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On Picard's problem via Nevanlinna theory

by

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Abstract. We consider the classical Picard problem for non-parabolic complete Kähler manifolds with non-negative Ricci curvature. Based on a global Green function approach, we give an affirmative answer to this problem under certain conditions by developing Nevanlinna theory. That is, we prove that every meromorphic function on such a manifold reduces to a constant if it omits three distinct values whenever the manifold satisfies a volume growth condition; and prove that every meromorphic function of non-polynomial growth on such a manifold may avoid two distinct values at most.

1. Introduction

1.1. Motivation. The celebrated Picard theorem asserts that a meromorphic function on $\mathbb{C}^m$ must be a constant if it omits three distinct values. Historically, numerous researchers have dedicated themselves to generalizing this theorem (see, e.g., [1, 13, 16, 14, 15, 20, 21, 25, 26, 31, 51]). The study of Picard's problem for manifolds may have originated from S. T. Yau's pioneering work [50] on Liouville's theorem, which establishes Liouville's property for harmonic functions on a Ricci non-negatively curved complete Riemannian manifold. His contribution inspires the following natural generalization of the classical Picard theorem.

Let (M, g) be a complete non-compact Kähler manifold with non-negative Ricci curvature (examples of such manifolds can be seen in [38, 43, 44]). The fundamental Picard's problem states that

Picard's Problem. *Is every meromorphic function on M necessarily a constant if it omits three distinct values?*

Remark 1. Let $p : \mathbb{D} \to \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ be the universal covering, where $\mathbb{D}$ is the unit disc in $\mathbb{C}$. By Liouville's theorem due to Yau [50], we may

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give an affirmative answer to Picard's problem if every holomorphic mapping $f : M \to \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ lifts to a holomorphic mapping $\tilde{f} : M \to \mathbb{D}$ via p. By the mapping lifting theorem, f admits a lifting $\tilde{f}$ if and only if

$$f_*(\pi_1(M, x_0)) \subseteq p_*(\pi_1(\mathbb{D}, \tilde{y}_0))$$

for some $x_0 \in M$ with $p(\tilde{y}_0) = y_0 = f(x_0)$. Since $\mathbb{D}$ is simply-connected, its fundamental group $\pi_1(\mathbb{D}, \tilde{y}_0)$ is trivial. This implies that $f_*(\pi_1(M, x_0))$ must be a trivial group if $\tilde{f}$ exists. However, $f_*(\pi_1(M, x_0))$ is non-trivial in most cases, so f cannot be lifted in general.

An early result in this direction dates back to 1970, when S. Kobayashi [20] proved several Picard-type theorems for holomorphic mappings between certain complex manifolds. In particular, he showed the following:

THEOREM A (Kobayashi, 1970). *Assume that there is a complex Lie group acting transitively on M. Then every meromorphic function on M must be a constant if it omits three distinct values.*

However, Kobayashi's result is restricted to those complex manifolds on which some complex Lie group acts transitively. In 1975, Goldberg–Ishihara–Petridis [15] treated a class of locally flat manifolds (see also N. Petridis [31]). They showed the following.

THEOREM B (Goldberg–Ishihara–Petridis, 1975). *If M is locally flat, then every holomorphic mapping $f : M \to \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ of bounded dilatation must be a constant.*

Nevanlinna theory [27] brings about a major advancement of Picard's theorem, which focuses on the value distribution of meromorphic mappings between complex manifolds. For details, we refer the reader to the works of L. V. Ahlfors [2], H. Cartan [8], Carlson–Griffths–King [7, 17], J. Noguchi [29, 30], E. Nochka [28], M. Ru [32, 33], B. Shiffman [40], B. Shabat [39], F. Sakai [34, 35], W. Stoll [41, 42], P. Vojta [48] and H. Wu [49]; see also [3, 4, 5, 10, 11, 12] and the references therein.

In 2010, A. Atsuji [4] gave a positive answer for a class of meromorphic functions with slower growth by introducing a Nevanlinna-type theory based on heat diffusion. We recall his main results. Let α, Δ stand for the Kähler form and Laplace–Beltrami operator on M, respectively. Let $f : M \to \mathbb{P}^1(\mathbb{C})$ be a meromorphic function. We equip $\mathbb{P}^1(\mathbb{C})$ with the Fubini–Study metric ω_{FS}. The Kählerness of M indicates that the Hilbert–Schmidt norm of the differential df with respect to the metrics $\alpha, \omega_{\mathrm{FS}}$ can be expressed as

$$\|df\|^2 = 4m \frac{f^*\omega_{\mathrm{FS}} \wedge \alpha^{m-1}}{\alpha^m} = \Delta \log \|f\|^2.$$

He showed the following.

THEOREM C (Atsuji, 2010). *Let $f : M \to \mathbb{P}^1(\mathbb{C})$ be a meromorphic function. Assume that f satisfies the growth condition*

$$\int_1^\infty e^{-\epsilon t^2} dt \int_{B(t)} \|df\|^2 \, dv < \infty$$

for any $\epsilon > 0$, where $B(t)$ is the geodesic ball centered at a fixed reference point o with radius t in M. Then f must be a constant if it omits three distinct values.

Atsuji's technique [4] (see [10]) employs stochastic calculus via Brownian motion. He introduced the Nevanlinna-type functions $\tilde{T}_f(t, \omega_{\mathrm{FS}})$, $\tilde{m}_f(t, a)$ and $\tilde{N}_f(t, a)$, where t denotes the time parameter of the Brownian motion X_t on M. By computing curvatures and applying Itô's formula (see [19]), he established an analogue of Nevanlinna theory. Note that M is stochastically complete (i.e., X_t is conservative) due to Grigor'yan's criterion (see [18]). To ensure $\tilde{T}_f(t, \omega_{\mathrm{FS}})$ and $\tilde{N}_f(t, a)$ are well-defined, the following conditions are required:

- $\tilde{T}_f(t, \omega_{\mathrm{FS}}) < \infty$ for $t > 0$;
- $\tilde{T}_f(t, \omega_{\mathrm{FS}}) \to \infty$ as $t \to \infty$;
- $\tilde{N}_f(t, a) = 0$ if f omits a.

Hence, f must satisfy a suitable growth assumption, which explains why a growth condition is included in Theorem C.

In the present paper, our goal is to study Picard's problem using the tools of Nevanlinna theory. Specifically, we extend the classical Nevanlinna theory to non-parabolic complete Kähler manifolds with non-negative Ricci curvature, and employ the Second Main Theorem to establish a Picard-type theorem.

We first recall the basic notions of non-parabolicity and Ricci curvature for Riemannian manifolds. Let M be a complete Riemannian manifold. We say that M is *non-parabolic* if it admits a positive global Green function, and *parabolic* otherwise. Equivalently, from the viewpoint of geometric analysis, M is non-parabolic if

$$G(x, y) = 2 \int_0^\infty p(t, x, y) \, dt < \infty$$

for $x \neq y$, and parabolic if this integral is divergent. Here, $p(t, x, y)$ denotes the transition density function of the Brownian motion X_t generated by the Laplace–Beltrami operator Δ on M, and which is also called the *heat kernel* of M. Note that when M is non-parabolic, $G(x, y)$ defines the unique minimal positive global Green function of $\Delta/2$ for M. Further details may be found in [22, 23, 45, 46].

Let R denote the Riemannian curvature tensor of M. The Ricci curvature tensor of M of dimension n is defined by

$$\mathrm{Ric}(X,Y)=\sum_{j=1}^{n}R(X,e_j,e_j,Y)$$

for $X,Y\in T_xM$ with $x\in M$, where $\{e_1,\dots,e_n\}$ is an orthonormal basis of T_xM. The Ricci curvature of M at x in the direction X is defined by

$$\mathrm{Ric}(X)=\frac{\mathrm{Ric}(X,X)}{\|X\|^2}.$$

We say that M has non-negative Ricci curvature at x if $\mathrm{Ric}(X)\geq 0$ for all $0\neq X\in T_xM$; and that M has non-negative Ricci curvature if it has non-negative Ricci curvature at each $x\in M$. Moreover, the sectional curvature of M at x along the section Π spanned by $X,Y\in T_xM$ is defined by

$$K(\Pi)=\frac{R(X,Y,Y,X)}{\|X\wedge Y\|^2}.$$

By the definitions, the sign of the sectional curvature at a point determines the sign of the Ricci curvature at the same point.

