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Fractional divisibility of spheres with partially generic sets of rotations

by

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Abstract. We say that an r-tuple $(\gamma_1, \ldots, \gamma_r)$ of special orthogonal $d \times d$ matrices *fractionally divides* the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$ if there is a non-constant function $f \in L^2(\mathbb{S}^{d-1})$ such that its translations by $\gamma_1, \ldots, \gamma_r$ sum to the constant-1 function. Our main result shows, informally speaking, that fractional divisibility is impossible if at least $r/2$ rotations are "generic".

1. Introduction. Let an integer $d \geq 2$ be fixed throughout the paper. Let $\mathrm{SO}(d)$ denote the group of $d \times d$ special orthogonal real matrices. The elements of this group are naturally identified with orientation-preserving isometries of the Euclidean unit sphere

$$\mathbb{S}^{d-1} := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_2 = 1\},$$

where the action $\gamma.\mathbf{x}$ of $\gamma \in \mathrm{SO}(d)$ on the (column) vector $\mathbf{x} \in \mathbb{S}^{d-1}$ is the matrix product $\gamma\mathbf{x} \in \mathbb{S}^{d-1}$. Also, $\mathrm{SO}(d)$ has the natural left action on the functions on $\mathbb{S}^{d-1}$, namely,

$$(1.1) \quad (\gamma.f)(\mathbf{v}) := f(\gamma^{-1}.\mathbf{v}) \quad \text{for } \gamma \in \mathrm{SO}(d),\ f : \mathbb{S}^{d-1} \to \mathbb{R} \text{ and } \mathbf{v} \in \mathbb{S}^{d-1}.$$

Let μ denote the *spherical measure* on $\mathbb{S}^{d-1}$, which can be defined as the $(d-1)$-dimensional Hausdorff measure with respect to the geodesic distance on the sphere. This measure is invariant under the action $\mathrm{SO}(d) \curvearrowright \mathbb{S}^{d-1}$, so the action in (1.1) preserves the Hilbert space $L^2(\mathbb{S}^{d-1}, \mu)$. Let $\mathcal{D}_{d,r}$ be the set of r-tuples $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_r) \in \mathrm{SO}(d)^r$ such that $\mathbb{S}^{d-1}$ is *fractionally* $\boldsymbol{\gamma}$*-divisible* (when we also say that $\boldsymbol{\gamma}$ *fractionally divides* $\mathbb{S}^{d-1}$), meaning that there is a function $f \in L^2(\mathbb{S}^{d-1}, \mu)$ such that $\sum_{i=1}^{r} \gamma_i.f$ is equal to 1 μ-a.e.

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and f differs from $1/r$ on a set of positive measure (or, equivalently, f is not a constant function μ-a.e.).

This notion was studied in Conley, Grebík and Pikhurko [CGP24] who proved in [CGP24, Theorem 1.1] that, for any $d, r \geq 2$, the set $\mathcal{D}_{d,r}$ cannot contain any *generic* r-tuple $\boldsymbol{\gamma} \in \mathrm{SO}(d)^r$, where "generic" is defined in the algebraic geometry sense. Here we do not give the exact definition but point out that the set of non-generic elements of the topological group $\mathrm{SO}(d)^r$ both is meager and has Haar measure 0; see [CGP24, Lemma 1.3]. Thus we get the following corollary.

Corollary 1.1 (Conley, Grebík and Pikhurko [CGP24]). *For any integers $d, r \geq 2$, the subset $\mathcal{D}_{d,r}$ of the compact group $\mathrm{SO}(d)^r$ is both meager and null.*

We say that an r-tuple of matrices $(\gamma_1, \ldots, \gamma_r) \in \mathrm{SO}(d)^r$ *measurably divides* $\mathbb{S}^{d-1}$ if there is a measurable subset A of $(\mathbb{S}^{d-1}, \mu)$ such that its translates $\gamma_1.A, \ldots, \gamma_r.A$ (defined in the obvious way) form a partition of $\mathbb{S}^{d-1}$, without a single point being omitted or multiply covered. The authors of [CGP24] were motivated by the question of Wagon [Wag93, Question 4.15] (which is Question 5.15 in the newer edition [TW16] of the book) whether there is a triple of rotations in SO(3) that measurably divides $\mathbb{S}^2$. (The related question whether the Axiom of Choice can be avoided here altogether, not even on a μ-null set, was earlier asked by Mycielski [Myc57a, Myc57b].) The obvious more general version of the question of Wagon, where we consider arbitrary pairs (r, d), has known answer for all $d \neq 3$ (see [CGP24, p. 26] for references to these results). However, the cases when $d = 3$ and $r \geq 3$ still remain open. Since measurable divisibility is a stronger version of fractional divisibility (namely, we additionally require that f is $\{0, 1\}$-valued and $\sum_{i=1}^{r} \gamma_i.f$ is 1 everywhere), [CGP24, Theorem 1.1] implies that any r-tuple of matrices achieving it must have entries that satisfy some non-trivial polynomial relation with integer coefficients.

