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Meromorphic functions whose action on their Julia sets is non-ergodic

by

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Abstract. Nevanlinna functions are meromorphic functions with a finite number of asymptotic values and no critical values. Keen and Kotus (1995) proved that if the orbits of all the asymptotic values accumulate on a compact set on which the function acts as a repeller, then the function acts ergodically on its Julia set. Chen et al. (2024) proved the action of the function on its Julia set is still ergodic if some, but not all of the asymptotic values land on infinity, and the remaining ones land on a compact repeller. In this paper, we complete the characterization of ergodicity for Nevanlinna functions by proving that if all the asymptotic values land on infinity, then the Julia set is the whole sphere and the action of the map there is non-ergodic.

1. Introduction. It is a theorem of McMullen [Mc] and Lyubich [Lyu1] that if f is a rational map of degree greater than one, and if $P(f)$ is its post-singular set (see the formal definition in the next section), one of two things holds: either the Julia set $J(f)$ is equal to the whole Riemann sphere and the action of f is ergodic or, for almost every z in $J(f)$, the spherical distance $d(f^n(z), P(f)) \to 0$ as $n \to \infty$; that is, the ω-limit set $\omega(z)$ is a subset of $P(f)$ that varies with z.

In the same vein, Bock [Boc] proved a dichotomy theorem for transcendental functions: either the Julia set is the whole sphere and for any set A of positive Lebesgue measure, the orbits of almost all points in $\mathbb{C}$ have infinitely many iterates that land in A, or the Julia set is not the whole sphere, and almost every point in the Julia set is attracted to the post-singular set. Unlike the rational case, though, when the Julia set is the whole sphere, it is not known when such functions are ergodic. Misiurewicz [Mis] has shown that when $f(z) = e^z$, the Julia set is the whole sphere, but Lyubich [Lyu2], following earlier work [Dev, GGS], has shown that its action is *not* ergodic.

2020 *Mathematics Subject Classification*: Primary 37F10; Secondary 30F05, 30D05, 37A30.
Key words and phrases: asymptotic values, Nevanlinna function, ergodic.
Received 27 December 2024; revised 26 January 2026.
Published online 23 September 2026.

More examples of entire maps that *are* ergodic can be found in [CW, WZ]. Skorulski [S1, S2], answered the ergodicity question for meromorphic maps of the form $f(z) = \lambda \tan z$ and $f(z) = \frac{ae^{z^p}+be^{-z^p}}{ce^{z^p}+de^{z^{-p}}}$ that have two asymptotic values.

A more general sort of affirmative result can be found in [KK]: If the orbits of the singular values either land on or accumulate on a compact repelling set, and if the map f satisfies an additional growth rate condition, then its Julia set is the whole sphere and it is ergodic.

In this paper, we study the ergodicity question for meromorphic functions whose singular set consists of finitely many asymptotic values and no critical values, the so-called *Nevanlinna functions.* The dynamical properties of these functions were first investigated in [DK], where it was shown that they share many of the dynamical properties of rational maps. In [CJK], the authors studied the ergodicity question for subfamilies of Nevanlinna functions for which some of the asymptotic values have orbits that land on infinity. They proved:

THEOREM A. *Let f be a Nevanlinna function. If some, but not all, of the asymptotic values of f have orbits that land on infinity, and if the remaining asymptotic values have ω-limit sets that are compact repellers, then the Julia set of f is the Riemann sphere, and f is ergodic there.*

In this paper we complete the characterization of ergodicity for Nevanlinna functions by proving the following:

MAIN THEOREM 1. *Let f be a Nevanlinna function with N asymptotic values, $\lambda_1, \ldots, \lambda_N$. If all N asymptotic values land on infinity, then the Julia set of f is the Riemann sphere, and the action of f is* not *ergodic there.*

In addition, under the same hypotheses for the asymptotic values of f, we prove the following theorems.

MAIN THEOREM 2. *For almost every point $z \in \mathbb{C}$, its ω-limit set satisfies*

$$\omega(z) = P_f = \bigcup_{i=1}^{N} \{\lambda_i, f(\lambda_i), \ldots, f^{p_i-1}(\lambda_i), \infty\}.$$

MAIN THEOREM 3. *There is no f-invariant finite measure absolutely continuous with respect to Lebesgue measure.*

Theorem A, together with Main Theorems 1–3, give a full answer to ergodicity questions for Nevanlinna functions whose Julia set is the whole sphere and whose asymptotic values land either on infinity or on a compact repeller. The questions of whether the conclusions of all three theorems hold for Nevanlinna functions with asymptotic values whose orbits are infinite and whose accumulation sets are unbounded are still open. Another open question is whether there exists a unique σ-finite f-invariant measure absolutely continuous with respect to Lebesgue measure if all the asymptotic values of f have finite orbits.

The paper is organized as follows. The first part of Section 2 contains basic definitions and standard theorems. The remainder of that section is devoted to a detailed review of the properties of Nevanlinna functions. In particular, the asymptotic values divide a neighborhood of infinity into asymptotic tracts on which the function is exponentially contracting. In order to prove the function is non-ergodic, we need to produce disjoint invariant wandering sets in these tracts, each of which has positive Lebesgue measure. This involves estimating the contraction on the asymptotic tracts. In order to make these estimates, we use some classical techniques. We introduce an "auxiliary" variable in each tract which converts it to a half plane in which we can make the estimates. Section 3 contains the heart of the proof of Main Theorem 1. Distinct invariant wandering sets are constructed and their Lebesgue measure is shown to be positive which proves the function is not ergodic. The final section contains the proofs of Main Theorems 2 and 3.

2. Preliminaries

2.1. Definitions and standard theorems. A meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is a local homeomorphism everywhere except at the set S_f of singular values. In this paper, we will focus on functions whose singular set is finite so that the singular values are isolated. We will assume this throughout the paper. For these functions, the singular values fall into two categories. Let $v \in \widehat{\mathbb{C}}$ be a singular value and let V be a neighborhood of v. Then

- if, for some component U of $f^{-1}(V)$, there is a $u \in U$ such that $f'(u) = 0$, then u is a *critical point* and $v = f(u) \in V$ is the corresponding *critical value*, or
- if, for some component U of $f^{-1}(V)$, $f : U \to V\setminus\{v\}$ is a universal covering map then v is a *logarithmic asymptotic value*. The component U is called an *asymptotic tract* for v. Any path $\gamma(t) \in U$ such that $\lim_{t\to 1} \gamma(t) = \infty$, $\lim_{t\to 1} f(\gamma(t)) = v$ is called an *asymptotic path* for v.

Note that the definition of an asymptotic tract depends on the choice of the neighborhood V. If V_1, V_2 are punctured neighborhoods of v and U_1 and U_2 are unbounded components of their preimages such that $U = U_1 \cap U_2 \neq \emptyset$, we call them *equivalent asymptotic tracts*. For readability, we will not distinguish between an "asymptotic tract" and its equivalence class.

In the proofs of our results we will repeatedly use the Koebe distortion theorems. Many proofs exist in the standard literature on conformal mappings (see e.g. [Z, Theorem 6.16]). For the reader's convenience, we state the theorems as we use them without proof.

THEOREM 2.1 (Koebe Distortion Theorem). *Let $f : D(z_0, r) \to \mathbb{C}$ be a univalent function. Then for any $\eta < 1$,*

(1) $|f'(z_0)|\dfrac{\eta r}{(1+\eta)^2} \le |f(z) - f(z_0)| \le |f'(z_0)|\dfrac{\eta r}{(1-\eta)^2}$, $z \in D(z_0, \eta r)$,

(2) *if* $T(\eta) = \dfrac{(1+\eta)^4}{(1-\eta)^4}$, *then* $\dfrac{|f'(z)|}{|f'(w)|} \le T(\eta)$ *for any* $z, w \in D(z_0, \eta r)$,

(3) $\left|\arg \dfrac{f'(z)}{f'(z_0)}\right| \le 2\ln\left|\dfrac{1+\eta}{1-\eta}\right|$ *for any* $z \in D(z_0, \eta r)$.

THEOREM 2.2. *Let $f : D(z_0, r) \to \mathbb{C}$ be a univalent function, and $\eta < 1$. Then, for any $A, B \subset D(z_0, \eta r)$,*

$$\frac{(1-\eta)^4}{(1+\eta)^4}\frac{m(A)}{m(B)} \le \frac{m(f(A))}{m(f(B))} \le \frac{(1+\eta)^4}{(1-\eta)^4}\frac{m(A)}{m(B)}.$$

2.2. Nevanlinna functions. An important tool in studying meromorphic functions with finitely many critical points and finitely many asymptotic values is that they can be characterized by their Schwarzian derivatives.