To facilitate the construction of examples for manifolds considered later, we introduce a criterion for non-parabolicity using volume growth. A sharp necessary condition by N. Varopoulos [47] states that if M is non-parabolic, then

$$\int_1^\infty \frac{t\,dt}{V(t)}<\infty, \tag{1.1}$$

where $V(t)$ denotes the Riemannian volume of the geodesic ball centered at a fixed reference point $o\in M$ with radius t. However, condition (1.1) is far from sufficient, and a counterexample was constructed in [47]. The first major result for sufficiency was due to N. Varopoulos [46] and Li–Yau [24]. Based on the heat kernel estimates, they proved that if M has non-negative Ricci curvature and (1.1) holds, then M is non-parabolic. Therefore, for M with non-negative Ricci curvature, M is non-parabolic if and only if (1.1) is satisfied.

Below, we assume that (M,g) is a non-parabolic complete non-compact Kähler manifold with non-negative Ricci curvature, of complex dimension m. Here, non-parabolicity means that M is non-parabolic as a Riemannian manifold. The Ricci form $\mathscr{R}$ of g on M is defined by

$$\mathscr{R}=-dd^c\log\det(g_{i\bar{j}})$$

with $dd^c=\sqrt{-1}\partial\overline{\partial}/(2\pi)$. Since M is Kähler, $\mathrm{Ric}\geq 0$ is equivalent to $\mathscr{R}\geq 0$. Our novel idea is to construct a family $\{\Delta(r)\}_{r>0}$ of exhausting precompact domains for M using the minimal positive global Green function for M. This

allows us to well-define Nevanlinna functions on $\Delta(r)$ and then establish a Second Main Theorem by estimating the gradient of the Green function for $\Delta(r)$. Applying the Second Main Theorem, we provide an affirmative answer to the Picard problem under certain conditions.

1.2. Main results. Let $G(o,x)$ be the minimal positive global Green function of $\Delta/2$ for M, where Δ is the Laplace–Beltrami operator on M, and o is a fixed reference point in M. By Li–Yau's estimate [24], there are constants $A, B > 0$ such that

$$A \int_{\rho(x)}^{\infty} \frac{t\,dt}{V(t)} \le G(o,x) \le B \int_{\rho(x)}^{\infty} \frac{t\,dt}{V(t)}$$

for all $x \in M$ and all $t > 0$, where $\rho(x)$ stands for the Riemannian distance from x to o, and $V(t)$ stands for the Riemannian volume of the geodesic ball in M centered at o with radius t. For $r > 0$, define

$$\Delta(r) = \left\{ x \in M : G(o,x) > A \int_{r}^{\infty} \frac{t\,dt}{V(t)} \right\}.$$

Then $o \in \Delta(r)$ for all $r > 0$ and the family $\{\Delta(r)\}_{r>0}$ exhausts M.

Let X be a complex projective manifold with $\dim_{\mathbb{C}} X \le m$, and (L,h) a Hermitian positive line bundle over X with Chern form $c_1(L,h) > 0$. We now take a divisor $D \in |L|$, where $|L|$ denotes the complete linear system of L. For a meromorphic mapping $f : M \to X$, we can define the Nevanlinna functions $T_f(r,L), m_f(r,D), N_f(r,D), \overline{N}_f(r,D)$ on $\Delta(r)$ (see Section 3.2). Let K_X be the canonical line bundle over X. The *characteristic function* of $\mathscr{R}$ is defined by

$$T(r,\mathscr{R}) = \frac{\pi^m}{(m-1)!} \int_{\Delta(r)} g_r(o,x) \mathscr{R} \wedge \alpha^{m-1},$$

where α is the Kähler form of M, and $g_r(o,x)$ is the Green function of $\Delta/2$ for $\Delta(r)$ with a pole at o satisfying the Dirichlet boundary condition. Set

$$H(r,\delta) = \frac{1}{r}\left(\frac{V(r)}{r}\right)^{1+\delta} \int_{r}^{\infty} \frac{t\,dt}{V(t)}. \tag{1.2}$$

The main result is the following Second Main Theorem.

THEOREM 1.1 (= Theorem 6.1). *Let M be a non-parabolic complete non-compact Kähler manifold with non-negative Ricci curvature. Let X be a complex projective manifold of complex dimension not greater than that of M. Let $D \in |L|$ be a reduced divisor of simple normal crossing type, where L is a positive line bundle over X. Let $f : M \to X$ be a differentiably non-degenerate meromorphic mapping. Then for any $\delta > 0$, there exists a subset*

$E_\delta \subseteq (0,\infty)$ of finite Lebesgue measure such that
$T_f(r,L)+T_f(r,K_X)+T(r,\mathscr{R}) \le \overline{N}_f(r,D)+O\left(\log^+ T_f(r,L)+\log H(r,\delta)\right)$
for all $r>0$ outside E_δ, where $H(r,\delta)$ is given by (1.2).

For a divisor $D \in |L|$, the *simple defect* of f with respect to D is defined by

$$\bar{\delta}_f(D) = 1 - \limsup_{r\to\infty} \frac{\overline{N}_f(r,D)}{T_f(r,L)}.$$

Set

$$\left[\frac{c_1(K_X^*)}{c_1(L)}\right] = \inf\{s \in \mathbb{R} : \eta < s\omega,\ \exists \eta \in c_1(K_X^*),\ \exists \omega \in c_1(L)\}.$$

We establish a defect relation:

Corollary 1.2 (= Corollary 6.2). *Assume the same conditions as in Theorem* 1.1. *Then*

$$\bar{\delta}_f(D) \le \left[\frac{c_1(K_X^*)}{c_1(L)}\right] - \liminf_{r\to\infty} \frac{T(r,\mathscr{R})}{T_f(r,L)} \le \left[\frac{c_1(K_X^*)}{c_1(L)}\right],$$

if one of the following conditions is satisfied:

(i) *M satisfies the volume growth condition*

$$\lim_{r\to\infty} \frac{\log\left(\dfrac{V(r)}{r^2}\displaystyle\int_r^\infty \frac{t\,dt}{V(t)}\right)}{\log r} = 0; \tag{1.3}$$

(ii) *f is of non-polynomial growth, i.e.,*

$$\lim_{r\to\infty} \frac{\log r}{T_f(r,L)} = 0.$$

Remark 2. Note that $r^2 = o(V(r))$ as $r\to\infty$, due to

$$\int_1^\infty \frac{t\,dt}{V(t)} < \infty.$$

The above growth condition (1.3) is relaxed in our setting. For instance, it holds when $V(r) = O(r^\mu)$ with $\mu > 2$, and also when $V(r) = O(r^\mu \log^\nu r)$ with $\mu > 2$ or $\mu = 2$ and $\nu > 1$.

As an application of Corollary 1.2, we consider Picard's problem under the following conditions. We treat the case where $X = \mathbb{P}^1(\mathbb{C})$ with the point line bundle (i.e., the hyperplane line bundle) $\mathscr{O}(1)$. Take $D = a_1 + \cdots + a_q$, where $a_1,\dots,a_q$ are q distinct points in $\mathbb{P}^1(\mathbb{C})$. Set $L = q\mathscr{O}(1)$. Then $D \in |L|$. A basic fact is that

$$\left[\frac{c_1(K^*_{\mathbb{P}^1(\mathbb{C})})}{c_1(\mathscr{O}(1))}\right] = 2,$$

which yields

$$\sum_{j=1}^{q} \bar{\delta}_f(a_j) \leq 2$$

whenever condition (i) or (ii) in Corollary 1.2 is satisfied. By the arguments above, we conclude the following results.