Fractional divisibility of the real line $\mathbb{R}$, better known under the name (*translational*) *tilings by a function*, has been extensively studied from the perspective of Fourier analysis, see the survey [KL21]. A key role in the seminal results [KL96, LW96] is played by periodic sets of translations; this, for a given period, corresponds to fractional division under the action of SO(2) on the circle $\mathbb{S}^1$. This special case, when $d = 2$, connects our investigation with this area; however, note that when $d > 2$ there is a stark difference between commutative and non-commutative divisibility; see for instance the references in [KL21, Section 6.4] and the results about translational tilings of tori in [GGRT23].

The main result of this note (Theorem 1.2) is to strengthen Corollary 1.1 by allowing an adversary to fix some number of matrices. With this in mind, we define $f^{\mathrm{top}}(d,r)$ (resp. $f^{\mathrm{Haar}}(d,r)$) to be the smallest integer ℓ such that for every $(r-\ell)$-tuple $\boldsymbol{\gamma}' = (\gamma_{\ell+1},\dots,\gamma_r) \in \mathrm{SO}(d)^{r-\ell}$ the section

$$\mathcal{D}_{d,r}(\boldsymbol{\gamma}') := \{(\gamma_1,\dots,\gamma_\ell) \in \mathrm{SO}(d)^\ell : (\gamma_1,\dots,\gamma_r) \in \mathcal{D}_{d,r}\}$$

(which consists of those ℓ-tuples $(\gamma_1,\dots,\gamma_\ell) \in \mathrm{SO}(d)^\ell$ such that the extended r-tuple $(\gamma_1,\dots,\gamma_r)$ fractionally divides $\mathbb{S}^{d-1}$) is a meager (resp. null) subset of the group $\mathrm{SO}(d)^\ell$. Thus, Corollary 1.1 gives that each of these functions is at most r. Note that by the Kuratowski–Ulam and Fubini–Tonelli Theorems, if $f^{\mathrm{top}}(d,r) \le \ell' \le r$ or $f^{\mathrm{Haar}}(d,r) \le \ell' \le r$, then ℓ' also satisfies the defining condition. Thus, the following theorem, the main result of this note, strengthens Corollary 1.1.

THEOREM 1.2. *For every $d \ge 2$ and $r \ge 2$, each of $f^{\mathrm{top}}(d,r)$ and $f^{\mathrm{Haar}}(d,r)$ is at most ℓ, where $\ell := \lfloor r/2 \rfloor$ if $d \ge 3$ and $\ell := 1$ if $d = 2$.*

Let us provide some lower bounds on these functions.

PROPOSITION 1.3. *We have $f^{\mathrm{top}}(d,r), f^{\mathrm{Haar}}(d,r) \ge 1$ for any $d,r \ge 2$. Also, $f^{\mathrm{top}}(d,4), f^{\mathrm{Haar}}(d,4) \ge 2$ for every odd $d \ge 3$.*

Thus, by Theorem 1.2 and Proposition 1.3, we know the following values:

$$\begin{aligned}
f^{\mathrm{top}}(2,r) &= f^{\mathrm{Haar}}(2,r) = 1, &&\text{any } r \ge 2,\\
f^{\mathrm{top}}(d,3) &= f^{\mathrm{Haar}}(d,3) = 1, &&\text{any } d \ge 2,\\
f^{\mathrm{top}}(d,4) &= f^{\mathrm{Haar}}(d,4) = 2, &&\text{any odd } d \ge 3.
\end{aligned}$$

There is little empirical data but, if the authors are to guess, it seems plausible that each of $f^{\mathrm{top}}(d,r)$ and $f^{\mathrm{Haar}}(d,r)$ is at most 2 for all $d \ge 3$.

2. Spherical harmonics. Our proofs rely on some basic facts about spherical harmonics. For a detailed introduction to this topic, we refer to the book by Groemer [Gro96]. Here, we only state the definitions and results that we need. Recall that an integer $d \ge 2$ is fixed throughout this paper. So the dependence on d is often omitted.

A polynomial $p \in \mathbb{R}[\mathbf{x}]$ in $\mathbf{x} = (x_1,\dots,x_d)$ is called *harmonic* if $\Delta p = 0$, where

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_d^2}$$

is the *Laplace operator*. A *spherical harmonic* is a function from $\mathbb{S}^{d-1}$ to the reals which is the restriction to $\mathbb{S}^{d-1}$ of a harmonic polynomial on $\mathbb{R}^d$. Let $\mathcal{H}$ be the vector space of all spherical harmonics. For an integer $n \ge 0$, let $\mathcal{H}_n \subseteq \mathcal{H}$ be the linear subspace consisting of all functions $f : \mathbb{S}^{d-1} \to \mathbb{R}$ that are the restrictions to $\mathbb{S}^{d-1}$ of some harmonic polynomial p which is homogeneous of degree n, where we regard the zero polynomial as homogeneous of any degree. By [Gro96, Lemma 3.1.3], the homogeneous degree-n polynomial p

is uniquely determined by $f \in \mathcal{H}_n$, so we may switch between these two representations without mention.