DEFINITION 1. If $f(z)$ is a meromorphic function, its *Schwarzian derivative* is

$$S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

The Schwarzian differential operator satisfies the chain rule condition,

$$S(f \circ g) = S(f)g'^2 + S(g),$$

from which it is easy to deduce that if f is a Möbius transformation, then $S(f) = 0$, so that $f \circ g$ and g have the same Schwarzian derivative.

Nevanlinna [N, Chap. XI, §3] shows how, given a finite set of points in the plane and finite or infinite branching data for these points, this data defines, up to post-composition with a Möbius transformation, a meromorphic function whose topological covering properties are determined by this data. He proves the following:

THEOREM 2.3. *The Schwarzian derivative of a meromorphic function with finitely many critical points and finitely many asymptotic values is a rational function. If there are no critical points, it is a polynomial. Conversely, if a meromorphic function has a rational Schwarzian derivative, it has finitely many critical points and finitely many asymptotic values. If the Schwarzian derivative is a polynomial of degree m, then the meromorphic function has $m + 2$ asymptotic values and no critical points.*

In the literature, meromorphic functions with polynomial Schwarzian are often called *Nevanlinna functions* (see e.g. [C, EM]).

In this paper, we continue our study of the properties of the dynamical systems these functions generate. Given a Nevanlinna function, we can define the *orbit* $\{f^n z\}$ for any point $z \in \mathbb{C}$. In general, these orbits are infinite, but if, for some $m \geq 0$, $f^m(z) = \infty$, then the orbit is finite. Such points are called *prepoles of order* m.

The standard defintions of stable (Fatou) and unstable (Julia) behavior for rational maps carry over to points with infinite orbits; the prepoles are unstable. In [DK], it is proved that the classification of stable behavior for Nevanlinna maps is the same as that for rational maps. In particular, if all singular points are unstable, the Julia set is the whole sphere.

The results in [KK] and [CJK] together show the following:

THEOREM. *If f is a Nevanlinna function with N asymptotic values $\lambda_1, \ldots, \lambda_N$, and if for some $0 \leq k < N$, λ_i, $i = 1, \ldots, k$, are prepoles* ([1]), *and if the ω-limit sets of the remaining $N - k$ asymptotic values are compact repellers, then the Julia set is the Riemann sphere and f is ergodic there.*

In this paper we complete the answer to the question of ergodicity of the action of Nevanlinna functions whose Julia set is the whole sphere by proving the following result:

MAIN THEOREM 1. *If f is a Nevanlinna function with N asymptotic values $\lambda_1, \ldots, \lambda_N$, and if* all *of them are prepoles, then its Julia set is the Riemann sphere $\widehat{\mathbb{C}}$ and f is not ergodic there.*

Before getting into the details we sketch the main ideas in the proof (see also [Lyu1, S1]). We construct two disjoint sets A and B such that

(1) both A and B have positive Lebesgue measure,
(2) both sets are wandering and their orbits are disjoint.

We form the sets

$$X_A = \bigcup_{n \geq 0} f^{-n}\Big(\bigcup_{n \geq 1} f^n(A)\Big) \quad \text{and} \quad X_B = \bigcup_{n \geq 0} f^{-n}\Big(\bigcup_{n \geq 1} f^n(B)\Big).$$

They are both invariant and both have positive Lebesgue measure; this proves that the map is non-ergodic.

For the construction of A and B, we use the fact that the map f has N asymptotic tracts and N asymptotic values, each of which is a prepole. Therefore, each asymptotic tract is expansively mapped over all N asymptotic tracts under some finite number of iterations. We fix an asymptotic tract and choose some quadrilaterals that, under an appropriate change of coordinate are squares. We then show that the Lebesgue measure of the set of points whose orbit always lands on the fixed asymptotic tract is positive and that the set wanders to ∞; these are "almost" Markov chains because the map is expanding. If we choose the quadrilaterals so that the distance

([1]) If some $\lambda_i = \infty$, it is a "prepole of order 0".

between them is large enough, their orbits will be disjoint since the expansion rate on the quadrilaterals is different.

2.3. The behavior of f in a neighborhood of infinity. The proof of the main theorem depends on a careful study of the behavior of the Nevanlinna function f in a neighborhood of infinity. (See [CJK, DK, H, L].)

Let f be a Nevanlinna function with Schwarzian derivative $S(f) = 2P(z)$, where the degree of the polynomial $P(z)$ is $N-2$ and the leading coefficient of $P(z)$ is the constant $a \in \mathbb{C}^*$. The Schwarzian equation is related to the linear equation $w'' + P(z)w = 0$: that is, if w_1, w_2 are linearly independent solutions of this equation, then

$$S\left(\frac{aw_1 + bw_2}{cw_1 + dw_2}\right) = 2P(z).$$

In the special case $N = 2$, the solutions w_1, w_2 can be chosen as exponentials whose asymptotic tracts are half-planes.

This linear equation is often studied by making changes of variable in the asymptotic tracts in order to estimate the behavior of f in them. The new variable turns the asymptotic tract into a half-plane and the Liouville transformation turns the Schwarzian equation into an approximation of the equation in the case where $N = 2$. The interested reader can find more details in the discussions in [CJK] and [L]. We follow the notation from those discussions.

Denote solutions to the congruence

$$\arg a + N\theta \equiv 0 \bmod 2\pi$$

by θ_i, where $1 \le i \le N$. The *critical rays* of f are the half-lines $L_i = te^{i\theta_i}$, $t > 0$. For a small $\epsilon_0 > 0$, define the sectors

$$S_i = \{z : |\arg z - \theta_i| \le 2\pi/N - \epsilon_0\}$$

for $i = 1, \dots, N$. Thus the sector S_i contains the critical ray L_i and is contained between the critical rays L_{i-1} and L_{i+1}. (By convention, all indices are taken modulo N.) If we denote the boundaries of the sectors by $B_i^{\pm}$, depending on the sign of $\arg z - \theta_i$, the critical ray L_i is contained in a critical wedge wed_i of angle $2\epsilon_0$ whose boundaries are the rays B_{i-1}^{+} and B_{i+1}^{-}.

In each sector S_i, the function f has a "truncated solution" which grows like $\exp(z^{N/2})$. More precisely, let $R \gg 0$ and $A_R = \{z : |z| > R\}$. Then in S_i, one can define the auxiliary variable

$$\mathcal{Z}_i(z) = \int_{Re^{i\theta_i}}^{z} \sqrt{P(s)}\, ds = \frac{2\sqrt{a}}{N} z^{N/2}(1 + o(1)), \qquad z \in S_i \cap A_R.$$

We choose the branch of $\sqrt{P(s)}$ so that in the sector S_i, for any z on the critical ray L_i, $\mathcal{Z}_i(z)$ is on the positive real line. The image of S_i under $\mathcal{Z}_i$

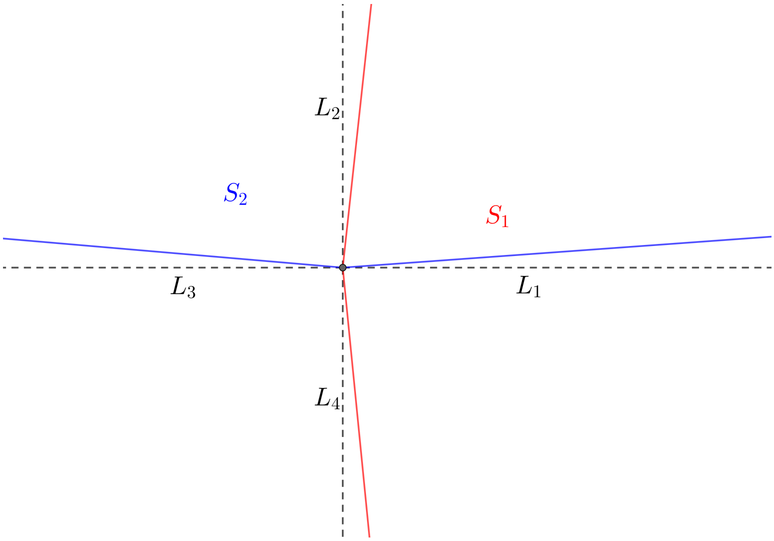


Fig. 1. The critical lines and sectors for $N = 4$

contains a sector in the $\mathcal{Z}_i$-plane,

$$\mathcal{S}_i = \{\mathcal{Z}_i = \mathcal{Z}_i(z) : |\arg \mathcal{Z}_i| < \pi - \epsilon\} \subset \mathcal{Z}_i(S_i)$$

for some small $\epsilon > 0$ depending on N and ϵ_0. In $\mathcal{S}_i$, we define

$$\mathcal{U}_i = \mathcal{S}_i \cap \{\mathcal{Z}_i : \Im \mathcal{Z}_i > c\} \quad \text{and} \quad \mathcal{L}_i = \mathcal{S}_i \cap \{\mathcal{Z}_i : \Im \mathcal{Z}_i < -c\}$$

for some $c \gg 0$.

