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Cardinal characteristics associated with small subsets of reals

by

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Abstract. Inspired by Bartoszyński's work on small sets, we introduce a new ideal defined by interval partitions on natural numbers and summable sequences of positive reals. Similarly, we present another ideal that relies on Bartoszyński's and Shelah's representation of F_σ measure zero sets. We show they are σ-ideals characterizing all small sets and F_σ measure zero sets. We also study the cardinal characteristics associated with the ideals introduced. We use them to describe invariants of measure, discuss their connection to Cichoń's diagram, and present related consistency results.

1. Introduction. The most important notion in this work is the notion of a *small set* recalled below in Definition 1.1. Introducing small sets is one of the main contributions of Bartoszyński's work [Bar88], responding to D. Fremlin's question of whether the cofinality of the covering number of the null ideal is uncountable. This remained an open question for almost two decades. It was in 2000 that Shelah [She00] finally provided a negative answer. Leading up to that, Bartoszyński [Bar88] offered a partial solution by showing that if $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}$, then $\mathrm{cf}(\mathrm{cov}(\mathcal{N})) > \aleph_0$. The notion of small sets turned out to be related to anti-localization cardinals (see, e.g., [CM23] for details and original references) and other combinatorial notions such as F_σ *measure zero sets*, see [BJ95a, Lem. 2.6.3].

To present Definition 1.1, given a formula ϕ, we use "$\exists^\infty n \in \omega : \phi$" to abbreviate "infinitely many natural numbers satisfy ϕ".

Definition 1.1 ([Bar88, BS18]). Let $X \subseteq {}^\omega 2$.

(1) X is *small* if there are sequences $\langle I_n, J_n \mid n \in \omega\rangle$ such that
   (a) $\{I_n \mid n \in \omega\} \subseteq [\omega]^{<\aleph_0}$ is a partition of a cofinite subset of ω,
   (b) $J_n \subseteq {}^{I_n}2$ for all n,

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   (c) $\sum_{n<\omega} \frac{|J_n|}{2^{|I_n|}} < \infty$,
   (d) $X \subseteq \{x \in {}^\omega 2 \mid \exists^\infty n \in \omega : x{\restriction}I_n \in J_n\}$.

(2) X is *small** if, in addition, the sets I_n are disjoint intervals, that is, if there exists a strictly increasing sequence $\{k_n \mid n \in \omega\}$ of integers such that $I_n = [k_n, k_{n+1})$ for $n \in \omega$.

Denote by $\mathcal{S}$ and $\mathcal{S}^\star$ the collection of all small sets and small* sets in ${}^\omega 2$, respectively. It is known that $\mathcal{S}^\star \subsetneq \mathcal{S}$ (see [BS18, Thm. 18]) and that every small set has measure zero, i.e., $\mathcal{S} \subseteq \mathcal{N}$ (see Notation 1.2). Bartoszyński [Bar88, Lem. 1.3] proved that none of them is an ideal.

Let $\mathbb{I}$ denote the family of all partitions of ω into finite non-empty intervals ([1]) and let

$$\ell^1_+ := \Big\{\varepsilon \in {}^\omega(0,\infty) \,\Big|\, \sum_{n\in\omega} \varepsilon_n < \infty\Big\}.$$

In this work, we offer a characterization of $\mathcal{S}^\star$ in terms of new σ-ideals $\mathcal{S}_{I,\varepsilon}$ (treating $\{\emptyset\}$ as a σ-ideal by convention) parametrized by $I \in \mathbb{I}$ and $\varepsilon \in \ell^1_+$, i.e,

$$\mathcal{S}^\star = \bigcup_{(I,\varepsilon)\in \mathbb{I}\times\ell^1_+} \mathcal{S}_{I,\varepsilon}.$$

Our ideal $\mathcal{S}_{I,\varepsilon}$ is motivated by the definition of small sets, recalled in our Definition 1.1. The introduction of $\mathcal{S}_{I,\varepsilon}$ itself and its properties are provided in Section 2.

Recall that given an ideal $\mathcal{I}$ of subsets of X such that $\{x\} \in \mathcal{I}$ for all $x \in X$, the *cardinal characteristics associated with* $\mathcal{I}$ are defined by

$$\begin{aligned}
\operatorname{add}(\mathcal{I}) &:= \min\Big\{|\mathcal{J}| \,\Big|\, \mathcal{J} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{J} \notin \mathcal{I}\Big\},\\
\operatorname{cov}(\mathcal{I}) &:= \min\Big\{|\mathcal{J}| \,\Big|\, \mathcal{J} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{J} = X\Big\},\\
\operatorname{non}(\mathcal{I}) &:= \min\{|A| \mid A \subseteq X \text{ and } A \notin \mathcal{I}\},\\
\operatorname{cof}(\mathcal{I}) &:= \min\{|\mathcal{J}| \mid \mathcal{J} \subseteq \mathcal{I} \text{ is cofinal in } \langle \mathcal{I}, \subseteq\rangle\}.
\end{aligned}$$

These cardinals are referred to as *additivity*, *covering*, *uniformity*, and *cofinality* of $\mathcal{I}$, respectively.

In order to present our main results, we review the basic conventions:

NOTATION 1.2.

(1) Given a formula ϕ, "$\forall^\infty n \in \omega : \phi$" means that all but finitely many natural numbers satisfy ϕ, and "$\exists^\infty n \in \omega : \phi$" means that infinitely many natural numbers satisfy ϕ.

([1]) For technical reasons, it is sometimes convenient to consider a partition of some cofinite subset of ω instead of a partition of ω. Thus, depending on the context, $I \in \mathbb{I}$ will sometimes mean that I is a partition of some cofinite subset of ω.

(2) Denote by $\mathcal{N}$ and $\mathcal{M}$ the σ-ideals of Lebesgue null sets and of meager sets in ${}^{\omega}2$, respectively, and let $\mathcal{E}$ be the σ-ideal generated by the closed measure zero subsets of ${}^{\omega}2$. It is well-known that $\mathcal{E} \subseteq \mathcal{N} \cap \mathcal{M}$. Moreover, it was proved that this inclusion is proper (see [BJ95a, Lemma 2.6.1]).
(3) $\mathfrak{c} := 2^{\aleph_0}$.
(4) For $f, g \in {}^{\omega}\omega$ define

$$f \leq^* g \quad \text{if and only if} \quad \forall^\infty n \in \omega : f(n) \leq g(n).$$

(5) We let

$$\mathfrak{b} := \min\{|F| \mid F \subseteq {}^{\omega}\omega \text{ and } \forall g \in {}^{\omega}\omega\ \exists f \in F : f \not\leq^* g\},$$
$$\mathfrak{d} := \min\{|D| \mid D \subseteq {}^{\omega}\omega \text{ and } \forall g \in {}^{\omega}\omega\ \exists f \in D : g \leq^* f\},$$

denote the *bounding number* and the *dominating number*, respectively.

One of our first main results exhibits that the covering and the uniformity of our ideal are related to the anti-localization cardinals $\mathfrak{b}^{\mathrm{aLc}}_{b,h}, \mathfrak{d}^{\mathrm{aLc}}_{b,h}$ (see Definition 4.3):

Theorem A (Theorem 5.4). *We have $\nu_i = \kappa_i$ for $i < 2$, where*

- $\nu_0 = \min\{\mathrm{cov}(\mathcal{S}_{I,\varepsilon}) \mid I \in \mathbb{I} \textit{ and } \varepsilon \in \ell^1_+\}$,
- $\nu_1 = \sup\{\mathrm{non}(\mathcal{S}_{I,\varepsilon}) \mid I \in \mathbb{I} \textit{ and } \varepsilon \in \ell^1_+\}$,
- $\kappa_0 = \min\left\{\mathfrak{b}^{\mathrm{aLc}}_{b,h} \,\middle|\, b, h \in {}^{\omega}\omega \textit{ and } \sum_{n\in\omega} \frac{h(n)}{b(n)} < \infty\right\}$,
- $\kappa_1 = \sup\left\{\mathfrak{d}^{\mathrm{aLc}}_{b,h} \,\middle|\, b, h \in {}^{\omega}\omega \textit{ and } \sum_{n\in\omega} \frac{h(n)}{b(n)} < \infty\right\}$.

As a consequence of Theorem A, we obtain an alternative characterization of $\mathrm{cov}(\mathcal{N})$ and $\mathrm{non}(\mathcal{N})$. We also establish an analogous characterization of $\mathrm{add}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{N})$.

Theorem B (Corollary 5.5, Theorem 4.16).

(a) *Assume that* $\mathrm{cov}(\mathcal{N}) < \mathfrak{b}$. *Then*

$$\mathrm{cov}(\mathcal{N}) = \min\{\mathrm{cov}(\mathcal{S}_{I,\varepsilon}) \mid I \in \mathbb{I} \textit{ and } \varepsilon \in \ell^1_+\}.$$

(b) *Assume that* $\mathrm{non}(\mathcal{N}) > \mathfrak{d}$. *Then*

$$\mathrm{non}(\mathcal{N}) = \sup\{\mathrm{non}(\mathcal{S}_{I,\varepsilon}) \mid I \in \mathbb{I} \textit{ and } \varepsilon \in \ell^1_+\}.$$

(c) *We have* $\mathrm{add}(\mathcal{N}) = \min\{\mathfrak{b}, \min\{\mathrm{add}(\mathcal{S}_{I,\varepsilon}) \mid I \in \mathbb{I} \textit{ and } \varepsilon \in \ell^1_+\}\}$.
(d) *We have* $\mathrm{cof}(\mathcal{N}) = \max\{\mathfrak{d}, \sup\{\mathrm{cof}(\mathcal{S}_{I,\varepsilon}) \mid I \in \mathbb{I} \textit{ and } \varepsilon \in \ell^1_+\}\}$.

Inspired by the combinatorial characterization of F_σ measure zero sets due to Bartoszyński and Shelah [BS92] stated below as Theorem 1.3, we also introduce a σ-ideal $\mathcal{E}_{I,\varepsilon}$ parametrized in the same way as $\mathcal{S}_{I,\varepsilon}$, which can be used to redescribe $\mathcal{E}$ as

$$\mathcal{E} = \bigcup_{(I,\varepsilon)\in\mathbb{I}\times\ell^1_+} \mathcal{E}_{I,\varepsilon}.$$

Details are developed in Section 2.

Theorem 1.3 ([BS92, Thm. 4.3]). *For every $X \subseteq {}^{\omega}2$, $X \in \mathcal{E}$ if and only if there is a partition $\langle I_n \mid n \in \omega \rangle$ of ω into finite, disjoint intervals and a sequence $\langle J_n \mid n \in \omega \rangle$ such that*

(a) $J_n \subseteq {}^{I_n}2$ *for all* n,
(b) $\sum_{n<\omega} \frac{|J_n|}{2^{|I_n|}} < \infty$,
(c) $X \subseteq \{x \in {}^{\omega}2 \mid \forall^{\infty} n : x{\restriction}I_n \in J_n\}$.

Note that it follows directly from Theorem 1.3 that $\mathcal{E} \subseteq \mathcal{S}^{\star}$.

We have found a connection between the localization cardinals $\mathfrak{b}^{\mathrm{Lc}}_{b,h}, \mathfrak{d}^{\mathrm{Lc}}_{b,h}$ (see Definition 4.3) and the covering and uniformity of $\mathcal{E}_{I,\varepsilon}$, similar to the connection established in Theorem A.

Theorem C (Theorem 5.7). *We have $\lambda_i = \theta_i$ for $i < 2$, where*

- $\theta_0 = \min\{\mathrm{cov}(\mathcal{E}_{I,\varepsilon}) \mid I \in \mathbb{I}$ *and* $\varepsilon \in \ell^1_+\}$,
- $\theta_1 = \sup\{\mathrm{non}(\mathcal{E}_{I,\varepsilon}) \mid I \in \mathbb{I}$ *and* $\varepsilon \in \ell^1_+\}$,
- $\lambda_0 = \min\left\{\mathfrak{d}^{\mathrm{Lc}}_{b,h} \;\middle|\; b, h \in {}^{\omega}\omega \text{ and } \sum_{n\in\omega} \frac{h(n)}{b(n)} < \infty\right\}$,
- $\lambda_1 = \sup\left\{\mathfrak{b}^{\mathrm{Lc}}_{b,h} \;\middle|\; b, h \in {}^{\omega}\omega \text{ and } \sum_{n\in\omega} \frac{h(n)}{b(n)} < \infty\right\}$.

Concerning the cardinals θ_i in the previous theorem, it turns out these are related to the covering and the uniformity of the ideal of null-additive sets (see Definition 5.8):

Theorem D (Proposition 5.10). *Let θ_i be as in Theorem C. Then*

(a) $\mathrm{non}(\mathcal{NA}) \leq \theta_1$,
(b) $\mathrm{cov}(\mathcal{NA}) \geq \theta_0$.

In Section 4, we clarify the relation between the ideals $\mathcal{E}_{I,\varepsilon}$ and $\mathcal{S}_{I,\varepsilon}$ as follows.

Theorem E (Proposition 4.2). *We have $\mathrm{cof}(\mathcal{S}_{I,\varepsilon}) = \mathrm{cof}(\mathcal{E}_{I,\varepsilon})$ and $\mathrm{add}(\mathcal{E}_{I,\varepsilon}) = \mathrm{add}(\mathcal{S}_{I,\varepsilon})$ for reasonable pairs $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$.*

In ZFC, we prove:

Theorem F. *In Cichoń's diagram (see Figure 1):*

(a) $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{E}_{I,\varepsilon}) = \mathrm{add}(\mathcal{S}_{I,\varepsilon}) \leq \mathrm{cof}(\mathcal{S}_{I,\varepsilon}) = \mathrm{cof}(\mathcal{E}_{I,\varepsilon}) \leq \mathrm{cof}(\mathcal{N})$,
(b) $\mathrm{cov}(\mathcal{N}) \leq \mathrm{cov}(\mathcal{S}) \leq \mathrm{cov}(\mathcal{S}_{I,\varepsilon}) \leq \mathrm{cov}(\mathcal{E}_{I,\varepsilon})$ *and*

$$\mathrm{non}(\mathcal{E}_{I,\varepsilon}) \leq \mathrm{non}(\mathcal{S}_{I,\varepsilon}) \leq \mathrm{non}(\mathcal{S}) \leq \mathrm{non}(\mathcal{N}),$$

(c) $\mathrm{cov}(\mathcal{E}) \leq \mathrm{cov}(\mathcal{E}_{I,\varepsilon})$ *and* $\mathrm{non}(\mathcal{E}_{I,\varepsilon}) \leq \mathrm{non}(\mathcal{E})$.

Item (a) in the theorem is obtained by combining Proposition 4.2 and Corollary 4.11, while items (b) and (c) follow from the inclusion relation between the ideals.

Regarding inequalities related to the cardinal characteristics associated with our ideals, Theorem F seems to be the most optimal: in Section 6, we

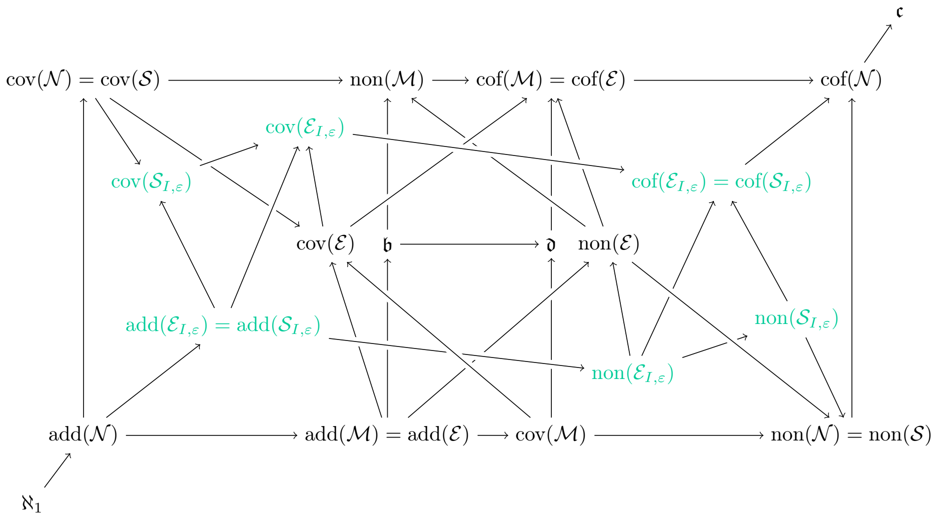

Fig. 1. Cichoń's diagram including the cardinal characteristics associated with our ideals, and $\mathrm{cov}(\mathcal{S}) = \mathrm{cov}(\mathcal{N})$, $\mathrm{non}(\mathcal{S}) = \mathrm{non}(\mathcal{N})$, $\mathrm{add}(\mathcal{E}) = \mathrm{add}(\mathcal{M})$, and $\mathrm{cof}(\mathcal{E}) = \mathrm{cof}(\mathcal{M})$ due to Bartoszyński and Shelah [BS92]. The relations hold for contributive pairs.

manage to show that, in most cases, no further inequalities can be proved with the cardinals in Cichoń's diagram.

2. Layering of small sets. The objective of this section is to present the basic properties of the two new ideals $\mathcal{S}_{I,\varepsilon}$ and $\mathcal{E}_{I,\varepsilon}$. Their definitions, as discussed in Section 1, are primarily based on the combinatorial description of small sets and F_σ measure zero sets. We examine $\mathcal{S}_{I,\varepsilon}$ and $\mathcal{E}_{I,\varepsilon}$ and in Theorem 2.3 we demonstrate that they are, in fact, σ-ideals. We also show that $\mathcal{S}^\star$ and $\mathcal{E}$ can be characterized using these σ-ideals.

The families $\mathcal{S}_{I,\varepsilon}$ and $\mathcal{E}_{I,\varepsilon}$ are defined as follows.

Definition 2.1.

(a) Let $(I,\varepsilon) \in \mathbb{I} \times \ell^1_+$. We define

$$\Sigma_{I,\varepsilon} = \left\{ \varphi \in \prod_{n\in\omega} \mathcal{P}({}^{I_n}2) \;\middle|\; \lim_{n\to\infty} \frac{|\varphi(n)|}{2^{|I_n|}\cdot\varepsilon_n} = 0 \right\}.$$

(b) For $\varphi \in \Sigma_{I,\varepsilon}$ we define

$$[\varphi]_\infty = \{x \in {}^\omega 2 \mid \exists^\infty n \in \omega : x{\restriction}I_n \in \varphi(n)\},$$
$$[\varphi]_* = \{x \in {}^\omega 2 \mid \forall^\infty n \in \omega : x{\restriction}I_n \in \varphi(n)\}.$$

(c) Given I and ε as in (a) we define

$$\mathcal{S}_{I,\varepsilon} = \{X \subseteq {}^\omega 2 \mid \exists \varphi \in \Sigma_{I,\varepsilon} : X \subseteq [\varphi]_\infty\},$$
$$\mathcal{E}_{I,\varepsilon} = \{X \subseteq {}^\omega 2 \mid \exists \varphi \in \Sigma_{I,\varepsilon} : X \subseteq [\varphi]_*\}.$$

Concerning Definition 2.1(a), we often use in the proofs the fact that

$$\lim_{n\to\infty} \frac{|\varphi(n)|}{2^{|I_n|}\cdot\varepsilon_n} = 0 \quad \text{if and only if} \quad \forall N<\omega\ \forall^\infty n\in\omega : \frac{|\varphi(n)|}{2^{|I_n|}} < \frac{\varepsilon_n}{N}.$$

We first show that the families $\mathcal{S}_{I,\varepsilon}$ and $\mathcal{E}_{I,\varepsilon}$ are σ-ideals (treating the trivial cases as σ-ideals). Moreover, these σ-ideals completely characterize the families $\mathcal{S}^\star$ and $\mathcal{E}$. Note that $\mathcal{S}_{I,\varepsilon}\subseteq\mathcal{S}^\star$ and $\mathcal{E}_{I,\varepsilon}\subseteq\mathcal{E}$ for any $I\in\mathbb{I}$ and $\varepsilon\in\ell^1_+$.

Next, we shall need a technical result.