COROLLARY 1.3. *Let M be a non-parabolic complete non-compact Kähler manifold with non-negative Ricci curvature. If M satisfies the volume growth condition* (1.3)*, then every meromorphic function on M must reduce to a constant if it omits three distinct values.*

COROLLARY 1.4. *Let M be a non-parabolic complete non-compact Kähler manifold with non-negative Ricci curvature. Then every meromorphic function of non-polynomial growth on M can omit two distinct values at most.*

Finally, we provide some examples for M where Corollaries 1.3 and 1.4 apply. By (1.1), we first notice that $M = \mathbb{C}^m$ with $m \geq 2$ is a non-parabolic complete non-compact Kähler manifold with zero Ricci curvature. Below, we give two non-trivial examples.

EXAMPLE 1. For an integer $l \geq 1$, we equip $\mathbb{P}^l(\mathbb{C})$ with the Fubini–Study metric ω_{FS}. Note that $\mathbb{P}^l(\mathbb{C})$ is a complete Kähler manifold with positive sectional curvature (and thus with positive Ricci curvature). We consider the product manifold $M = \mathbb{C}^k \times \mathbb{P}^l(\mathbb{C})$ with $k \geq 2$, equipped with the product metric induced from ω_{FS} and the standard Euclidean metric on $\mathbb{C}^k$. Since $\mathbb{C}^k$ is flat, M is Ricci non-negatively curved under this product metric. Again, since M has volume growth $O(r^{2k}) \geq O(r^4)$ as $r \to \infty$, we deduce that (1.1) is satisfied. Hence, M is a non-parabolic complete non-compact Kähler manifold with non-negative Ricci curvature. It is not hard to see that there exist a number of non-constant meromorphic functions on M. For instance, we can take $f(z, \zeta) = g(z) + h(\zeta)$ or $f(z, \zeta) = g(z)h(\zeta)$, where $g(z)$ is any meromorphic function on $\mathbb{C}^k$, and $h(\zeta)$ is any rational function on $\mathbb{P}^l(\mathbb{C})$.

EXAMPLE 2. Consider the product manifold $M = \mathbb{C}^k \times T^l$ with $k \geq 2$ and $l \geq 1$, where $T^l = \mathbb{C}^l/\Lambda$ denotes the l-dimensional complex torus equipped with the metric inherited from the standard Euclidean metric on $\mathbb{C}^l$. Then M is a complete non-compact Kähler manifold with zero Ricci curvature under the induced product metric. As $k \geq 2$, a similar argument to that in Example 1 also shows the non-parabolicity of M. The existence of non-constant meromorphic functions on M can be confirmed easily. For instance, for $l = 1$, we can take $f(z, w) = g(z) + h(w)$ or $f(z, w) = g(z)h(w)$, where $g(z)$ is any meromorphic function on $\mathbb{C}^k$, and $h(w)$ is any elliptic function with periodic lattice Λ on $\mathbb{C}$.

More generally, one can take $M = M_1 \times M_2$, where M_1 is any complete compact Kähler manifold with non-negative Ricci curvature, and M_2 is a complete non-compact Kähler manifold with non-negative Ricci curvature and volume growth satisfying (1.1).

2. Some facts from geometric analysis

2.1. Volume comparison theorem. A *space form* is a complete simply-connected Riemannian manifold with constant sectional curvature. Let M^K denote the n-dimensional space form with constant sectional curvature K, and let $V(K,r)$ denote the Riemannian volume of a geodesic ball with radius r in M^K. Let (M,g) denote a complete Riemannian manifold of dimension n with Ricci curvature tensor Ric_M, and let $V(r)$ denote the Riemannian volume of a geodesic ball centered at a fixed reference point o with radius r in M.

The well-known Bishop–Gromov volume comparison theorem (see [6]) states the following.

THEOREM 2.1. *If* $\mathrm{Ric}_M \geq (n-1)Kg$, *then the volume ratio* $V(r)/V(K,r)$ *is a non-increasing function of* $r > 0$, *and tends to* 1 *as* $r \to 0$. *Hence,*

$$V(r) \leq V(K,r) \quad \text{for all } r > 0.$$

When $M^K = \mathbb{R}^n$, we obtain

COROLLARY 2.2. *We have*

$$V(r) \leq \omega_n r^n \quad \text{for all } r > 0,$$

where ω_n *denotes the standard Euclidean volume of the unit ball in* $\mathbb{R}^n$.

When $\mathrm{Ric}_M \geq 0$, Calabi–Yau (see [37]) showed

THEOREM 2.3. *Assume that* M *is non-compact. If* $\mathrm{Ric}_M \geq 0$, *then* M *has an infinite volume. More precisely, for any* $\epsilon_0 > 0$, *there exists a constant* $c = c(o,n,\epsilon_0) > 0$ *such that*

$$V(r) \geq cr \quad \text{for all } r \geq \epsilon_0.$$

2.2. Estimates of heat kernels. Let M be a complete Riemannian manifold of dimension n, with Laplace–Beltrami operator Δ. Let $\rho(x,y)$ stand for the Riemannian distance between $x, y \in M$, and let $V_x(r)$ denote the Riemannian volume of the geodesic ball in M centered at x with radius r. The heat kernel $p(t,x,y)$ of M associated to Δ is defined as the minimal positive fundamental solution of the heat equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) u(t,x) = 0.$$

Li–Yau [24] (see also [37]) established a two-sided estimate of $p(t,x,y)$.

THEOREM 2.4. *Assume that M has non-negative Ricci curvature. Then for any $0 < \epsilon < 1$, there exist constants $C_1 = C_1(\epsilon, n), C_2 = C_2(\epsilon, n) > 0$ such that*

$$\frac{C_1}{V_x(\sqrt{t})} e^{-\frac{\rho(x,y)^2}{(4-\epsilon)t}} \leq p(t,x,y) \leq \frac{C_2}{V_x(\sqrt{t})} e^{-\frac{\rho(x,y)^2}{(4+\epsilon)t}}$$

for all $x \in M$ and all $t > 0$.

Set

$$G(x,y) = 2 \int_0^\infty p(t,x,y)\, dt.$$

This integral is convergent if and only if M is non-parabolic. When M is non-parabolic, $G(x,y)$ is the unique minimal positive global Green function of $\Delta/2$ for M, i.e.,

$$\begin{cases} -\frac{1}{2}\Delta_y G(x,y) = \delta_x(y), & x, y \in M, \\ G(x,y) > 0, & x, y \in M, \\ \lim\limits_{\rho(x,y)\to\infty} G(x,y) = 0. \end{cases}$$

where δ_x is the Dirac delta function at x.

When M is non-parabolic with non-negative Ricci curvature, Li–Yau [24] (see also [37]) gave two-sided bounds of $G(x,y)$:

THEOREM 2.5. *Assume that M has non-negative Ricci curvature. If M is non-parabolic, then there exist constants $A, B > 0$ depending only on n such that*

$$A \int_{\rho(x,y)}^\infty \frac{t\, dt}{V_x(t)} \leq G(x,y) \leq B \int_{\rho(x,y)}^\infty \frac{t\, dt}{V_x(t)}$$

for all $x, y \in M$.