For $z \in S_i$ and $\mathcal{Z}_i = \mathcal{Z}_i(z)$, we use the Liouville transformation to transform the linear equation to one in the variable $\mathcal{Z}_i$ that has a solution on $\mathcal{S}_i$ defined by $F(\mathcal{Z}_i(z)) = f(z)$. The map $F(\mathcal{Z}_i)$ can be approximately expressed as

$$f(z) = F(\mathcal{Z}_i) = \frac{A_i e^{i\mathcal{Z}_i} + B_i e^{-i\mathcal{Z}_i}}{C_i e^{i\mathcal{Z}_i} + D_i e^{-i\mathcal{Z}_i}}. \tag{2.1}$$

The sets $\mathcal{U}_i$ and $\mathcal{L}_i$ are respectively asymptotic tracts for the asymptotic values B_i/D_i and A_i/C_i of $F(\mathcal{Z}_i)$. Therefore, since

$$T_i = \mathcal{Z}_i^{-1}(\mathcal{U}_i) \quad \text{and} \quad T_{i-1} = \mathcal{Z}_i^{-1}(\mathcal{L}_i)$$

are mapped by f to punctured neighborhoods of the asymptotic values λ_i and λ_{i-1} of f, λ_i and λ_{i-1} are also the asymptotic values of F; that is ([2]),

$$B_i/D_i = \lambda_i \quad \text{and} \quad A_i/C_i = \lambda_{i-1}.$$

([2]) Since this holds for each pair of asymptotic values λ_i, λ_{i-1}, it is clear that both cannot be infinite.

By equation (2.1), on the asymptotic tracts $\mathcal{U}_i$ and $\mathcal{L}_i$, the map $F(\mathcal{Z}_i)$ can be expressed as a composition

$$F(\mathcal{Z}_i) = \begin{cases} M_{i,U} \circ E_U & \text{for } \mathcal{Z}_i \in U_i, \\ M_{i,L} \circ E_L & \text{for } \mathcal{Z}_i \in L_i, \end{cases}$$

where

$$E_U(\mathcal{Z}_i) = e^{2i\mathcal{Z}_i}, \quad E_L(\mathcal{Z}_i) = e^{-2i\mathcal{Z}_i},$$
$$M_{i,U}(\xi) = \frac{A_i\xi + B_i}{C_i\xi + D_i}, \quad \text{and} \quad M_{i,L}(\xi) = \frac{A_i + B_i\xi}{C_i + D_i\xi}.$$

Note that E_U and E_L are infinite-to-one universal covering maps of $\mathcal{U}_i$ and $\mathcal{L}_i$ onto the punctured disk $D^* = D^*(0, e^{-2c})$. Both the Möbius maps $M_{i,U}$ and $M_{i,L}$ map $D = D^* \cup \{0\}$ injectively onto neighborhoods N_i and N_{i-1} of the asymptotic values λ_i and λ_{i-1}. Thus we obtain the following factorizations of the truncated solutions in T_i and T_{i-1}:

$$f = M_{i,U} \circ E_U \circ \mathcal{Z}_i \quad \text{and} \quad f = M_{i,L} \circ E_L \circ \mathcal{Z}_i.$$

By hypothesis, f^{k_i-1} and $f^{k_{i-1}-1}$ map N_i and N_{i-1} to neighborhoods P_i and P_{i-1} of the poles p_i and p_{i-1} and so f maps both P_i and P_{i-1} to a neighborhood of infinity. Because it is simpler to estimate maps between finite regions, we introduce a transformation that maps a neighborhood of infinity into a neighborhood of the origin. To this end, let $I(z) = 1/z$ and set

$$f = I \circ g, \quad \text{where} \quad I = 1/z,\ g(z) = 1/f(z),$$

so that the map $g(z)$ maps both P_i and P_{i-1} into a neighborhood of the origin. The maps

$$h_{i,U}(\xi) = g \circ f^{k_i-1} \circ M_{i,U}(\xi) \quad \text{and} \quad h_{i,L}(\xi) = g \circ f^{k_{i-1}-1} \circ M_{i,L}(\xi)$$

both fix the origin and so map the disk $D(0, e^{-c})$ into a neighborhood of 0.

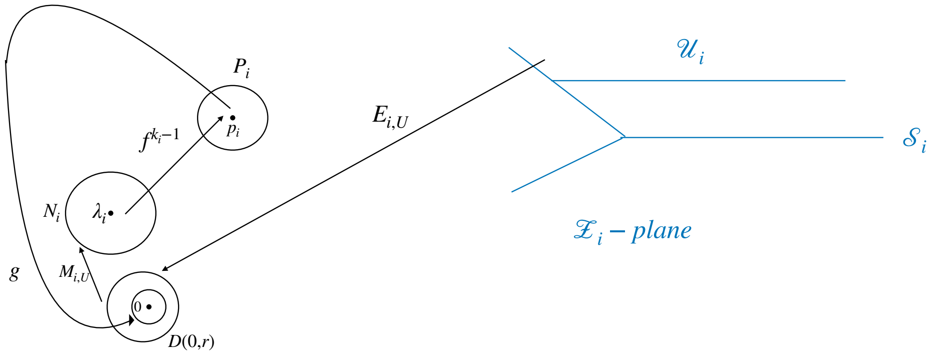
Next, define the following maps on T_i and T_{i-1} in the sector S_i respectively by

$$\varphi_{i,U}(z) = f^{k_i+1}(z) = I \circ h_{i,U} \circ E_U(\mathcal{Z}_i), \quad \varphi_{i,L}(z) = f^{k_{i-1}+1}(z) = I \circ h_{i,L} \circ E_L(\mathcal{Z}_i).$$

The maps $\varphi_{i,U}$ and $\varphi_{i,L}$ are compositions of a Möbius map, a univalent map and the exponential map, and they map the respective asymptotic tracts T_i and T_{i-1} onto a neighborhood Ω of ∞. See Figures 2 and 3. Next, for $z \in T_i \cup T_{i-1} \subset S_i$, we define the maps

$$\Phi_i(z) = \begin{cases} \varphi_{i,U}(z), & z \in T_i, \\ \varphi_{i,L}(z), & z \in T_{i-1}. \end{cases}$$

Note that $T_i \subset (S_i \cap S_{i+1}) \cup \text{wed}_i$. If $z \in S_i \cap S_{i+1}$, there are two choices of the auxiliary variables $\mathcal{Z}_i$, $\mathcal{Z}_{i+1}$ for z. By the choice of the branch of $\sqrt{P(z)}$,



$$h_{i,U} \circ E_{i,U} = g \circ f^{k_i-1} \circ M_{i,U} \circ E_{i,U}$$

Fig. 2. The decomposition of $h_{i,U} \circ E_{i,U}$ as a map from the auxiliary plane to the dynamic plane

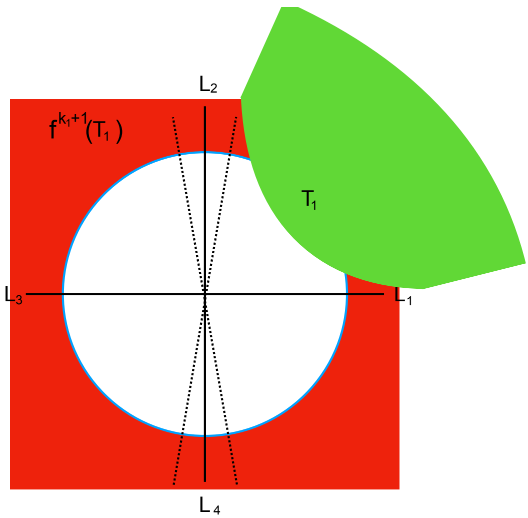


Fig. 3. The map of the asymptotic tract T_i (green) and its image under f^{k_i+1} (red)

$\mathcal{Z}_i(z) = -\mathcal{Z}_{i+1}(z)$ is in the upper half-plane. Therefore, if

$$\xi = E_{i,U}(\mathcal{Z}_i(z)) = E_{i+1,L}(\mathcal{Z}_{i+1}(z))$$

then $h_{i,U}(\xi) = h_{i+1,L}(\xi)$; that is, $\varphi_{i,U}(z) = \phi_{i+1,L}(z)$ so that the map Φ_i is well-defined.