Lemma 2.2. *For any $\varepsilon\in\ell^1_+$ there is a non-decreasing $\delta\in{}^\omega(0,\infty)$ such that $\delta\to\infty$ and $\sum_{j\in\omega}\delta_j\varepsilon_j<\infty$.*

Proof. Let $s=\sum_{j\in\omega}\varepsilon_j$. Define an increasing sequence $\langle n_i \mid i\in\omega\rangle$ as follows: let $n_0=0$ and for $i\geq 1$ let

$$n_i = \min\left\{k>n_{i-1} \,\middle|\, \sum_{n\geq k} 2^i\varepsilon_n < \frac{s}{2^i}\right\}.$$

For $j\in[n_i,n_{i+1})$ define $\delta_j=2^i$. To conclude the proof it is enough to show

$$\lim_{i\to\infty}\sum_{j<n_i}\delta_j\varepsilon_j<\infty.$$

For any $i\in\omega$ we have

$$\sum_{j<n_i}\delta_j\varepsilon_j = \sum_{k<i}\sum_{j=n_k}^{n_{k+1}-1} 2^k\varepsilon_j < \sum_{k<i}\frac{s}{2^k} < 2s.$$

The sequence $a_i=\sum_{j<n_i}\delta_j\varepsilon_j$ is monotone and bounded from above, hence convergent. ∎

From now on, whenever $a<b\leq\omega$, we let $[a,b)$ denote the integer interval $\{n\in\omega \mid a\leq n<b\}$.

Theorem 2.3.

(a) *The family $\mathcal{S}_{I,\varepsilon}$ is a σ-ideal for any $(I,\varepsilon)\in\mathbb{I}\times\ell^1_+$.*
(b) *The family $\mathcal{E}_{I,\varepsilon}$ is a σ-ideal for any $(I,\varepsilon)\in\mathbb{I}\times\ell^1_+$.*
(c) *We have $X\in\mathcal{S}^\star$ if and only if there is a pair $(I,\varepsilon)\in\mathbb{I}\times\ell^1_+$ such that $X\in\mathcal{S}_{I,\varepsilon}$. In particular,*

$$\mathcal{S}^\star = \bigcup_{(I,\varepsilon)\in\mathbb{I}\times\ell^1_+}\mathcal{S}_{I,\varepsilon}.$$

(d) *We have $X\in\mathcal{E}$ if and only if there is a pair $(I,\varepsilon)\in\mathbb{I}\times\ell^1_+$ such that $X\in\mathcal{E}_{I,\varepsilon}$. In particular,*

$$\mathcal{E} = \bigcup_{(I,\varepsilon)\in\mathbb{I}\times\ell^1_+}\mathcal{E}_{I,\varepsilon}.$$

Proof. To prove (a) and (b) it suffices to prove the following result:

CLAIM 2.4. *Given* $\{\varphi^n \mid n\in\omega\} \subseteq \Sigma_{I,\varepsilon}$ *there is* $\varphi \in \Sigma_{I,\varepsilon}$ *such that*

$$\forall n\in\omega\ \forall^\infty j\in\omega : \varphi^n(j) \subseteq \varphi(j).$$

Proof. Define a sequence $\langle k_n \mid n\in\omega\rangle$ as follows: $k_0 = 0$ and for $n \geq 1$,

$$k_n = \min\left\{ m > k_{n-1} \,\middle|\, \forall i < n\ \forall j \geq m : \frac{|\varphi^i(j)|}{2^{|I_j|}} < \frac{\varepsilon_j}{n^2} \right\}.$$

Now define φ by

$$\varphi(j) = \bigcup_{i<n} \varphi^i(j)$$

for all $j \in [k_n, k_{n+1})$. Notice that

- for all $j \in [k_n, k_{n+1})$,

$$\frac{|\varphi(j)|}{2^{|I_j|}} \leq \sum_{i=0}^{n-1} \frac{|\varphi^i(j)|}{2^{|I_j|}} < n \cdot \frac{\varepsilon_j}{n^2} = \frac{\varepsilon_j}{n},$$

- $\forall n\in\omega\ \forall^\infty j\in\omega : \varphi^n(j) \subseteq \varphi(j)$. ∎

(c) The direction from right to left is straightforward. We prove the reverse direction. Let $Y \in \mathcal{S}^\star$. Choose $I \in \mathbb{I}$ and $\varphi \in \prod_{n\in\omega} \mathcal{P}({}^{I_n}2)$ such that $\sum_{n\in\omega} \frac{|\varphi(n)|}{2^{|I_n|}} < \infty$ and $Y \subseteq \{x \in {}^\omega 2 \mid \exists^\infty n : x{\restriction}I_n \in \varphi(n)\}$. Applying Lemma 2.2 we find $\delta \in {}^\omega(0,\infty)$ such that $\delta \to \infty$ and

$$\sum_{n\in\omega} \delta_n \frac{|\varphi(n)|}{2^{|I_n|}} < \infty.$$

For any $n \in \omega$ define

$$\varepsilon_n = \delta_n \frac{|\varphi(n)|}{2^{|I_n|}}.$$

Then $\varepsilon \in \ell^1_+$ and

$$\frac{|\varphi(n)|}{2^{|I_n|}\cdot \varepsilon_n} = \frac{1}{\delta_n} \to 0.$$

Thus, $\varphi \in \Sigma_{I,\varepsilon}$ and $Y \in \mathcal{S}_{I,\varepsilon}$.

(d) Proceed as in (c), using Theorem 1.3. ∎

For many I,ε we have $\mathcal{S}_{I,\varepsilon} = \{\emptyset\}$ or $\mathcal{E}_{I,\varepsilon} = \{\emptyset\}$. Since we are interested in non-trivial σ-ideals, we provide the following characterizations, Lemmas 2.5 and 2.8, to avoid the trivial cases.

LEMMA 2.5. *The following statements are equivalent for any* $(I,\varepsilon) \in \mathbb{I}\times \ell^1_+$*:*

(1) $\mathcal{S}_{I,\varepsilon} \neq \{\emptyset\}$.
(2) $\exists A \in [\omega]^\omega : \lim_{n\in A} 2^{|I_n|}\cdot \varepsilon_n = \infty$.

Proof. We shall use the following representation of (2):

$$\exists A \in [\omega]^\omega\ \forall N \in \omega\ \forall^\infty n \in A : N \cdot 2^{-|I_n|} < \varepsilon_n.$$

(1)⇒(2). Suppose ¬(2). Assume that for any $A \in [\omega]^\omega$ there is $N \in \omega$ such that

$$\exists^\infty n \in A : \frac{1}{2^{|I_n|}} \geq \frac{\varepsilon_n}{N}.$$

Let $\varphi \in \Sigma_{I,\varepsilon}$ be such that $\emptyset \neq [\varphi]_\infty \in \mathcal{S}_{I,\varepsilon}$. There exist infinitely many n such that $\varphi(n) \neq \emptyset$. Let $A = \{n \in \omega \mid \varphi(n) \neq \emptyset\}$. Since $|\varphi(n)| \geq 1$ for any $n \in A$, we have $\frac{|\varphi(n)|}{2^{|I_n|}} \geq \frac{1}{2^{|I_n|}}$ for all $n \in A$. Therefore,

$$\exists^\infty n \in \omega : \frac{|\varphi(n)|}{2^{|I_n|}} \geq \frac{\varepsilon_n}{N}.$$

Consequently, $\varphi \notin \Sigma_{I,\varepsilon}$, which is a contradiction.

(2)⇒(1). Define $\varphi \in \Sigma_{I,\varepsilon}$ with $|\varphi(n)| = 1$ for all $n \in A$ and 0 otherwise. We will show that $\varphi \in \Sigma_{I,\varepsilon}$. Let $N \in \omega$. Then by (2),

$$\forall^\infty n \in A : \frac{|\varphi(n)|}{2^{|I_n|}} = \frac{1}{2^{|I_n|}} < \frac{\varepsilon_n}{N} \quad \text{and} \quad \forall^\infty n \in \omega \setminus A : \frac{|\varphi(n)|}{2^{|I_n|}} = 0 < \frac{\varepsilon_n}{N}.$$

Therefore, $\varphi \in \Sigma_{I,\varepsilon}$, and clearly, $[\varphi]_\infty \neq \emptyset$. ∎

Any pair $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$ satisfying the conditions in Lemma 2.5 is called *$\mathcal{S}^\star$-contributive.*

In view of Lemma 2.5, we have the following result.

COROLLARY 2.6. *Let $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$. If the sequence $\langle 2^{|I_n|} \cdot \varepsilon_n \mid n \in \omega\rangle$ is bounded, then $\mathcal{S}_{I,\varepsilon} = \{\emptyset\}$.*

REMARK 2.7. Let $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$.

(a) If the sequence $\langle |I_n| \mid n \in \omega\rangle$ is bounded, then $\mathcal{S}_{I,\varepsilon} = \mathcal{E}_{I,\varepsilon} = \{\emptyset\}$. This is a consequence of Corollary 2.6.

(b) Let $\varphi \in \Sigma_{I,\varepsilon}$. If there is a set $A \in [\omega]^\omega$ such that $\langle |I_n| \mid n \in A\rangle$ is bounded, then $\forall^\infty n \in A : \varphi(n) = \emptyset$. Indeed, if there existed infinitely many $n \in A$ such that $\varphi(n) \neq \emptyset$, then for infinitely many $n \in A$,

$$\frac{|\varphi(n)|}{2^{|I_n|} \cdot \varepsilon_n} \geq \frac{|\varphi(n)|}{2^B \cdot \varepsilon_n} \geq \frac{1}{2^B} \cdot \frac{1}{\varepsilon_n} \to \infty,$$

where B is a bound of $\langle |I_n| \mid n \in A\rangle$. Since a subsequence of $\frac{|\varphi(n)|}{2^{|I_n|} \cdot \varepsilon_n}$ tends to infinity, $\lim_{n\to\infty} \frac{|\varphi(n)|}{2^{|I_n|} \cdot \varepsilon_n} = 0$ cannot hold.

Just as in Lemma 2.5, we have the following result for $\mathcal{E}_{I,\varepsilon}$:

LEMMA 2.8. *The following statements are equivalent for any $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$:*

(1) $\mathcal{E}_{I,\varepsilon} \neq \{\emptyset\}$.
(2) $\lim_{n\to\infty} 2^{|I_n|} \cdot \varepsilon_n = \infty$.

Proof. Similar to the proof of Lemma 2.5. ∎

A pair $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$ satisfying conditions in Lemma 2.8 is called $\mathcal{E}$-*contributive*. Notice that any $\mathcal{E}$-contributive (I, ε) is automatically also $\mathcal{S}^\star$-contributive.

The inclusions between the considered ideals are depicted in Figure 2.

$$\begin{tikzcd}
\mathcal{M} & \mathcal{M} \cap \mathcal{N} \arrow[l] \arrow[d] & \mathcal{E} \arrow[l] \arrow[d] & \mathcal{E}_{I,\varepsilon} \arrow[l] \arrow[d] \\
& \mathcal{N} & \mathcal{S}^\star \arrow[l] & \mathcal{S}_{I,\varepsilon} \arrow[l]
\end{tikzcd}$$

Fig. 2. Inclusions

Many inclusions in Figure 2 are proper or cannot be added, as one can see in the following result. Moreover, it was shown in [Bar88] that the inclusion $\mathcal{S}^\star \subseteq \mathcal{N}$ is proper.

Proposition 2.9. *The following hold for any $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$:*

(a) $\mathcal{S}_{I,\varepsilon} \subsetneq \mathcal{S}^\star$.
(b) $\mathcal{E}_{I,\varepsilon} \subsetneq \mathcal{E}$.

Moreover, if (I, ε) is $\mathcal{S}^\star$-contributive then

(c) $\mathcal{E}_{I,\varepsilon} \subsetneq \mathcal{S}_{I,\varepsilon}$.
(d) $\mathcal{S}_{I,\varepsilon} \not\subseteq \mathcal{E}$.

Proof. (a) This follows from the fact that $\mathcal{S}_{I,\varepsilon}$ is a σ-ideal by Theorem 2.3 (recall that $\mathcal{S}^\star$ is not an ideal: see [Bar88, Lem. 1.3] or [BS18, BJ95a]).

(b) Assume that $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$ is $\mathcal{E}$-contributive and let $k \in {}^{\uparrow\omega}\omega$ be such that $I_n = [k_n, k_{n+1})$. Let $a \in {}^{\uparrow\omega}\omega$ be such that

$$\forall n \in \omega : k_{a_{n+1}} > 2k_{a_n+1} - k_{a_n}.$$

For each $n \in \omega$, let $I'_n := [k_{a_n}, k_{a_{n+1}})$ and

$$\psi(n) := \{x \in {}^{I'_n}2 \mid x{\restriction}(I'_n \smallsetminus I_{a_n}) = \mathbf{0}\}.$$

Then it follows from the definition of the function a that $|\psi(n)| = 2^{|I_{a_n}|} = 2^{k_{a_n+1} - k_{a_n}} < 2^{k_{a_{n+1}} - k_{a_n+1}}$ so we have

$$\frac{|\psi(n)|}{2^{|I'_n|}} < \frac{2^{k_{a_{n+1}} - k_{a_n+1}}}{2^{k_{a_{n+1}} - k_{a_n}}} = 2^{-(k_{a_n+1} - k_{a_n})}.$$

Therefore, $[\psi]_* \in \mathcal{E}$.

Notice that for any $\varphi \in \Sigma_{I,\varepsilon}$ there is a function $x \in [\psi]_*$ such that $x{\restriction}I_{a_n} \notin \varphi(a_n)$ for all n, thus $x \notin [\varphi]_*$.

(c) If $\mathcal{E}_{I,\varepsilon} = \{\emptyset\}$ for some I, ε, there is nothing to prove. So, assume that $\mathcal{E}_{I,\varepsilon} \neq \{\emptyset\}$. Take any infinite and co-infinite $B \subseteq \omega$ and define $\varphi(n) = \{\mathbf{0}\}$ on B ($\mathbf{0}$ represents a constant zero sequence on any I_n), and $\varphi(n) = \emptyset$ on $\omega \smallsetminus B$. Then clearly $\varphi \in \Sigma_{I,\varepsilon}$ and $\emptyset \neq [\varphi]_\infty \in \mathcal{S}_{I,\varepsilon}$. It remains to see that $[\varphi]_\infty \notin \mathcal{E}_{I,\varepsilon}$. To this end, we will show that $[\varphi]_\infty \not\subseteq [\psi]_*$ for any $\psi \in \Sigma_{I,\varepsilon}$.

So, let $\psi \in \Sigma_{I,\varepsilon}$. The set $v := \{n \in \omega \smallsetminus B \mid {}^{I_n}2 \neq \psi(n)\}$ is infinite. Define $x \in {}^{\omega}2$ by

$$x{\restriction}I_n = \begin{cases} \mathbf{0} & \text{if } n \in \omega \smallsetminus v, \\ \text{any } a \in {}^{I_n}2 \smallsetminus \psi(n) & \text{if } n \in v. \end{cases}$$

Therefore $x \in [\varphi]_\infty$ and $x \notin [\psi]_*$.

(d) Choose any $\varphi \in \Sigma_{I,\varepsilon}$ such that $[\varphi]_\infty \neq \emptyset$. We will show that $[\varphi]_\infty \notin \mathcal{E}$, i.e., for any $J \in \mathbb{I}$ and any $\psi \in \prod_{n\in\omega} \mathcal{P}({}^{J_n}2)$ with $\sum_{n\in\omega} \frac{|\psi(n)|}{2^{|J_n|}} < \infty$, we have $[\varphi]_\infty \not\subseteq [\psi]_*$ (see Theorem 1.3). Thus, fix such J and ψ.

Case 1. If $\exists^\infty n : \psi(n) = \emptyset$, then $[\psi]_* = \emptyset$ and consequently $\emptyset \neq [\varphi]_\infty \not\subseteq [\psi]_*$.

Case 2. If $\forall^\infty n : \psi(n) \neq \emptyset$ then $\forall^\infty n : \psi(n) \subsetneq {}^{J_n}2$ by $\sum_{n\in\omega} \frac{|\psi(n)|}{2^{|J_n|}} < \infty$. Define increasing sequences $\langle k_n \mid n \in \omega\rangle$, $\langle j_n \mid n \in \omega\rangle$ such that

$$\bigcup_{n\in\omega} I_{k_n} \cap \bigcup_{n\in\omega} J_{j_n} = \emptyset,$$

and $\varphi(k_n)$ is non-empty for all $n \in \omega$. Consider the set

$$X = \{x \in {}^{\omega}2 \mid \forall n \in \omega : x{\restriction}I_{k_n+1} \in \varphi(k_n+1)\}.$$

Clearly, $X \subseteq [\varphi]_\infty$. On the other hand, pick $x \in {}^{\omega}2$ such that $x{\restriction}I_{k_n+1} \in \varphi(k_n+1)$ and $x{\restriction}J_{j_n} \notin \psi(n)$ for all $n \in \omega$. Hence, $X \not\subseteq [\psi]_*$. ■

3. Monotonicity. In the current section we investigate what requirement on two pairs (I,ε) and (I',ε') has to be posed in order to achieve $\mathcal{S}_{I,\varepsilon} \subseteq \mathcal{S}_{I',\varepsilon'}$, and similarly for $\mathcal{E}_{I,\varepsilon} \subseteq \mathcal{E}_{I',\varepsilon'}$. Before, we have to review the terminology on relational systems.

The standard sources for the terminology on relational systems are [Bla10, Voj93]. We say that $\mathbf{R} = \langle X, Y, \sqsubset\rangle$ is a *relational system* if it consists of two non-empty sets X and Y and a relation $\sqsubset$, a subset of $X \times Y$, for which:

(1) a set $F \subseteq X$ is $\mathbf{R}$-*bounded* if $\exists y \in Y\ \forall x \in F : x \sqsubset y$,
(2) a set $D \subseteq Y$ is $\mathbf{R}$-*dominating* if $\forall x \in X\ \exists y \in D : x \sqsubset y$.

We associate two cardinal characteristics with the relational system $\mathbf{R}$:

$$\mathfrak{b}(\mathbf{R}) := \min\{|F| \mid F \subseteq X \text{ is } \mathbf{R}\text{-unbounded}\},$$

the *unbounding number* of $\mathbf{R}$, and

$$\mathfrak{d}(\mathbf{R}) := \min\{|D| \mid D \subseteq Y \text{ is } \mathbf{R}\text{-dominating}\},$$

the *dominating number* of $\mathbf{R}$.

The dual of $\mathbf{R}$ is defined by $\mathbf{R}^\perp := \langle Y, X, \sqsubset^\perp\rangle$, where $y \sqsubset^\perp x$ if and only if $x \not\sqsubset y$. Note that $\mathfrak{b}(\mathbf{R}^\perp) = \mathfrak{d}(\mathbf{R})$ and $\mathfrak{d}(\mathbf{R}^\perp) = \mathfrak{b}(\mathbf{R})$.

The cardinal $\mathfrak{b}(\mathbf{R})$ may be undefined, in which case we write $\mathfrak{b}(\mathbf{R}) = \infty$, likewise for $\mathfrak{d}(\mathbf{R})$. Concretely, $\mathfrak{b}(\mathbf{R}) = \infty$ if and only if $\mathfrak{d}(\mathbf{R}) = 1$; and $\mathfrak{d}(\mathbf{R}) = \infty$ if and only if $\mathfrak{b}(\mathbf{R}) = 1$.

Cardinal characteristics associated with ideals can be characterized by relational systems:

Example 3.1. For $\mathcal{I} \subseteq \mathcal{P}(X)$, define the following relational systems:

(1) $\mathcal{I} := \langle \mathcal{I}, \mathcal{I}, \subseteq \rangle$, which is a directed partial order when $\mathcal{I}$ is closed under unions (e.g., an ideal),
(2) $\mathbf{C}_{\mathcal{I}} := \langle X, \mathcal{I}, \in \rangle$.

Whenever $\mathcal{I}$ is an ideal on X, we have

(a) $\mathfrak{b}(\mathcal{I}) = \mathrm{add}(\mathcal{I})$,
(b) $\mathfrak{d}(\mathcal{I}) = \mathrm{cof}(\mathcal{I})$,
(c) $\mathfrak{d}(\mathbf{C}_{\mathcal{I}}) = \mathrm{cov}(\mathcal{I})$,
(d) $\mathfrak{b}(\mathbf{C}_{\mathcal{I}}) = \mathrm{non}(\mathcal{I})$.

We shall extensively use a so-called Tukey connection ([2]), which is a practical tool to determine relations between cardinal characteristics.