In order to state some further properties of spherical harmonics, we need the following definitions. Recall that μ denotes the spherical measure on $\mathbb{S}^{d-1}$. Let the shorthand a.e. stand for μ-almost everywhere. Let σ_d denote the total measure of the sphere:

$$\sigma_d := \mu(\mathbb{S}^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

By [Gro96, Lemma 1.3.1], the density ρ of the push-forward of μ under the projection to any coordinate axis is

$$\rho(t) := \begin{cases} \sigma_{d-1}\,(1-t^2)^{(d-3)/2}, & -1 < t < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

An important role is played by the *Gegenbauer polynomials* $(P_0, P_1, \ldots)$ which are obtained from $(1, t, t^2, \ldots)$ by the Gram–Schmidt orthonormalization process on $L^2([-1,1], \rho(t)\,\mathrm{d}t)$, except they are normalized to assume value 1 at $t = 1$ (instead of being unit vectors in the L^2-norm). In the special case $d = 3$ (when ρ is the constant function), we get the *Legendre polynomials*. Of course, the degree of P_n is exactly n.

We are now ready to collect some properties that we will need (with references to the statements in [Gro96] that imply them).

Lemma 2.1. *Let $d \geq 2$. For every integer $n \geq 0$ the following holds:*

(i) *The space $\mathcal{H}_n$ is invariant under the action of* $\mathrm{SO}(d)$ ([Gro96, Proposition 3.2.4]).

(ii) *The dimension of $\mathcal{H}_n$ is*

$$N_n := \binom{d+n-1}{n} - \binom{d+n-3}{n-2},$$

where we agree that $\binom{d+n-3}{n-2} = 0$ for $n = 0$ or 1 ([Gro96, Theorem 3.1.4]).

(iii) *For every $\mathbf{v} \in \mathbb{S}^{d-1}$, the function $P_n^{\mathbf{v}} : \mathbb{S}^{d-1} \to \mathbb{R}$, defined by*

$$P_n^{\mathbf{v}}(\mathbf{x}) := P_n(\mathbf{v} \cdot \mathbf{x}) \quad \text{for } \mathbf{x} \in \mathbb{S}^{d-1}, \tag{2.2}$$

belongs to $\mathcal{H}_n$, where $\mathbf{v} \cdot \mathbf{x} := \sum_{i=1}^{d} v_i x_i$ denotes the scalar product of vectors in $\mathbb{R}^d$ ([Gro96, Theorem 3.3.3]).

(iv) *There is a choice of $\mathbf{v}_1, \ldots, \mathbf{v}_{N_n} \in \mathbb{S}^{d-1}$ such that the functions $P_n^{\mathbf{v}_i}$, $i \in [N_n]$, form a basis of the vector space $\mathcal{H}_n$* ([Gro96, Theorem 3.3.14]).

(v) *For all $\mathbf{u}, \mathbf{v} \in \mathbb{S}^{d-1}$, we have $\langle P_n^{\mathbf{u}}, P_n^{\mathbf{v}} \rangle = \frac{\sigma_d}{N_n} P_n(\mathbf{u} \cdot \mathbf{v})$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $L^2(\mathbb{S}^{d-1}, \mu)$* ([Gro96, Proposition 3.3.6 and Theorem 3.4.1]).

(vi) *The subspaces* $\mathcal{H}_0, \mathcal{H}_1, \ldots$ *of* $L^2(\mathbb{S}^{d-1}, \mu)$ *are pairwise orthogonal* ([Gro96, Theorem 3.2.1]) *and the linear span of their union is dense in* $L^2(\mathbb{S}^{d-1}, \mu)$ ([Gro96, Corollary 3.2.7]).

3. Proofs. We will need the following characterization of the set $\mathcal{D}_{d,r}$.

LEMMA 3.1. *Let* $d, r \geq 2$ *and take any* $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_r) \in \mathrm{SO}(d)^r$. *For an integer* $n \geq 1$, *define*

$$\mathbf{L}_n(g) := \sum_{s=1}^{r} \gamma_s.g \quad \text{for } g \in \mathcal{H}_n, \tag{3.1}$$

and note that $\mathbf{L}_n(g) \in \mathcal{H}_n$ *by Lemma* 2.1(i). *Then the* r*-tuple* $\boldsymbol{\gamma}$ *fractionally divides* $\mathbb{S}^{d-1}$ *if and only if there is an integer* $n \geq 1$ *such that the linear map* $\mathbf{L}_n : \mathcal{H}_n \to \mathcal{H}_n$ *is not invertible.*