Recall that the fixed constant $c \gg 0$ is the large constant that defines the sets $\mathcal{L}_i$ and $\mathcal{U}_i$. Assume it is chosen large enough so that for all $1 \leq i \leq N$,

(1) $f^j(\lambda_i) \notin \bigcup_{i=1}^{N} T_i$, $j = 0, 1 \ldots, k_i - 1$;
(2) $h_{i,U}(z), h_{i,L}(z)$ are univalent on the filled disk $D(0,r) = D^*(0,r) \cup \{0\}$, where $r = e^{-c}$.

Define the constants

$$m_i = \frac{1}{|h'_{i,U}(0)|}, \quad m'_i = \frac{1}{|h'_{i,L}(0)|}, \qquad 1 \le i \le N.$$

and $m = \min_i \{m_i, m'_i\}$, $M = \max_i \{m_i, m'_i\}$.

2.4. Basic calculations. Choose an $\alpha_0 > c$, and for integers $k > 0$ ([3]), where N is the number of asymptotic values of f, define the following sequences of real numbers:

$$\alpha_k = e^{N\alpha_{k-1}}.$$

LEMMA 2.1. *If $z \in T_i$ and $\mathcal{Z}_i = \mathcal{Z}_i(z) = x + iy$ with $y > \alpha_0$, then*

$$|\Phi_i(z)| \ge m_i(e^{2y} - 2e^{2c} + e^{4c-2y}), \tag{2.2}$$

$$|\Phi_i(z)| \le m_i(e^{2y} + 2e^{2c} + e^{4c-2y}). \tag{2.3}$$

Proof. Suppose that $z \in T_i$ satisfies $\mathcal{Z}_i = \mathcal{Z}_i(z) = x + iy$ with $y > c$, then

$$\xi = E_U(\mathcal{Z}_i) = e^{2i\mathcal{Z}_i} \in D^*(0, r),$$
$$|\xi| = e^{-2y} = \rho r \quad \text{and} \quad \rho = e^{2c-2y} \in (0, 1).$$

Since the map $h_{i,U}$ is univalent on $D(0, r)$ and $h_{i,U}(0) = 0$, Koebe's distortion theorem implies

$$\frac{|h'_{i,U}(0)|\rho r}{(1+\rho)^2} \le |h_{i,U}(\xi)| \le \frac{|h'_{i,U}(0)|\rho r}{(1-\rho)^2}.$$

Since $I(z) = 1/z$,

$$\frac{(1-\rho)^2}{\rho r} \frac{1}{|h'_{i,U}(0)|} \le |I(h_{i,U}(\xi))| \le \frac{(1+\rho)^2}{\rho r} \frac{1}{|h'_{i,U}(0)|}.$$

Note that

$$\frac{(1-\rho)^2}{\rho r} = \frac{1}{\rho r} - \frac{2}{r} + \frac{\rho}{r} = e^{2y} - 2e^{2c} + e^{4c-2y},$$
$$\frac{(1+\rho)^2}{\rho r} = e^{2y} + 2e^{2c} + e^{4c-2y}.$$

Inequalities (2.2) and (2.3) follow, and the proof of the lemma is complete. ∎

LEMMA 2.2. *Suppose $z_1 \in T_i$ and $z_2 \in T_j$ where $1 \le i, j \le N$ (not necessarily distinct), and that $\Phi_i(z_1) \in S_{i'}$ and $\Phi_j(z_2)$ are in $S_{j'}$. Set $\mathcal{Z}_i(z_1) = x_1 + iy_1$ and $\mathcal{Z}_j(z_2) = x_2 + iy_2$. Assume that N_0 is sufficiently large and that for $k > N_0$, we have $y_1 \ge \alpha_k$, $y_2 \ge y_1 + 2\alpha_{k-1}$. If*

$$|(\arg(\Phi_j(z_2)) - \theta_{j'})| \ge \frac{4}{N\alpha_k}, \text{ or equivalently } |\sin(\mathcal{Z}_{j'}(\Phi_j(z_2)))| \ge \frac{2}{\alpha_k}, \tag{2.4}$$

then $\Im(\mathcal{Z}_{j'}(\Phi_j(z_2))) \ge \Im(\mathcal{Z}_{i'}(\Phi_i(z_1))) + 2\alpha_{k+1}$.

([3]) We use k here as an index and k_i as the order of the prepole λ_i. This should not cause confusion.

Proof. By hypothesis $z_1 \in T_i$ and $z_2 \in T_j$ with $y_2 > y_1 > \alpha_k > \alpha_0$. Inequality (2.3) implies that

$$\Im(\mathcal{Z}_{i'}(\Phi_i(z_1)) \leq |\mathcal{Z}_{i'}(\Phi_i(z_1))| \leq \frac{2|a|^{N/2}}{N}(m_i(e^{2y_1}+2e^{2c}+e^{4c-2y_1}))^{N/2}. \tag{2.5}$$

Moreover, by (2.2) and (2.4), when k is large enough,

$$\begin{aligned} \Im(\mathcal{Z}_{j'}(\Phi_j(z_2))) &\geq |(\Phi_j(z_2)\sin(\arg(\mathcal{Z}_j(\Phi_j(z_2)))| \\ &\geq \frac{2|a|^{N/2}}{N}(m_j(e^{2y_2}-2e^{2c}+e^{4c-2y_2}))^{N/2}\frac{2}{\alpha_k}. \end{aligned} \tag{2.6}$$

Note that $y_2 > y_1 + 2\alpha_{k-1}$, and $\alpha_k = e^{N\alpha_{k-1}}$, so that

$$\begin{aligned} &(m_j(e^{2y_2}-2e^{2c}+e^{4c-2y_2}))^{N/2}\frac{2}{\alpha_k} \\ &\quad > 2\bigg(m_j\frac{e^{2y_1+4\alpha_{k-1}}-2e^{2c}+e^{4c-2y_2}}{e^{2\alpha_{k-1}}}\bigg)^{N/2} \\ &\quad = 2\bigg(m_j\bigg(e^{2y_1+2\alpha_{k-1}}-\frac{2e^{2c}-e^{4c-2y_2}}{e^{2\alpha_{k-1}}}\bigg)\bigg)^{N/2} \\ &\quad > (m_i(e^{2y_1}+2e^{2c}+e^{4c-2y_1}))^{N/2}+N\alpha_{k+1}/|a|^{N/2} \end{aligned} \tag{2.7}$$

for k large enough, since $\alpha_k \to \infty$ and $y_1 \geq \alpha_k$. Therefore

$$\Im(\mathcal{Z}_{j'}((\Phi_j(z_2)))) \geq \Im(\mathcal{Z}_{i'}(\Phi_i(z_1))) + 2\alpha_{k+1}. \ \blacksquare$$

Define a set of horizontal strips in $\mathcal{U}_i$, indexed by $k > 0$, to be

$$\mathrm{Hor}_k^i = \{\mathcal{Z}_i = x+iy \in \mathcal{S}_i : \alpha_k + 2\alpha_{k-1} \leq y \leq \alpha_{k+1} - 2\alpha_k)\} \cap \mathcal{U}_i.$$

The pull-backs of these strips,

$$H_k^i = \mathcal{Z}_i^{-1}(\mathrm{Hor}_k^i) \subset T_i,$$

are strips in T_i.

The image $\Phi_i(H_k^i)$ is an annular region A_i in a neighborhood of infinity of the z-plane that, for all $j = 1,\ldots,N$, intersects the sectors S_j, and in particular, the asymptotic tracts T_j, the critical lines L_j and the wedges wed_j. See Figure 3. Thus, the maps $\mathcal{Z}_j$'s are well-defined for all $1 \leq j \leq N$. Define the map $\Psi_{i,j}$ on the $\mathcal{Z}_i$-plane by the functional equation

$$\mathcal{Z}_j \circ \Phi_i = \Psi_{i,j} \circ \mathcal{Z}_i.$$

The maps $\Psi_{i,j}$ are infinite-to-one. To create regions of injectivity, divide each horizontal strip Hor_k^i into infinitely many rectangles $\mathrm{Rect}_{k,n}^i$ of width π and define N disjoint subrectangles $\mathrm{Rect}_{j,k,n}^i$ in each rectangle $\mathrm{Rect}_{k,n}^i$ by

$$\mathrm{Rect}_{j,k,n}^i = \{\mathcal{Z}_i = x+iy : x_{j+1}^i + n\pi + 3/(N\alpha_k) \leq x \leq x_j^i + n\pi - 3/(N\alpha_k)\},$$

where x_j^i satisfies $2x_j^i + \arg(h'_{i,U}(0)) = -\theta_j$. Recall that θ_j is the jth solution of $\arg a + N\theta \equiv 0 \bmod 2\pi$.