3. Nevanlinna functions and First Main Theorem. Let (M, g) be a non-parabolic complete non-compact Kähler manifold with non-negative Ricci curvature, of complex dimension m. Recall that the Kähler form α of M associated to the metric g is defined by

$$\alpha = \frac{\sqrt{-1}}{\pi} \sum_{i,j=1}^m g_{i\bar{j}} dz_i \wedge d\bar{z}_j.$$

3.1. Construction of $\Delta(r)$. Fix a reference point $o \in M$. Let $V(r)$ stand for the Riemannian volume of the geodesic ball $B(r)$ in M centered at o with radius r. Recall that the non-parabolicity of M implies that

$$\int_1^\infty \frac{t\, dt}{V(t)} < \infty.$$

Thus, we have the unique minimal positive global Green function $G(o,x)$ of the half Laplace–Beltrami operator $\Delta/2$ for M, which can be written as

$$G(o,x) = 2\int_0^\infty p(t,o,x)\,dt,$$

where $p(t,o,x)$ is the heat kernel of M. Let $\rho(x)$ be the Riemannian distance from x to o. By Theorem 2.5, there exist constants $A, B > 0$ such that

$$A\int_{\rho(x)}^\infty \frac{t\,dt}{V(t)} \le G(o,x) \le B\int_{\rho(x)}^\infty \frac{t\,dt}{V(t)} \tag{3.1}$$

for all $x \in M$. For $r > 0$, define

$$\Delta(r) = \left\{x \in M : G(o,x) > A\int_r^\infty \frac{t\,dt}{V(t)}\right\}.$$

Since

$$\lim_{x\to o} G(o,x) = \infty, \qquad \lim_{\rho(x)\to\infty} G(o,x) = 0,$$

one can conclude immediately that $\Delta(r)$ is a precompact domain containing o with $\overline{\Delta(r_1)} \subseteq \Delta(r_2)$ whenever $r_1 < r_2$ and with

$$\lim_{r\to 0}\Delta(r) = \emptyset, \qquad \lim_{r\to\infty}\Delta(r) = M.$$

Hence, the family $\{\Delta(r)\}_{r>0}$ exhausts M, i.e., for any sequence $\{r_n\}_{n=1}^\infty$ with $0 < r_1 < r_2 < \cdots \to \infty$, we have

$$\bigcup_{n=1}^\infty \Delta(r_n) = M, \qquad \emptyset \ne \Delta(r_1) \subseteq \overline{\Delta(r_1)} \subseteq \Delta(r_2) \subseteq \overline{\Delta(r_2)} \subseteq \cdots.$$

The boundary $\partial\Delta(r)$ of $\Delta(r)$ can be described as

$$\partial\Delta(r) = \left\{x \in M : G(o,x) = A\int_r^\infty \frac{t\,dt}{V(t)}\right\}.$$

By Sard's theorem, $\partial\Delta(r)$ is smooth for almost all $r > 0$.

Set

$$g_r(o,x) = G(o,x) - A\int_r^\infty \frac{t\,dt}{V(t)}.$$

Evidently, $g_r(o,x)$ defines the positive Green function of $\Delta/2$ for $\Delta(r)$ with a pole at o satisfying the Dirichlet boundary condition, i.e.,

$$\begin{cases} -\frac{1}{2}\Delta g_r(o,x) = \delta_o(x), & x \in \Delta(r), \\ g_r(o,x) = 0, & x \in \partial\Delta(r). \end{cases}$$

Let π_r stand for the harmonic measure on $\partial\Delta(r)$ with respect to o, which can be written as

$$d\pi_r = \frac{1}{2}\,\frac{\partial g_r(o,x)}{\partial\vec{\nu}}d\sigma_r,$$

where $\partial/\partial\vec{\nu}$ is the inward normal derivative on $\partial\Delta(r)$, and $d\sigma_r$ is the induced Riemannian area element of $\partial\Delta(r)$.

3.2. Nevanlinna functions. In what follows, we introduce Nevanlinna functions. Let $f : M \to X$ be a meromorphic mapping, where X is a complex projective manifold. Let (L,h) be a Hermitian holomorphic line bundle over X, with Chern form

$$c_1(L,h) := -dd^c \log h,$$

where

$$d = \partial + \overline{\partial}, \qquad d^c = \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial)$$

so that

$$dd^c = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}.$$

Fix a divisor $D \in |L|$, where $|L|$ is the complete linear system of L. Let s_D be the canonical section associated to D, i.e., s_D is a holomorphic section of L over X with zero divisor D. The *characteristic function, proximity function, counting function* and *simple counting function* of f are respectively defined by

$$\begin{aligned}
T_f(r,L) &= -\frac{1}{4}\int\limits_{\Delta(r)} g_r(o,x)\Delta\log(h\circ f)\,dv,\\
m_f(r,D) &= \int\limits_{\partial\Delta(r)} \log\frac{1}{\|s_D\circ f\|}\,d\pi_r,\\
N_f(r,D) &= \frac{\pi^m}{(m-1)!}\int\limits_{f^*D\cap\Delta(r)} g_r(o,x)\alpha^{m-1},\\
\overline{N}_f(r,D) &= \frac{\pi^m}{(m-1)!}\int\limits_{f^{-1}(D)\cap\Delta(r)} g_r(o,x)\alpha^{m-1},
\end{aligned}$$

where dv denotes the Riemannian volume element of M.

Locally, write $s_D = \tilde{s}_D e$, where e is a local holomorphic frame of L and $\tilde{s}_D$ is a holomorphic function. Using the Poincaré–Lelong formula (see, e.g., [7]), we get

$$[D] = dd^c[\log|\tilde{s}_D|^2]$$

in the sense of currents. So, we obtain alternative expressions of $T_f(r,L)$ and

$N_f(r,D)$:

$$T_f(r,L) = \frac{\pi^m}{(m-1)!}\int\limits_{\Delta(r)} g_r(o,x) f^* c_1(L,h)\wedge\alpha^{m-1}$$
$$= \frac{1}{4}\int\limits_{\Delta(r)} g_r(o,x)\|df\|^2\,dv$$

and

$$N_f(r,D) = \frac{\pi^m}{(m-1)!}\int\limits_{\Delta(r)} g_r(o,x) dd^c[\log|\tilde{s}_D\circ f|^2]\wedge\alpha^{m-1}$$
$$= \frac{1}{4}\int\limits_{\Delta(r)} g_r(o,x)\Delta\log|\tilde{s}_D\circ f|^2\,dv.$$

3.3. First Main Theorem. To establish the First Main Theorem for f, we need the Green–Dynkin formula (see [5, 11, 12]).

LEMMA 3.1 (Green–Dynkin formula). *Let ϕ be a $\mathscr{C}^2$ function on M outside a polar set of singularities at most. Assume that $\phi(o)\neq\infty$. Then*

$$\int\limits_{\partial\Delta(r)}\phi\,d\pi_r - \phi(o) = \frac{1}{2}\int\limits_{\Delta(r)} g_r(o,x)\Delta\phi\,dv.$$

Assume that $f(o)\notin \operatorname{Supp} D$. Applying the Green–Dynkin formula to $\log\|s_D\circ f\|$, we are led to

$$m_f(r,D) - \log\frac{1}{\|s_D\circ f(o)\|} = \frac{1}{2}\int\limits_{\Delta(r)} g_r(o,x)\Delta\log\frac{1}{\|s_D\circ f\|}\,dv$$
$$= -\frac{1}{4}\int\limits_{\Delta(r)} g_r(o,x)\Delta\log(h\circ f)\,dv - \frac{1}{4}\int\limits_{\Delta(r)} g_r(o,x)\Delta\log|\tilde{s}_D\circ f|^2\,dv$$
$$= T_f(r,L) - N_f(r,D).$$

To conclude, we obtain

THEOREM 3.2 (First Main Theorem). *Assume that $f(o)\notin\operatorname{Supp} D$. Then*

$$T_f(r,L) + \log\frac{1}{\|s_D\circ f(o)\|} = m_f(r,D) + N_f(r,D).$$

4. Gradient estimates of Green functions. Given now a non-parabolic complete non-compact Kähler manifold (M,g) with non-negative Ricci curvature, of complex dimension m, to establish a gradient estimate of $g_r(o,x)$ we need some lemmas.

