{\alpha_\gamma}) + \epsilon/6.
\end{aligned}$$

And since $\nu_{x_{\alpha_\gamma}}(x_{\alpha_\gamma}, z_{\alpha_\gamma}) = 0$, we have

$$\begin{aligned}
\text{Var}(\nu_{x_{\alpha_\gamma}})(U \setminus \{y_{\alpha_\gamma}\}, \{z_{\alpha_\gamma}\}) & \leq \text{Var}(\nu_{x_{\alpha_\gamma}})((K \setminus I_{\alpha_\gamma}) \cup \{x_{\alpha_\gamma}\}, \{z_{\alpha_\gamma}\}) \\
& \leq \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus I_{\alpha_\gamma}, \{z_{\alpha_\gamma}\}) + |\nu_{x_{\alpha_\gamma}}(x_{\alpha_\gamma}, z_{\alpha_\gamma})| \\
& = \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus I_{\alpha_\gamma}, \{z_{\alpha_\gamma}\}).
\end{aligned}$$

Therefore, recalling $\nu_{x_{\alpha_\gamma}}(y_{\alpha_\gamma}, z_{\alpha_\gamma}) = 0$ and condition (Fc), we can write

$$\begin{aligned}
|\nu_{x_{\alpha_\gamma}}(U, V)| & \leq |\nu_{x_{\alpha_\gamma}}(y_{\alpha_\gamma}, z_{\alpha_\gamma})| + |\nu_{x_{\alpha_\gamma}}(\{y_{\alpha_\gamma}\}, V \setminus \{z_{\alpha_\gamma}\})| \\
& \quad + |\nu_{x_{\alpha_\gamma}}(U \setminus \{y_{\alpha_\gamma}\}, \{z_{\alpha_\gamma}\})| + |\nu_{x_{\alpha_\gamma}}(U \setminus \{y_{\alpha_\gamma}\}, V \setminus \{z_{\alpha_\gamma}\})| \\
& \leq \text{Var}(\nu_{x_{\alpha_\gamma}})(\{y_{\alpha_\gamma}\}, V \setminus \{z_{\alpha_\gamma}\}) + \text{Var}(\nu_{x_{\alpha_\gamma}})(U \setminus \{y_{\alpha_\gamma}\}, \{z_{\alpha_\gamma}\}) \\
& \quad + \text{Var}(\nu_{x_{\alpha_\gamma}})(U \setminus \{y_{\alpha_\gamma}\}, V \setminus \{z_{\alpha_\gamma}\}) \\
& \leq \text{Var}(\nu_{x_{\alpha_\gamma}})(\{y_{\alpha_\gamma}\}, K \setminus I_{\alpha_\gamma}) + \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus I_{\alpha_\gamma}, \{z_{\alpha_\gamma}\}) \\
& \quad + \text{Var}(\nu_{x_{\alpha_\gamma}})(K \setminus I_{\alpha_\gamma}, K \setminus I_{\alpha_\gamma}) + \epsilon/6 < \epsilon/3.
\end{aligned}$$

On the other hand, by condition (Fd), we have

$$\begin{aligned}
& |\nu_{y_{\alpha_\gamma}}(U \setminus \{y_{\alpha_\gamma}\}, V \setminus \{z_{\alpha_\gamma}\})| \\
& \leq \text{Var}(\nu_{y_{\alpha_\gamma}})(U \setminus \{y_{\alpha_\gamma}\}, V \setminus \{z_{\alpha_\gamma}\}) \\
& \leq \text{Var}(\nu_{y_{\alpha_\gamma}})((K \setminus J_{\alpha_\gamma}) \cup \{x_{\alpha_\gamma}\}, K \setminus J_{\alpha_\gamma}) \\
& \leq \text{Var}(\nu_{y_{\alpha_\gamma}})(K \setminus J_{\alpha_\gamma}, K \setminus J_{\alpha_\gamma}) + \text{Var}(\nu_{y_{\alpha_\gamma}})(\{x_{\alpha_\gamma}\}, K \setminus J_{\alpha_\gamma}) \\
& \leq \text{Var}(\nu_{y_{\alpha_\gamma}})(K \setminus J_{\alpha_\gamma}, K \setminus J_{\alpha_\gamma}) + \epsilon/6
\end{aligned}$$

and recalling that $\nu_{y_{\alpha_\gamma}}(x_{\alpha_\gamma}, z_{\alpha_\gamma}) = 0$, we can write