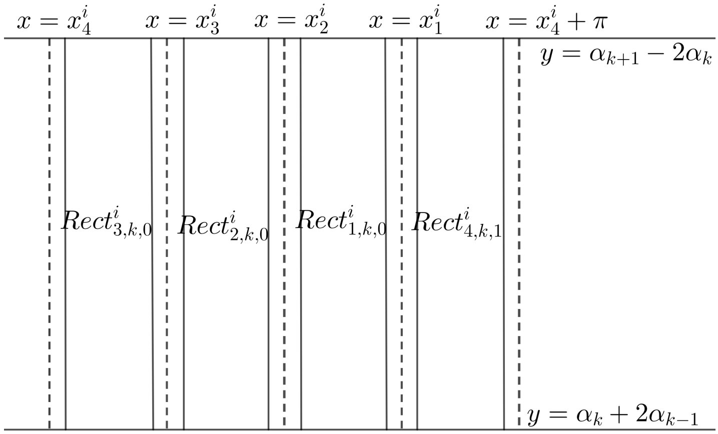


Fig. 4. Hor_k^i for $N = 4$

REMARK 1. Under the map $\mathcal{Z}_j^{-1} \circ \Psi_{i,j}$, each rectangle $\mathrm{Rect}_{k,n}^i$ maps injectively onto the annulus A_i in the z-plane. The vertical lines $x = x_j^i + n\pi$ in Hor_k^i are preimages of the critical rays L_j. The rectangles $\mathrm{Rect}_{j,k,n}^i$ in each $\mathrm{Rect}_{k,n}^i$ injectively map onto the disjoint asymptotic tracts in the annulus.

The next lemma states that in the $\mathcal{Z}_i$-plane, for any point $x + iy$ in these sub-rectangles $\mathrm{Rect}_{j,k,n}^i$, the inequality (2.4) in Lemma 2.2 is satisfied. In particular, the sub-rectangles are mapped away from the critical rays and into an asymptotic tract.

LEMMA 2.3. *There exists an integer N_0 such that if $k > N_0$, and if $\mathcal{Z}_i(z) \in \mathrm{Rect}_{j,k,n}^i$ for some i, then for each n and for all $j = 1, \ldots, N$,*

$$|\arg(\Psi_{i,j}(z))| \geq 2/\alpha_k.$$

Proof. For the proof, fix i and n and, for readability, omit writing them. Set

$$\begin{aligned} M_j &= \{\mathcal{Z} = x_j + iy : y \geq \alpha_0\}, \\ L_{j,k} &= \left\{\mathcal{Z} = \left(x_{j+1} + \frac{3}{N\alpha_k}\right) + iy : y > \alpha_0\right\}, \\ R_{j,k} &= \left\{\mathcal{Z} = \left(x_j - \frac{3}{N\alpha_k}\right) + iy : y > \alpha_0\right\}. \end{aligned}$$

Note that the vertical lines $L_{j,k}$ and $R_{j,k}$ contain the respective left and right vertical borders of $\mathrm{Rect}_{j,k}$ and the line M_j lies in the complementary region between $R_{j,k}$ and $L_{j-1,k}$.

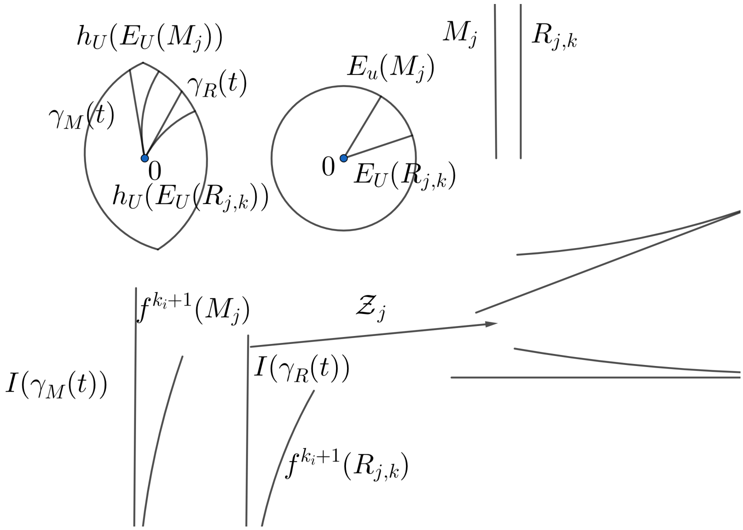


Fig. 5. Rays and curves in Lemma 2.3

We claim that

(2.8) if $\mathcal{Z} \in L_{j,k}$ then

$$\theta_{j+1} - \arg(\Phi_i(\mathcal{Z}_i^{-1}(\mathcal{Z}))) > \frac{4}{N\alpha_k} \quad \text{and} \quad \arg(\Psi_{i,j+1}(\mathcal{Z})) \leq -\frac{2}{\alpha_k};$$

if $\mathcal{Z} \in R_{j,k}$, then

$$\arg(\Phi_i(\mathcal{Z}_i^{-1}(\mathcal{Z}))) - \theta_j > \frac{4}{N\alpha_k} \quad \text{and} \quad \arg(\Psi_{i,j}(\mathcal{Z})) \geq \frac{2}{\alpha_k}.$$

This says that both $\Phi(\mathcal{Z}^{-1}(L_{j,k}))$ and $\Phi(\mathcal{Z}^{-1}(R_{j,k}))$ are bounded away from the images of the critical rays L_i and L_{i+1}; equivalently, the images of $L_{j,k}$ and $R_{j,k}$ under Ψ are bounded away from the real line, as are the points in $\mathrm{Rect}_{j,k}$, which proves the lemma.

To prove the claim, first consider the images of the half-lines $R_{j,k}$ and M_j under E_U:

$$E_U(R_{j,k}) = e^{-2y} e^{2i(x_j - \frac{3}{N\alpha_k})} \quad \text{and} \quad E_U(M_j) = e^{-2y} e^{2i(x_j)}, \qquad y \geq \alpha_0.$$

They are line segments meeting at 0 and the angle between them satisfies

$$\arg(E_U(R_{j,k})) - \arg(E_U(M_j)) = -\frac{6}{N\alpha_k}.$$

Next, since $h_U : D(0,\rho) \to \mathbb{C}$ is conformal, the images $h_U(E_U(M_j))$ and $h_U(E_U(R_{j,k}))$ are curves with the initial point 0. Let $\gamma_M(t)$ and $\gamma_R(t)$ be the lines tangent to these curves at 0, respectively. Then for $t \geq 0$,

$$\begin{aligned} \gamma_M(t) &= te^{i(\arg(h'_U(0))+2x_j)}, \\ \gamma_R(t) &= te^{i(\arg(h'_U(0))+2x_j-\frac{6}{N\alpha_k})}. \end{aligned} \tag{2.9}$$

That is, the angle between $\gamma_M(t)$ and $\gamma_R(t)$ is equal to $6/(N\alpha_k)$.

Suppose $\mathcal{Z}$ is on the line $R_{j,k}$, $\xi = E_{i,U}(\mathcal{Z})$, and $|\xi| = \rho r$ with $|\rho| < e^{2c-2\alpha_k} \ll 1$. Then, if k is large enough, by Koebe's distortion theorem,

$$\left|\arg(h_U(\xi)) - \left(\arg h'_U(0) + 2x_j - \frac{6}{N\alpha_k}\right)\right| < 2\ln\left(\frac{1+\rho}{1-\rho}\right) < \frac{1}{N\alpha_k}.$$

That is, $h_U(\xi) = r_0 e^{i\theta}$ for some $r_0 > 0$, where

$$\theta - (\arg h'_U(0) + 2x_j) \geq \frac{4}{N\alpha_k}. \tag{2.10}$$

If $w = r_0 e^{i(\arg h'_U(0)+2x_j)}$, then

$$I(w) = \frac{1}{r_0} e^{-i(\arg h'_U(0)+2x_j)} \quad \text{and} \quad \Phi_i(z) = I(h_U(\xi)) = e^{-i\theta}/r_0.$$

By the definition of x_j, $\arg(I(w)) = \theta_j$, and by (2.10), $\arg(\Phi_i(\mathcal{Z}_i^{-1}(\mathcal{Z}))) - \theta_j > 4/(N\alpha_k)$. Note that since $\mathcal{Z}$ is an arbitrary point in $R_{j,k}$, by the definitions of θ_j and $I(\gamma_M(t))$ it is mapped to the positive real line under $\mathcal{Z}_j$. Since $\Psi_{i,j} = \mathcal{Z}_j \circ \Phi_i$, we conclude

$$\arg(\Psi_{i,j}(\mathcal{Z})) \geq 2/\alpha_k.$$

One can check, using similar arguments, that at each point z such that $\mathcal{Z}(z)$ is on the line $L_{j,k}$,

$$\theta_{j+1} - \arg(\Phi_i(\mathcal{Z}_i^{-1}(\mathcal{Z}))) \geq 4/(N\alpha_k).$$