Definition 3.2. Let $\mathbf{R} = \langle X, Y, \sqsubset \rangle$ and $\mathbf{R}' = \langle X', Y', \sqsubset' \rangle$ be two relational systems. We say that

$$(\Psi_-, \Psi_+) \colon \mathbf{R} \to \mathbf{R}'$$

is a *Tukey connection from* $\mathbf{R}$ *into* $\mathbf{R}'$ if $\Psi_- \colon X \to X'$ and $\Psi_+ \colon Y' \to Y$ are functions such that

$$\forall x \in X \ \forall y' \in Y' : \Psi_-(x) \sqsubset' y' \Rightarrow x \sqsubset \Psi_+(y').$$

The *Tukey order* between relational systems is defined by $\mathbf{R} \preceq_{\mathrm{T}} \mathbf{R}'$ if and only if there is a Tukey connection from $\mathbf{R}$ into $\mathbf{R}'$. *Tukey equivalence* is defined by $\mathbf{R} \cong_{\mathrm{T}} \mathbf{R}'$ if and only if $\mathbf{R} \preceq_{\mathrm{T}} \mathbf{R}'$ and $\mathbf{R}' \preceq_{\mathrm{T}} \mathbf{R}$.

The crucial assertion connecting Tukey order and cardinal invariants is the following one.

Fact 3.3. *Let* $\mathbf{R} = \langle X, Y, \sqsubset \rangle$ *and* $\mathbf{R}' = \langle X', Y', \sqsubset' \rangle$ *be relational systems. Then*

(a) $\mathbf{R} \preceq_{\mathrm{T}} \mathbf{R}'$ *implies* $(\mathbf{R}')^{\perp} \preceq_{\mathrm{T}} \mathbf{R}^{\perp}$,
(b) $\mathbf{R} \preceq_{\mathrm{T}} \mathbf{R}'$ *implies* $\mathfrak{b}(\mathbf{R}') \leq \mathfrak{b}(\mathbf{R})$ *and* $\mathfrak{d}(\mathbf{R}) \leq \mathfrak{d}(\mathbf{R}')$,
(c) $\mathbf{R} \cong_{\mathrm{T}} \mathbf{R}'$ *implies* $\mathfrak{b}(\mathbf{R}') = \mathfrak{b}(\mathbf{R})$ *and* $\mathfrak{d}(\mathbf{R}) = \mathfrak{d}(\mathbf{R}')$. ■

Hence, we can proceed to a question about the role of the parameters in the families $\mathcal{S}_{I,\varepsilon}$, i.e., what kind of (natural) relation between partitions or elements of ℓ^1_+ guarantees the relation (inclusion) between the families $\mathcal{S}_{I,\varepsilon}$. The same question may be asked for $\mathcal{E}_{I,\varepsilon}$.

([2]) Some authors use "Galois–Tukey connection".

Definition 3.4. Define the following relation on $\mathbb{I}$:

$$I \sqsubseteq J \quad \text{if and only if} \quad \forall^\infty n < \omega\ \exists m < \omega : I_m \subseteq J_n.$$

Note that $\sqsubseteq$ is a directed preorder on $\mathbb{I}$, so we think of $\mathbb{I}$ as the relational system with the relation $\sqsubseteq$. Also notice that we can consider the family of all partitions of all cofinite subsets of ω instead of $\mathbb{I}$ ([3]). In [Bla10] it is proved that $\mathbb{I} \cong_{\mathrm{T}} {}^\omega\omega$.

The second relation on $\mathbb{I}$ is the refinement relation $\sqsubseteq_{\mathrm{R}}$ defined, for $I, J \in \mathbb{I}$, by

$$I \sqsubseteq_{\mathrm{R}} J \quad \text{if and only if} \quad \forall^\infty n \in \omega\ \exists F \in [\omega]^{<\omega} : J_n = \bigcup_{k\in F} I_k,$$

i.e., all but finitely many J_n's are finite unions of some I_k's. Note that $I \sqsubseteq_{\mathrm{R}} J$ implies $\forall^\infty k\ \exists n : I_k \subseteq J_n$. Also, if $I \sqsubseteq_{\mathrm{R}} J$, then $I \sqsubseteq J$. For $I \sqsubseteq J$ we define $\mathrm{Sub}(I, J, n) = \{k \in \omega \mid I_k \subseteq J_n\}$.

Unfortunately, as we shall see in Observation 3.6, none of these relations is sufficient for inclusions between the σ-ideals. However, after considering some reasonable additional assumptions they might come in useful. On the other hand, we show that the standard relation $\leq^*$ between elements of ℓ^1_+ yields the inclusion between the families $\mathcal{S}_{I,\varepsilon}$ and $\mathcal{E}_{I,\varepsilon}$ without the need for additional assumptions.

Proposition 3.5. *Let $\varepsilon, \varepsilon' \in \ell^1_+$ and $I, J \in \mathbb{I}$.*

(a) *If $\varepsilon \leq^* \varepsilon'$, then $\mathcal{S}_{I,\varepsilon} \subseteq \mathcal{S}_{I,\varepsilon'}$ and $\mathcal{E}_{I,\varepsilon} \subseteq \mathcal{E}_{I,\varepsilon'}$.*
(b) *Let $k \geq 1$. If $\varepsilon \leq^* \varepsilon' \leq^* k\cdot\varepsilon$, then $\mathcal{S}_{I,\varepsilon} = \mathcal{S}_{I,\varepsilon'}$ and $\mathcal{E}_{I,\varepsilon} = \mathcal{E}_{I,\varepsilon'}$.*
(c) *If $I \sqsubseteq J$, ε is decreasing and $\forall^\infty n : \min \mathrm{Sub}(I, J, n) \geq n$ ([4]), then $\mathcal{E}_{I,\varepsilon} \subseteq \mathcal{E}_{J,\varepsilon}$.*
(d) *If $I \sqsubseteq_{\mathrm{R}} J$ and $\forall^\infty n : \sum_{k\in \mathrm{Sub}(I,J,n)} \varepsilon_k \leq \varepsilon_n$, then $\mathcal{S}_{I,\varepsilon} \subseteq \mathcal{S}_{J,\varepsilon}$.*

Proof. (a) Observe that in this case we have

$$\forall^\infty n : \frac{|\varphi(n)|}{2^{|I_n|}\cdot \varepsilon'_n} \leq \frac{|\varphi(n)|}{2^{|I_n|}\cdot \varepsilon_n},$$

therefore $\Sigma_{I,\varepsilon} \subseteq \Sigma_{I,\varepsilon'}$.

(b) The "$\subseteq$" part follows from (a). For the "$\supseteq$" part, without loss of generality let $\varphi \in \Sigma_{I,k\cdot\varepsilon}$. We will show that $\varphi \in \Sigma_{I,\varepsilon}$. We have

$$\lim_{n\to\infty} \frac{|\varphi(n)|}{2^{|I_n|}\cdot k\cdot \varepsilon_n} = \frac{1}{k}\cdot \lim_{n\to\infty} \frac{|\varphi(n)|}{2^{|I_n|}\cdot \varepsilon_n} = 0.$$

([3]) This is a simple consequence of the fact that $\mathbb{I}$ is a $\sqsubseteq$-cofinal subset in such a family. Hence, its bounding and dominating cardinal characteristics are the same as $\mathfrak{b}(\mathbb{I})$ and $\mathfrak{d}(\mathbb{I})$, respectively. Thus, most of the time, by $\mathbb{I}$ we mean the partition of some cofinite subset of ω.

([4]) This happens, e.g., when $\exists^\infty n : |\{k \mid J_n \cap I_k \neq \emptyset\}| \geq 2$.

(c) Let $\varphi \in \Sigma_{I,\varepsilon}$. For every n define

$$\begin{aligned}\psi(n) &= \{x \in {}^{J_n}2 \mid \forall k \in \mathrm{Sub}(I,J,n) : x{\restriction}I_k \in \varphi(k)\}\\ &= \{x \in {}^{J_n}2 \mid \forall k : I_k \subseteq J_n \Rightarrow x{\restriction}I_k \in \varphi(k)\}.\end{aligned}$$

Since $\varepsilon \to 0$, we have $\forall^\infty k : \varepsilon_k < 1$. Furthermore, ε is decreasing and so

$$\prod_{k\in\mathrm{Sub}(I,J,n)} \varepsilon_k \le \min\{\varepsilon_k \mid k \in \mathrm{Sub}(I,J,n)\} \le \varepsilon_n.$$

Thus for all but finitely many $n \in \omega$ we have

$$\begin{aligned}\frac{|\psi(n)|}{2^{|J_n|}\cdot\varepsilon_n} &= \frac{\prod_{k\in\mathrm{Sub}(I,J,n)}|\varphi(k)|\cdot 2^{|J_n|-\sum_{j\in\mathrm{Sub}(I,J,n)}|I_j|}}{2^{|J_n|}\cdot\varepsilon_n}\\ &= \frac{1}{\varepsilon_n}\cdot\prod_{k\in\mathrm{Sub}(I,J,n)}\frac{|\varphi(k)|}{2^{|I_k|}} \le \prod_{k\in\mathrm{Sub}(I,J,n)}\frac{|\varphi(k)|}{2^{|I_k|}\cdot\varepsilon_k} \to 0.\end{aligned}$$

The last term converges to zero since $\varphi \in \Sigma_{I,\varepsilon}$. Therefore, $\psi \in \Sigma_{J,\varepsilon}$. Note that $[\varphi]_* \subseteq [\psi]_*$. Indeed, if $x \in {}^\omega 2$ is such that $\forall^\infty k : x{\restriction}I_k \in \varphi(k)$, then $\forall^\infty n : x{\restriction}J_n \in \psi(n)$.

(d) We will show that if ε decreases sufficiently fast, then $I \sqsubseteq_{\mathrm{R}} J$ implies $S_{I,\varepsilon} \subseteq S_{J,\varepsilon}$. So, let $\varepsilon \in \ell^1_+$ be such that $\forall^\infty n : \sum_{k\in\mathrm{Sub}(I,J,n)}\varepsilon_k \le \varepsilon_n$. For any $\varphi \in \Sigma_{I,\varepsilon}$ set

$$\psi(n) = \{s \in {}^{J_n}2 \mid \exists k \in \mathrm{Sub}(I,J,n) : s{\restriction}I_k \in \varphi(k)\},$$

i.e., $\psi(n)$ is the set of extensions of functions from $\bigcup_{k\in\mathrm{Sub}(I,J,n)}\varphi(k)$ on J_n.

First, note that for any $N \in \omega$ and all but finitely many $n \in \omega$,

$$\begin{aligned}\frac{|\psi(n)|}{2^{|J_n|}} &= \sum_{k\in\mathrm{Sub}(I,J,n)}\frac{|\varphi(k)|\cdot 2^{|J_n\setminus I_k|}}{2^{|I_k|}\cdot 2^{|J_n\setminus I_k|}}\\ &= \sum_{k\in\mathrm{Sub}(I,J,n)}\frac{|\varphi(k)|}{2^{|I_k|}} < \frac{\sum_{k\in\mathrm{Sub}(I,J,n)}\varepsilon_k}{N},\end{aligned}$$

while the last expression is at most ε_n/N, and thus $\psi \in \Sigma_{J,\varepsilon}$.

Notice that $[\varphi]_\infty \subseteq [\psi]_\infty$. Indeed, if $x \in {}^\omega 2$ is such that $\exists^\infty k : x{\restriction}I_k \in \varphi(k)$, consider $\langle n_k \mid k \in \omega\rangle$ such that $I_k \subseteq J_{n_k}$ for all but finitely many $k \in \omega$ (there is such a sequence by $I \sqsubseteq_{\mathrm{R}} J$). Then $x{\restriction}J_{n_k}$ is an extension of $x{\restriction}I_k \in \varphi(k)$ on J_{n_k}, that is, $x{\restriction}J_{n_k} \in \psi(n_k)$ for all but finitely many $k \in \omega$. Therefore, $\exists^\infty n : x{\restriction}J_n \in \psi(n)$.

Since for any $\varphi \in \Sigma_{I,\varepsilon}$ we found $\psi \in \Sigma_{J,\varepsilon}$ with $[\varphi]_\infty \subseteq [\psi]_\infty$, we have $\mathcal{S}_{I,\varepsilon} \subseteq \mathcal{S}_{J,\varepsilon}$. ∎

In Proposition 3.5(c)–(d), the relations $\sqsubseteq$ and $\sqsubseteq_R$ are not sufficient to guarantee the corresponding conclusions.

OBSERVATION 3.6. *There are I, J, ε such that $I \sqsubseteq J$ and $\mathcal{S}_{I,\varepsilon} \not\subseteq \mathcal{S}_{J,\varepsilon}$. The same is true for the relation $\sqsubseteq$ with $\mathcal{E}_{I,\varepsilon}$, and for the relation $\sqsubseteq_R$ with both $\mathcal{S}_{I,\varepsilon}$ and $\mathcal{E}_{I,\varepsilon}$.*

Proof. The assertions for $\sqsubseteq_R$ can be proven similarly to those for $\sqsubseteq$. Thus we prove just the latter.

We shall show first that given any directed preorder $\preceq$ on $\mathbb{I}$, it is not necessarily true that $I \preceq J$ implies $\mathcal{S}_{I,\varepsilon} \subseteq \mathcal{S}_{J,\varepsilon}$ for any $\varepsilon \in \ell^1_+$. Assume that the implication holds. By the proof of [Bar88, Lem. 1.3], there are $X, Y \in \mathcal{S}^\star$ such that $X \cup Y \notin \mathcal{S}^\star$. By Theorem 2.3(c), choose $I, I' \in \mathbb{I}$ and $\varepsilon, \varepsilon' \in \ell^1_+$ such that $X \in \mathcal{S}_{I,\varepsilon}$ and $Y \in \mathcal{S}_{I',\varepsilon'}$. Define $\delta \in \ell^1_+$ by $\delta_n = \max\{\varepsilon_n, \varepsilon'_n\}$ for each n. Then $\Sigma_{I,\varepsilon} \subseteq \Sigma_{I,\delta}$, $\Sigma_{I',\varepsilon'} \subseteq \Sigma_{I',\delta}$, and $X \in \mathcal{S}_{I,\delta}$, $Y \in \mathcal{S}_{I',\delta}$.

On the other hand, since $\preceq$ is a directed preorder, we can find a partition J such that $I, I' \preceq J$. By the assumption, $\mathcal{S}_{I,\delta}, \mathcal{S}_{I',\delta} \subseteq \mathcal{S}_{J,\delta}$, so we get $X, Y \in \mathcal{S}_{J,\delta}$. Therefore, $X \cup Y \in \mathcal{S}_{J,\delta} \subseteq \mathcal{S}^\star$, which is a contradiction.

Regarding $\mathcal{E}_{I,\varepsilon}$, we shall show that $I \sqsubseteq J$ does not necessarily imply $\mathcal{E}_{I,\varepsilon} \subseteq \mathcal{E}_{J,\varepsilon}$. That is, there are $I, J \in \mathbb{I}$ with $I \sqsubseteq J$ and $\varepsilon \in \ell^1_+$ such that (I, ε) is $\mathcal{E}$-contributive and (J, ε) is not $\mathcal{E}$-contributive (even $\mathcal{S}^\star$-contributive). Thus, $\{\emptyset\} \neq \mathcal{E}_{I,\varepsilon} \not\subseteq \mathcal{E}_{J,\varepsilon} = \{\emptyset\}$. To show this, construct inductively I such that

$$\forall^\infty n : 2^{-|I_n|} < \min\left\{\frac{2^{-|I_{n-1}|}}{n}, 2^{-(n+1)}\right\}.$$

It is easy to construct such a partition – each step takes sufficiently long I_n. Put $\varepsilon_n = 2^{-|I_{n-1}|}$ for $n \geq 2$. Let I_0 be an arbitrary finite interval (initial in ω) of length at least 2, $|I_1| = 2$, $\varepsilon_0 = 1$ and $\varepsilon_1 = \frac{1}{2}$. Note that $\varepsilon_n < 2^{-n}$ for all but finitely many $n \in \omega$, thus, $\varepsilon \in \ell^1_+$. Also, note that $\forall^\infty n : 2^{-|I_n|} < \frac{\varepsilon_n}{n}$, therefore, by Lemma 2.8 we see that (I, ε) is $\mathcal{E}$-contributive. Define J such that $I_0 = J_0 \cup J_1$ and $J_n = I_{n-1}$ for all $n \in \omega$. Clearly, $I \sqsubseteq J$ (even $J \sqsubseteq I$). Moreover, $\forall^\infty n : 2^{-|J_n|} = 2^{-|I_{n-1}|} = \varepsilon_n$. Thus, by Lemmas 2.8 and 2.5 we have $\mathcal{E}_{J,\varepsilon} = \mathcal{S}_{J,\varepsilon} = \{\emptyset\}$. ∎

The refinement relation has minimal usefulness regarding cardinal invariants since its bounding and dominating numbers are trivial. Denote by $\mathbf{D}_\mathrm{R}$ the relational system $\langle \mathbb{I}, \mathbb{I}, \sqsubseteq_\mathrm{R} \rangle$.

PROPOSITION 3.7. *$\mathfrak{b}(\mathbf{D}_\mathrm{R}) = 2$ and $\mathfrak{d}(\mathbf{D}_\mathrm{R}) = \mathfrak{c}$.*

Proof. First, it is very easy to see that the relation $\sqsubseteq_\mathrm{R}$ is not a directed preorder. To see this, let $f \in {}^\omega\omega$ be an increasing function. Put $I_n = [f(2n), f(2n+2))$, $J_0 = [f(0), f(3))$, and for $n \geq 1$ put $J_n = [f(2n+1), f(2n+3))$. We get overlapping partitions for which we cannot find a common upper bound. Hence, $\mathfrak{b}(\mathbf{D}_\mathrm{R}) = 2$.

We shall formulate previous facts as a claim since we use it in the next construction. First, for any $I = \{[i_k, i_{k+1}) \mid k \in \omega\} \in \mathbb{I}$ define the set of endpoints of its intervals $\mathrm{ep}(I) = \{i_k \mid k \in \omega\}$.

CLAIM 3.8. *If $I, J \in \mathbb{I}$ satisfy $|\mathrm{ep}(I) \cap \mathrm{ep}(J)| < \omega$, then no $K \in \mathbb{I}$ is an $\sqsubseteq_{\mathrm{R}}$-upper bound for both I and J, i.e., for any $K^I, K^J \in \mathbb{I}$ such that $I \sqsubseteq_{\mathrm{R}} K^I$ and $J \sqsubseteq_{\mathrm{R}} K^J$ we have $K^I \neq K^J$.*

Proof. Assume not. Let $K \in \mathbb{I}$ be such that $I, J \sqsubseteq_{\mathrm{R}} K$. Then

$$\exists m_0 \in \omega\ \forall m > m_0 : K_m \text{ is a union of some } I_k\text{'s},$$
$$\exists m_1 \in \omega\ \forall m > m_1 : K_m \text{ is a union of some } J_n\text{'s}.$$

Since $|\mathrm{ep}(I) \cap \mathrm{ep}(J)| < \omega$ we also have

$$\exists M \in \omega\ \forall k, n > M :$$
$$\min(I_k) \neq \min(J_n), \max(J_n) \text{ and } \max(I_k) \neq \min(J_n), \max(J_n).$$

Hence,

$$\exists m_2 \in \omega\ \forall m > m_2 : \min \mathrm{Sub}(I, K, m) > M \text{ and } \min \mathrm{Sub}(J, K, m) > M.$$

Now consider any $m > \max\{m_0, m_1, m_2\}$. For this m, K_m is a union of J_n's for $n \in \mathrm{Sub}(J, K, m)$. Since $m > m_0$, K_m is also a union of I_k's for $k \in \mathrm{Sub}(I, K, m)$. But this is not possible by the fact that $m > m_2$, which implies that for any $k \in \mathrm{Sub}(I, K, m)$ and $n \in \mathrm{Sub}(J, K, m)$, I_k and J_n do not share common endpoints. ∎

We will show that $\mathfrak{d}(\sqsubseteq_{\mathrm{R}}) = \mathfrak{c}$. Let $\mathcal{D}$ be a $\sqsubseteq_{\mathrm{R}}$-dominating family in $\mathbb{I}$. Consider an arbitrary AD family $\mathcal{A}$ of cardinality $\mathfrak{c}$. Assign to any $A = \{a_i \in \omega \mid i \in \omega\} \in \mathcal{A}$ (naturally enumerated in an increasing way) a partition $I_A = \{[a_i, a_{i+1}) \mid i \in \omega\}$ and denote $\mathcal{I}_{\mathcal{A}} = \{I_A \mid A \in \mathcal{A}\}$. Note that I_A and I_B satisfy $|\mathrm{ep}(I_A) \cap \mathrm{ep}(I_B)| < \omega$ whenever $A \neq B$. By Claim 3.8, no $K \in \mathbb{I}$ is a bound for two distinct elements of $\mathcal{I}_{\mathcal{A}}$, let alone infinitely many of them. That is, for any $A \in \mathcal{A}$ there is $K_A \in \mathcal{D}$, a bound for I_A, and by Claim 3.8, this K_A is an upper bound for no other I_B (i.e., for $B \neq A$). Thus $|\{K_A \mid A \in \mathcal{A}\}| = \mathfrak{c}$ and as $\{K_A \mid A \in \mathcal{A}\} \subseteq \mathcal{D}$, also $|\mathcal{D}| = \mathfrak{c}$. ∎

We shall establish a basic tool for showing inclusions between the σ-ideals under study, i.e., we show that the inclusion strongly depends on coordinatewise inclusions.