$$\begin{aligned}
\text{Var}(\nu_{y_{\alpha_\gamma}})((U \setminus \{y_{\alpha_\gamma}\}, \{z_{\alpha_\gamma}\}) & \leq \text{Var}(\nu_{y_{\alpha_\gamma}})((K \setminus J_{\alpha_\gamma}) \cup \{x_{\alpha_\gamma}\}, \{z_{\alpha_\gamma}\}) \\
& \leq \text{Var}(\nu_{y_{\alpha_\gamma}})(K \setminus J_{\alpha_\gamma}, \{z_{\alpha_\gamma}\}) + |\nu_{y_{\alpha_\gamma}}(x_{\alpha_\gamma}, z_{\alpha_\gamma})| \\
& = \text{Var}(\nu_{y_{\alpha_\gamma}})(K \setminus J_{\alpha_\gamma}, \{z_{\alpha_\gamma}\}).
\end{aligned}$$

Hence, since $|\nu_{y_{\alpha_\gamma}}(y_{\alpha_\gamma}, z_{\alpha_\gamma})| \geq \epsilon$, the relations above and condition (Fc)

imply

$$\begin{aligned}
|\nu_{y_{\alpha_\gamma}}(U, V)| &\geq |\nu_{y_{\alpha_\gamma}}(y_{\alpha_\gamma}, z_{\alpha_\gamma})| - (|\nu_{y_{\alpha_\gamma}}(\{y_{\alpha_\gamma}\}, V \setminus \{z_{\alpha_\gamma}\})| \\
&\quad + |\nu_{y_{\alpha_\gamma}}(U \setminus \{y_{\alpha_\gamma}\}, \{z_{\alpha_\gamma}\})| + |\nu_{y_{\alpha_\gamma}}(U \setminus \{y_{\alpha_\gamma}\}, V \setminus \{z_{\alpha_\gamma}\})|) \\
&\geq \epsilon - (\text{Var}(\nu_{y_{\alpha_\gamma}})(\{y_{\alpha_\gamma}\}, V \setminus \{z_{\alpha_\gamma}\}) \\
&\quad + \text{Var}(\nu_{y_{\alpha_\gamma}})(U \setminus \{y_{\alpha_\gamma}\}, \{z_{\alpha_\gamma}\}) + \text{Var}(\nu_{y_{\alpha_\gamma}})(U \setminus \{y_{\alpha_\gamma}\}, V \setminus \{z_{\alpha_\gamma}\})) \\
&\geq \epsilon - (\text{Var}(\nu_{y_{\alpha_\gamma}})(\{y_{\alpha_\gamma}\}, K \setminus J_{\alpha_\gamma}) \\
&\quad + \text{Var}(\nu_{y_{\alpha_\gamma}})(K \setminus J_{\alpha_\gamma}, \{z_{\alpha_\gamma}\}) + \text{Var}(\nu_{y_{\alpha_\gamma}})(K \setminus J_{\alpha_\gamma}, K \setminus J_{\alpha_\gamma}) + \epsilon/6) \\
&> \epsilon - \epsilon/3 = 2\epsilon/3.
\end{aligned}$$

Now we notice that both nets $\{x_{\alpha_\gamma}\}_{\gamma \in \Gamma}$ and $\{y_{\alpha_\gamma}\}_{\gamma \in \Gamma}$ converge to a . Since ν is weak*-continuous and both U and V are clopen compact subsets of L , the preceding inequalities yield

$$\begin{aligned}
|\nu_a(U, V)| &= \lim_{\gamma \rightarrow \infty} |\nu_{x_{\alpha_\gamma}}(U, V)| \leq \epsilon/3 \\
&< 2\epsilon/3 \leq \lim_{\gamma \rightarrow \infty} |\nu_{y_{\alpha_\gamma}}(U, V)| = |\nu_a(U, V)|,
\end{aligned}$$

which is a contradiction. ■

5. Proof of the main theorem. We are now ready to establish the main result of this work. Our argument relies on the principal theorems from Section 4, which are gradually employed in the process of dismantling a bilinear operator $G : C_0(L) \times C_0(L) \rightarrow C_0(L)$ until only its most trivial components remain. As in the previous section, L denotes a collapsing space (see Definition 2.2), and $K = L \cup \{\infty\}$ stands for its one-point compactification. For any bilinear operator $G : C_0(L) \times C_0(L) \rightarrow C_0(L)$, we denote by $\hat{G} : K \rightarrow \mathcal{F}_0(K, K)$ its representing function as in Theorem 4.2, that is, the function satisfying

$$G(f, g)(t) = \sum_{x \in K} \left(\sum_{y \in K} \hat{G}_t(x, y) g(y) \right) f(x).$$

We note that, by thinking of G as a map acting on $\ell_\infty(L) \times \ell_\infty(L)$, one can explicitly compute the values of its representing function $\hat{G}$ via

$$\hat{G}_t(x, y) = G(\chi_x, \chi_y)(t).$$

This provides a concrete way of evaluating $\hat{G}_t(x, y)$ for any $x, y \in L$ and $t \in K$.