Proof. Note that $\mathbf{L}_n$ is a linear operator on $\mathcal{H}_n$. Let $\mathbf{L}_n^* : \mathcal{H}_n \to \mathcal{H}_n$ denote its adjoint operator with respect to the scalar product inherited from $L^2(\mathbb{S}^{d-1}, \mu)$. Since the action of $\mathrm{SO}(d)$ preserves the measure μ, we deduce that $\mathbf{L}_n^*(g) = \sum_{s=1}^{r} \gamma_s^{-1}.g$ for $g \in \mathcal{H}_n$. For a unit vector $\mathbf{v} \in \mathbb{S}^{d-1}$, define

$$G^{\mathbf{v}}_{n,\boldsymbol{\gamma}} := \mathbf{L}_n^*(P_n^{\mathbf{v}}). \tag{3.2}$$

Note that $G^{\mathbf{v}}_{n,\boldsymbol{\gamma}} = \sum_{s=1}^{r} P_n^{\gamma_s^{-1}.\mathbf{v}}$, since for all $\gamma \in \mathrm{SO}(d)$ and $\mathbf{x} \in \mathbb{S}^{d-1}$,

$$\big(\gamma^{-1}.P_n^{\mathbf{v}}\big)(\mathbf{x}) = P_n^{\mathbf{v}}(\gamma.\mathbf{x}) = P_n(\mathbf{v} \cdot (\gamma.\mathbf{x})) = P_n((\gamma^{-1}.\mathbf{v}) \cdot \mathbf{x}) = \big(P_n^{\gamma^{-1}.\mathbf{v}}\big)(\mathbf{x}).$$

Suppose first that there is a non-constant $f \in L^2(\mathbb{S}^{d-1}, \mu)$ such that $\sum_{s=1}^{r} \gamma_s.f = 1$ a.e. By Lemma 2.1(vi), we have the direct orthogonal sum $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$. Thus, we can uniquely write $f = \sum_{n=0}^{\infty} F_n$ in $L^2(\mathbb{S}^{d-1}, \mu)$ with $F_n \in \mathcal{H}_n$ for every $n \geq 0$. Since the action of $\mathrm{SO}(d)$ preserves each space $\mathcal{H}_n$ by Lemma 2.1(i) as well as the scalar product on $L^2(\mathbb{S}^{d-1}, \mu)$, we know that $\gamma.f = \sum_{n=0}^{\infty} \gamma.F_n$ is the harmonic expansion of $\gamma.f \in L^2(\mathbb{S}^{d-1}, \mu)$.

Since the sum $\sum_{s=1}^{r} \gamma_s.f$ is a constant function 1 a.e. and $\mathcal{H}_0$ is exactly the set of constant functions on $\mathbb{S}^{d-1}$, we deduce by Lemma 2.1(vi) and the invariance of the scalar product under $\mathrm{SO}(d)$ that, for every integer $n \geq 1$ and every vector $\mathbf{v} \in \mathbb{S}^{d-1}$,

$$\begin{aligned} 0 = \langle P_n^{\mathbf{v}}, 1\rangle &= \langle P_n^{\mathbf{v}}, \gamma_1.f + \cdots + \gamma_r.f\rangle = \langle P_n^{\mathbf{v}}, \gamma_1.F_n + \cdots + \gamma_r.F_n\rangle \\ &= \langle P_n^{\mathbf{v}}, \mathbf{L}_n(F_n)\rangle = \langle \mathbf{L}_n^*(P_n^{\mathbf{v}}), F_n\rangle = \langle G^{\mathbf{v}}_{n,\boldsymbol{\gamma}}, F_n\rangle. \end{aligned}$$

Since $f : \mathbb{S}^{d-1} \to \mathbb{R}$ is not a constant function μ-a.e., there is $n \geq 1$ such that $F_n \neq 0$. Thus the functions $G^{\mathbf{v}}_{n,\boldsymbol{\gamma}} \in \mathcal{H}_n$ for $\mathbf{v} \in \mathbb{S}^{d-1}$ do not span the whole space. It follows from Lemma 2.1(iv) that $\mathbf{L}_n^*$ is not surjective and thus not invertible; of course, the same applies to $\mathbf{L}_n$.

Conversely, suppose that $\mathbf{L}_n$ is not invertible for some $n \geq 1$. By the finite dimensionality of $\mathcal{H}_n$ there is a non-zero polynomial $g \in \mathcal{H}_n$ with $\mathbf{L}_n(g) = 0$.

The last identity means that $\sum_{s=1}^{r}\gamma_s.g = 0$. Thus, for example, $f := 1/r+g$ shows that $\boldsymbol{\gamma}$ fractionally divides $\mathbb{S}^{d-1}$. ∎

Remark 3.2. Note that the function $f = 1/r + g$ obtained at the end of the proof of Lemma 3.1 is a polynomial, so the identity $\sum_{s=1}^{r}\gamma_s.f = 1$ holds everywhere without any exceptions. Also, by taking $f := 1/r+cg$ with some constant c satisfying $0 < c < \frac{1}{r}\|g\|_\infty^{-1}$, we can additionally ensure that $0 < f < 1$ everywhere. Thus, by Lemma 3.1, the definition of fractional divisibility will not be affected if we impose these extra restrictions on f.