LEMMA 3.9. *Let $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$, and $\varphi, \psi \in \Sigma_{I,\varepsilon}$. The following statements are equivalent:*

(1) $\forall^\infty n \in \omega : \varphi(n) \subseteq \psi(n)$.
(2) $[\varphi]_\infty \subseteq [\psi]_\infty$.

Proof. (1)⇒(2) is clear: $x \in [\varphi]_\infty$ if and only if $\exists^\infty n : x{\restriction}I_n \in \varphi(n)$. Since $\forall^\infty n : \varphi(n) \subseteq \psi(n)$, we have $\exists^\infty n : x{\restriction}I_n \in \psi(n)$.

¬(1)⇒¬(2). By the assumption there exists an increasing sequence $\langle k_n \mid n \in \omega\rangle$ such that $\varphi(k_n) \setminus \psi(k_n) \neq \emptyset$. Consider any $x \in {}^\omega 2$ such that $x{\restriction}I_{k_n} \in \varphi(k_n) \setminus \psi(k_n)$ for all $n \in \omega$, and $x{\restriction}I_m \notin \psi(m)$ for all but finitely many $m \in \omega \setminus \{k_n \mid n \in \omega\}$. Then clearly $x \in [\varphi]_\infty$ but $x \notin [\psi]_\infty$. ∎

REMARK 3.10. Note that for any $\varphi \in \Sigma_{I,\varepsilon}$ we have $[\varphi]_* \neq \emptyset$ if and only if $\forall^\infty n \in \omega : \varphi(n) \neq \emptyset$.

Adding more assumptions in Lemma 3.9 yields the following assertion, which can be proven in a similar fashion.

LEMMA 3.11. *Let $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$, and let $\varphi, \psi \in \Sigma_{I,\varepsilon}$ be such that $\forall^\infty n \in \omega : \varphi(n) \neq \emptyset$. The following statements are equivalent:*

(1) $\forall^\infty n \in \omega : \varphi(n) \subseteq \psi(n)$.
(2) $[\varphi]_* \subseteq [\psi]_*$.
(3) $[\varphi]_\infty \subseteq [\psi]_\infty$.

REMARK 3.12. Note that for the implication (1)⇒(2) in Lemma 3.11 we do not need to assume that $\varphi(n) \neq \emptyset$ for all but finitely many n's.

In connection with Proposition 3.5(a)–(b), we shall demonstrate that for distinct $\varepsilon, \varepsilon'$ we may have distinct $\mathcal{S}_{I,\varepsilon}, \mathcal{S}_{I,\varepsilon'}$, and similarly for $\mathcal{E}_{I,\varepsilon}$. That is, it is not necessarily true that $\mathcal{S}_{I,\varepsilon} = \mathcal{S}_{I,\varepsilon'}$ and $\mathcal{E}_{I,\varepsilon} = \mathcal{E}_{I,\varepsilon'}$ for $\varepsilon \neq^* \varepsilon'$.

The above fact can be easily proven by considering $I, \varepsilon, \varepsilon'$ such that (I, ε) is not $\mathcal{S}^\star$-contributive (resp. not $\mathcal{E}$-contributive) but (I, ε') is $\mathcal{E}$-contributive (resp. $\mathcal{S}^\star$-contributive). Indeed, consider I such that $|I_n| = n$ and define $\varepsilon, \varepsilon'$ by $\varepsilon_n = \frac{1}{2^n}$ and $\varepsilon'_n = \frac{n}{2^n}$ for every $n \in \omega$. Then for any $A \in [\omega]^\omega$ we have $\forall^\infty n \in A : 2^{|I_n|} \cdot \varepsilon_n = \frac{2^n}{2^n} = 1$, thus $\lim_{n \in A} 2^{|I_n|} \cdot \varepsilon_n \neq \infty$. By Lemma 2.5, (I, ε) is not $\mathcal{S}^\star$-contributive. On the other hand, $\lim_{n\to\infty} 2^{|I_n|} \cdot \varepsilon'_n = \frac{2^{|I_n|} \cdot n}{2^{|I_n|}} = n \to \infty$. By Lemma 2.8 we deduce that (I, ε') is $\mathcal{E}$-contributive.

The question is whether $\mathcal{S}_{I,\varepsilon}$ or $\mathcal{E}_{I,\varepsilon}$, once they are non-trivial, can be different for different ε. We will answer this question in the following example.

OBSERVATION 3.13. *We have $\mathcal{S}_{I,\varepsilon} \neq \mathcal{S}_{I,\varepsilon'}$ and $\mathcal{E}_{I,\varepsilon} \neq \mathcal{E}_{I,\varepsilon'}$ for any I with $|I_n| \geq (n+1)^3$ and $\varepsilon, \varepsilon'$ defined by*

$$\varepsilon_n = \frac{1}{(n+1)^2} + \frac{n}{2^n} \quad \textit{and} \quad \varepsilon'_n = \frac{1}{(n+1)^3}.$$

Proof. The proof for $\mathcal{E}_{I,\varepsilon}$ is similar to that for $\mathcal{S}_{I,\varepsilon}$. Thus we shall prove the assertion just for $\mathcal{S}_{I,\varepsilon}$.

Note that both (I, ε) and (I, ε') are $\mathcal{E}$-contributive. Let

$$\xi_n = \max\left\{m \in \omega \,\middle|\, \frac{m}{2^{|I_n|}} < \frac{1}{(n+1)^2}\right\}.$$

CLAIM 3.14. $\forall K \in \omega\ \exists n_K \in \omega\ \forall n > n_K : \frac{\xi_n}{2^{|I_n|} \cdot K} \geq \frac{1}{(n+1)^3}$.

Proof. Notice that for all $n \in \omega$ we have

$$\frac{\xi_n}{2^{|I_n|}} \geq \frac{1}{(n+1)^2} - \frac{1}{2^{|I_n|}} \geq \frac{1}{(n+1)^2} - \frac{1}{2^n}.$$

It is easy to show that $\forall K \in \omega\ \forall^\infty n \in \omega : \frac{1}{(n+1)^2} - \frac{1}{2^n} \geq \frac{K}{(n+1)^3}$, since for positive n, this inequality is equivalent to

$$2^n(n-K+1) \geq (n+1)^3,$$

and 2^n increases faster than $(n+1)^3$. ∎

We may assume that the sequence $\langle n_K \mid K \in \omega \rangle$ is increasing. Define φ such that $|\varphi(i)| = \lceil \xi_i/K \rceil$ for $i \in [n_K, n_{K+1})$. Note that $\forall^\infty n : \varphi(n) \neq \emptyset$. We will show that $\varphi \in \Sigma_{I,\varepsilon}$.

Let $N \in \omega$. Then for all but finitely many $K > N$ and all $i \in [n_K, n_{K+1})$,

$$\frac{|\varphi(i)|}{2^{|I_i|}} = \frac{\lceil \xi_i/K \rceil}{2^{|I_i|}} \leq \frac{\xi_i/K + 1}{2^{|I_i|}} = \frac{\xi_i}{2^{|I_i|} \cdot K} + \frac{1}{2^{|I_i|}} < \frac{1}{(i+1)^2 \cdot N} + \frac{i}{2^i \cdot N}.$$

Thus, by the remark below Definition 2.1, $\varphi \in \Sigma_{I,\varepsilon}$.

On the other hand, for any $K \in \omega$ and any $i \in [n_K, n_{K+1})$ we have

$$\frac{|\varphi(i)|}{2^{|I_i|}} = \frac{\lceil \xi_i/K \rceil}{2^{|I_i|}} \geq \frac{\xi_i}{2^{|I_i|} \cdot K} \geq \frac{1}{(i+1)^3}.$$

This yields $[\varphi]_\infty \notin \mathcal{S}_{I,\varepsilon'}$, since otherwise there is $\psi \in \Sigma_{I,\varepsilon'}$ with $[\varphi]_\infty \subseteq [\psi]_\infty$. However, by Lemma 3.9 we have $\varphi(n) \subseteq \psi(n)$ for all but finitely many n, and we obtain

$$\frac{|\psi(i)|}{2^{|I_i|}} \geq \frac{|\varphi(i)|}{2^{|I_i|}} \geq \frac{1}{(i+1)^3} = \varepsilon_i. \ ∎$$

4. Additivity and cofinality of the new ideals. In this section, we will study the cardinals $\mathrm{add}(\mathcal{E}_{I,\varepsilon})$ and $\mathrm{add}(\mathcal{S}_{I,\varepsilon})$. We will be mostly concerned with proving Theorems B, E, and F(a).

We begin with the following combinatorial result that will be used to prove Theorem E.

LEMMA 4.1. *Let $(I,\varepsilon) \in \mathbb{I} \times \ell^1_+$ be an $\mathcal{E}$-contributive pair. Then for any $\varphi \in \Sigma_{I,\varepsilon}$ there is $\bar\varphi \in \Sigma_{I,\varepsilon}$ such that $\varphi(n) \subseteq \bar\varphi(n)$ and $\bar\varphi(n) \neq \emptyset$ for all but finitely many $n \in \omega$.*

Proof. If $\varphi(n) = \emptyset$ take $\bar\varphi(n) = \{\mathbf{0}\}$. Otherwise put $\bar\varphi(n) = \varphi(n)$. We will show that $\bar\varphi \in \Sigma_{I,\varepsilon}$. Putting $v = \{n \in \omega \mid \varphi(n) = \emptyset\}$ we have

$$\forall n \in \omega \setminus v : \frac{|\bar\varphi(n)|}{2^{|I_n|}} = \frac{|\varphi(n)|}{2^{|I_n|}} \quad \text{and} \quad \forall n \in v : \frac{|\bar\varphi(n)|}{2^{|I_n|}} = \frac{1}{2^{|I_n|}}.$$

Let $N \in \omega$. By the fact that $\varphi \in \Sigma_{I,\varepsilon}$ and by Lemma 2.8 we have

$$\forall^\infty n \in \omega \setminus v : \frac{|\bar\varphi(n)|}{2^{|I_n|}} < \frac{\varepsilon_n}{N} \quad \text{and} \quad \forall^\infty n \in v : \frac{|\bar\varphi(n)|}{2^{|I_n|}} = \frac{1}{2^{|I_n|}} < \frac{\varepsilon_n}{N}.$$

Consequently, $\forall^\infty n \in \omega : \frac{|\bar\varphi(n)|}{2^{|I_n|}} < \frac{\varepsilon_n}{N}$. ∎

Note that we may assume that the smallest base of $\mathcal{E}_{I,\varepsilon} \neq \{\emptyset\}$ consists only of sets $[\varphi]_*$ such that $\forall^\infty n : \varphi(n) \neq \emptyset$.

Proposition 4.2. *For any $\mathcal{E}$-contributive $(I,\varepsilon)\in\mathbb{I}\times\ell^1_+$ we have $\mathcal{S}_{I,\varepsilon}\cong_{\mathrm{T}}\mathcal{E}_{I,\varepsilon}$. In particular,* $\mathrm{cof}(\mathcal{S}_{I,\varepsilon})=\mathrm{cof}(\mathcal{E}_{I,\varepsilon})$ *and* $\mathrm{add}(\mathcal{E}_{I,\varepsilon})=\mathrm{add}(\mathcal{S}_{I,\varepsilon})$.

Proof. "$\preceq_{\mathrm{T}}$". For any $X\in\mathcal{S}_{I,\varepsilon}$ there is $\varphi_X\in\Sigma_{I,\varepsilon}$ such that $X\subseteq[\varphi_X]_\infty$ and $\forall^\infty n:\varphi_X(n)\neq\emptyset$, see Lemma 4.1. Define $\Psi_-:\mathcal{S}_{I,\varepsilon}\to\mathcal{E}_{I,\varepsilon}$ by $\Psi_-(X)=[\varphi_X]_*$. For any $Y\in\mathcal{E}_{I,\varepsilon}$ there is $\psi_Y\in\Sigma_{I,\varepsilon}$ such that $Y\subseteq[\psi_Y]_*$. Define $\Psi_+:\mathcal{E}_{I,\varepsilon}\to\mathcal{S}_{I,\varepsilon}$ by $\Psi_+(Y)=[\psi_Y]_\infty$.

Let $X\in\mathcal{S}_{I,\varepsilon}$ and $Y\in\mathcal{E}_{I,\varepsilon}$ be such that $\Psi_-(X)=[\varphi_X]_*\subseteq Y$. Since $Y\subseteq[\psi_Y]_*$, we have $[\varphi_X]_*\subseteq[\psi_Y]_*$. By Lemma 3.11 we have $X\subseteq[\varphi_X]_\infty\subseteq[\psi_Y]_\infty=\Psi_+(Y)$.

"$\succeq_{\mathrm{T}}$". To see this just switch the roles of Ψ_- and Ψ_+ from the first part of this proof. ■

It is well-known that $\mathrm{add}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{N})$ can be characterized using the notion of slaloms. We shall show the connection between localization and anti-localization cardinals based on slaloms and the σ-ideals under consideration, but first, we need the basic terminology from localization and anti-localization theory. For more details and further references, see [CM23].

Definition 4.3. Given a sequence $b=\langle b(n)\mid n\in\omega\rangle$ of non-empty sets and $h\colon\omega\to\omega$, define

$$\prod b:=\prod_{n\in\omega}b(n),$$
$$\mathcal{S}(b,h):=\prod_{n\in\omega}[b(n)]^{\leq h(n)}.$$

For two functions $x\in\prod b$ and $\varphi\in\mathcal{S}(b,h)$ write

$$x\in^*\varphi\quad\text{if and only if}\quad\forall^\infty n\in\omega:x(n)\in\varphi(n),$$

and

$$x\in^\infty\varphi\quad\text{if and only if}\quad\exists^\infty n\in\omega:x(n)\in\varphi(n).$$

The negations of $x\in^*\varphi$ and $x\in^\infty\varphi$ are denoted by $x\notin^*\varphi$ and $x\notin^\infty\varphi$ respectively, i.e., $x\notin^*\varphi$ if and only if $\exists^\infty n\in\omega:x(n)\notin\varphi(n)$, and $x\notin^\infty\varphi$ if and only if $\forall^\infty n\in\omega:x(n)\notin\varphi(n)$.

We set

$$\mathfrak{b}^{\mathrm{Lc}}_{b,h}=\min\Big\{|F|\,\Big|\,F\subseteq\prod b\ \&\ \neg\exists\varphi\in\mathcal{S}(b,h)\ \forall x\in F:x\in^*\varphi\Big\},$$
$$\mathfrak{d}^{\mathrm{Lc}}_{b,h}=\min\Big\{|R|\,\Big|\,R\subseteq\mathcal{S}(b,h)\ \&\ \forall x\in\prod b\ \exists\varphi\in R:x\in^*\varphi\Big\},$$
$$\mathfrak{b}^{\mathrm{aLc}}_{b,h}=\min\Big\{|S|\,\Big|\,S\subseteq\mathcal{S}(b,h)\ \&\ \neg\exists x\in\prod b\ \forall\varphi\in S:x\notin^\infty\varphi\Big\},$$
$$\mathfrak{d}^{\mathrm{aLc}}_{b,h}=\min\Big\{|E|\,\Big|\,E\subseteq\prod b\ \&\ \forall\varphi\in\mathcal{S}(b,h)\ \exists x\in E:x\notin^\infty\varphi\Big\}.$$

Denote $\mathbf{Lc}(b,h) := \langle \prod b, \mathcal{S}(b,h), \in^* \rangle$ and $\mathbf{aLc}(b,h) := \langle \mathcal{S}(b,h), \prod b, \not\ni^\infty \rangle$. So we have $\mathfrak{b}^{\mathrm{Lc}}_{b,h} = \mathfrak{b}(\mathbf{Lc}(b,h))$, $\mathfrak{b}^{\mathrm{aLc}}_{b,h} = \mathfrak{b}(\mathbf{aLc}(b,h))$, and similarly for the dominating numbers.

If b is the constant sequence ω, we use the notation $\mathbf{Lc}(\omega,h)$, $\mathbf{aLc}(\omega,h)$ and we denote the associated cardinal characteristics by $\mathfrak{b}^{\mathrm{Lc}}_{\omega,h}$, $\mathfrak{d}^{\mathrm{Lc}}_{\omega,h}$, $\mathfrak{b}^{\mathrm{aLc}}_{\omega,h}$, $\mathfrak{d}^{\mathrm{aLc}}_{\omega,h}$.

Theorem 4.4 (Bartoszyński [Bar84], see also [CM23]). *If $h \in {}^\omega\omega$ and $\lim_{n\to\infty} h(n) = \infty$, then $\mathfrak{b}^{\mathrm{Lc}}_{\omega,h} = \mathrm{add}(\mathcal{N})$ and $\mathfrak{d}^{\mathrm{Lc}}_{\omega,h} = \mathrm{cof}(\mathcal{N})$.*

Theorem 4.5 (Bartoszyński [Bar87], Miller [Mil82]). *If $h \in {}^\omega\omega$ and $h \geq^* 1$, then $\mathfrak{b}^{\mathrm{aLc}}_{\omega,h} = \mathrm{non}(\mathcal{M})$ and $\mathfrak{d}^{\mathrm{aLc}}_{\omega,h} = \mathrm{cov}(\mathcal{M})$.*

Before we state the relationships between the relevant σ-ideals and localization cardinals, we need to expand the terminology concerning relational systems. Let $\mathbf{R} = \langle X, Y, \sqsubset \rangle$, $\mathbf{R}' = \langle X', Y', \sqsubset' \rangle$ be relational systems. We define the *sequential composition* $(\mathbf{R};\mathbf{R}') = \langle X \times (X')^Y, Y \times Y', \sqsubset^\bullet \rangle$, where the binary relation $(x,f) \sqsubset^\bullet (a,b)$ means $x \sqsubset a$ and $f(a) \sqsubset' b$. More about this way of creating a relational system from existing relational systems can be found in [Bla10]. Also, we will use the following abstraction of certain types of sequences, allowing us to think about complex objects as natural numbers. Here, we let ${}^{\uparrow\omega}\omega = \{f \in {}^\omega\omega \mid f \text{ is increasing}\}$.

Remark 4.6. Let $I \in \mathbb{I}$. For any $a \in {}^{\uparrow\omega}\omega$, the set

$$\prod_{n\in\omega}\Big(\prod_{m\in[a(n),a(n+1))} \mathcal{P}({}^{I_m}2)\Big)$$

can be thought of as a subset of ${}^\omega\omega$. Indeed, denote $\lambda_i = |\mathcal{P}({}^{I_i}2)| < \omega$. Then for each $a \in {}^{\uparrow\omega}\omega$ and $n \in \omega$ there are $\kappa_{a,n} := \prod_{i=a(n)}^{a(n+1)-1} \lambda_i$ finite sequences t having the domain $[a(n), a(n+1))$ such that $t(i) \subseteq {}^{I_i}2$ for each $i \in [a(n), a(n+1))$. Enumerate these sequences in an arbitrary way to get $\{t^{a,n}_K \mid K < \kappa_{a,n}\}$. We define $\varPhi\colon \bigcup_{a\in{}^{\uparrow\omega}\omega} \prod_{n\in\omega}(\prod_{m\in[a(n),a(n+1))} \mathcal{P}({}^{I_m}2)) \to {}^\omega\omega$ as follows. For $f \in \bigcup_{a\in{}^{\uparrow\omega}\omega} \prod_{n\in\omega}(\prod_{m\in[a(n),a(n+1))} \mathcal{P}({}^{I_m}2))$ there is a unique a' such that $f \in \prod_{n\in\omega}(\prod_{m\in[a'(n),a'(n+1))} \mathcal{P}({}^{I_m}2))$. We let $\varPhi(f)(n) = K$ if and only if $f(n) = t^{a',n}_K$, i.e., if $f(n)$ is the K-th element with respect to the above-mentioned enumeration (5).