This completes the proof of the claim and hence the lemma. ∎

Lemma 2.4. *Suppose $k > N_0$. If $\mathcal{Z}_i \in \mathrm{Rect}^i_{j,k,n}$ for some n, then $\Psi_{i,j}(\mathcal{Z}_i) \in \mathrm{Hor}^j_{k+1}$.*

Proof. Suppose $k > 0$ and $\mathcal{Z}_i \in \mathrm{Rect}^i_{j,k,n}$. Then if $\mathcal{Z}_i = x + iy$,

$$\alpha_k + 2\alpha_{k-1} \leq y \leq \alpha_{k+1} - 2\alpha_k.$$

By (2.3),

$$\begin{aligned} \Im\Psi_{i,j}(\mathcal{Z}_i) < |\Psi_{i,j}(\mathcal{Z}_i)| &\leq \frac{2\sqrt{|a|}}{N}(m_i(e^{2\alpha_{k+1}-2\alpha_k} + 2e^{2c} + e^{4c+2y}))^{N/2} \\ &< \alpha_{k+2} - 2\alpha_{k+1}. \end{aligned}$$

Since $\mathcal{Z}_i \in \mathrm{Rect}^i_{j,k,n}$, by Lemma 2.3, $|\arg(\Phi_{i,j}(\mathcal{Z}))| > 2/\alpha_k$. Because $\mathcal{Z}_i \in \mathrm{Rect}^i_{j,k,n}$, we have $\Im\mathcal{Z}_j \geq \alpha_k + 2\alpha_{k-1}$. Thus Lemma 2.2 implies

$$\Im(\Psi_{i,j}(z)) \geq 2\alpha_{k+1} > \alpha_{k+1} + 2\alpha_k.$$

Therefore $\Psi_{i,j}(z)) \in \mathrm{Hor}^j_{k+1}$. ∎

3. Disjoint wandering sets of positive Lebesgue measure. To construct the wandering sets it is more convenient to work with the maps $\Psi_{i,j}$ on the auxiliary planes and then pull back. The first step is to estimate the expansion factor.

PROPOSITION 3.1. *Fix the point $\mathcal{Z}_i^*$ in the $\mathcal{Z}_i$ plane and assume that $\Psi_{i,j}(\mathcal{Z}_i^*) \in \mathrm{Hor}_k^j$. Then*

$$|(\Psi_{i,j})'(\mathcal{Z}_i^*)| \geq \frac{\alpha_k}{4\pi}.$$

Proof. Set $\mathcal{Z}_j^* = \Psi_{i,j}(\mathcal{Z}_i^*) \in \mathrm{Hor}_k^j$. Then $\alpha_k + 2\alpha_{k-1} \leq \Im(\mathcal{Z}_j^*) \leq \alpha_{k+1} - 2\alpha_k$.

Let $D_k = D(\mathcal{Z}_j^*, \alpha_k/2)$. By our choices of c and α_0, the orbits of all the λ_i's are outside the preimage $\mathcal{Z}_j^{-1}(D_k)$. It follows that there is a univalent branch G of $\Psi_{i,j}^{-1}$ defined on D_k. By Koebe's $\frac{1}{4}$-theorem,

$$D\left(\mathcal{Z}_i^*, \frac{|G'(\mathcal{Z}_j^*)|}{4}\,\frac{\alpha_k}{2}\right) \subset G(D_k).$$

Because $\Psi_{i,j}$ is univalent on $G(D_k)$, the radius of $G(D_k)$ is less than $\pi/2$; that is,

$$\frac{|G'(\mathcal{Z}_j^*)|}{4}\,\frac{\alpha_k}{2} \leq \frac{\pi}{2}.$$

This implies that

$$|\Psi_{i,j}'(\mathcal{Z}_i^*)| \geq \frac{\alpha_k}{4\pi}. \ \blacksquare$$

Define a family $\mathcal{G}^i = \{S_{m,n}^i : m, n \in \mathbb{Z}\}$, where $S_{m,n}^i$ is a square bounded by the straight lines in the $\mathcal{Z}_i$-plane given by

$$x = x_N^i + n\pi, \quad x = x_N^i + (n+1)\pi, \quad y = m\pi, \quad y = (m+1)\pi.$$

For each $k > 0$, let $\mathcal{G}_k^i \subset \mathcal{G}^i$ be the collection of $S_{m,n}^i \subset \mathrm{Hor}_k^i$. It will be convenient to use the index δ^i where $S_{\delta^i} = S_{m,n}^i$. Note that since x_N^i is fixed, each such square lies between the vertical lines

$$V_n = V(x_N^i + n\pi) = \{\mathcal{Z} = x_N^i + n\pi + iy,\ y > \alpha_0\},$$
$$V_{n+1} = V(x_N^i + (n+1)\pi) = \{\mathcal{Z} = x_N^i + (n+1)\pi + iy),\ y > \alpha_0\}.$$

Denote $\mathcal{G} = \bigcup_{i=1}^N \bigcup_{k>0} \mathcal{G}_k^i$.

For a square $S_{\delta^i} \in \mathcal{G}_k^i$, define the rectangles $\mathcal{R}_{\delta^i}^j = S_{\delta^i} \cap \mathrm{Rect}_{j,k,n}^i$. It follows from the definition of Hor_k^i that

$$\frac{m(S_{\delta^i} \setminus \bigcup_{j=1}^N \mathcal{R}_{\delta^i}^j)}{m(S_{\delta^i})} \leq \frac{6}{\alpha_k}.$$

By Lemmas 2.3 and 2.4, for each $j = 1, \ldots, N$, $\Psi_{i,j}$ maps the rectangle $\mathcal{R}_{\delta^i}^j$ onto a simply connected domain contained in the horizontal strip Hor_{k+1}^j.

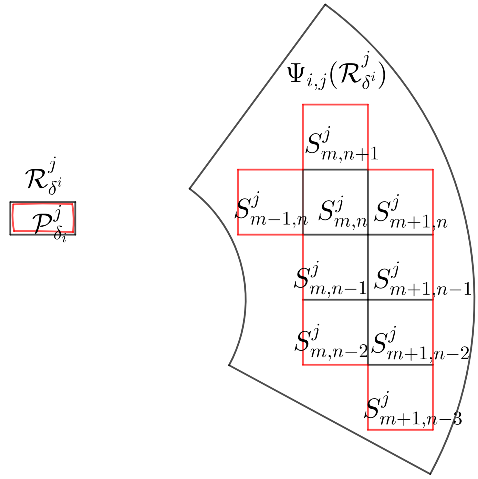


Fig. 6. The definition of $\mathcal{P}^j_{\delta^i}$

Let $\mathcal{U}^j(S_{\delta^i})$ denote the union of all the squares S_{δ^j} entirely contained in the interior of $\Psi_{i,j}(\mathcal{R}^j_{\delta^i})$:

$$\mathcal{U}^j(S_{\delta^i}) = \bigcup S_{\delta^j} \quad \text{where} \quad S_{\delta^j} \subset \text{Interior}(\Psi_{i,j}(\mathcal{R}^j_{\delta^i})) \subset \text{Hor}^j_{k+1}\,.$$

Note that because all the squares have side length π,

$$\text{dist}(\partial\Psi_{i,j}(\mathcal{R}^j_{\delta^i}), \partial\mathcal{U}^j(S_{\delta^i})) \leq \sqrt{2}\pi.$$

Otherwise, more squares could be added inside $\Psi_{i,j}(R^i_{\delta_j})$.