LEMMA 4.1. *We have*

$$g_r(o,x) = A\int_t^r \frac{s\,ds}{V(s)}, \qquad x \in \partial\Delta(t),$$

for all $0 < t \le r$, *where* A *is given by* (3.1).

Proof. Using the definition of the Green function for $\Delta(r)$, it is immediate that for $0 < t \le r$,

$$\begin{aligned} g_r(o,x) &= G(o,x) - A\int_r^\infty \frac{t\,dt}{V(t)} \\ &= G(o,x) - A\int_t^\infty \frac{s\,ds}{V(s)}ds + A\int_t^r \frac{s\,ds}{V(s)} \\ &= g_t(o,x) + A\int_t^r \frac{s\,ds}{V(s)}. \end{aligned}$$

Since $g_t(o,x) = 0$ for $x \in \partial\Delta(t)$, we obtain

$$g_r(o,x) = A\int_t^r \frac{s\,ds}{V(s)}, \qquad x \in \partial\Delta(t). \ \blacksquare$$

Let ∇ denote the gradient operator on any Riemannian manifold. Cheng–Yau [9] proved the following result.

LEMMA 4.2. *Let* N *be a complete Riemannian manifold of dimension* $n \ge 2$. *Let* $B(x_0,r)$ *be any geodesic ball centered at* x_0 *with radius* r *in* N. *Then there exists a constant* $C_n > 0$ *depending only on* n *such that*

$$\frac{\|\nabla u(x)\|}{u(x)} \le \frac{C_n r^2}{r^2 - d(x_0,x)^2}\left(|\kappa(r)| + \frac{1}{d(x_0,x)}\right)$$

for any non-negative harmonic function u *on* $B(x_0,r)$, *where* $d(x_0,x)$ *is the Riemannian distance between* x_0 *and* x, *and* $\kappa(r)$ *is the lower bound of the Ricci curvature of* $B(x_0,r)$.

We establish an upper estimate of $\|\nabla g_r(o,x)\|$:

THEOREM 4.3. *There exists a constant* $c_1 > 0$ *independent of* r *such that*

$$\|\nabla g_r(o,x)\| \le \frac{c_1}{r}\int_r^\infty \frac{t\,dt}{V(t)}, \qquad x \in \partial\Delta(r).$$

Proof. By the curvature assumption, $\mathrm{Ric}_M \ge 0$. Thus, it follows from Lemma 4.2 (letting $r \to \infty$) and (3.1) that (see also [36, Remark 5])

$$\|\nabla G(o,x)\| \le \frac{c_0}{\rho(x)}G(o,x) \le \frac{c_0 B}{\rho(x)}\int_{\rho(x)}^\infty \frac{t\,dt}{V(t)}$$

for some large constant $c_0 > 0$ which depends only on the dimension m. By (3.1) again,

$$\int_{\rho(x)}^{\infty} \frac{t\,dt}{V(t)} \le \int_{r}^{\infty} \frac{t\,dt}{V(t)}, \qquad x \in \partial\Delta(r),$$

which gives $\rho(x) \ge r$ for $x \in \partial\Delta(r)$. Set $c_1 = c_0 B$. Then we conclude that

$$\|\nabla g_r(o,x)\| = \|\nabla G(o,x)\| \le \frac{c_1}{r} \int_{r}^{\infty} \frac{t\,dt}{V(t)}, \qquad x \in \partial\Delta(r). \ \blacksquare$$

Since

$$\frac{\partial g_r(o,x)}{\partial \vec{\nu}} = \|\nabla g_r(o,x)\|, \qquad x \in \partial\Delta(r),$$

we derive an upper estimate of $d\pi_r$:

COROLLARY 4.4. *There exists a constant $c_2 > 0$ independent of r such that*

$$d\pi_r \le \frac{c_2}{r} \int_{r}^{\infty} \frac{t\,dt}{V(t)} d\sigma_r,$$

where $d\sigma_r$ is the induced Riemannian area element of $\partial\Delta(r)$.

5. Calculus lemma and logarithmic derivative lemma. Let now (M, g) be a non-parabolic complete non-compact Kähler manifold with non-negative Ricci curvature, of complex dimension m.

5.1. Calculus lemma. We need the following Borel's growth lemma (see [30, 33]).

LEMMA 5.1 (Borel's growth lemma). *Let $u \ge 0$ be a non-decreasing function on (r_0, ∞) with $r_0 \ge 0$. Then for any $\delta > 0$, there exists a subset $E_\delta \subseteq (r_0, \infty)$ of finite Lebesgue measure such that*

$$u'(r) \le u(r)^{1+\delta}$$

for all $r > r_0$ outside E_δ.

Proof. The conclusion is obviously true for $u \equiv 0$. Below, assume that $u \not\equiv 0$. Since $u \ge 0$ is non-decreasing, there is a number $r_1 > r_0$ such that $u(r_1) > 0$. By monotonicity, the limit $\eta = \lim_{r\to\infty} u(r)$ exists, possibly infinite. If $\eta = \infty$, then $\eta^{-1} = 0$. Set

$$E_\delta = \{r \in (r_0, \infty) : u'(r) > u(r)^{1+\delta}\}.$$

Note that $u'(r)$ exists for almost all $r \in (r_0, \infty)$. Then

$$\begin{aligned}\int\limits_{E_\delta} dr &\le \int\limits_{r_0}^{r_1} dr + \int\limits_{r_1}^{\infty} \frac{u'(r)}{u(r)^{1+\delta}} dr \\ &= \frac{1}{\delta u(r_1)^\delta} - \frac{1}{\delta \eta^\delta} + r_1 - r_0 < \infty.\end{aligned}$$

This completes the proof. ∎

We establish the following Calculus Lemma.

THEOREM 5.2 (Calculus Lemma). *Let $k \ge 0$ be a locally integrable function on M. Assume that k is locally bounded at o. Then there is a constant $C > 0$ independent of r such that for any $\delta > 0$, there exists a subset $E_\delta \subseteq (0, \infty)$ of finite Lebesgue measure such that*

$$\int\limits_{\partial\Delta(r)} k \, d\pi_r \le C H(r, \delta) \Big(\int\limits_{\Delta(r)} g_r(o, x) k \, dv \Big)^{(1+\delta)^2}$$

for all $r > 0$ outside E_δ, where $H(r, \delta)$ is given by (1.2).

Proof. Invoking Lemma 4.1, we arrive at

$$\begin{aligned}\int\limits_{\Delta(r)} g_r(o, x) k \, dv &= \int\limits_0^r dt \int\limits_{\partial\Delta(t)} g_r(o, x) k \, d\sigma_t \\ &= A \int\limits_0^r \Big(\int\limits_t^r \frac{s \, ds}{V(s)} \Big) dt \int\limits_{\partial\Delta(t)} k \, d\sigma_t.\end{aligned}$$

Set

$$\Lambda(r) = A \int\limits_0^r \Big(\int\limits_t^r \frac{s \, ds}{V(s)} \Big) dt \int\limits_{\partial\Delta(t)} k \, d\sigma_t.$$

A simple computation leads to

$$\Lambda'(r) = \frac{d\Lambda(r)}{dr} = \frac{Ar}{V(r)} \int\limits_0^r dt \int\limits_{\partial\Delta(t)} k \, d\sigma_t.$$

Further, we have

$$\frac{d}{dr} \Big(\frac{V(r) \Lambda'(r)}{r} \Big) = A \int\limits_{\partial\Delta(r)} k \, d\sigma_r.$$