The proof of the following result is based on [BJ95a, Thm. 5.1.2]:

(5) Notice that this can be done systematically. For example, enumerate the sets $\mathcal{P}\left({}^{I_n}2\right) = \{Y_i \mid i < \lambda_n\}$. To every sequence S having the domain $[a(n), a(n+1))$ such that $t(i) \subseteq {}^{I_i}2$ for each $i \in [a(n), a(n+1))$ assign the vector $\boldsymbol{c} = \langle c_0, \dots, c_{a(n+1)-a(n)-1} \rangle$, where the terms c_i are such that $t(i) = Y_{c_i}$ for each $i < a(n+1) - a(n) - 1$. Then assign K to $f(n)$ if and only if the vector $\boldsymbol{c}$ corresponding to $f(n)$ is the Kth with respect to the lexicographical ordering.

THEOREM 4.7. *Let $h \in {}^{\omega}\omega$ be defined by $h(n) = (n+1)^2$ for all $n \in \omega$. For any $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$ the following hold:*

(a) *If (I, ε) is $\mathcal{E}$-contributive, then $\mathcal{E}_{I,\varepsilon} \preceq_{\mathrm{T}} (\langle {}^{\uparrow\omega}\omega, {}^{\uparrow\omega}\omega, \leq^* \rangle; \mathbf{Lc}(\omega, h))$.*
(b) *If (I, ε) is $\mathcal{S}$-contributive, then $\mathcal{S}_{I,\varepsilon} \preceq_{\mathrm{T}} (\langle {}^{\uparrow\omega}\omega, {}^{\uparrow\omega}\omega, \leq^* \rangle; \mathbf{Lc}(\omega, h))$.*

Proof. (a) We need to find $\Psi_- \colon \mathcal{E}_{I,\varepsilon} \to {}^{\uparrow\omega}\omega \times {}^{{}^{\uparrow\omega}\omega}({}^{\omega}\omega)$ and $\Psi_+ \colon {}^{\uparrow\omega}\omega \times \mathcal{S}(\omega, h) \to \mathcal{E}_{I,\varepsilon}$. First, we are going to define Ψ_-. For $X \in \mathcal{E}_{I,\varepsilon}$ there is $\varphi_X \in \Sigma_{I,\varepsilon}$ such that $X \subseteq [\varphi_X]_*$. Define $k^X = \langle k^X_n \mid n \in \omega \rangle$ by $k^X_0 = 0$ and for $n \geq 1$,

$$k^X_n = \min \left\{ m > k^X_{n-1} \,\middle|\, \forall j \geq m : \frac{|\varphi_X(j)|}{2^{|I_j|}} < \frac{\varepsilon_j}{(n+1)^3} \right\}.$$

Notice that the existence of k^X is guaranteed by the fact that $\varphi_X \in \Sigma_{I,\varepsilon}$.

We will define a function

$$F^X \colon {}^{\uparrow\omega}\omega \to \bigcup_{a \in {}^{\uparrow\omega}\omega} \prod_{n \in \omega} \Big(\prod_{m \in [a(n), a(n+1))} \mathcal{P}({}^{I_m}2) \Big)$$

as follows. For any $b \in {}^{\uparrow\omega}\omega$ let us set $F^X(b)(n) = \varphi_X \restriction [b(n), b(n+1))$ for all $n \in \omega$. Put $\Psi_-(X) = (k^X, \Phi \circ F^X)$, where Φ is the mapping from Remark 4.6.

Now, we are going to define Ψ_+. Let $b \in {}^{\uparrow\omega}\omega$ and $S \in \mathcal{S}(\omega, h)$. We define $\bar{S}_b \in \mathcal{S}(\omega, h)$ by setting $\bar{S}_b(n)$ to be

$$\left\{ t^{b,n}_K \,\middle|\, K \in S(n) \cap \kappa_{b,n} \text{ and } \forall j \in [b(n), b(n+1)) : \frac{|t^{b,n}_K(j)|}{2^{|I_j|}} < \frac{\varepsilon_j}{(n+1)^3} \right\},$$

where the function $\kappa_{b,n}$ and the enumeration $t^{b,n}_K$ are from Remark 4.6.

For every $n \in \omega$ and every $j \in [b(n), b(n+1))$ define

$$\psi_{b,S}(j) = \bigcup_{t \in \bar{S}_b(n)} t(j). \tag{4.8}$$

Note that $\psi_{b,S} \in \Sigma_{I,\varepsilon}$ since for $j \in [b(n), b(n+1))$ we have

$$\frac{|\psi_{b,S}(j)|}{2^{|I_j|}} < |\bar{S}_b(n)| \cdot \frac{\varepsilon_j}{(n+1)^3} \leq |S(n)| \cdot \frac{\varepsilon_j}{(n+1)^3} \leq (n+1)^2 \cdot \frac{\varepsilon_j}{(n+1)^3} \leq \frac{\varepsilon_j}{n+1}.$$

Thus, we define $\Psi_+(b, S) = [\psi_{b,S}]_*$.

Let $X \in \mathcal{E}_{I,\varepsilon}$ and $(b, S) \in {}^{\uparrow\omega}\omega \times \mathcal{S}(\omega, h)$ be such that $\Psi_-(X) \sqsubset^\bullet (b, S)$, i.e., $k^X \leq^* b$ and $(\Phi \circ F^X)(b) \in^* S$. We shall show that $X \subseteq \Psi_+(b, S)$.

Since $k^X \leq^* b$ it follows that

$$\forall^\infty n \in \omega \; \forall j \geq b(n) : \frac{|\varphi_X(j)|}{2^{|I_j|}} < \frac{\varepsilon_j}{(n+1)^3}. \tag{4.9}$$

Denote by $\langle K_n \mid n \in \omega \rangle$ the sequence $(\Phi \circ F^X)(b)$. By the fact that $\varphi_X \in \Sigma_{I,\varepsilon}$, it follows that $\varphi_X \restriction [b(n), b(n+1)) \in \prod_{m \in [b(n), b(n+1))} \mathcal{P}({}^{I_m}2)$ for any $n \in \omega$. Therefore, we have

(i) $\forall n \in \omega : K_n < \kappa_{b,n}$,
(ii) $\forall n \in \omega : F^X(b)(n) = \varphi_X \restriction [b(n), b(n+1)) = t^{b,n}_{K_n}$,
(iii) $\forall n \in \omega\ \forall j \in [b(n), b(n+1)) : \varphi_X(j) = t^{b,n}_{K_n}(j)$.

Thus, it follows from $(\Phi \circ F^X)(b) = \langle K_n \mid n \in \omega\rangle \in^* S$ that $\forall^\infty n \in \omega : K_n \in S(n) \cap \kappa_{b,n}$, and by (4.9) and (iii) we have $\forall^\infty n \in \omega : t^{b,n}_{K_n} = \varphi_X \restriction [b(n), b(n+1)) \in \bar{S}_b(n)$. Consequently, $\forall^\infty j \in \omega : \varphi_X(j) \subseteq \psi_{b,S}(j)$ by (4.8). Applying Lemma 3.11 we get $X \subseteq [\varphi_X]_* \subseteq [\psi_{b,S}]_* = \Psi_+(b, S)$.

(b) The proof is almost identical to that of (a): we just need to replace $\mathcal{E}_{I,\varepsilon}$ with $\mathcal{S}_{I,\varepsilon}$, $[\cdot]_*$ with $[\cdot]_\infty$ and Lemma 3.11 with Lemma 3.9. ∎

The following result can be found in [Bla10].

FACT 4.10. *Let* $\mathbf{R} = \langle X, Y, \sqsubset\rangle$ *and* $\mathbf{R}' = \langle X', Y', \sqsubset'\rangle$ *be relational systems. Then* $\mathfrak{d}(\mathbf{R}; \mathbf{R}') = \mathfrak{d}(\mathbf{R}) \cdot \mathfrak{d}(\mathbf{R}')$ *and* $\mathfrak{b}(\mathbf{R}; \mathbf{R}') = \min\{\mathfrak{b}(\mathbf{R}), \mathfrak{b}(\mathbf{R}')\}$.

Recall that by Proposition 4.2, $\operatorname{cof}(\mathcal{S}_{I,\varepsilon}) = \operatorname{cof}(\mathcal{E}_{I,\varepsilon})$ and $\operatorname{add}(\mathcal{E}_{I,\varepsilon}) = \operatorname{add}(\mathcal{S}_{I,\varepsilon})$ for all $\mathcal{E}$-contributive pairs (I, ε). However, the next result holds for all (I, ε).

COROLLARY 4.11. *For any* $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$ *the following hold:*

(a) $\operatorname{add}(\mathcal{N}) \le \operatorname{add}(\mathcal{E}_{I,\varepsilon}), \operatorname{add}(\mathcal{S}_{I,\varepsilon})$.
(b) $\operatorname{cof}(\mathcal{E}_{I,\varepsilon}), \operatorname{cof}(\mathcal{S}_{I,\varepsilon}) \le \operatorname{cof}(\mathcal{N})$.

Proof. We shall prove the assertion for $\mathcal{E}_{I,\varepsilon}$. The proof for $\mathcal{S}_{I,\varepsilon}$ is similar.

If (I, ε) is not $\mathcal{E}$-contributive then $\operatorname{add}(\mathcal{N}) \le \infty = \operatorname{add}(\mathcal{E}_{I,\varepsilon})$ and $\operatorname{cof}(\mathcal{E}_{I,\varepsilon}) = 1 \le \operatorname{cof}(\mathcal{N})$. For $\mathcal{E}$-contributive pairs we proceed as follows.

(a) By Theorem 4.7(a) and Fact 4.10 we have $\min\{\mathfrak{b}, \mathfrak{b}^{\mathrm{Lc}}_{\omega,h}\} \le \operatorname{add}(\mathcal{E}_{I,\varepsilon})$. Applying Theorem 4.4 and the inequality $\operatorname{add}(\mathcal{N}) \le \mathfrak{b}$ we get $\min\{\mathfrak{b}, \mathfrak{b}^{\mathrm{Lc}}_{\omega,h}\} = \min\{\mathfrak{b}, \operatorname{add}(\mathcal{N})\} = \operatorname{add}(\mathcal{N})$.

(b) By Theorem 4.7(a) and Fact 4.10 we have $\operatorname{cof}(\mathcal{E}_{I,\varepsilon}) \le \mathfrak{d} \cdot \mathfrak{d}^{\mathrm{Lc}}_{\omega,h}$. Applying Theorem 4.4 and the fact that $\mathfrak{d} \le \operatorname{cof}(\mathcal{N})$ we get $\mathfrak{d} \cdot \mathfrak{d}^{\mathrm{Lc}}_{\omega,h} = \mathfrak{d} \cdot \operatorname{cof}(\mathcal{N}) = \operatorname{cof}(\mathcal{N})$. ∎

COROLLARY 4.12. *For any* $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$ *the following hold:*

(a) $\operatorname{add}(\mathcal{N}) \le \min\{\operatorname{add}(\mathcal{S}_{I,\varepsilon}) \mid I \in \mathbb{I},\ \varepsilon \in \ell^1_+\}$.
(b) $\operatorname{cof}(\mathcal{N}) \ge \sup\{\operatorname{cof}(\mathcal{S}_{I,\varepsilon}) \mid I \in \mathbb{I},\ \varepsilon \in \ell^1_+\}$.

One can ask whether the inequalities in Corollary 4.12 could be changed to equalities. The answer is positive after assuming some additional inequalities between cardinal invariants. We start with two auxiliary results. The next result describes the null sets.

LEMMA 4.13 (see, e.g., [BJ95a, Lem. 2.5.1]). *For any null set* $G \subseteq {}^\omega 2$ *there is a sequence* $\langle F_n \mid n \in \omega\rangle \in \prod_{n\in\omega} \mathcal{P}({}^n 2)$ *with* $\sum_{n\in\omega} \frac{|F_n|}{2^n} < \infty$ *such* *that*

$$G \subseteq \{x \in {}^{\omega}2 \mid \exists^{\infty} n : x{\restriction}n \in F_n\}.$$

In order to be brief, we shall use the following way of coding of small sets:

DEFINITION 4.14 ([CM23, Def. 6.5]). We say that $\mathbf{t} = (L, \varepsilon, \varphi, \psi)$ is a 2-*small** *coding* if

(T1) $(L, \varepsilon) \in \mathbb{I} \times \ell^1_+$, and we denote $L_{2k} = [n_k, m_k)$ and $L_{2k+1} = [m_k, n_{k+1})$ (so $n_k < m_k < n_{k+1}$), and define $I := \langle I_k \mid k < \omega\rangle$ and $I' := \langle I'_k \mid k < \omega\rangle$ by $I_k := [n_k, n_{k+1})$ and $I'_k := [m_k, m_{k+1})$.

(T2) $\varphi \in \Sigma_{I,\varepsilon}$, $\psi \in \Sigma_{I',\varepsilon}$,

The following is a strengthening of a similar result in [Bar88]. Our proof is a modification of the corresponding reasoning in [Bar88] and follows the proof of Lemma 6.6 in [CM23].

LEMMA 4.15. *Let $\{G_\alpha \mid \alpha < \kappa\} \subseteq \mathcal{N}$, let $\varepsilon \in \ell^1_+$ be decreasing and let $D \subseteq \mathbb{I}$ be dominating in $\mathbb{I}$. If $\kappa < \mathfrak{b}$, then there is some $L \in D$ and some 2-small* codings $\mathbf{t}^\alpha = (L, \varepsilon, \varphi^\alpha, \psi^\alpha)$ for $\alpha < \kappa$ such that*

$$G_\alpha \subseteq [\varphi^\alpha]_\infty \cup [\psi^\alpha]_\infty \quad \textit{for every } \alpha < \kappa.$$

Proof. By Lemma 4.13 there are sequences $\langle F^\alpha_n \mid n \in \omega\rangle$ such that $G_\alpha \subseteq \{x \in {}^{\omega}2 \mid \exists^\infty n : x{\restriction}n \in F^\alpha_n\}$ for each $\alpha < \kappa$. Define sequences n^α_k, m^α_k as follows: $n^\alpha_0 = 0$,

$$m^\alpha_k = \min\left\{ j > n^\alpha_k \,\middle|\, 2^{n^\alpha_k} \cdot \sum_{i=j}^{\infty} \frac{|F^\alpha_i|}{2^i} < \frac{\varepsilon_k}{k} \right\},$$

$$n^\alpha_{k+1} = \min\left\{ j > m^\alpha_k \,\middle|\, 2^{m^\alpha_k} \cdot \sum_{i=j}^{\infty} \frac{|F^\alpha_i|}{2^i} < \frac{\varepsilon_k}{k} \right\},$$

for each $\alpha < \kappa$, $k \in \omega$.

Now we let $J^\alpha = \{[n^\alpha_{2k}, n^\alpha_{2(k+1)}) \mid k \in \omega\}$ and since $\kappa < \mathfrak{b}$ we can find $L \in D$ such that $J^\alpha \sqsubseteq L$ for each $\alpha < \kappa$. Denote $L_{2k} = [n_k, m_k)$ and $L_{2k+1} = [m_k, n_{k+1})$. Now define $I = \{[n_k, n_{k+1}) \mid k \in \omega\}$ and $I' = \{[m_k, m_{k+1}) \mid k \in \omega\}$. By the definition of the $\sqsubseteq$-relation, for any $\alpha < \kappa$ there is an $i_\alpha \in \omega$ such that

$$\forall k > i_\alpha\ \exists j, j' \geq k : [n^\alpha_j, m^\alpha_j) \subseteq L_{2k} \ \text{ and } \ [n^\alpha_{j'}, m^\alpha_{j'+1}) \subseteq L_{2k+1}.$$

Note that we may assume that $j, j' \geq k$, since by the definition of J^α, there is a k_α such that $\forall k > k_\alpha : [n^\alpha_{2l}, n^\alpha_{2l+2}) \subseteq L_k$ for some l, and hence $[n^\alpha_{2l}, m^\alpha_{2l}), [n^\alpha_{2l+1}, m^\alpha_{2l+1}) \subseteq L_k$. Then the smallest (with respect to indices) possible subinterval for L_{k+1} is $[n^\alpha_{2l+2}, m^\alpha_{2l+2})$, for L_{k+2} it is $[n^\alpha_{2l+4}, m^\alpha_{2l+4})$ etc. Clearly, indices of subintervals grow at least twice as fast as indices of L_k's. So, eventually j's must catch k's up.

Now, for any $\alpha < \kappa$ define sequences $\varphi^\alpha, \psi^\alpha$ as follows: $\varphi^\alpha(k) = \psi^\alpha(k) = \emptyset$ for $k \le i_\alpha$, otherwise $\varphi^\alpha(k)$ is defined as

$$\{s \in {}^{I_k}2 \mid \exists i \in [m_k, n_{k+1})\ \exists t \in F_i^\alpha : s\restriction(\mathrm{dom}(t) \cap I_k) = t\restriction(\mathrm{dom}(t) \cap I_k)\},$$

and $\psi^\alpha(k)$ as

$$\{s \in {}^{I'_k}2 \mid \exists i \in [n_{k+1}, m_{k+1})\ \exists t \in F_i^\alpha : s\restriction(\mathrm{dom}(t) \cap I'_k) = t\restriction(\mathrm{dom}(t) \cap I'_k)\}.$$

For $k \le i_\alpha$ we have $\frac{|\varphi^\alpha(k)|}{2^{|I_k|}} = 0$. For $k > i_\alpha$,

$$[n_j^\alpha, m_j^\alpha) \subseteq [n_k, m_k) = L_{2k} \quad \text{and} \quad [m_{j'}^\alpha, n_{j'+1}^\alpha) \subseteq [m_k, n_{k+1}) = L_{2k+1}$$

for some $j, j' \ge k$. That is, $n_k \le n_j^\alpha < m_j^\alpha \le m_k$ and $m_k \le m_{j'}^\alpha < n_{j'+1}^\alpha \le n_{k+1}$. Therefore, since for any k we have

$$|\varphi^\alpha(k)| \le \sum_{i=m_k}^{n_{k+1}} |F_i^\alpha| \cdot 2^{|I_k|-(i-n_k)}$$

(for any $i \in [m_k, n_{k+1})$ there are $2^{|I_k|-(i-n_k)}$ extensions of F_i^α inside $[n_k, n_{k+1})$), one can easily see that for all but finitely many $k \in \omega$ we have

$$\frac{|\varphi^\alpha(k)|}{2^{|I_k|}} \le 2^{n_k} \cdot \sum_{i=m_k}^{n_{k+1}} \frac{|F_i^\alpha|}{2^i} \le 2^{n_j^\alpha} \cdot \sum_{i=m_j^\alpha}^{n_{k+1}} \frac{|F_i^\alpha|}{2^i} < \frac{\varepsilon_j}{j} \le \frac{\varepsilon_k}{j} \le \frac{\varepsilon_k}{k}.$$

Hence, $\frac{|\varphi^\alpha(k)|}{2^{|I_k|}\cdot\varepsilon_k} \le \frac{1}{k} \to 0$. Therefore $\varphi^\alpha \in \Sigma_{I,\varepsilon}$. In a similar way we can prove $\psi^\alpha \in \Sigma_{I',\varepsilon}$.

It remains to show $G_\alpha \subseteq [\varphi^\alpha]_\infty \cup [\psi^\alpha]_\infty$. Let $x \in G_\alpha$ and $X = \{n \in \omega \mid x\restriction n \in F_n^\alpha\}$. By our assumption about G_α we have $|X| = \omega$. Then one of the sets $X \cap \bigcup_{k\in\omega}[m_k, n_{k+1})$ or $X \cap \bigcup_{k\in\omega}[n_{k+1}, m_{k+1})$ is infinite. Without loss of generality assume the first case, i.e., for infinitely many n we have $n \in [m_k, n_{k+1})$ for some k and $x\restriction n \in F_n^\alpha$. For those k's, by the definition of $\varphi^\alpha(k)$ there is $s \in \varphi^\alpha(k)$ such that $s = x\restriction[n_k, n_{k+1}) = x\restriction I_k$. So, we have $x\restriction I_k \in \varphi^\alpha(k)$. ■

We show that we can prove the reverse inequalities to those in Corollary 4.12 under additional assumptions.