For each $j = 1, \ldots, N$, let $\mathcal{P}^j_{\delta^i} = \mathcal{P}^j_{\delta^i}(S_{\delta^i}) = \Psi^{-1}_{i,j}(\mathcal{U}^j(S_{\delta^j_0}))$. Since it is the pullback of a union of squares, the following inequality shows that each $\mathcal{P}^j_{\delta^i}$ is a domain whose area is very close to that of the rectangle $\mathcal{R}^j_{\delta^i}$; see Figure 6. Proposition 3.1 shows that the expansion factor for $\Psi_{i,j}$ on S_{δ^i} is at least $\alpha_k/(4\pi)$, which implies that

$$\text{dist}(\partial\mathcal{P}^j_{\delta^i}, \partial\mathcal{R}^j_{\delta^i}) < \frac{\text{dist}(\partial\Psi_{i,j}(\mathcal{R}^j_{\delta^i}), \partial\mathcal{U}^j(S_{\delta^i}))}{\frac{\alpha_k}{4\pi}} \leq \frac{4\pi^2\sqrt{2}}{\alpha_k},$$

so that the set $\mathcal{R}^j_{\delta^i} \setminus \mathcal{P}^j_{\delta^i}$ is contained in the $4\pi^2\sqrt{2}/\alpha_k$-neighborhood of $\partial\mathcal{R}^j_{\delta^i}$. Thus for the rectangles $\mathcal{R}^j_{\delta^i}$,

$$\frac{m(\mathcal{R}^j_{\delta^i} \setminus \mathcal{P}^j_{\delta^i})}{m(\mathcal{R}^j_{\delta^i})} \leq \frac{4\sqrt{2}\pi^2}{\alpha_k}$$

and for the full square S_{δ^i},

$$\frac{m(S_{\delta_0^i} \setminus \bigcup_{i=j}^N \mathcal{P}_{\delta^i}^j)}{m(S_{\delta_0^i})} \le \frac{4\sqrt{2}\pi^2 + 6}{\alpha_k}. \tag{3.1}$$

Thus each $\mathcal{P}_{\delta_j}^i$ is "almost" equal to the rectangle $\mathcal{R}_{\delta^i}^j$ and $\bigcup_{i=1}^N \mathcal{P}_{\delta^i}^j$ is "almost" equal to the square S_{δ^i}. The set $\mathcal{U}^j(S_{\delta_0^i}) = \Psi_{i,j}(\bigcup_{j=1}^N \mathcal{P}_{\delta_j}^i)$ is a union of actual squares, which by Lemma 2.2 lie in Hor_{k+1}^j.

The process of using the maps $\Psi_{i,j}$ and their inverses to push forward and pull back can be iterated any number of times starting from any square S_{δ^i} in some Hor_k^i. The Lebesgue measure of the resulting pullbacks to S_{δ^i} needs to be estimated. In order to do this, we need to introduce some more notation for iterated maps.

Given $l \in \mathbb{N}$, let $\boldsymbol{\iota} = (\delta^{i_0}, \delta^{i_1}, \ldots, \delta^{i_l})$, where $i_0, i_1, \ldots, i_l \in \{1, \ldots, N\}$. Let $\boldsymbol{\sigma} = \{j_0, \ldots, j_l\}$ where $j_0, \ldots, j_l \in \{1, \ldots, N\}$ and $j_0 = i_1, \ldots, j_{l-1} = i_l$. Denote the composition $\Psi_{i_1,i_2} \circ \Psi_{i_0,i_1}$ by Ψ_{i_0,i_1,i_2} and for each l, inductively set

$$\Psi^l = \Psi_{i_0,i_1,\ldots i_l}.$$

Define $\mathcal{P}_{\boldsymbol{\iota}}^{\boldsymbol{\sigma}}$ as the set consisting of all points in the square $S_{\delta^{i_0}} \subset \mathcal{Z}_{i_0}$ whose orbit under Ψ^l lies in the sequence of quadrilaterals $\mathcal{P}_{\delta_j}^{i_{j+1}}(S_{\delta_{j+1}^i})$ for $j = 0, \ldots, l-1$.

Denote the family of indices $(\boldsymbol{\iota}, \boldsymbol{\sigma})$ for which $\mathcal{P}_{\boldsymbol{\iota}}^{\boldsymbol{\sigma}}$ has a non-empty interior, and is contained in $\mathrm{Hor}_k^{i_0}$, by $\mathcal{I}_k^l$. For readability, identify the index $(\boldsymbol{\iota}, \boldsymbol{\sigma}) \in \mathcal{I}_k^l$ with the set of points in $\mathcal{P}_{\boldsymbol{\iota}}^{\boldsymbol{\sigma}}$.

Let N_0 be chosen as in Lemma 2.3 and assume $k > N_0$. Set $\mathcal{I}^l = \bigcup_{k \ge N_0} \mathcal{I}_k^l$. By definition, for each $\mathcal{P}_{\boldsymbol{\iota}}^{\boldsymbol{\sigma}} \in \mathcal{I}^l$, its image under Ψ^{l+1} is a union of squares in $\mathrm{Hor}_{k+l+1}^{j_l}$.

Set

$$Y^l = \bigcup_{(\boldsymbol{\iota},\boldsymbol{\sigma}) \in \mathcal{I}^l} \mathcal{P}_{\boldsymbol{\iota}}^{\boldsymbol{\sigma}}.$$

Note that, by definition, $Y^{l+1} \subset Y^l$. Let

$$Y^\infty = \bigcap_{l=0}^\infty Y^l. \tag{3.2}$$

Lemma 3.1. *There exists a constant $C > 0$ such that if $K \in \mathcal{I}_k^l$, then*

$$\frac{m(K \setminus Y^{l+1})}{m(K)} \le \frac{C}{\alpha_{|k|+l+1}}.$$

Proof. By definition, if K is the set defined by $\mathcal{P}_{\boldsymbol{\iota}}^{\boldsymbol{\sigma}} \in \mathcal{I}_k^l$, then $\Psi^{l+1}(K)$ is a union of squares $\bigcup_{\delta^j \in \Delta} S_{\delta^j}$ in $\mathrm{Hor}_{k+1+l}^{j_l}$ for some family Δ. Denote this union by $W = \Psi^{l+1}(K)$. Recall that each square S_{δ^j} contains the sets $\mathcal{P}^i_{\delta^j}$, $i = 1, \dots, N$, each of which is mapped to a square. Therefore,

$$\Psi^{l+1}(W \cap Y^{l+1}) = \bigcup_{j_l \in \Delta} \bigcup_{i=1}^{N} \mathcal{P}^i_{S_{\delta^{j_l}}}.$$

The union of the orbits of the asymptotic values λ_i is a finite set. Since the points $\mathcal{Z}^{-1}_{j_1}(\mathrm{Hor}^{j_l}_{k+1+l})$ lie in the asymptotic tract of λ_i, we can choose k so that they are far from its orbit. In fact, we can choose it so that there is a neighborhood of $\mathcal{Z}^{-1}_{j_1}(\mathrm{Hor}^{j_l}_{k+1+l})$ of radius 2π disjoint from the orbit of asymptotic values. It follows that the branches of $\Psi^{-(l+1)}$ are well-defined on this neighborhood.

Denote the branch that maps W back to K by Ξ and apply Koebe's distortion theorem. It says that for $\zeta, \xi \in W$, there exists a constant C_0 independent of Ξ such that

$$\frac{|\Xi'(\zeta)|}{|\Xi'(\xi)|} \le C_0.$$

Then

$$\frac{m(W \setminus Y^{l+1})}{m(W)} = \frac{m(\Xi(\cup(S_{\delta^{j_l}} \setminus \bigcup_{i=1}^N \mathcal{P}^i_{\delta_{j_l}})))}{m(\Xi(\cup S_{\delta^{j_l}}))}$$
$$\le C_0^2 \frac{\sum m(S_{\delta_{j_l}} \setminus \bigcup_{i=1}^N \mathcal{P}^i_{\delta^{j_l}})}{\sum m((S_{\delta^{j_1}})} \le \frac{C}{\alpha_{|k|+l+1}}$$

for some constant $C = (4\sqrt{2}\pi^2 + 6)C_0^2 > 0$. Note that the last inequality follows from (3.1). ∎

Lemma 3.2. *If $N_0 > 0$ is the integer defined in Lemma 2.3, then for any square $S_{\delta^i} \in \bigcup_{k \ge N_0} \mathcal{G}^i_k$, we have $m(S_{\delta^i} \cap Y^\infty) > 0$.*

Proof. Suppose that $S = S_{\delta^i} \in \mathrm{Hor}^i_k$ for some $k > N_0$. For each $l \ge 0$, set $S^l = S \cap Y^l$. According to the previous lemma,

$$m(S^l \setminus S^{l+1}) \le \frac{m(S^l)C}{\alpha_{|k|+l}}.$$

Equivalently,

$$m(S^{l+1}) \ge m(S^l)\left(1 - \frac{C}{\alpha_{|k|+l}}\right).$$

Since

$$\sum_{l=0}^{\infty} \frac{1}{\alpha_{|k|+l}} \le \sum_{l=0}^{\infty} \frac{1}{\alpha^l_{|k|}}$$

and the right side is convergent, it follows that

$$0 < C_1 = \prod_{l=0}^{\infty} \left(1 - \frac{C}{\alpha_{|k|+l}}\right) < \infty.$$

Therefore, for $S^\infty = S^0 \cap Y^\infty$,

$$m(S^\infty) \geq m(S) \prod_{l=0}^{\infty} \left(1 - \frac{C}{\alpha_{|k|+l}}\right) \geq C_1 m(S) > 0. \ \blacksquare$$

Now we are ready to complete the proof of Main Theorem 1.