Employing Borel's growth lemma to the left-hand side of the above equality twice, for $V(r)\Lambda'(r)/r$ and for $\Lambda'(r)$, we conclude that for any $\delta > 0$, there exists a subset $E_\delta \subseteq (0, \infty)$ of finite Lebesgue measure such that

$$\int\limits_{\partial\Delta(r)} k \, d\sigma_r \le \frac{1}{A} \Big(\frac{V(r)}{r} \Big)^{1+\delta} \Lambda(r)^{(1+\delta)^2}$$

for all $r > 0$ outside E_δ. On the other hand, Corollary 4.4 implies that there exists a constant $c > 0$ independent of r such that

$$d\pi_r \le \frac{c}{r} \int_r^\infty \frac{t\,dt}{V(t)}\, d\sigma_r.$$

Set $C = c/A$. Combining the above, we have

$$\begin{aligned}\int_{\partial\Delta(r)} k\, d\pi_r &\le \frac{c}{Ar}\left(\frac{V(r)}{r}\right)^{1+\delta} \int_r^\infty \frac{t\,dt}{V(t)} \Lambda(r)^{(1+\delta)^2} \\ &= CH(r,\delta)\Lambda(r)^{(1+\delta)^2}\end{aligned}$$

for all $r > 0$ outside E_δ. The proof is complete. ∎

5.2. Logarithmic Derivative Lemma. Let ψ be a meromorphic function on M. The norm of the gradient $\nabla\psi$ is defined by

$$\|\nabla\psi\|^2 = 2\sum_{i,j=1}^m g^{i\bar{j}} \frac{\partial\psi}{\partial z_i}\overline{\frac{\partial\psi}{\partial z_j}}$$

in a local holomorphic coordinate $z = (z_1, \dots, z_m)$, where $(g^{i\bar{j}})$ is the inverse of $(g_{i\bar{j}})$. Define the Nevanlinna characteristic function of ψ by

$$T(r,\psi) = m(r,\psi) + N(r,\psi),$$

where

$$\begin{aligned} m(r,\psi) &= \int_{\partial\Delta(r)} \log^+ |\psi|\, d\pi_r, \\ N(r,\psi) &= \frac{\pi^m}{(m-1)!} \int_{\psi^*\infty\cap\Delta(r)} g_r(o,x)\alpha^{m-1}. \end{aligned}$$

For any $a \in \mathbb{C}$, it is not hard to deduce that

$$T\left(r, \frac{1}{\psi - a}\right) = T(r,\psi) + O(1).$$

Put a singular metric on $\mathbb{P}^1(\mathbb{C})$:

$$\Psi = \frac{1}{|\zeta|^2(1+\log^2|\zeta|)} \frac{\sqrt{-1}}{4\pi^2} d\zeta \wedge d\bar{\zeta},$$

which satisfies

$$\int_{\mathbb{P}^1(\mathbb{C})} \Psi = 1.$$

Lemma 5.3. *We have*

$$\frac{1}{4\pi} \int_{\Delta(r)} g_r(o,x) \frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}\, dv \le T(r,\psi) + O(1).$$

Proof. A direct computation gives

$$\frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)} = 4m\pi\frac{\psi^*\Psi\wedge\alpha^{m-1}}{\alpha^m}.$$

Hence, we conclude from Fubini's theorem that

$$\begin{aligned}\frac{1}{4\pi}\int\limits_{\Delta(r)} g_r(o,x)\frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}\,dv &= m\int\limits_{\Delta(r)} g_r(o,x)\frac{\psi^*\Psi\wedge\alpha^{m-1}}{\alpha^m}\,dv\\ &= \frac{\pi^m}{(m-1)!}\int\limits_{\mathbb{P}^1(\mathbb{C})}\Psi(\zeta)\int\limits_{\psi^*\zeta\cap\Delta(r)} g_r(o,x)\alpha^{m-1}\\ &= \int\limits_{\mathbb{P}^1(\mathbb{C})} N\bigg(r,\frac{1}{\psi-\zeta}\bigg)\Psi(\zeta)\\ &\le \int\limits_{\mathbb{P}^1(\mathbb{C})}(T(r,\psi)+O(1))\Psi = T(r,\psi)+O(1).\ \blacksquare\end{aligned}$$

LEMMA 5.4. *Let $\psi\not\equiv 0$ be a meromorphic function on M. Then for any $\delta>0$, there exists a subset $E_\delta\subseteq(0,\infty)$ of finite Lebesgue measure such that*

$$\begin{aligned}&\int\limits_{\partial\Delta(r)}\log^+\frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}\,d\pi_r\\ &\qquad\le(1+\delta)^2\log^+T(r,\psi)+\log H(r,\delta)+O(1)\end{aligned}$$

for all $r>0$ outside E_δ, where $H(r,\delta)$ is given by (1.2).

Proof. Since π_r is a harmonic measure on $\partial\Delta(r)$, it is a probability measure. Using the concavity of log, we deduce that

$$\begin{aligned}\int\limits_{\partial\Delta(r)}\log^+\frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}\,d\pi_r &\le \log\int\limits_{\partial\Delta(r)}\bigg(1+\frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}\bigg)\,d\pi_r\\ &\le \log^+\int\limits_{\partial\Delta(r)}\frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}\,d\pi_r+O(1).\end{aligned}$$

From this together with Theorem 5.2 and Lemma 5.3, we conclude that for any $\delta>0$, there exists a subset $E_\delta\subseteq(0,\infty)$ of finite Lebesgue measure such that

$$\begin{aligned}&\log^+\int\limits_{\partial\Delta(r)}\frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}\,d\pi_r\\ &\quad\le(1+\delta)^2\log^+\int\limits_{\Delta(r)} g_r(o,x)\frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}\,dv+\log H(r,\delta)+O(1)\\ &\quad\le(1+\delta)^2\log^+T(r,\psi)+\log H(r,\delta)+O(1)\end{aligned}$$

for all $r>0$ outside E_δ. ∎

Define

$$m\left(r, \frac{\|\nabla\psi\|}{|\psi|}\right) = \int\limits_{\partial\Delta(r)} \log^+ \frac{\|\nabla\psi\|}{|\psi|}\, d\pi_r.$$

With the above preparations, we will prove the following Logarithmic Derivative Lemma.

THEOREM 5.5 (Logarithmic Derivative Lemma). *Let $\psi \not\equiv 0$ be a meromorphic function on M. Then for any $\delta > 0$, there exists a subset $E_\delta \subseteq (0,\infty)$ of finite Lebesgue measure such that*

$$m\left(r, \frac{\|\nabla\psi\|}{|\psi|}\right) \le \frac{2+(1+\delta)^2}{2} \log^+ T(r,\psi) + \frac{1}{2}\log H(r,\delta)$$

for all $r > 0$ outside E_δ, where $H(r,\delta)$ is given by (1.2).

Proof. The concavity of log gives

$$\begin{aligned}
m\left(r, \frac{\|\nabla\psi\|}{|\psi|}\right) &\le \frac{1}{2} \int\limits_{\partial\Delta(r)} \log^+ \frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}\, d\pi_r \\
&\quad + \frac{1}{2} \int\limits_{\partial\Delta(r)} \log(1+\log^2|\psi|)\, d\pi_r \\
&= \frac{1}{2} \int\limits_{\partial\Delta(r)} \log^+ \frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}\, d\pi_r \\
&\quad + \frac{1}{2} \int\limits_{\partial\Delta(r)} \log\left(1+\left(\log^+|\psi| + \log^+\frac{1}{|\psi|}\right)^2\right) d\pi_r \\
&\le \frac{1}{2} \int\limits_{\partial\Delta(r)} \log^+ \frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}\, d\pi_r \\
&\quad + \log \int\limits_{\partial\Delta(r)} \left(\log^+|\psi| + \log^+\frac{1}{|\psi|}\right) d\pi_r + O(1) \\
&\le \frac{1}{2} \int\limits_{\partial\Delta(r)} \log^+ \frac{\|\nabla\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}\, d\pi_r + \log^+ T(r,\psi) + O(1).
\end{aligned}$$

By Lemma 5.4, for any $\delta > 0$, there exists a subset $E_\delta \subseteq (0,\infty)$ of finite Lebesgue measure such that

$$m\left(r, \frac{\|\nabla\psi\|}{|\psi|}\right) \le \frac{2+(1+\delta)^2}{2} \log^+ T(r,\psi) + \frac{\delta}{2}\log H(r,\delta)$$

for all $r > 0$ outside E_δ, where $H(r,\delta)$ is given by (1.2). ∎

6. Second Main Theorem and defect relation. We let (M, g) be a non-parabolic complete non-compact Kähler manifold with non-negative Ricci curvature, of complex dimension m, and let X be a complex projective manifold of complex dimension not greater than m. Let (L, h) be a positive holomorphic line bundle, and $D \in |L|$ a reduced divisor of simple normal crossing type. Recall that

$$T(r, \mathscr{R}) = \frac{\pi^m}{(m-1)!} \int\limits_{\Delta(r)} g_r(o, x) \mathscr{R} \wedge \alpha^{m-1},$$

where $\mathscr{R} = -dd^c \log \det(g_{i\bar{j}})$ is the Ricci form of g on M.