Theorem 4.16. *The following equalities hold:*

(a) $\mathrm{add}(\mathcal{N}) = \min\{\mathfrak{b}, \min\{\mathrm{add}(\mathcal{S}_{I,\varepsilon}) \mid I \in \mathbb{I}$ *and* $\varepsilon \in \ell^1_+\}\}$.
(b) $\mathrm{cof}(\mathcal{N}) = \max\{\mathfrak{d}, \sup\{\mathrm{cof}(\mathcal{S}_{I,\varepsilon}) \mid I \in \mathbb{I}$ *and* $\varepsilon \in \ell^1_+\}\}$.

Proof. (a) The inequality "$\le$" is clear from Corollary 4.12. So we shall prove $\mathrm{add}(\mathcal{N}) \ge \min\{\mathrm{add}(\mathcal{S}_{I,\varepsilon}) \mid I \in \mathbb{I}, \varepsilon \in \ell^1_+\}$. To see this, let $\mathcal{A} = \{A_\alpha \mid \alpha < \kappa\} \subseteq \mathcal{N}$ be a witness for $\mathrm{add}(\mathcal{N})$, i.e., $|\mathcal{A}| = \mathrm{add}(\mathcal{N})$ and $\bigcup_{\alpha<\kappa} A_\alpha \notin \mathcal{N}$. By Lemma 4.15 there are $\varepsilon \in \ell^1_+$, $I, I' \in \mathbb{I}$ and also $\varphi^\alpha \in \Sigma_{I,\varepsilon}$, $\psi^\alpha \in \Sigma_{I',\varepsilon}$ such that $A_\alpha \subseteq [\varphi^\alpha]_\infty \cup [\psi^\alpha]_\infty$ for each $\alpha < \kappa$. Clearly, one of the sets $\bigcup_{\alpha<\kappa}[\varphi^\alpha]_\infty$ or $\bigcup_{\alpha<\kappa}[\psi^\alpha]_\infty$ is not null, otherwise $\bigcup_{\alpha<\kappa}[\varphi^\alpha]_\infty \cup [\psi^\alpha]_\infty \supseteq \bigcup_{\alpha<\kappa} A_\alpha$ would be null, which is not true by the assumption. Recall that $\mathcal{S}_{I,\varepsilon} \subseteq \mathcal{N}$ for any $I \in \mathbb{I}$, $\varepsilon \in \ell^1_+$. Consequently, $\bigcup_{\alpha<\kappa}[\varphi^\alpha]_\infty \notin \mathcal{S}_{I,\varepsilon}$ or $\bigcup_{\alpha<\kappa}[\psi^\alpha]_\infty \notin \mathcal{S}_{I',\varepsilon}$. Without loss of generality assume the first case. Then $\{[\varphi^\alpha]_\infty \mid \alpha < \kappa\}$ is a family of sets from $\mathcal{S}_{I,\varepsilon}$ such that $\bigcup\{[\varphi^\alpha]_\infty \mid \alpha < \kappa\} \notin \mathcal{S}_{I,\varepsilon}$ and it has cardinality $\leq \operatorname{add}(\mathcal{N})$. We get $\operatorname{add}(\mathcal{S}_{I,\varepsilon}) \leq |\{[\varphi^\alpha]_\infty \mid \alpha < \kappa\}| \leq \operatorname{add}(\mathcal{N})$.

(b) Since $\sup\{\operatorname{cof}(\mathcal{S}_{I,\varepsilon}) \mid I \in \mathbb{I}, \varepsilon \in \ell^1_+\} \leq \operatorname{cof}(\mathcal{N})$, we need to prove only the other inequality. Pick a dominating family D in $\mathbb{I}$ such that $|L_k| \geq k+1$, $k \in \omega$, for all $L \in D$ and such that $|D| = \mathfrak{d}$. For each $L \in D$ define partitions I_L and I'_L as in Definition 4.14(T1). Note that $|I_{L,k}| \geq 2(k+1)$ and $|I'_{L,k}| \geq 2(k+1)$ for each k. For each $L \in D$ denote by $\mathcal{W}_L$ and $\mathcal{W}'_L$ witnesses for $\operatorname{cof}(\mathcal{S}_{I_L,2^{-k-1}})$ and $\operatorname{cof}(\mathcal{S}_{I'_L,2^{-k-1}})$, respectively. Note that $(I_L, 2^{-k-1})$ and $(I'_L, 2^{-k-1})$ are $\mathcal{E}$-contributive by Lemma 2.8(2) since $\lim_{k\to\infty} 2^{|I_{L,k}|} \cdot 2^{-k-1} \geq 2^{2k+2} \cdot 2^{-k-1} = 2^{k+1} \to \infty$. Similarly $\lim_{k\to\infty} 2^{|I'_{L,k}|} \cdot \varepsilon_k \to \infty$. Define

$$\mathcal{A} = \bigcup_{L\in D} \{A_0 \cup A_1 \mid A_0 \in \mathcal{W}_L,\ A_1 \in \mathcal{W}'_L\}.$$

Let $\mathcal{B} \subseteq \mathcal{N}$ be a base of $\mathcal{N}$. By Lemma 4.15 for each $B \in \mathcal{B}$ there is $L \in D$ such that $B = B_0 \cup B_1$, where $B_0 \in \mathcal{S}_{I_L,2^{-k-1}}$ and $B_1 \in \mathcal{S}_{I'_L,2^{-k-1}}$, and consequently (since $\mathcal{W}_L$ and $\mathcal{W}'_L$ are bases) there are $A_0^B \in \mathcal{W}_L$ and $A_1^B \in \mathcal{W}'_L$ such that $B \subseteq A_0^B \cup A_1^B \in \mathcal{A}$. Thus, $\mathcal{A}$ is a base of $\mathcal{N}$. Note that

$$|\mathcal{A}| \leq |D| \cdot \sup\{|\mathcal{W}_L| \cdot |\mathcal{W}'_L| \mid L \in D\} \leq \mathfrak{d} \cdot \sup\{\operatorname{cof}(\mathcal{S}_{J,\varepsilon}) \mid J \in \mathbb{I},\ \varepsilon \in \ell^1_+\}.$$

Since $\mathfrak{d} < \operatorname{cof}(\mathcal{N}) \leq |\mathcal{A}|$, we see that $|\mathcal{A}|$ is at most

$$\max\{\mathfrak{d}, \sup\{\operatorname{cof}(\mathcal{S}_{J,\varepsilon}) \mid J \in \mathbb{I}, \varepsilon \in \ell^1_+\}\} = \sup\{\operatorname{cof}(\mathcal{S}_{J,\varepsilon}) \mid J \in \mathbb{I},\ \varepsilon \in \ell^1_+\}.$$

Consequently, $\operatorname{cof}(\mathcal{N}) \leq |\mathcal{A}| \leq \sup\{\operatorname{cof}(\mathcal{S}_{J,\varepsilon}) \mid J \in \mathbb{I},\ \varepsilon \in \ell^1_+\}$. ∎

5. Covering and uniformity of the new ideals. This section is mainly devoted to proving Theorems A and C. We begin with the following observation concerning localization and anti-localization cardinals.

Remark 5.1. The localization and anti-localization cardinals in Definition 4.3 depend on the parameter $b = \langle b(n) \mid n \in \omega\rangle$. The following lemma shows that only the cardinalities of $b(n)$'s are important for investigating localization and anti-localization cardinals. Therefore, we can formulate all our results assuming $b \in {}^\omega\omega$ (identifying each natural number $b(n)$ with $\{0, 1, \ldots, b(n)-1\}$). However, in the proofs we usually work with a suitable sequence of sets instead of a sequence of natural numbers. The next result is due to [CM19].

Lemma 5.2. *Let b, b' be sequences of length ω of non-empty sets and let $h, h' \in {}^\omega\omega$. If for all but finitely many $n \in \omega$, $|b(n)| \leq |b'(n)|$, and $h' \leq^* h$, then $\mathbf{Lc}(b, h) \preceq_{\mathrm{T}} \mathbf{Lc}(b', h')$ and $\mathbf{aLc}(b', h') \preceq_{\mathrm{T}} \mathbf{aLc}(b, h)$. In particular,*

(a) $\mathfrak{b}^{\mathrm{Lc}}_{b',h'} \leq \mathfrak{b}^{\mathrm{Lc}}_{b,h}$ *and* $\mathfrak{d}^{\mathrm{Lc}}_{b,h} \leq \mathfrak{d}^{\mathrm{Lc}}_{b',h'}$,
(b) $\mathfrak{b}^{\mathrm{aLc}}_{b,h} \leq \mathfrak{b}^{\mathrm{aLc}}_{b',h'}$ *and* $\mathfrak{d}^{\mathrm{aLc}}_{b',h'} \leq \mathfrak{d}^{\mathrm{aLc}}_{b,h}$.

The inequalities of Theorem A will be inferred from the following result.

LEMMA 5.3.

(a) *Let* $(I,\varepsilon) \in \mathbb{I} \times \ell^1_+$. *Then there are* $h, b \in {}^{\omega}\omega$ *such that* $\sum_{n\in\omega} \frac{h(n)}{b(n)} < \infty$ *and* $\mathbf{aLc}(b,h)^{\perp} \preceq_{\mathrm{T}} \mathbf{C}_{\mathcal{S}_{I,\varepsilon}}$. *In particular,* $\mathfrak{b}^{\mathrm{aLc}}_{b,h} \leq \mathrm{cov}(\mathcal{S}_{I,\varepsilon})$ *and* $\mathrm{non}(\mathcal{S}_{I,\varepsilon}) \leq \mathfrak{d}^{\mathrm{aLc}}_{b,h}$.
(b) *Let* $h, b \in {}^{\omega}\omega$ *be such that* $\sum_{n\in\omega} \frac{h(n)}{b(n)} < \infty$. *Then there is* $(I,\varepsilon) \in \mathbb{I} \times \ell^1_+$ *such that* $\mathrm{cov}(\mathcal{S}_{I,\varepsilon}) \leq \mathfrak{b}^{\mathrm{aLc}}_{b,h}$ *and* $\mathfrak{d}^{\mathrm{aLc}}_{b,h} \leq \mathrm{non}(\mathcal{S}_{I,\varepsilon})$.

Proof. (a) If I, ε are not $\mathcal{S}^{\star}$-contributive then pick arbitrary h, b such that the corresponding relational system $\mathbf{aLc}(b,h)$ is defined, and the proof is complete. Otherwise, for $n \in \omega$ define $b(n) = {}^{I_n}2$ and $h(n) = \lfloor |b(n)| \cdot \varepsilon_n \rfloor$. Notice that $\sum_{n\in\omega} \frac{h(n)}{|b(n)|} < \infty$ because $\varepsilon \in \ell^1_+$. So it remains to prove that $\mathbf{aLc}(b,h)^{\perp} \preceq_{\mathrm{T}} \mathbf{C}_{\mathcal{S}_{I,\varepsilon}}$.

Let $X \in \mathcal{S}_{I,\varepsilon}$. Choose $\varphi_X \in \Sigma_{I,\varepsilon}$ such that $X \subseteq [\varphi_X]_\infty$. Since

$$\lim_{n\to\infty} \frac{|\varphi_X(n)|}{2^{|I_n|} \cdot \varepsilon_n} = 0,$$

we can find n_0 such that $|\varphi_X(n)| < 2^{|I_n|} \cdot \varepsilon_n$ for all $n > n_0$. Consequently, since $|\varphi(n)| \in \omega$ for all n, we deduce that $\forall n > n_0 : |\varphi_X(n)| \leq \lfloor 2^{|I_n|} \cdot \varepsilon_n \rfloor$. Next, for $n \in \omega$ define

$$\varphi^*_X(n) = \begin{cases} \emptyset, & n \leq n_0, \\ \varphi_X(n), & n > n_0. \end{cases}$$

It is clear that $\varphi^*_X \in \mathcal{S}(b,h)$ and $X \subseteq [\varphi^*_X]_\infty$, so put $\Psi_+(X) = \varphi^*_X$. On the other hand, define $\Psi_- \colon \prod b \to {}^{\omega}2$ by $\Psi_-(s) = s_0^\frown s_1^\frown s_2^\frown \cdots$. It is clear that $\Psi_-(s) \in X$ implies $s \in^\infty \Psi_+(X)$, which guarantees that (Ψ_-, Ψ_+) is the desired Tukey connection.

(b) We will show that there are $(I,\varepsilon) \in \mathbb{I} \times \ell^1_+$ and a sequence b' with $|b'(n)| \leq b(n)$ for all $n \in \omega$ such that $\mathrm{cov}(\mathcal{S}_{I,\varepsilon}) \leq \mathfrak{b}^{\mathrm{aLc}}_{b',h}$ and $\mathfrak{d}^{\mathrm{aLc}}_{b',h} \leq \mathrm{non}(\mathcal{S}_{I,\varepsilon})$. Consequently, by Lemma 5.2,

$$\mathrm{cov}(\mathcal{S}_{I,\varepsilon}) \leq \mathfrak{b}^{\mathrm{aLc}}_{b',h} \leq \mathfrak{b}^{\mathrm{aLc}}_{b,h} \quad \text{and} \quad \mathfrak{d}^{\mathrm{aLc}}_{b,h} \leq \mathfrak{d}^{\mathrm{aLc}}_{b',h} \leq \mathrm{non}(\mathcal{S}_{I,\varepsilon}).$$

Let $\langle I_n \mid n \in \omega \rangle$ be an interval partition of ω such that $|I_n| = \lfloor \log_2 b(n) \rfloor$. For $n \in \omega$ define $b'(n) = {}^{I_n}2$. Note that

$$|b'(n)| = 2^{\lfloor \log_2 b(n) \rfloor} \leq 2^{\log_2 b(n)} = b(n) < 2^{|I_n|+1} = 2 \cdot |b'(n)|,$$

and thus also $\frac{b(n)}{2} < |b'(n)|$ for all $n \in \omega$. Therefore,

$$\sum_{n\in\omega} \frac{h(n)}{|b'(n)|} \leq \sum_{n\in\omega} \frac{2h(n)}{b(n)} < \infty.$$

Now, we will find ε such that $\sum_{n\in\omega} \varepsilon_n < \infty$ and $\lim_{n\to\infty} \frac{h(n)}{|b'(n)|\cdot\varepsilon_n} = 0$. By Lemma 2.2 there is $\delta \to \infty$ such that $\sum_{n\in\omega} \frac{h(n)}{|b'(n)|}\delta_n < \infty$. Put $\varepsilon_n := \frac{h(n)}{|b'(n)|}\delta_n$ for every $n \in \omega$.

We continue by showing that $\mathbf{C}_{\mathcal{S}_{I,\varepsilon}} \preceq_{\mathrm{T}} \mathbf{aLc}(b',h)^{\perp}$, i.e., $\langle {}^{\omega}2, \mathcal{S}_{I,\varepsilon}, \in\rangle \preceq_{\mathrm{T}} \langle \prod b', \mathcal{S}(b',h), \in^{\infty}\rangle$. We need to define $\Psi_-\colon {}^{\omega}2 \to \prod b'$ and $\Psi_+\colon \mathcal{S}(b',h) \to \mathcal{S}_{I,\varepsilon}$.

For $x \in {}^{\omega}2$, define $\Psi_-(x)(n) := x{\restriction}I_n$. Note that $\mathcal{S}(b',h) \subseteq \Sigma_{I,\varepsilon}$. Indeed, if $\varphi \in \mathcal{S}(b',h)$ then $\varphi(n) \subseteq {}^{I_n}2$ and $\frac{|\varphi(n)|}{2^{|I_n|}\cdot\varepsilon_n} = \frac{|\varphi(n)|}{|b'(n)|\cdot\varepsilon_n} \leq \frac{h(n)}{|b'(n)|\cdot\varepsilon_n} = \frac{1}{\delta_n} \to 0$. Thus, we define Ψ_+ naturally by $\Psi_+(\varphi) = [\varphi]_\infty$ for any $\varphi \in \mathcal{S}(b',h)$.

Let $x \in {}^{\omega}2$ and $\varphi \in \mathcal{S}(b',h)$ be such that $\Psi_-(x) \in^{\infty} \varphi$. Then, clearly, $x \in \Psi_+(\varphi) = [\varphi]_\infty$. ∎

We are ready to show Theorem A:

Theorem 5.4. *The following equalities hold:*

(a)
$$\min\{\mathrm{cov}(\mathcal{S}_{I,\varepsilon}) \mid I \in \mathbb{I} \textit{ and } \varepsilon \in \ell^1_+\} = \min\left\{\mathfrak{b}^{\mathrm{aLc}}_{b,h} \,\middle|\, b,h \in {}^{\omega}\omega \textit{ and } \sum_{n\in\omega}\frac{h(n)}{b(n)} < \infty\right\}.$$

(b)
$$\sup\{\mathrm{non}(\mathcal{S}_{I,\varepsilon}) \mid I \in \mathbb{I} \textit{ and } \varepsilon \in \ell^1_+\} = \sup\left\{\mathfrak{d}^{\mathrm{aLc}}_{b,h} \,\middle|\, b,h \in {}^{\omega}\omega \textit{ and } \sum_{n\in\omega}\frac{h(n)}{b(n)} < \infty\right\}.$$

Proof. By Lemma 5.3(a) we have the inequalities "$\geq$" in (a) and "$\leq$" in (b). On the other hand, by applying Lemma 5.3(b), we have "$\leq$" in (a) and "$\geq$" in (b). ∎

As a direct consequence of the theorem above, we provide another characterization of $\mathrm{cov}(\mathcal{N})$ and $\mathrm{non}(\mathcal{N})$ using the σ-ideals $\mathcal{S}_{I,\varepsilon}$:

Corollary 5.5.

(a) *Assume that* $\mathrm{cov}(\mathcal{N}) < \mathfrak{b}$. *Then*
$$\mathrm{cov}(\mathcal{N}) = \min\{\mathrm{cov}(\mathcal{S}_{I,\varepsilon}) \mid I \in \mathbb{I} \textit{ and } \varepsilon \in \ell^1_+\}.$$

(b) *Assume that* $\mathrm{non}(\mathcal{N}) > \mathfrak{d}$. *Then*
$$\mathrm{non}(\mathcal{N}) = \sup\{\mathrm{non}(\mathcal{S}_{I,\varepsilon}) \mid I \in \mathbb{I} \textit{ and } \varepsilon \in \ell^1_+\}.$$

Proof. By [Bar88], we have the following (see also [CM23, Thm. 6.1] for more details):

(a) If $\mathrm{cov}(\mathcal{N}) < \mathfrak{b}$ then $\mathrm{cov}(\mathcal{N}) = \min\{\mathfrak{b}^{\mathrm{aLc}}_{b,h} \mid b, h \in {}^{\omega}\omega \text{ and } \sum_{n\in\omega} \frac{h(n)}{b(n)} < \infty\}$.

(b) If $\mathrm{non}(\mathcal{N}) > \mathfrak{d}$ then $\mathrm{non}(\mathcal{N}) = \sup\{\mathfrak{d}^{\mathrm{aLc}}_{b,h} \mid b, h \in {}^{\omega}\omega \text{ and } \sum_{n\in\omega} \frac{h(n)}{b(n)} < \infty\}$. To conclude the proof, it is enough to apply Theorem 5.4. ■

Lemma 5.3 focuses on the family $\mathcal{S}_{I,\varepsilon}$. We obtain similar results for $\mathcal{E}_{I,\varepsilon}$.

LEMMA 5.6.

(a) *Let $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$. Then there are $h, b \in {}^{\omega}\omega$ such that $\sum_{n\in\omega} \frac{h(n)}{b(n)} < \infty$ and $\mathbf{Lc}(b, h) \preceq_{\mathrm{T}} \mathbf{C}_{\mathcal{E}_{I,\varepsilon}}$ In particular, $\mathrm{non}(\mathcal{E}_{I,\varepsilon}) \leq \mathfrak{b}^{\mathrm{Lc}}_{b,h}$ and $\mathfrak{d}^{\mathrm{Lc}}_{b,h} \leq \mathrm{cov}(\mathcal{E}_{I,\varepsilon})$.*

(b) *Let $h, b \in {}^{\omega}\omega$ be such that $\sum_{n\in\omega} \frac{h(n)}{b(n)} < \infty$. Then there is $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$ such that $\mathfrak{b}^{\mathrm{Lc}}_{b,h} \leq \mathrm{non}(\mathcal{E}_{I,\varepsilon})$ and $\mathrm{cov}(\mathcal{E}_{I,\varepsilon}) \leq \mathfrak{d}^{\mathrm{Lc}}_{b,h}$.*

Proof. Similar to the proof of Lemma 5.3. ■

Theorem C follows directly from the result mentioned above.