As $C_0(L)^*$ is isometrically isomorphic to $\ell_1(L)$ (see [20, Corollary 19.7.7]), given functionals $\varphi, \psi \in C_0(L)^*$ there exist unique sequences $(a_z)_{z \in L}, (b_z)_{z \in L} \in \ell_1(L)$ such that $\varphi(f) = \sum_{z \in L} a_z f(z)$ and $\psi(f) = \sum_{z \in L} b_z f(z)$ for all $f \in C_0(L)$. For the purpose of the proof, we will use this explicit representation

to define the cross-interaction operator

$$H(f, g)(t) = \sum_{w \in L} (a_w f(w)g(t) + b_w f(t)g(w)),$$

which corresponds to $H(f, g)(t) = \varphi(f)g(t) + f(t)\psi(g)$ in its abstract form.

Proof of Theorem 1.1. Let $G : C_0(L) \times C_0(L) \rightarrow C_0(L)$ be a bilinear operator. According to Theorem 4.6, there exist $(a_z)_{z \in L}, (b_z)_{z \in L} \in \ell_1(L)$ such that, for each $z \in L$, there exists $\gamma_z < \omega_1$ such that, for all $x, y \in L \setminus L_{\gamma_z}$,

$$\hat{G}_x(z, y) = \begin{cases} a_z & \text{if } x = y, \\ 0 & \text{if } x \neq y, \end{cases} \quad \hat{G}_x(y, z) = \begin{cases} b_z & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Define the cross-interaction operator $H : C_0(L) \times C_0(L) \rightarrow C_0(L)$ by

$$H(f, g)(t) = \sum_{w \in L} (a_w f(w)g(t) + b_w f(t)g(w)),$$

which is well-defined since $(a_w)_{w \in L}, (b_w)_{w \in L} \in \ell_1(L)$. Set $F = G - H$.

Fix $z \in L$, and let γ_z be as above, increasing it if necessary so that $z \in L_{\gamma_z}$. For $x, y \in L \setminus L_{\gamma_z}$ we have

$$\begin{aligned} \hat{F}_x(z, y) &= \hat{G}_x(z, y) - \hat{H}_x(z, y) = \hat{G}_x(z, y) - a_z \chi_y(x), \\ \hat{F}_x(y, z) &= \hat{G}_x(y, z) - \hat{H}_x(y, z) = \hat{G}_x(y, z) - b_z \chi_y(x), \end{aligned}$$

and thus $\hat{F}_x(z, y) = \hat{F}_x(y, z) = 0$.

By Theorem 4.11, there exist $r \in \mathbb{R}$ and $\rho < \omega_1$ such that $\hat{F}_y(y, y) = r$ for all $y \in L \setminus L_\rho$. Define the scaled multiplication operator $M : C_0(L) \times C_0(L) \rightarrow C_0(L)$ by $M(f, g)(t) = r f(t)g(t)$, and set $S = F - M$.

Towards a contradiction, assume that S does not have separable image. Then, for each $\alpha < \omega_1$, there exist $f_\alpha, g_\alpha \in C_0(L)$ such that $S(f_\alpha, g_\alpha) \notin C_0(L_\alpha)$. Hence, there exists $x_\alpha \in L \setminus L_\alpha$ with $S(f_\alpha, g_\alpha)(x_\alpha) \neq 0$. We form the sequence $\{x_\alpha\}_{\alpha < \omega_1}$ and assume, after passing to a suitable refinement, that $\{x_\alpha\}_{\alpha < \omega_1}$ is a sequence of pairwise distinct points.

For each $\alpha < \omega_1$, we have

$$S(f_\alpha, g_\alpha)(x_\alpha) = \sum_{y \in K} \left(\sum_{z \in K} \hat{S}_{x_\alpha}(y, z) g_\alpha(z) \right) f_\alpha(y) \neq 0,$$

so we may fix $y_\alpha, z_\alpha \in L$ with $\hat{S}_{x_\alpha}(y_\alpha, z_\alpha) \neq 0$ and form the sequences $\{y_\alpha\}_{\alpha < \omega_1}$ and $\{z_\alpha\}_{\alpha < \omega_1}$.