Proof of Theorem 1.2. Recall that we are given arbitrary matrices

$$\gamma_{\ell+1},\ldots,\gamma_r \in \mathrm{SO}(d)$$

and we would like to show that the corresponding section of $\mathcal{D}_{d,r}$ is "small". Let Z_n consist of those $\boldsymbol{\gamma} = (\gamma_1,\ldots,\gamma_\ell)$ in $\mathrm{SO}(d)^\ell$ for which $\mathbf{L}_n$ is non-invertible, where $\mathbf{L}_n$ is the linear map from Lemma 3.1 defined with respect to the extended r-tuple $\boldsymbol{\gamma}' := (\gamma_1,\ldots,\gamma_r)$. Since the classes of meager and null subsets are closed under countable unions, it is enough by Lemma 3.1 to prove that, for every n, the set Z_n is both null and meager.

First, suppose that $d \geq 3$.

By Lemma 2.1(iv), choose $\mathbf{v}_1,\ldots,\mathbf{v}_{N_n} \in \mathbb{S}^{d-1}$ such that the functions $P_n^{\mathbf{v}_i}$, $i \in [N_n]$, form a basis of the vector space $\mathcal{H}_n$. Note that the linear map $\mathbf{L}_n$ is non-invertible if and only if $G_{n,\boldsymbol{\gamma}'}^{\mathbf{v}_i}$ for $i \in [N_n]$ are linearly dependent, where the functions $G_{n,\boldsymbol{\gamma}'}^{\mathbf{v}} := \mathbf{L}_n^*(P_n^{\mathbf{v}})$, for $\mathbf{v} \in \mathbb{S}^{d-1}$, are the same as in the proof of Lemma 3.1. Consider the $N_n \times N_n$ matrices $L = L(\boldsymbol{\gamma})$ and M with entries

$$\begin{aligned} L_{ij} &:= \frac{1}{\sigma_d}\langle G_{n,\boldsymbol{\gamma}'}^{\mathbf{v}_i}, P_n^{\mathbf{v}_j}\rangle, \\ M_{ij} &:= \frac{1}{\sigma_d}\langle P_n^{\mathbf{v}_i}, P_n^{\mathbf{v}_j}\rangle \quad \text{ for } i,j \in [N_n]. \end{aligned}$$

Since M is up to a scalar factor the Gram matrix of the basis $\{P_n^{\mathbf{v}_i} : i \in [N_n]\}$ of $\mathcal{H}_n$, we have $\det M \neq 0$. Write the vectors $G_{n,\boldsymbol{\gamma}'}^{\mathbf{v}_i}$ in this basis,

$$(G_{n,\boldsymbol{\gamma}'}^{\mathbf{v}_1},\ldots,G_{n,\boldsymbol{\gamma}'}^{\mathbf{v}_{N_n}})^T = A\,(P_n^{\mathbf{v}_1},\ldots,P_n^{\mathbf{v}_{N_n}})^T$$

for some $N_n \times N_n$ matrix A. Then L is the matrix product AM. Thus, $\det L$ is zero if and only if $G_{n,\boldsymbol{\gamma}'}^{\mathbf{v}_i}$ for $i \in [N_n]$ are linearly dependent. By Lemma 2.1(v), we have, for all $i,j \in [N_n]$,

$$\begin{aligned} L_{ij} &= \frac{1}{\sigma_d}\sum_{s=1}^{r}\langle P_n^{\gamma_s^{-1}.\mathbf{v}_i}, P_n^{\mathbf{v}_j}\rangle \\ &= \frac{1}{N_n}\sum_{s=1}^{r} P_n((\gamma_s^{-1}.\mathbf{v}_i)\cdot\mathbf{v}_j) = \frac{1}{N_n}\sum_{s=1}^{r} P_n(\mathbf{v}_i\cdot(\gamma_s.\mathbf{v}_j)). \end{aligned}$$

For fixed $\mathbf{v}_1, \ldots, \mathbf{v}_{N_n}$, this represents each L_{ij} as a polynomial in the $d^2\ell$ entries of the matrices $\gamma_1, \ldots, \gamma_\ell$. Thus, $\det L$ is also such a polynomial.