Let $Z = \sum_j \mu_j Z_j$ be a divisor, where $Z_j's$ are prime divisors. The *reduced form* of Z is defined by $\mathrm{Red}(Z) := \sum_j Z_j$.

Let $D = D_1 + \cdots + D_q$ be the irreducible decomposition of D. Equip every holomorphic line bundle $\mathscr{O}(D_j)$ with a Hermitian metric h_j. This induces the Hermitian metric $h = h_1 \otimes \cdots \otimes h_q$ on L. Pick $s_j \in H^0(X, \mathscr{O}(D_j))$ such that $(s_j) = D_j$ with $\|s_j\| < 1$. On X, we define a singular volume form

$$\Phi = \frac{\Omega}{\prod_{j=1}^q \|s_j\|^2}, \qquad \Omega = \bigwedge^n c_1(L, h).$$

Let $f : M \to X$ be a differentiably non-degenerate meromorphic mapping. Set

$$f^*\Phi \wedge \alpha^{m-n} = \xi \alpha^m.$$

It is clear that

$$\alpha^m = m! \det(g_{i\bar{j}}) \bigwedge_{j=1}^m \frac{\sqrt{-1}}{\pi} dz_j \wedge d\bar{z}_j.$$

Further, we have

$$dd^c[\log \xi] \geq f^* c_1(L, h_L) - f^* \mathrm{Ric}(\Omega) + \mathscr{R} - [\mathrm{Red}(f^* D)]$$

in the sense of currents. This yields

$$\frac{1}{4} \int\limits_{\Delta(r)} g_r(o, x) \Delta \log \xi \, dv \geq T_f(r, L) + T_f(r, K_X) + T(r, \mathscr{R}) - \overline{N}_f(r, D). \tag{6.1}$$

We now state the main theorem of this paper, i.e., the Second Main Theorem.

THEOREM 6.1 (Second Main Theorem). *Let M be a non-parabolic complete non-compact Kähler manifold with non-negative Ricci curvature. Let X be a complex projective manifold of complex dimension not greater than that of M. Let $D \in |L|$ be a reduced divisor of simple normal crossing type, where L is a positive line bundle over X. Let $f : M \to X$ be a differentiably*

non-degenerate meromorphic mapping. Then for any $\delta > 0$, there exists a subset $E_\delta \subseteq (0,\infty)$ of finite Lebesgue measure such that

$$T_f(r,L) + T_f(r,K_X) + T(r,\mathscr{R}) \le \overline{N}_f(r,D) + O\big(\log^+ T_f(r,L) + \log H(r,\delta)\big)$$

for all $r > 0$ outside E_δ, where $H(r,\delta)$ is given by (1.2).

Proof. Since D has the simple normal crossing type, there exist a finite open covering $\{U_\lambda\}$ of X and finitely many rational functions $w_{\lambda 1},\dots,w_{\lambda n}$ on X for each λ, such that $w_{\lambda 1},\dots,w_{\lambda n}$ are holomorphic on U_λ with

$$\begin{aligned} dw_{\lambda 1}\wedge\cdots\wedge dw_{\lambda n}(x) &\neq 0, \quad \forall x \in U_\lambda,\\ D\cap U_\lambda &= \{w_{\lambda 1}\cdots w_{\lambda h_\lambda} = 0\}, \quad \exists h_\lambda \le n. \end{aligned}$$

Moreover, we can require that $\mathscr{O}(D_j)|_{U_\lambda} \cong U_\lambda \times \mathbb{C}$ for all λ, j. On U_λ, write

$$\Phi = \frac{e_\lambda}{|w_{\lambda 1}|^2\cdots|w_{\lambda h_\lambda}|^2}\bigwedge_{k=1}^{n}\frac{\sqrt{-1}}{\pi}dw_{\lambda k}\wedge d\bar{w}_{\lambda k},$$

where e_λ is a positive smooth function on U_λ. Let $\{\phi_\lambda\}$ be a partition of unity subordinate to $\{U_\lambda\}$. Set

$$\Phi_\lambda = \frac{\phi_\lambda e_\lambda}{|w_{\lambda 1}|^2\cdots|w_{\lambda h_\lambda}|^2}\bigwedge_{k=1}^{n}\frac{\sqrt{-1}}{\pi}dw_{\lambda k}\wedge d\bar{w}_{\lambda k}.$$

Again, put $f_{\lambda k} = w_{\lambda k}\circ f$. On $f^{-1}(U_\lambda)$, we have

$$\begin{aligned} f^*\Phi_\lambda &= \frac{\phi_\lambda\circ f\cdot e_\lambda\circ f}{|f_{\lambda 1}|^2\cdots|f_{\lambda h_\lambda}|^2}\bigwedge_{k=1}^{n}\frac{\sqrt{-1}}{\pi}df_{\lambda k}\wedge d\bar{f}_{\lambda k}\\ &= \phi_\lambda\circ f\cdot e_\lambda\circ f\sum_{1\le i_1\neq\cdots\neq i_n\le m}\frac{\big|\frac{\partial f_{\lambda 1}}{\partial z_{i_1}}\big|^2}{|f_{\lambda 1}|^2}\cdots\frac{\big|\frac{\partial f_{\lambda h_\lambda}}{\partial z_{i_{h_\lambda}}}\big|^2}{|f_{\lambda h_\lambda}|^2}\left|\frac{\partial f_{\lambda(h_\lambda+1)}}{\partial z_{i_{h_\lambda+1}}}\right|^2\\ &\quad\cdots\left|\frac{\partial f_{\lambda n}}{\partial z_{i_n}}\right|^2\left(\frac{\sqrt{-1}}{\pi}\right)^n dz_{i_1}\wedge d\bar{z}_{i_1}\wedge\cdots\wedge dz_{i_n}\wedge d\bar{z}_{i_n}. \end{aligned}$$

Fix an $x_0 \in M$. We can take a local holomorphic coordinate $z = (z_1,\dots,z_m)$ around x_0 and a local holomorphic coordinate $\zeta = (\zeta_1,\dots,\zeta_n)$ around $f(x_0)$ such that

$$\begin{aligned} \alpha|_{x_0} &= \frac{\sqrt{-1}}{\pi}\sum_{j=1}^{m}dz_j\wedge d\bar{z}_j,\\ c_1(L,h)\big|_{f(x_0)} &= \frac{\sqrt{-1}}{\pi}\sum_{j=1}^{n}d\zeta_j\wedge d\bar{\zeta}_j. \end{aligned}$$