Suppose that $z_\alpha = z$ for uncountably many α . After a simultaneous refinement, we may assume that $\hat{S}_{x_\alpha}(y_\alpha, z) \neq 0$ for each $\alpha < \omega_1$. If the set $\{y_\alpha : \alpha < \omega_1\}$ is countable, this contradicts Proposition 4.4. If $\{y_\alpha : \alpha < \omega_1\}$ is uncountable, let $\gamma_z < \omega_1$ be such that $z \in L_{\gamma_z}$ and $\hat{F}_x(y, z) = \hat{F}_x(z, y) = 0$ for all $x, y \in L \setminus L_{\gamma_z}$. Pick $\alpha < \omega_1$ such that $x_\alpha, y_\alpha \in L \setminus L_{\gamma_z}$. Since $\hat{M}_x(y, z)$

can be nonzero only if $x = y = z$, we obtain

$$\hat{S}_{x_\alpha}(y_\alpha, z) = \hat{F}_{x_\alpha}(y_\alpha, z) - \hat{M}_{x_\alpha}(y_\alpha, z) = 0,$$

a contradiction. By the same reasoning, there is no refinement with $y_\alpha = y$ for uncountably many α .

Now we claim that there exists a simultaneous refinement such that $\{x_\alpha\}_{\alpha < \omega_1}$, $\{y_\alpha\}_{\alpha < \omega_1}$, and $\{z_\alpha\}_{\alpha < \omega_1}$ are sequences of pairwise distinct points (recall that $\{x_\alpha\}_{\alpha < \omega_1}$ was already chosen to be pairwise distinct). Indeed, proceeding by transfinite induction on $\gamma < \omega_1$, assume that sequences $\{x_{\alpha_\xi}\}_{\xi < \gamma}$, $\{y_{\alpha_\xi}\}_{\xi < \gamma}$ and $\{z_{\alpha_\xi}\}_{\xi < \gamma}$ of pairwise distinct points are obtained. Let

$$A_y = \bigcup_{\xi < \gamma} \{\alpha < \omega_1 : y_\alpha = y_{\alpha_\xi}\}, \quad A_z = \bigcup_{\xi < \gamma} \{\alpha < \omega_1 : z_\alpha = z_{\alpha_\xi}\}.$$

These are countable according to the discussion above, so fix $\alpha_\gamma < \omega_1$ strictly greater than $\sup(A_y \cup A_z \cup \{\alpha_\xi : \xi < \gamma\})$. This completes the inductive step, which establishes our claim.

Next, assuming that $\{x_\alpha\}_{\alpha < \omega_1}$, $\{y_\alpha\}_{\alpha < \omega_1}$, and $\{z_\alpha\}_{\alpha < \omega_1}$ are sequences of pairwise distinct points, if $x_\alpha \notin \{y_\alpha, z_\alpha\}$ for uncountably many α , we contradict Theorem 4.10. Hence, after another simultaneous refinement, either $x_\alpha = y_\alpha$ for all $\alpha < \omega_1$, or $x_\alpha = z_\alpha$ for all $\alpha < \omega_1$. Without loss of generality assume the former. If $z_\alpha \neq y_\alpha$ for uncountably many α , we contradict Theorem 4.13. Therefore, after a final simultaneous refinement, we may assume $x_\alpha = y_\alpha = z_\alpha$ for all α . But then, for $y_\alpha \in L \setminus L_\rho$ (with ρ fixed above),

$$\hat{S}_{y_\alpha}(y_\alpha, y_\alpha) = \hat{F}_{y_\alpha}(y_\alpha, y_\alpha) - \hat{M}_{y_\alpha}(y_\alpha, y_\alpha) = r - r = 0,$$

which is a contradiction. Thus, S has separable image.

We now establish the uniqueness of the decomposition. Suppose that there exist $r' \in \mathbb{R}$, $(a'_w)_{w \in L}, (b'_w)_{w \in L} \in \ell_1(L)$, and a bilinear operator $S' : C_0(L) \times C_0(L) \rightarrow C_0(L)$ with separable image such that

$$G(f, g)(t) = r' f(t)g(t) + \sum_{w \in L} (a'_w f(w)g(t) + b'_w f(t)g(w)) + S'(f, g)(t),$$

yielding another representation of G . By Proposition 2.1, there exists $\eta < \omega_1$ such that the images of S and S' are contained in $C_0(L_\eta)$. Fix $t \in L$ and choose $s \in L \setminus L_\eta$ with $s \neq t$. A direct computation in both representations yields

$$\begin{aligned} a'_t &= G(\chi_t, \chi_s)(s) = a_t, & b'_t &= G(\chi_s, \chi_t)(s) = b_t, \\ r' &= G(\chi_s, \chi_s)(s) - a_s - b_s = r. \end{aligned}$$

This implies $(a_w)_{w \in L} = (a'_w)_{w \in L}$, $(b_w)_{w \in L} = (b'_w)_{w \in L}$, and consequently $S = S'$. The proof of Theorem 1.1 is complete. ■

Acknowledgements. The author would like to thank the referee for their careful reading of the manuscript and for providing comments that led to significant improvements in the final version of the paper, including a reformulation of the introduction and important corrections throughout the text.