Let us see what happens if we let $\gamma_1, \ldots, \gamma_\ell$ be equal to $\gamma_{\ell+1}$. The linear map $\mathbf{L}_n$ sends $g \in \mathcal{H}_n$ to

$$\sum_{s=1}^{r} \gamma_s.g = (\ell+1)\gamma_{\ell+1}.g + \sum_{s=\ell+2}^{r} \gamma_s.g \in \mathcal{H}_n.$$

Looking at the norm coming from $L^2(\mathbb{S}^{d-1}, \mu) \supseteq \mathcal{H}_n$ and using the fact that the action of $\mathrm{SO}(d)$ on this space preserves the norm, we deduce by the triangle inequality that, for all $g \in \mathcal{H}_n \setminus \{0\}$,

$$\|\mathbf{L}_n(g)\|_2 \geq (\ell+1)\|\gamma_{\ell+1}.g\|_2 - \sum_{s=\ell+2}^{r} \|\gamma_s.g\|_2 = (2\ell+2-r)\,\|g\|_2 > 0.$$

By the finite-dimensionality of $\mathcal{H}_n$, the linear map $\mathbf{L}_n$ is invertible. Furthermore, note that the invertible adjoint map $\mathbf{L}_n^*$ sends the basis vectors $P_n^{\mathbf{v}_1}, \ldots, P_n^{\mathbf{v}_{N_n}}$ to $G_{n,\boldsymbol{\gamma}'}^{\mathbf{v}_1}, \ldots, G_{n,\boldsymbol{\gamma}'}^{\mathbf{v}_{N_n}}$ respectively. Thus, the latter vectors are linearly independent (which was observed earlier to be equivalent to $\det L \neq 0$). Thus, the polynomial $\det L$ is non-zero when evaluated at the constant ℓ-tuple $(\gamma_{\ell+1}, \ldots, \gamma_{\ell+1}) \in \mathrm{SO}(d)^\ell$.

We are now ready to show that, for each $n \geq 1$, the set Z_n (that consists of those $(\gamma_1, \ldots, \gamma_\ell) \in \mathrm{SO}(d)^\ell$ on which $\det L$ vanishes) is both null and meager. Informally speaking, this is true since the zero set of a non-vanishing polynomial $\det L$ on the irreducible variety $\mathrm{SO}(d)^\ell$ must have strictly smaller dimension than the variety itself. For the sake of completeness, we present a proof of this intuitively clear result (which is obtained by an easy adaptation of the proof of [CGP24, Lemma 1.3]).

Clearly, Z_n is a closed subset of $\mathrm{SO}(d)^\ell$ so, in particular, it is measurable. First, we prove that Z_n is a null set with respect to the Haar measure ν on $\mathrm{SO}(d)^\ell$. Let us recall how the Haar measure ν can be constructed for the group $\Gamma := \mathrm{SO}(d)^\ell$ (and, in fact, for any real Lie group), following the presentation in [Kna02, Sections VIII.1–2]. Namely, choose some linear basis for the Lie algebra $(\mathfrak{so}(d))^\ell$ viewed as the tangent space $T_{(I_d, \ldots, I_d)}$ at the identity $(I_d, \ldots, I_d) \in \mathrm{SO}(d)^\ell$ and, using the translations of these vectors, turn them into left-invariant vector fields $X_1, \ldots, X_m$. (Note that the Lie algebra $(\mathfrak{so}(d))^\ell$, which consists of all ℓ-tuples of skew-symmetric matrices, has dimension $m = \binom{d}{2}\ell$ as a vector space.) For each $\boldsymbol{\gamma} \in \Gamma$, let $e_1(\boldsymbol{\gamma}), \ldots, e_m(\boldsymbol{\gamma}) \in T_{\boldsymbol{\gamma}}^*$ be the dual basis to $(X_1(\boldsymbol{\gamma}), \ldots, X_m(\boldsymbol{\gamma}))$. Then $\omega = e_1 \wedge \cdots \wedge e_m$ (the skew-symmetric product) is a smooth m-form on Γ, which is positive and left-invariant and thus defines a Borel left-invariant non-zero measure on Γ ([Kna02, Theorem 8.21]). By uniqueness, this has to be a multiple of the Haar measure ν. In particular, any smooth submanifold of Γ of dimension (as a manifold) less than m has zero Haar measure ([Kna02, (8.25)]).

The variety $\mathrm{SO}(d)^\ell$ is irreducible and has dimension $m = \binom{d}{2}\ell$ as a real algebraic variety. This is a well-known fact for $\ell = 1$ and the proof for an arbitrary ℓ is carefully spelled out in [CGP24, Lemma 8.2].

The set $Z_n \subsetneq \mathrm{SO}(d)^\ell$, as an algebraic variety, has dimension smaller than m which follows from the definition of the *algebraic dimension* of a variety as the maximum length of a strictly nested chain of non-empty irreducible varieties. (Indeed, a chain for Z_n extends to a larger chain for $\mathrm{SO}(d)^\ell$ by adding $\mathrm{SO}(d)^\ell \supsetneq Z_n$ at the top.) The standard results in the theory of (semi-)algebraic sets imply that every bounded variety of algebraic dimension k in some $\mathbb{R}^n$ admits a finite partition, with each part being a smooth semialgebraic manifold in $\mathbb{R}^n$ of dimension at most k; see e.g. [BPR06, Theorem 5.38]. Apply this result to each of finitely many irreducible components Z of Z_n. The dimension s of each resulting manifold S is at most $\dim Z$. Indeed, pick a point $\mathbf{s} \in S$ and the projection from S on some s coordinates which is a homeomorphism around $\mathbf{s}$. Observe that these s coordinates are algebraically independent in the function field $\mathbb{R}(Z)$ (because a non-zero polynomial on $\mathbb{R}^s$ cannot vanish on a non-empty open set). By [Has07, Proposition 7.27], the algebraic dimension of Z is at least s, as claimed.