THEOREM 5.7. *The following equalities hold:*

(a)
$$\min\{\mathrm{cov}(\mathcal{E}_{I,\varepsilon}) \mid I \in \mathbb{I} \text{ and } \varepsilon \in \ell^1_+\} = \min\left\{\mathfrak{d}^{\mathrm{Lc}}_{b,h} \,\middle|\, b, h \in {}^{\omega}\omega \text{ and } \sum_{n\in\omega} \frac{h(n)}{b(n)} < \infty\right\}.$$

(b)
$$\sup\{\mathrm{non}(\mathcal{E}_{I,\varepsilon}) \mid I \in \mathbb{I} \text{ and } \varepsilon \in \ell^1_+\} = \sup\left\{\mathfrak{b}^{\mathrm{Lc}}_{b,h} \,\middle|\, b, h \in {}^{\omega}\omega \text{ and } \sum_{n\in\omega} \frac{h(n)}{b(n)} < \infty\right\}.$$

Proof. The proof is analogous to the proof of Theorem 5.4. ■

The last provable result we are going to mention in this paper is establishing a connection between the uniformity and the covering of $\mathcal{E}_{I,\varepsilon}$ and the uniformity and the covering of the σ-ideal of null-additive sets. The sum on ${}^{\omega}2$ is the coordinatewise sum modulo 2, and the addition of sets is defined in a natural way.

DEFINITION 5.8. A set $X \subseteq {}^{\omega}2$ is *null-additive* if $X + A \in \mathcal{N}$ for all $A \in \mathcal{N}$. Denote by $\mathcal{NA}$ the σ-ideal of the null-additive sets in ${}^{\omega}2$.

We employ the following characterization of $\mathcal{NA}$ due to Shelah [She95].

THEOREM 5.9 ([She95], see also [BJ95a, Thm. 2.7.18(3)]). *A set $X \subseteq {}^{\omega}2$ is null-additive if and only if for all $\langle I_n \mid n \in \omega\rangle \in \mathbb{I}$ there is $\varphi \in \prod_{n\in\omega} \mathcal{P}({}^{I_n}2)$ such that*

(a) $\forall n \in \omega : |\varphi(n)| \leq n$,
(b) $\forall x \in X\ \forall^{\infty} n \in \omega : x{\restriction}I_n \in \varphi(n)$.

Using the previous characterization of $\mathcal{NA}$, we easily get

Proposition 5.10.

(a) $\mathrm{non}(\mathcal{NA}) \leq \sup\{\mathrm{non}(\mathcal{E}_{I,\varepsilon}) \mid I \in \mathbb{I} \text{ and } \varepsilon \in \ell^1_+\}$.
(b) $\mathrm{cov}(\mathcal{NA}) \geq \min\{\mathrm{cov}(\mathcal{E}_{I,\varepsilon}) \mid I \in \mathbb{I} \text{ and } \varepsilon \in \ell^1_+\}$.

Proof. We will show that there are $I \in \mathbb{I}, \varepsilon \in \ell^1_+$, such that $\mathrm{non}(\mathcal{NA}) \leq \mathrm{non}(\mathcal{E}_{I,\varepsilon})$ and $\mathrm{cov}(\mathcal{E}_{I,\varepsilon}) \leq \mathrm{cov}(\mathcal{NA})$. Let $I \in \mathbb{I}$ be such that $|I_0|$ is an arbitrary natural number and $|I_n| \geq \log_2(n^2{\cdot}2^n)$ (i.e., $2^{|I_n|} \geq n^2{\cdot}2^n$) for all $n \in \omega\smallsetminus\{0\}$. Define $\varepsilon \in \ell^1_+$ by $\varepsilon_n = 2^{-n}$ for all $n \in \omega$. Note that such a pair is $\mathcal{E}$-contributive by Lemma 2.8(2), since

$$\frac{1}{2^{|I_n|}\cdot\varepsilon_n} \leq \frac{1}{n^2\cdot 2^n\cdot 2^{-n}} = \frac{1}{n^2} \to 0.$$

We shall show $\mathbf{C}_{\mathcal{E}_{I,\varepsilon}} \preceq_{\mathrm{T}} \mathbf{C}_{\mathcal{NA}}$. Define $\Psi_-\colon {}^{\omega}2 \to {}^{\omega}2$ by putting $\Psi_- = \mathrm{id}_{{}^{\omega}2}$. Define $\Psi_+\colon \mathcal{NA} \to \mathcal{E}_{I,\varepsilon}$ as follows. For any $X \in \mathcal{NA}$, there is $\varphi_X \in \prod_{n\in\omega}({}^{I_n}2)$ satisfying conditions (a) and (b) of Theorem 5.9. Note that $\varphi_X \in \Sigma_{I,\varepsilon}$ since

$$\frac{|\varphi_X(n)|}{2^{|I_n|}\cdot\varepsilon_n} \leq \frac{n}{2^{|I_n|}\cdot\varepsilon_n} \leq \frac{n}{n^2\cdot 2^n\cdot 2^{-n}} = \frac{1}{n} \to 0.$$

Hence, put $\Psi_+(X) = [\varphi_X]_*$.

Let $y \in {}^{\omega}2$ and $X \in \mathcal{NA}$ be such that $\Psi_-(y) = y \in X$. Then, of course, $y \in [\varphi_X]_* = \Psi_+(X)$ because $X \subseteq [\varphi_X]_*$. ∎

6. Consistency results. In this section, different forcing models are used to show that ZFC cannot prove more inequalities for the cardinal characteristics associated with our new ideals with the cardinals in Cichoń's diagram (see Figure 1). As usual, we start by looking at the behavior of the cardinal characteristics associated with our ideals in well-known forcing models.

In the following items we skip some details, most of which, however, can be found in the references. We consider just contributive (in fact, $\mathcal{E}$-contributive) pairs (I,ε) throughout the whole section.

(M1) Let λ be an infinite cardinal such that $\lambda^{\aleph_0} = \lambda$ and let h and b be in ${}^{\omega}\omega$ satisfying the assumptions of Lemma 5.3(b). Then $\mathbb{C}_\lambda$ forces $\mathrm{non}(\mathcal{M}) = \mathfrak{b}^{\mathrm{aLc}}_{b,h} = \aleph_1 < \mathrm{cov}(\mathcal{M}) = \mathfrak{d}^{\mathrm{aLc}}_{b,h} = \mathfrak{c} = \lambda$ (see, e.g., [Bre09]). In particular, $\mathbb{C}_\lambda$ forces $\mathrm{add}(\mathcal{E}_{I,\varepsilon}) = \mathrm{non}(\mathcal{E}_{I,\varepsilon}) = \aleph_1$ and $\mathrm{cov}(\mathcal{E}_{I,\varepsilon}) = \mathrm{cof}(\mathcal{E}_{I,\varepsilon}) = \lambda$ for any (I,ε). On the other hand, by using Lemma 5.3(b), we can find $I_{b,h} \in \mathbb{I}$ and $\varepsilon_{b,h} \in \ell^1_+$ such that $\mathfrak{d}^{\mathrm{aLc}}_{b,h} \leq \mathrm{non}(\mathcal{S}_{I_{b,h},\varepsilon_{b,h}})$ and $\mathrm{cov}(\mathcal{S}_{I_{b,h},\varepsilon_{b,h}}) \leq \mathfrak{b}^{\mathrm{aLc}}_{b,h}$. Hence, in $V^{\mathbb{C}_\lambda}$, $\mathrm{non}(\mathcal{S}_{I_{b,h},\varepsilon_{b,h}}) = \lambda$ and $\mathrm{cov}(\mathcal{S}_{I_{b,h},\varepsilon_{b,h}}) = \aleph_1$.

(M2) Let $\aleph_1 \leq \kappa \leq \lambda = \lambda^{\aleph_0}$ with κ regular and let $\varrho, \rho \in \omega^\omega$ be such that $\sum_{i<\omega}\frac{\rho(i)^i}{\varrho(i)} < \infty$. Then by Lemma 5.3(b), there are $I_{\varrho,\rho^{\mathrm{id}}} \in \mathbb{I}$ and $\varepsilon_{\varrho,\rho^{\mathrm{id}}} \in \ell^1_+$

such that $\mathrm{cov}(\mathcal{S}_{I_{\varrho,\rho^{\mathrm{id}}},\varepsilon_{\varrho,\rho^{\mathrm{id}}}}) \leq \mathfrak{b}^{\mathrm{aLc}}_{\varrho,\rho^{\mathrm{id}}}$ and $\mathfrak{d}^{\mathrm{aLc}}_{\varrho,\rho^{\mathrm{id}}} \leq \mathrm{non}(\mathcal{S}_{I_{\varrho,\rho^{\mathrm{id}}},\varepsilon_{\varrho,\rho^{\mathrm{id}}}})$ (6). Now let $\mathbb{E}_\pi$ be a FS iteration of eventually different real forcing of length $\pi = \lambda\kappa$. Then, in $V^{\mathbb{E}_\pi}$, $\mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \aleph_1 \leq \mathrm{non}(\mathcal{E}) = \mathrm{cov}(\mathcal{E}) = \kappa \leq \mathfrak{d} = \lambda$ (see, e.g., [Car24]). Moreover, by [CM19, Lem. 4.24], $\mathbb{E}_\pi$ forces that $\mathfrak{b}^{\mathrm{aLc}}_{\varrho,\rho^{\mathrm{id}}} = \aleph_1$ and $\mathfrak{d}^{\mathrm{aLc}}_{\varrho,\rho^{\mathrm{id}}} \leq \lambda$. Hence, in $V^{\mathbb{E}_\pi}$, $\mathrm{cov}(\mathcal{S}_{I_{\varrho,\rho^{\mathrm{id}}},\varepsilon_{\varrho,\rho^{\mathrm{id}}}}) = \aleph_1$ and $\mathrm{non}(\mathcal{S}_{I_{\varrho,\rho^{\mathrm{id}}},\varepsilon_{\varrho,\rho^{\mathrm{id}}}}) = \lambda$. On the other hand, given $I \in \mathbb{I}$ and $\varepsilon \in \ell^1_+$, by Lemma 5.6(a), we can find $b_{I,\varepsilon}$ and $h_{I,\varepsilon}$ such that $\mathrm{non}(\mathcal{E}_{I,\varepsilon}) \leq \mathfrak{b}^{\mathrm{Lc}}_{b_{I,\varepsilon},h_{I,\varepsilon}}$ and $\mathfrak{d}^{\mathrm{Lc}}_{b_{I,\varepsilon},h_{I,\varepsilon}} \leq \mathrm{cov}(\mathcal{E}_{I,\varepsilon})$. Then, in $V^{\mathbb{E}_\pi}$, we have $\mathrm{non}(\mathcal{E}_{I,\varepsilon}) = \aleph_1$ and $\mathrm{cov}(\mathcal{E}_{I,\varepsilon}) = \mathfrak{c} = \lambda$ because $\mathbb{E}_\pi$ forces $\mathfrak{b}^{\mathrm{Lc}}_{b,h} = \aleph_1$ and $\mathfrak{d}^{\mathrm{Lc}}_{b,h} = \lambda$ for all $b, h \in {}^\omega\omega$ with $0 < h < b$ since $\mathbb{E}$ is σ-centered (see [CM19, Lem. 4.24]).

(M3) Assume $\aleph_1 \leq \kappa \leq \lambda = \lambda^{\aleph_0}$ with κ regular. Let $\mathbb{B}_\pi$ be a FS iteration of random forcing of length $\pi = \lambda\kappa$. Then, in $V^{\mathbb{B}_\pi}$, $\mathrm{non}(\mathcal{E}) = \mathfrak{b} = \aleph_1 \leq \mathrm{cov}(\mathcal{N}) = \mathrm{non}(\mathcal{N}) = \kappa \leq \mathrm{cov}(\mathcal{E}) = \mathfrak{d} = \mathfrak{c} = \lambda$ (see [BS92] and also [Car23, Thm. 5.6]).

In particular, in $V^{\mathbb{B}_\pi}$, we have $\mathrm{add}(\mathcal{S}_{I,\varepsilon}) = \mathrm{non}(\mathcal{E}_{I,\varepsilon}) = \aleph_1$, and $\mathrm{cov}(\mathcal{E}_{I,\varepsilon}) = \mathrm{cof}(\mathcal{S}_{I,\varepsilon}) = \mathfrak{c} = \lambda$ for any (I, ε).

(M4) Assume $\aleph_1 \leq \kappa \leq \lambda = \lambda^{\aleph_0}$ with κ regular. Let $\mathbb{D}_\pi$ be a FS iteration of Hechler forcing of length $\pi = \lambda\kappa$. Then, in $V^{\mathbb{D}_\pi}$, $\mathrm{cov}(\mathcal{N}) = \aleph_1 \leq \mathfrak{d} = \mathfrak{b} = \kappa \leq \mathrm{non}(\mathcal{N}) = \mathfrak{c} = \lambda$ (see, e.g., [Mej13, Thm. 5]). On the other hand, given $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$, by Lemma 5.6(a), we can find $b_{I,\varepsilon}$ and $h_{I,\varepsilon}$ such that $\mathrm{non}(\mathcal{E}_{I,\varepsilon}) \leq \mathfrak{b}^{\mathrm{Lc}}_{b_{I,\varepsilon},h_{I,\varepsilon}}$ and $\mathfrak{d}^{\mathrm{Lc}}_{b_{I,\varepsilon},h_{I,\varepsilon}} \leq \mathrm{cov}(\mathcal{E}_{I,\varepsilon})$. Then, in $V^{\mathbb{D}_\pi}$, we have $\mathrm{add}(\mathcal{S}_{I,\varepsilon}) = \mathrm{non}(\mathcal{E}_{I,\varepsilon}) = \aleph_1$, and $\mathrm{cof}(\mathcal{S}_{I,\varepsilon}) = \mathrm{cov}(\mathcal{E}_{I,\varepsilon}) = \lambda = \mathfrak{c}$ because $\mathbb{D}_\pi$ forces $\mathfrak{b}^{\mathrm{Lc}}_{b_{I,\varepsilon},h_{I,\varepsilon}} = \aleph_1$ and $\mathfrak{d}^{\mathrm{Lc}}_{b_{I,\varepsilon},h_{I,\varepsilon}} = \lambda$ by [CM19, Lem. 4.24].

As in (M2), we can find (I', ε') such that $\mathrm{cov}(\mathcal{S}_{I',\varepsilon'}) = \aleph_1$ and $\mathrm{non}(\mathcal{S}_{I',\varepsilon'}) = \lambda$ in $V^{\mathbb{D}_\pi}$.

(M5) Assume $\aleph_1 \leq \kappa \leq \lambda = \lambda^{\aleph_0}$ with κ regular. Let h and b be in ${}^\omega\omega$ satisfying the assumptions of Lemma 5.6(b) and let $\mathbb{P}$ be a FS iteration of the (b, h)-localization forcing due to Brendle and Mejía [BM14]. Then, in $V^{\mathbb{P}}$, $\mathfrak{b} = \mathrm{cov}(\mathcal{N}) = \aleph_1 \leq \mathfrak{b}^{\mathrm{Lc}}_{b,h} = \mathrm{non}(\mathcal{E}) = \mathrm{cov}(\mathcal{E}) = \mathfrak{d}^{\mathrm{Lc}}_{b,h} = \kappa \leq \mathfrak{d} = \mathfrak{c} = \lambda$ (see [Car23] or see also [Car24]). By Lemma 5.6(b), we can find $(I_{b,h}, \varepsilon_{b,h}) \in \mathbb{I} \times \ell^1_+$ such that $\mathfrak{b}^{\mathrm{Lc}}_{b,h} \leq \mathrm{non}(\mathcal{E}_{I_{b,h},\varepsilon_{b,h}})$ and $\mathrm{cov}(\mathcal{E}_{I_{b,h},\varepsilon_{b,h}}) \leq \mathfrak{d}^{\mathrm{Lc}}_{b,h}$. Hence, in the generic extension, $\mathrm{non}(\mathcal{E}_{I_{b,h},\varepsilon_{b,h}}) = \mathrm{cov}(\mathcal{E}_{I_{b,h},\varepsilon_{b,h}}) = \kappa$.

(M6) This model is obtained by forcing with $\mathbb{A}_\kappa$ (random algebra for adding κ random reals) with $\mathrm{cf}(\kappa) > \aleph_0$ over a model of CH. It satisfies $\mathrm{non}(\mathcal{N}) = \mathfrak{d} = \aleph_1$ and $\mathrm{cov}(\mathcal{N}) = \mathfrak{c} = \kappa$ (see, e.g., [BJ95a, Model 7.6.8]). In this model, $\mathrm{non}(\mathcal{S}_{I,\varepsilon}) = \mathrm{non}(\mathcal{E}_{I,\varepsilon}) = \aleph_1$ and $\mathrm{cov}(\mathcal{S}_{I,\varepsilon}) = \mathrm{cov}(\mathcal{E}_{I,\varepsilon}) = \kappa$ for any (I, ε).

(6) The function ρ^{id} is defined by $\rho^{\mathrm{id}}(i) = \rho(i)^i$ for all $i \in \omega$.

(M7) The dual random model is obtained by forcing with $\mathbb{B}_{\omega_1}$ over a model for MA $+\ \mathfrak{c} \geq \aleph_2$. It satisfies $\mathrm{non}(\mathcal{N}) = \aleph_1$ and $\mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \mathfrak{c} = \aleph_2$ (see, e.g., [BJ95a, Model 7.6.7]). In this model, $\mathrm{non}(\mathcal{S}_{I,\varepsilon}) = \mathrm{non}(\mathcal{E}_{I,\varepsilon}) = \aleph_1$ and $\mathrm{cov}(\mathcal{S}_{I,\varepsilon}) = \mathrm{cov}(\mathcal{E}_{I,\varepsilon}) = \aleph_2$ for any (I, ε).

(M8) Let $h, b \in {}^\omega\omega$ be such that $\sum_{n\in\omega} \frac{h(n)}{b(n)} < \infty$. Let $\mathbb{M}_{\omega_2}$ be a CS iteration of length ω_2 of Mathias forcing. Then, in $V^{\mathbb{M}_{\omega_2}}$, $\mathrm{add}(\mathcal{E}) = \mathrm{cov}(\mathcal{E}) = \aleph_1$, and $\mathrm{non}(\mathcal{E}) = \mathrm{cof}(\mathcal{E}) = \mathfrak{c} = \aleph_2$ (see e.g [Car24, Lem. 2.4]). It is also well-known that $\mathbb{M}_{\omega_2}$ forces $\mathfrak{b} = \aleph_2$ (see, e.g., [BJ95a, Sec. 7.4.A].

Moreover, $\mathbb{M}$ satisfies the Laver property (see [BJ95b, Sec. 7.4.A]), therefore we get $\mathfrak{d}^{\mathrm{Lc}}_{b,h} = \aleph_1$ in $V^{\mathbb{M}_{\omega_2}}$ (see [Kad00, Theorem 4.3]); consequently, in $V^{\mathbb{M}_{\omega_2}}$, by applying Lemma 5.6(b), we can find $(I_{b,h}, \varepsilon_{b,h}) \in \mathbb{I} \times \ell^1_+$ such that $\mathrm{cov}(\mathcal{E}_{I_{b,h},\varepsilon_{b,h}}) \leq \mathfrak{d}^{\mathrm{Lc}}_{b,h}$. Hence, we obtain $\mathrm{cov}(\mathcal{E}_{I_{b,h},\varepsilon_{b,h}}) = \aleph_1$.