Set $f^*\Phi_\lambda \wedge \alpha^{m-n} = \xi_\lambda \alpha^m$. Then $\xi = \sum_\lambda \xi_\lambda$ and

$$\begin{aligned}\xi_\lambda\big|_{x_0} &= \phi_\lambda \circ f \cdot e_\lambda \circ f \sum_{1\le i_1\ne\cdots\ne i_n\le m} \frac{\left|\frac{\partial f_{\lambda 1}}{\partial z_{i_1}}\right|^2}{|f_{\lambda 1}|^2}\cdots\frac{\left|\frac{\partial f_{\lambda h_\lambda}}{\partial z_{i_{h_\lambda}}}\right|^2}{|f_{\lambda h_\lambda}|^2}\left|\frac{\partial f_{\lambda(h_\lambda+1)}}{\partial z_{i_{h_\lambda+1}}}\right|^2\cdots\left|\frac{\partial f_{\lambda n}}{\partial z_{i_n}}\right|^2\\ &\le \phi_\lambda \circ f \cdot e_\lambda \circ f \sum_{1\le i_1\ne\cdots\ne i_n\le m} \frac{\|\nabla f_{\lambda 1}\|^2}{|f_{\lambda 1}|^2}\cdots\frac{\|\nabla f_{\lambda h_\lambda}\|^2}{|f_{\lambda h_\lambda}|^2}\\ &\quad\cdot\|\nabla f_{\lambda(h_\lambda+1)}\|^2\cdots\|\nabla f_{\lambda n}\|^2.\end{aligned}$$

Define a non-negative function ϱ on M by

$$f^*c_1(L,h)\wedge\alpha^{m-1} = \varrho\alpha^m. \tag{6.2}$$

Moreover, put $f_j = \zeta_j \circ f$ for $1 \le j \le n$. Then

$$f^*c_1(L,h)\wedge\alpha^{m-1}|_{x_0} = \frac{(m-1)!}{2}\sum_{j=1}^m \|\nabla f_j\|^2\alpha^m,$$

which yields

$$\begin{aligned}\varrho|_{x_0} &= (m-1)!\sum_{i=1}^n\sum_{j=1}^m\left|\frac{\partial f_i}{\partial z_j}\right|^2\\ &= \frac{(m-1)!}{2}\sum_{j=1}^n\|\nabla f_j\|^2.\end{aligned}$$

Putting the above together, we are led to

$$\xi_\lambda \le \frac{\phi_\lambda\circ f\cdot e_\lambda\circ f\cdot(2\varrho)^{n-h_\lambda}}{(m-1)!^{n-h_\lambda}}\sum_{1\le i_1\ne\cdots\ne i_n\le m}\frac{\|\nabla f_{\lambda 1}\|^2}{|f_{\lambda 1}|^2}\cdots\frac{\|\nabla f_{\lambda h_\lambda}\|^2}{|f_{\lambda h_\lambda}|^2}$$

on $f^{-1}(U_\lambda)$. Since $\phi_\lambda \circ f \cdot e_\lambda \circ f$ is bounded on M and

$$\log^+\xi \le \sum_\lambda \log^+\xi_\lambda + O(1),$$

we obtain

$$\log^+\xi \le O\bigg(\log^+\varrho + \sum_{k,\lambda}\log^+\frac{\|\nabla f_{\lambda k}\|}{|f_{\lambda k}|}+1\bigg). \tag{6.3}$$

By Lemma 3.1,

$$\frac{1}{2}\int\limits_{\Delta(r)} g_r(o,x)\Delta\log\xi\,dv = \int\limits_{\partial\Delta(r)}\log\xi\,d\pi_r + O(1). \tag{6.4}$$

Combining (6.3) with (6.4) and using Theorem 5.5, we obtain

$$\begin{aligned}\frac{1}{2}\int\limits_{\Delta(r)} g_r(o,x)\Delta\log\xi\,dv &\le O\Big(\sum_{k,\lambda} m\Big(r,\frac{\|\nabla f_{\lambda k}\|}{|f_{\lambda k}|}\Big)+\log^+\int\limits_{\partial\Delta(r)}\varrho\,d\pi_r+1\Big)\\ &\le O\Big(\sum_{k,\lambda}\log^+ T(r,f_{\lambda k})+\log^+\int\limits_{\partial\Delta(r)}\varrho\,d\pi_r+1\Big)\\ &\le O\Big(\log^+ T_f(r,L)+\log^+\int\limits_{\partial\Delta(r)}\varrho\,d\pi_r+1\Big).\end{aligned}$$

By Theorem 5.2 and (6.2), for any $\delta>0$, there exists a subset $E_\delta\subseteq(0,\infty)$ of finite Lebesgue measure such that

$$\log^+\int\limits_{\partial\Delta(r)}\varrho\,d\pi_r\le(1+\delta)^2\log^+ T_f(r,L)+\log H(r,\delta)+O(1)$$

for all $r>0$ outside E_δ. Hence, we conclude that

$$\frac{1}{4}\int\limits_{\Delta(r)} g_r(o,x)\Delta\log\xi\,dv\le O\big(\log^+ T_f(r,L)+\log H(r,\delta)\big)$$

for all $r>0$ outside E_δ. From this with (6.1), the conclusion follows. ∎

Recall that the *simple defect* of f with respect to D is defined by

$$\bar\delta_f(D)=1-\limsup_{r\to\infty}\frac{\overline{N}_f(r,D)}{T_f(r,L)}.$$

By the First Main Theorem, we have

$$0\le\bar\delta_f(D)\le 1.$$

Making use of Theorem 6.1, we immediately get

Corollary 6.2 (Defect Relation). *Assume the same conditions as in Theorem* 6.1. *Then*

$$\bar\delta_f(D)\le\left[\frac{c_1(K_X^*)}{c_1(L)}\right]-\liminf_{r\to\infty}\frac{T(r,\mathscr{R})}{T_f(r,L)}\le\left[\frac{c_1(K_X^*)}{c_1(L)}\right],$$

if one of the following conditions is satisfied:

(i) *M satisfies the volume growth condition*

$$\lim_{r\to\infty}\frac{\log\Big(\dfrac{V(r)}{r^2}\displaystyle\int_r^\infty\frac{t\,dt}{V(t)}\Big)}{\log r}=0;$$

(ii) *f is of non-polynomial growth, i.e.,*

$$\lim_{r\to\infty}\frac{\log r}{T_f(r,L)}=0.$$

Proof. We first show that the corollary holds if condition (i) is satisfied. By Corollary 2.2, we have $V(r) \le \omega_{2m} r^{2m}$. Thus, we deduce that

$$\begin{aligned}\frac{\log H(r,\delta)}{\log r} &= \frac{\log\left(\dfrac{V(r)}{r^2}\displaystyle\int_r^\infty \frac{t\,dt}{V(t)}\right)}{\log r} + \frac{\delta \log \frac{V(r)}{r}}{\log r} \\ &\le \frac{\log\left(\dfrac{V(r)}{r^2}\displaystyle\int_r^\infty \frac{t\,dt}{V(t)}\right)}{\log r} + (2m-1)\omega_{2m}\delta.\end{aligned}$$

Using (i), we get

$$\limsup_{r\to\infty} \frac{\log H(r,\delta)}{\log r} \le (2m-1)\omega_{2m}\delta.$$

Since each non-constant meromorphic mapping has growth $O(\log r)$ at least, we get the conclusion if (i) is satisfied due to Theorem 6.1. For $r > 1$, we obtain

$$\begin{aligned}H(r,\delta) &= \frac{1}{r}\left(\frac{V(r)}{r}\right)^{1+\delta}\int_r^\infty \frac{t\,dt}{V(t)} \le \frac{1}{r}\left(\frac{\omega_{2m} r^{2m}}{r}\right)^{1+\delta}\int_1^\infty \frac{t\,dt}{V(t)} \\ &\le c\omega_{2m}^{1+\delta} r^{(2m-1)(1+\delta)-1},\end{aligned}$$

where

$$c = \int_1^\infty \frac{t\,dt}{V(t)}.$$

Thus, (ii) yields

$$\lim_{r\to\infty} \frac{\log H(r,\delta)}{T_f(r,L)} = 0.$$

This completes the proof. ∎

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