Thus, we have covered Z_n by finitely many manifolds of dimension less than m. As observed earlier, each such manifold has Haar measure 0 (by [Kna02, (8.25)]). We conclude that the Haar measure of Z_n is indeed zero.

Let us show that Z_n is meager in $\mathrm{SO}(d)^\ell$. Since Z_n is closed, it is enough to show that the relative interior U of $Z_n \subseteq \mathrm{SO}(d)^\ell$ is empty. Suppose to the contrary that $U \neq \emptyset$. Since the compact group $\mathrm{SO}(d)^\ell$ acts transitively on itself by homeomorphisms, finitely many translates of U cover the whole group. As the Haar measure ν is invariant under this action, we see that $\nu(U) > 0$. However, this contradicts the identity $\nu(Z_n) = 0$ that we have already proved.

This proves the theorem for $d \geq 3$.

Finally, suppose that $d = 2$. While the above arguments (for $d \geq 3$) also apply for $d = 2$, we get in fact the stronger conclusion that each section is countable by explicitly writing the action of $\mathrm{SO}(2)$ on $L^2(\mathbb{S}^1, \mu)$. Here, we identify $\mathrm{SO}(2)$ with the circle $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ (where we take reals modulo 2π) by mapping each rotation matrix $\left(\begin{smallmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{smallmatrix}\right) \in \mathrm{SO}(2)$ to its angle $\phi \in \mathbb{T}$. Take any integer $n \geq 1$. It is well-known (see e.g. [Gro96, Section 3.1]) that the dimension of $\mathcal{H}_n$ is 2 and we can take $c, s \in \mathcal{H}_n$ to be its basis vectors, where c, s are the (unique) homogeneous degree-n polynomials that satisfy $c(\cos\alpha, \sin\alpha) := \cos(n\alpha)$ and $s(\cos\alpha, \sin\alpha) := \sin(n\alpha)$ for every $\alpha \in [0, 2\pi)$.

Then the action of $\phi \in \mathbb{T}$ on $\mathcal{H}_n$ in the basis (c, s) is given by the matrix

$$M_\phi := \begin{pmatrix} \cos(n\phi) & -\sin(n\phi) \\ \sin(n\phi) & \cos(n\phi) \end{pmatrix}.$$

Indeed, the action of $\phi \in \mathbb{T}$ on, for example, $c \in \mathcal{H}_n$ is the function

$$\cos(n(x - \phi)) = \cos(n\phi)c(x) + \sin(n\phi)s(x), \qquad x \in \mathbb{T}.$$

Recall that we are given $\gamma_2, \ldots, \gamma_r \in \mathrm{SO}(2)$. Let the linear map $g \mapsto \sum_{s=2}^{r} \gamma_s.g$ from $\mathcal{H}_n$ to itself be given in the basis (c, s) by the matrix $K = (K_{i,j})_{i,j=1}^2$. We vary only $\gamma_1 = M_\phi$ for $\phi \in \mathbb{T}$. Let $x := \cos(n\phi)$ and $y := \sin(n\phi) = \pm\sqrt{1 - x^2}$. Suppose that $y = \sqrt{1 - x^2}$ with the other case being analogous. Consider the linear map $\mathbf{L}_n$ that sends $g \in \mathcal{H}_n$ to $\sum_{s=1}^{r} \gamma_s.g$. As in the case $d \geq 3$, we will be done if we show that the set of those $\phi \in \mathbb{T}$ for which $\mathbf{L}_n c$ and $\mathbf{L}_n s$ are linearly dependent is both null and meager. The matrix of $\mathbf{L}_n$ in the basis (c, s) is $M_\phi + K$. Its determinant is

$$\begin{aligned} \det(M_\phi + K) &= (x + K_{1,1})(x + K_{2,2}) \\ &\quad - (-\sqrt{1 - x^2} + K_{1,2})(\sqrt{1 - x^2} + K_{2,1}) \\ &= \sqrt{1 - x^2}(K_{2,1} - K_{1,2}) + x(K_{1,1} + K_{2,2}) + \det K + 1. \end{aligned}$$