(M9) Let $h, b \in {}^\omega\omega$ be such that $\sum_{n\in\omega} \frac{h(n)}{b(n)} < \infty$. Let $\mathbb{MI}_{\omega_2}$ be a CS iteration of length ω_2 of Miller forcing. Then, in $V^{\mathbb{MI}_{\omega_2}}$, $\mathrm{add}(\mathcal{E}) = \mathrm{cov}(\mathcal{E}) = \mathrm{non}(\mathcal{E}) = \aleph_1$, and $\mathrm{cof}(\mathcal{E}) = \aleph_2$ (see e.g [Car24, Lem. 2.5]). In particular, in $V^{\mathbb{MI}_{\omega_2}}$, $\mathrm{non}(\mathcal{E}_{I,\varepsilon}) = \aleph_1$ for any (I, ε). By arguing as in (M8), we can find $(I_{b,h}, \varepsilon_{b,h})$ such that $\mathrm{cov}(\mathcal{S}_{I_{b,h},\varepsilon_{b,h}}) = \mathrm{cov}(\mathcal{E}_{I_{b,h},\varepsilon_{b,h}}) = \aleph_1$ because Miller forcing satisfies the Laver property.

In what follows, we aim to enhance (M5) by establishing the consistency of $\mathfrak{b} < \mathrm{non}(\mathcal{E}_{I,\varepsilon})$ for any given I and ε. To achieve this objective, we will introduce a specific forcing notion, denoted by $\mathbb{P}_{I,\varepsilon}$, which serves to increase the value of $\mathrm{non}(\mathcal{E}_{I,\varepsilon})$ without adding dominating reals.

Definition 6.1. Let $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$. Let $\mathbb{P}_{I,\varepsilon}$ be a poset whose conditions are triples of the form (s, N, F) such that s is a finite sequence with $s(n) \subseteq {}^{I_n}2$ for $n \in |s|$, $N < \omega$, and $F \in [{}^\omega 2]^{<\omega}$ satisfies $\forall n \geq |s| : N\cdot|F| \leq 2^{|I_n|}\cdot\varepsilon_n$. We order $\mathbb{P}_{I,\varepsilon}$ by $(t, M, H) \leq (s, N, F)$ if $s \subseteq t$, $M \geq N$, and $F \subseteq H$, and the following requirements are met:

- $\forall x \in F\, \forall n \in |t| \smallsetminus |s| : x{\restriction}I_n \in t(n)$,
- $\forall n \in |t| \smallsetminus |s|\ : N \cdot |t(n)| \leq 2^{|I_n|} \cdot \varepsilon_n$.

Let $G \subseteq \mathbb{P}$ be a $\mathbb{P}_{I,\varepsilon}$-generic over V; in $V[G]$, define

$$\varphi_{\mathrm{gen}} = \bigcup \{s \mid \exists N, F : (s, N, F) \in G\}.$$

Then $\varphi_{\mathrm{gen}} \in \Sigma_{I,\varepsilon}$ and for each $x \in {}^\omega 2 \cap V$ we have $\forall^\infty n \in \omega : x{\restriction}I_n \in \varphi_{\mathrm{gen}}(n)$.

Lemma 6.2. *Let $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$. Then $\mathbb{P}_{I,\varepsilon}$ is σ-linked, so it is ccc.*

Proof. Let $s \in \mathrm{seq}_{<\omega}({}^{I_n}2)$ and $N < \omega$. Then the set

$$P_{I,\varepsilon}(s, N) = \{(s', N', F) \in \mathbb{P}_{I,\varepsilon} \mid s = s' \,\&\, N = N'\}$$

is linked and $\bigcup_{s\in\mathrm{seq}_{<\omega}({}^{I_n}2),\, N<\omega} P_{I,\varepsilon}(s, N)$ is dense in $\mathbb{P}_{I,\varepsilon}$. ∎

In order to prove $\mathfrak{b} < \mathrm{non}(\mathcal{E}_{I,\varepsilon})$, we use the notion of *ultrafilter-limits* introduced in [GMS16], and the notion of *filter-linkedness* introduced by Mejía [Mej19], with improvements in [BCM21, Section 3].

Given a poset $\mathbb{P}$, the $\mathbb{P}$-name $\dot{G}$ usually denotes the canonical name of the $\mathbb{P}$-generic set. If $\bar{p} = \langle p_n \mid n < \omega\rangle$ is a sequence in $\mathbb{P}$, denote by $\dot{W}_{\mathbb{P}}(\bar{p})$ the $\mathbb{P}$-name of $\{n < \omega \mid p_n \in \dot{G}\}$. When the forcing is understood from the context, we write just $\dot{W}(\bar{p})$.

Definition 6.3 ([GMS16, CMR25]). Let $\mathbb{P}$ be a poset, let $D \subseteq \mathcal{P}(\omega)$ be a non-principal ultrafilter, and let μ be an infinite cardinal.

(1) A set $Q \subseteq \mathbb{P}$ has *D-limits* if there is a function $\lim^D \colon {}^{\omega}Q \to \mathbb{P}$ and a $\mathbb{P}$-name $\dot{D}'$ of an ultrafilter extending D such that, for any $\bar{q} = \langle q_i \mid i < \omega\rangle \in {}^{\omega}Q$,
$$\lim{}^D \bar{q} \Vdash \dot{W}(\bar{q}) \in \dot{D}'.$$
(2) A set $Q \subseteq \mathbb{P}$ has *uf-limits* if it has D-limits for any ultrafilter D.
(3) The poset $\mathbb{P}$ is *μ-D-*lim*-linked* if $\mathbb{P} = \bigcup_{\alpha<\mu} Q_\alpha$ where each Q_α has D-limits. We say that $\mathbb{P}$ is *uniformly μ-D-*lim*-linked* if, additionally, the $\mathbb{P}$-name $\dot{D}'$ from (1) only depends on D (and not on Q_α, although we have different limits for each Q_α).
(4) The poset $\mathbb{P}$ is *μ-uf-*lim*-linked* if $\mathbb{P} = \bigcup_{\alpha<\mu} Q_\alpha$ where each Q_α has uf-limits. We say that $\mathbb{P}$ is *uniformly μ-uf-*lim*-linked* if, additionally, for any ultrafilter D on ω, the $\mathbb{P}$-name $\dot{D}'$ from (1) depends only on D.

For not adding dominating reals, the following notion is weaker than the previous one.

Definition 6.4 ([Mej19]). Let $\mathbb{P}$ be a poset and let F be a filter on ω. A set $Q \subseteq \mathbb{P}$ is *F-linked* if, for any $\bar{p} = \langle p_n \mid n < \omega\rangle \in {}^{\omega}Q$, there is some $q \in \mathbb{P}$ forcing that $F \cup \{\dot{W}(\bar{p})\}$ generates a filter on ω. We say that Q is *uf-linked* (*ultrafilter-linked*) if it is F-linked for any filter F on ω containing the *Fréchet filter* $\mathrm{Fr} := \{\omega \smallsetminus a \mid a \in [\omega]^{<\aleph_0}\}$.

For an infinite cardinal μ, $\mathbb{P}$ is *μ-F-linked* if $\mathbb{P} = \bigcup_{\alpha<\mu} Q_\alpha$ for some F-linked Q_α ($\alpha < \mu$). When these Q_α are uf-linked, we say that $\mathbb{P}$ is *μ-uf-linked.*

Example 6.5.

(1) Random forcing is σ-uf-linked [Mej19], but it may not be σ-uf-lim-linked (compare [BCM21, Rem. 3.10]).
(2) Any poset of size $\leq\kappa$ is κ-Fr-linked (witnessed by its singletons). In particular, Cohen forcing is σ-Fr-linked.

It is clear that any uf-lim-linked set $Q \subseteq \mathbb{P}$ is uf-linked, which implies that Q is Fr-linked.

Theorem 6.6 ([Mej19, Thm. 3.30]). *Let $\kappa \geq \aleph_1$ be a regular cardinal. If $\mathbb{P}$ is a κ-*Fr*-linked poset, then $\mathbb{P}$ forces $\mathfrak{b} \leq \kappa$.*

Proof. Although the conclusion of [Mej19, Thm. 3.30] is different, the same proof works. ∎

Example 6.7. The following are the instances of μ-uf-lim-linked posets we use in our applications:

(1) Any poset of size μ is uniformly μ-uf-lim-linked (because singletons are uf-lim-linked). In particular, Cohen forcing is uniformly σ-uf-lim-linked.
(2) The standard eventually different real forcing notion is uniformly σ-uf-lim-linked (see [GMS16, BCM21]).

We aim to show that $\mathbb{P}_{I,\varepsilon}$ is uniformly σ-uf-lim-linked, witnessed by

$$P_{s,N,m} = \{(t, M, F) \in \mathbb{P}_{I,\varepsilon} \mid s = t,\ N = M \text{ and } |F| \leq m\}$$

for $s \in \mathrm{seq}_{<\omega}({}^{I_n}2)$ and $N < \omega$. Let D be an ultrafilter on ω, and $\bar{p} = \langle p_n \mid n \in \omega\rangle$ be a sequence in $P_{s,N,m}$ with $p_n = (s, N, F_n)$. Consider the lexicographic order $\lhd$ on ${}^{\omega}2$, and let $\{x_{n,k} \mid k < m_n\}$ be a $\lhd$-increasing enumeration of F_n where $m_n \leq m$. Next, find a unique $m_* \leq m$ such that $A := \{n \in \omega \mid m_n = m_*\} \in D$. For each $k < m_*$, define $x_k := \lim^D_n x_{n,k}$ in ${}^{\omega}2$ where $x_k(i)$ is the unique member of $\{0,1\}$ such that $\{n \in A \mid x_{n,k}(i) = x_k(i)\} \in D$ (this coincides with the topological D-limit). Therefore, we can think of $F := \{x_k \mid k < m_*\}$ as the D-limit of $\langle F_n \mid n < \omega\rangle$, so we define $\lim^D \bar{p} := (s, N, F)$. Note that $\lim^D \bar{p} \in P_{s,N,m}$.

Lemma 6.8. *The poset $\mathbb{P}_{I,\varepsilon}$ is uniformly σ-uf-*lim*-linked: For any ultrafilter D on ω, there is a $\mathbb{P}_{I,\varepsilon}$-name of an ultrafilter $\dot{D}'$ on ω extending D such that, for any $s \in \mathrm{seq}_{<\omega}(\mathcal{P}({}^{I_n}2))$, $N < \omega$, $m < \omega$ and $\bar{p} \in P_{s,N,m}$, we have $\lim^D \bar{p} \Vdash \dot{W}(\bar{p}) \in \dot{D}'$.*

To prove the lemma, it suffices to show the following result:

Claim 6.9. *Assume that $M < \omega$,*

$$\{(s_k, N_k, m_k) \mid k < M\} \subseteq \mathrm{seq}_{<\omega}({}^{I_n}2) \times \omega \times \omega,$$

$\langle \bar{p}^k \mid k < M\rangle$ is such that each $\bar{p}^k = \langle p_{k,n} \mid n < \omega\rangle$ is a sequence in P_{s_k,N_k,m_k}, q_k is the D-limit of $\bar{p}^k$ for each $k < M$, and $q \in \mathbb{P}_{I,\varepsilon}$ is stronger than each q_k. Then, for any $a \in D$, there are $n < \omega$ and $q' \leq q$ stronger than $p_{k,n}$ for all $k < M$ (i.e. q' forces $a \cap \bigcap_{k<M} \dot{W}(\bar{p}^k) \neq \emptyset$).

Proof. Write the forcing conditions as $p_{k,n} = (s_k, N_k, F_{k,n})$ and $q_k = (s_k, N_k, F_k)$, where each $F_k = \{x^k_j \mid j < m_{*,k}\}$ is the D-limit of $F_{k,n} = \{x^{k,n}_j \mid j < m_{*,k}\}$ (increasing $\lhd$-enumeration) with $m_{*,k} \leq m_k$. Assume that $q = (s, N, F) \leq q_k$ in $\mathbb{P}_{I,\varepsilon}$ for all $k < M$. Next, by strengthening q if necessary, we assume that $\forall n \geq |s| : |F| + \sum_{k<M} m_{*,k} \leq \frac{2^{|I_n|}\cdot\varepsilon_n}{N}$. Then

$$\forall j < m_{*,k}\ \forall x \in F\ \forall n \in |s| \smallsetminus |s_k| : x^k_j{\restriction}I_n \in s(n),$$

so

$$b_k := \{n < \omega \mid \forall j < m_{*,k}\ \forall x \in F\ \forall n \in |s| \setminus |s_k| : x_j^{k,n} \restriction I_n \in s(n)\} \in D.$$

Hence, $a \cap \bigcap_{k<M} b_k \neq \emptyset$, so choose $n \in a \cap \bigcap_{k<M} b_k$ and put $q' = (s, N, F')$, where $F' := F \cup \bigcup_{k<M} F_{k,n}$. This is a condition in $\mathbb{P}_{I,\varepsilon}$ because $\forall n \geq |s|$: $|F'| \leq |F| + \sum_{k<M} m_{*,k} \leq \frac{2^{|I_n|} \cdot \varepsilon_n}{N}$. Furthermore, q' is stronger than q and $p_{n,k}$ for any $k < M$. ■

THEOREM 6.10. *Let $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$. Given regular uncountable $\kappa < \lambda = \lambda^{<\lambda}$, there is a ccc poset $\mathbb{P}$ forcing* $\mathrm{non}(\mathcal{E}_{I,\varepsilon}) = \lambda = \mathfrak{c}$ *and* $\mathfrak{b} = \kappa$.

Proof. Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}} \mid \alpha < \lambda \rangle$ be a FS iteration of ccc forcing such that

- for α even, $\Vdash_\alpha \dot{\mathbb{Q}} = \mathbb{P}_{I,\varepsilon}$, the forcing defined in Definition 6.1,
- for α odd, $\Vdash_\alpha \dot{\mathbb{Q}}$ is a subposet of Hechler forcing of size $<\kappa$.

First, notice that $\mathbb{P}$ forces $\lambda \geq \mathfrak{c}$. Guarantee that we go through all such small subposets of Hechler forcing by a book-keeping argument. Then it is not hard to see that $\mathbb{P}$ forces $\kappa \leq \mathfrak{b}$. On the other hand, since the iterands of the FS iteration that determine $\mathbb{P}$ are κ-Fr-linked (when α is even, this follows by Lemma 6.8, and when α is odd, this follows by Example 6.5(2)), by applying Theorem 6.6 we deduce that $\mathbb{P}$ forces $\mathfrak{b} \leq \kappa$.

As we iteratively add slaloms $\varphi_{\mathrm{gen}} \in \Sigma_{I,\varepsilon}$ which for all but finitely many $n \in \omega$ satisfy $x \restriction I_n \in \varphi_{\mathrm{gen}}(n)$ for any $x \in {}^\omega\omega$ in the ground model, we see that $\mathbb{P}$ forces $\mathrm{non}(\mathcal{E}_{I,\varepsilon}) \geq \lambda$, so along with $\Vdash_{\mathbb{P}} \lambda \geq \mathfrak{c}$, we conclude that $\mathbb{P}$ forces $\mathrm{non}(\mathcal{E}_{I,\varepsilon}) \geq \lambda \geq \mathfrak{c} \geq \mathrm{non}(\mathcal{E}_{I,\varepsilon})$. ■

As a direct consequence, we obtain the following result:

COROLLARY 6.11. *It is consistent with* ZFC *that* $\mathrm{non}(\mathcal{E}_{I,\varepsilon}) > \mathfrak{b}$ *for any* $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$.

7. Open questions. We discuss some open questions from this study. With regard to Figure 1 and items (M1)–(M9), we do not know the following.

Despite knowing that consistently $\mathrm{add}(\mathcal{N}) < \mathrm{non}(\mathcal{E}_{I,\varepsilon})$ for all $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$ by Corollary 6.11, we still ask:

QUESTION 7.1. *Is it consistent that* $\mathrm{add}(\mathcal{E}_{I,\varepsilon}) < \mathrm{non}(\mathcal{E}_{I,\varepsilon})$ *for some* (*or for all*) $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$*? Dually,* $\mathrm{cov}(\mathcal{E}_{I,\varepsilon}) < \mathrm{cof}(\mathcal{E}_{I,\varepsilon})$*?*

In relation to (M5), we ask:

QUESTION 7.2. *Is it consistent that* $\mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{E}_{I,\varepsilon})$ *for all* $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$*? Dually, the same question about the inequality* $\mathrm{cov}(\mathcal{E}_{I,\varepsilon}) < \mathrm{non}(\mathcal{N})$.

One way to attack Questions 7.1–7.2 is to see whether $\mathbb{P}_{I,\varepsilon}$ satisfies the (ϱ, ρ)-linked property for some suitable functions $\rho, \sigma \in {}^\omega\omega$ (see [OK14] for more details), but this is not yet clear.

Question 7.3. *Is it consistent that* $\mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathcal{E}_{I,\varepsilon})$ *for some* (*or for all*) $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$*? Dually,* $\mathrm{cof}(\mathcal{E}_{I,\varepsilon}) < \mathrm{cof}(\mathcal{N})$*?*

Question 7.4. *Is it consistent that* $\mathrm{cov}(\mathcal{N}) < \mathrm{cov}(\mathcal{S}_{I,\varepsilon})$ *for some* (*or for all*) $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$*?*

Question 7.5. *Is it consistent that* $\mathrm{cov}(\mathcal{E}_{I,\varepsilon})$ *and* $\mathrm{cov}(\mathcal{S}_{I,\varepsilon})$ *are different for all* $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$*? The same is asked for uniformities.*

It is known that forcing notions with the Laver property do not increase the covering of $\mathcal{E}$ (see, e.g., [Car24, Lem. 2.2]), so we pose the question:

Question 7.6. *Do forcing notions satisfying the Laver property keep the cardinal invariant* $\mathrm{cov}(\mathcal{E}_{I,\varepsilon})$ *small?*

Regarding Proposition 5.10, the following is still unknown:

Question 7.7. *Is* $\mathrm{non}(\mathcal{NA}) < \sup\{\mathrm{non}(\mathcal{E}_{I,\varepsilon}) \mid I \in \mathbb{I} \text{ and } \varepsilon \in \ell^1_+\}$ *consistent? Dually, is* $\mathrm{cov}(\mathcal{NA}) > \min\{\mathrm{cov}(\mathcal{E}_{I,\varepsilon}) \mid I \in \mathbb{I} \text{ and } \varepsilon \in \ell^1_+\}$ *consistent?*

In [Mej13], Mejía built a forcing model with four cardinal characteristics associated with $\mathcal{N}$ that are pairwise different. The first author [Car23] produced a similar model for $\mathcal{E}$, and recently Brendle, Mejía, and the first author [BCM25] constructed such a model for $\mathcal{SN}$. In this situation, we inquire:

Question 7.8. *Is it consistent that for some* (*or for all*) $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$*, the four cardinal characteristics associated with* $\mathcal{S}_{I,\varepsilon}$ *are pairwise different? The same question for* $\mathcal{E}_{I,\varepsilon}$*.*

In Proposition 3.5(d), we have shown that if $I \sqsubseteq_{\mathrm{R}} J$ and an additional assumption on I, J, ε is true, then $\mathcal{S}_{I,\varepsilon} \subseteq \mathcal{S}_{J,\varepsilon}$.

Question 7.9. *Is there any reasonable relation* R *such that if* IRJ*, then* $\mathcal{S}_{I,\varepsilon} \subseteq \mathcal{S}_{J,\varepsilon}$*? The same question for* $\mathcal{E}_{I,\varepsilon}$*.*

Note that by Observation 3.6, the relation R cannot be a directed preorder.

By Proposition 2.9, $\mathcal{S}_{I,\varepsilon} \smallsetminus \mathcal{E} \neq \emptyset$. There is a natural question about the structure of $\mathcal{S}_{I,\varepsilon}$ inside $\mathcal{E}$. We know $\mathcal{E}_{I,\varepsilon} \subseteq \mathcal{S}_{I,\varepsilon} \cap \mathcal{E}$, but the reverse inclusion is not clear.

Question 7.10. *Does* $\mathcal{E}_{I,\varepsilon} = \mathcal{S}_{I,\varepsilon} \cap \mathcal{E}$ *hold true for any* $\mathcal{S}^\star$*-contributive* $(I, \varepsilon) \in \mathbb{I} \times \ell^1_+$*?*

In [GM25], for an ideal J on ω, the families $\mathcal{N}^*_J$ and $\mathcal{N}_J$ such that $\mathcal{E} \subseteq \mathcal{N}^*_J \subseteq \mathcal{N}_J \subseteq \mathcal{N}$ are introduced and investigated.

Question 7.11. *What are the relations between the* σ*-ideals* $\mathcal{S}_{I,\varepsilon}$*,* $\mathcal{N}^*_J$*, and* $\mathcal{N}_J$*?*

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