Funding. This research was supported by the Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), grant no. 2023/12916-1.

References

- [1] G. Androulakis and T. Schlumprecht, *Strictly singular, non-compact operators exist on the space of Gowers and Maurey*, J. London Math. Soc. 64 (2001), 655–674.
- [2] S. A. Argyros and R. G. Haydon, *A hereditarily indecomposable $\mathcal{L}_\infty$ -space that solves the scalar-plus-compact problem*, Acta Math. 206 (2011), 1–54.
- [3] R. Blei, *Analysis in Integer and Fractional Dimensions*, Cambridge Stud. Adv. Math. 71, Cambridge Univ. Press, Cambridge, 2001.
- [4] L. Candido, *Few operators on Banach spaces $C_0(L \times L)$* , Topology Appl. 375 (2025), art. 109577, 24 pp.
- [5] L. Candido, *On Banach spaces of the form $C_0(\alpha \times L)$ with few operators*, Banach J. Math. Anal. 15 (2021), no. 2, art. 41, 23 pp.
- [6] L. Candido and P. Koszmider, *On complemented copies of $c_0(\omega_1)$ in $C(K^n)$ spaces*, Studia Math. 233 (2016), 209–226.
- [7] N. Dunford and J. T. Schwartz, *Linear Operators, Part I: General Theory*, 4th printing, Interscience, New York, 1967.
- [8] W. T. Gowers and B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. 6 (1993), 851–874.
- [9] P. Koszmider and N. J. Laustsen, *A Banach space induced by an almost disjoint family, admitting only few operators and decompositions*, Adv. Math. 381 (2021), art. 107613, 39 pp.
- [10] P. Koszmider, *A survey on Banach spaces $C(K)$ with few operators*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. 104 (2010), 309–326.
- [11] P. Koszmider, *Banach spaces of continuous functions with few operators*, Math. Ann. 330 (2004), 151–183.
- [12] P. Koszmider, *On decompositions of Banach spaces of continuous functions on Mrówka’s spaces*, Proc. Amer. Math. Soc. 133 (2005), 2137–2146.
- [13] P. Koszmider and P. Zieliński, *Complementation and decompositions in weakly Lindelöf Banach spaces*, J. Math. Anal. Appl. 376 (2011), 329–341.
- [14] K. Kunen, *Set Theory: An Introduction to Independence Proofs*, North-Holland, Amsterdam, 1980.
- [15] J. Lindenstrauss, *Some open problems in Banach space theory*, in: Séminaire Choquet. Initiation à l’analyse, 15e année (1975/76), Secrétariat Math., Paris, 1977, exp. 18, 9 pp.
- [16] B. Maurey, *Banach spaces with few operators*, in: Handbook of the Geometry of Banach Spaces, Vol. 2, North-Holland, Amsterdam, 2003, 1247–1297.
- [17] A. J. Ostaszewski, *On countably compact, perfectly normal spaces*, J. London Math. Soc. 14 (1976), 505–516.

- [18] G. Plebanek, *A construction of a Banach space $C(K)$ with few operators*, Topology Appl. 143 (2004), 217–239.
- [19] R. A. Ryan, *Introduction to Tensor Products of Banach Spaces*, Springer Monogr. Math., Springer, London, 2002.
- [20] Z. Semadeni, *Banach Spaces of Continuous Functions, Vol. I*, Monograf. Mat. 55, PWN, Warszawa, 1971.
- [21] S. Shelah, *A Banach space with few operators*, Israel J. Math. 30 (1978), 181–191.
- [22] S. Shelah and J. Steprāns, *A Banach space on which there are few operators*, Proc. Amer. Math. Soc. 104 (1988), 101–105.

Leandro Candido

Instituto de Ciência e Tecnologia, Departamento de Matemática

Universidade Federal de São Paulo–UNIFESP

CEP 12247014, São José dos Campos–SP, Brasil

E-mail: leandro.candido@unifesp.br