It is enough to show that the set of $x \in [-1, 1]$ for which this is 0 is countable, since the map $\phi \mapsto \cos(n\phi)$ has countable pre-images (in fact, each has at most $2n$ elements). The equation $\det(M_\phi + K) = 0$ implies that

$$(K_{1,2} - K_{2,1})^2(1 - x^2) = (x(K_{1,1} + K_{2,2}) + \det K + 1)^2.$$

For this polynomial identity to be satisfied for an infinite set of x's, it must be the case that the coefficients at all powers of x match. By looking at the terms linear in x, we conclude that $(K_{1,1} + K_{2,2})(\det K + 1) = 0$. One of the factors is 0 and either case implies that $K_{1,2} = K_{2,1}$ which in turn implies that $\det K = -1$ and $K_{1,1} + K_{2,2} = 0$. Thus $K = \left(\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right)$ for some $a, b \in \mathbb{R}$. Note that K is a linear combination of some matrices M_ψ. As each matrix M_ψ has equal diagonal entries and opposite off-diagonal entries, we conclude that $a = b = 0$, contradicting $\det K = -1$. Thus, the set of x's for which $\mathbf{L}_n c, \mathbf{L}_n s \in \mathcal{H}_n$ are linearly dependent is in fact finite and so is its pre-image in $\mathbb{T}$, as desired. This finishes the proof of the theorem. ∎

Proof of Proposition 1.3. We start by showing that $f^{\mathrm{top}}(d, r) > 0$ and $f^{\mathrm{Haar}}(d, r) > 0$ for all $d, r \geq 2$. By definition, this is the same as showing that we can always find an r-tuple of rotations that fractionally divides $\mathbb{S}^{d-1}$. Given arbitrary $d, r \geq 2$, take $\gamma_1, \ldots, \gamma_r \in \mathrm{SO}(d)$ that are the rotations of the (x_1, x_2)-plane by angles $2\pi m/r$ for $m \in \{0, \ldots, r - 1\}$ (and fix every other standard basis vector). Define $f : \mathbb{S}^{d-1} \to \{0, 1\}$ by $f(\mathbf{x}) := 1$ if there are $\rho > 0$ and $\theta \in [0, 2\pi/r)$ such that $(x_1, x_2) = (\rho\cos\theta, \rho\sin\theta)$ and let $f(\mathbf{x}) := 0$

otherwise. This f shows that $(\gamma_1, \ldots, \gamma_r) \in \mathcal{D}_{d,r}$, so each of $f^{\mathrm{top}}(d,r)$ and $f^{\mathrm{Haar}}(d,r)$ is at least 1.

Let us consider the case when $d = 2m + 1 \geq 3$ is odd and $r = 4$. Let $D(c_1, \ldots, c_d)$ denote the diagonal matrix with $c_1, \ldots, c_d$ on the diagonal and let $c^{(k)}$ denote the sequence where c is repeated k times.

Let $\gamma_2, \gamma_3, \gamma_4$ be the diagonal matrices

$$D((-1)^{(2m-1)}, -1, 1),\ D((-1)^{(2m-1)}, 1, -1) \text{ and } D((1)^{(2m-1)}, -1, -1).$$

We see that their sum is $D((-1)^{(d)})$, the diagonal matrix with all diagonal entries equal to -1. We can naturally identify the space $\mathbb{R}^d$ with $\mathcal{H}_1$ by sending $\mathbf{u} \in \mathbb{R}^d$ to the linear map $\mathbf{x} \mapsto \mathbf{x} \cdot \mathbf{u}$. Under this identification the action of $\gamma \in \mathrm{SO}(d)$ on $\mathcal{H}_1$ is given by the usual matrix product. Thus, $\sum_{s=2}^{4} \gamma_s . g = -g$ for every $g \in \mathcal{H}_1$.

Let us show that the section $\mathcal{D}_{d,4}(\gamma_2, \gamma_3, \gamma_4)$ is the whole group $\mathrm{SO}(d)$. Take any $\gamma_1 \in \mathrm{SO}(d)$. Let $\mathbf{u} \in \mathbb{S}^{d-1}$ be a fixed point of γ_1, which exists since d is odd. Then the function $g : \mathbb{S}^{d-1} \to \mathbb{R}$ which sends $\mathbf{x}$ to $\mathbf{u} \cdot \mathbf{x}$ is a non-zero element of $\mathcal{H}_1$ with $\gamma_1 . g = g$ and thus $\sum_{s=1}^{4} \gamma_s . g = 0$. We can now invoke Lemma 3.1 (or just observe that e.g. $f = \frac{1}{4} + \frac{1}{4} g$ is a non-constant function that shows fractional divisibility). ■

4. Concluding remarks. Our definition of fractional divisibility allows f to be an arbitrary function in $L^2(\mathbb{S}^{d-1}, \mu)$. Of course, by varying this requirement and/or adding constraints such that the values of f are restricted to $[0, 1]$, we get potentially different notions. Since the main result of this note (Theorem 1.2) is about the non-existence of f, we picked the largest natural class of functions for which our proof works. This definition is rather natural; also, it admits an alternative characterization (as given by Lemma 3.1) and is not affected when we add some extra restrictions on f (as discussed in Remark 3.2).

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