Proof of Main Theorem 1. Let $k \geq N_0$, where N_0 is the large constant defined in Lemma 2.4. Let $\mathcal{I}^l$ be defined as above. From the definitions of $\mathrm{Hor}_k^{i_0}$, and by Lemma 3.2, we can find two squares $S_{\delta^{i_0}} \neq S'_{\delta^{i_0}} \in \mathrm{Hor}_k^{i_0}$ such that

- $m(S_{\delta^{i_0}} \cap Y^\infty) > 0$ and $m(S'_{\delta^{i_0}} \cap Y^\infty) > 0$;
- every pair of points $(\mathcal{Z}, \mathcal{Z}') \in (S_{\delta^{i_0}}, S'_{\delta^{i_0}})$ satisfies $|\Im \mathcal{Z}| - |\Im \mathcal{Z}'| \geq 2\alpha_{k-1}$.

Let $W_1 = S_{\delta^i} \cap Y^\infty$ and $W_2 = S'_{\delta^i} \cap Y^\infty$. By construction, for any $l \geq 0$,

(1) $\Psi^l(W_1) \subset \mathrm{Hor}_{k+l}^{i_1}$, $\Psi^l(W_2) \subset \mathrm{Hor}_{k+l}^{i_2}$ for some $i_1, i_2 \in \{1, \ldots, N\}$;
(2) for all pairs $(\mathcal{Z}_{i_1} \in \Psi^l(W_1), \mathcal{Z}_{i_2} \in \Psi^l(W_2))$, $|\Im \mathcal{Z}_{i_1}| - |\Im \mathcal{Z}_{i_2}| \geq 2\alpha_{k+l-1}$.

The first assertion shows that both W_1 and W_2 are wandering sets and the second shows that their forward orbits are disjoint. Note that $\mathcal{Z}_{i_0}$ is analytic, so both $\mathcal{Z}_{i_0}^{-1}(W_1), \mathcal{Z}_{i_0}^{-1}(W_2)$ have positive Lebesgue measure.

Let

$$E_1 = \bigcup_{m=0}^{\infty} f^{-m}\Big(\bigcup_{n=0}^{\infty} f^n(\mathcal{Z}_{i_0}^{-1}(W_1))\Big) \supset \mathcal{Z}_{i_0}^{-1}(W_1),$$
$$E_2 = \bigcup_{m=0}^{\infty} f^{-m}\Big(\bigcup_{n=0}^{\infty} f^n(\mathcal{Z}_{i_0}^{-1}(W_2))\Big) \supset \mathcal{Z}_{i_0}^{-1}(W_2).$$

Both have positive Lebesgue measure in the Julia set of f. They are mutually disjoint and completely invariant. This implies that f acts non-ergodically on its Julia set, completing the proof of the Main Theorem. $\blacksquare$

4. Main Theorems 2 and 3. Let $P_f = \bigcup_{i=1}^{N} \{\lambda_i, f(\lambda_i), \ldots, f^{p_i-1}(\lambda_i), \infty\}$ be the post-singular set. As we proved above, f is not ergodic. From the extended dichotomy of Bock [Boc], we know that for almost every point $z \in \mathbb{C}$, $\lim_{n\to\infty} d(f^n(z), P_f) \to 0$. This implies that the ω-limit set $\omega(z)$ is contained in P_f and in terms of the auxiliary variables, it says that for each such z, there exists a fixed $i \in \{1, \ldots, N\}$, a sequence $n_k \to \infty$, and an n_{k_0} such that for all $n_k \geq n_{k_0}$, $f^{n_k}(z)$ is in the asymptotic tract T_i and $\Im \mathcal{Z}_i(f^{n_k}(z)) \to \infty$.

The set Y^∞ was defined in (3.2) as a subset of $\bigcup_{i=1}^N \mathcal{Z}_i$ and we showed in the proof of Lemma 3.2 that it has positive Lebesgue measure. In order to prove Main Theorem 2, which is a result about points in $\mathbb{C}$, we need to pull Y^∞ back to $\mathbb{C}$. To do this, define $E^\infty = \bigcup_{i=1}^N \mathcal{Z}_i^{-1}(Y^\infty \cap \mathcal{Z}_i)$ and set $E = \bigcup_{n\in\mathbb{N}} f^{-n}(E^\infty)$.

Since $E^\infty \subset E$ and $m(Y) > 0$, pulling back gives $m(E) > 0$. In fact, more is true.

Lemma 4.1. *The set E in $\mathbb{C}$ has full Lebesgue measure, that is, $m(\mathbb{C}\setminus E) = 0$.*

Proof. Assume the contrary, $m(\mathbb{C} \setminus E) > 0$, and let $z \in \mathbb{C} \setminus E$ be a Lebsgue density point. As above, there is an $i \in \{1, \dots, N\}$ and a sequence n_k such that $\Im\mathcal{Z}_i(f^{n_k}(z)) \to \infty$ as $n_k \to \infty$. Let k_0 be the smallest integer such that $f^{k_0}(z)$ lies in the asymptotic tract T_i and $\Im\mathcal{Z}_i(f^{n_{k_0}}(z)) > \alpha_0$.

Let $l_k < \infty$ be a sequence such that

$$\alpha_{l_k} - 2\alpha_{l_k-1} \le \Im\mathcal{Z}_i(f^{n_k}(z)) \le \alpha_{l_k+1} - 2\alpha_{l_k}.$$

Note that these inequalities define a set of strips $\widetilde{\mathrm{Hor}}{}^i_{l_k}$ that is different from but intersects the strips $\mathrm{Hor}^i_{n_k}$.

Let $\mathcal{Z}^{n_k} = \mathcal{Z}_i(f^{n_k}(z))$, and let F_{n_k} be the branch of $f^{-n_k}\circ\mathcal{Z}_i^{-1}$ that maps $\mathcal{Z}^{n_k}$ to z. Let Q^{n_k} be the square centered at $\mathcal{Z}^{n_k}$ with side-length $8\alpha_{l_k-1}$ (4). Then

$$m\Big(Q^{n_k} \cap \bigcup_k \mathrm{Hor}^i_{l_k}\Big) \ge \tfrac{1}{2} m(Q^{n_k}).$$

Note that Q^{n_k} intersects $\mathrm{Hor}^i_{l_k}$ and/or $\mathrm{Hor}^i_{l_k-1}$. The minimum on the left of the above inequality occurs, for example, when $\mathcal{Z}^{n_k}$ falls on the mid-line of the gap between $\mathrm{Hor}^i_{l_k}$ and $\mathrm{Hor}^i_{l_k-1}$ because the height of the gap is $4\alpha_{l_k-1}$.

Let

$$\widetilde{Q}^{n_k} = \{z \in Q^{n_k} : z \in S \subset Q^{n_k} \cap (\mathrm{Hor}^i_{l_k} \cup \mathrm{Hor}^i_{l_k-1}) \text{ for squares } S \in \mathcal{G}^i\}.$$

By the above,

$$m(\widetilde{Q}^{n_k}) \ge \tfrac{1}{4} m(Q^{n_k}).$$

Since Lemma 3.2 implies that for any square $S \subset \bigcup_k^\infty \mathrm{Hor}^i_{l_k}$ there is a constant $C > 0$ such that $m(S \cap Y^\infty) \ge Cm(S)$, it follows that

$$m(\widetilde{Q}^{n_k} \cap Y) \ge m(\widetilde{Q}^{n_k} \cap Y^\infty) \ge \tfrac{C}{4} m(Q^{n_k}).$$

Let $\widetilde{D}_{n_k} = D(\mathcal{Z}^{n_k}, 8\sqrt{2}\alpha_{l_{k-1}}) \supset \widetilde{Q}^{n_k}$. Recall that $z = F_{n_k}(\mathcal{Z}^{n_k}) = f^{-n_k} \circ (\mathcal{Z}_i)^{-1}(\mathcal{Z}^{n_k})$ and set

$$A = |(\mathcal{Z}_i \circ f^{n_{k_0}})'(z)| = |(F^{-1}_{n_{k_0}})(\mathcal{Z}^{n_k})'| > 0.$$

(4) This is not one of the squares in $\mathcal{G}^i_{n_k}$.

Let $U_k = F_{n_k}(\widetilde{Q}^{n_k})$ and denote the respective inscribed and circumscribed circles for U_k by $D(z, r_k)$ and $D(z, R_k)$. Since F_{n_k} is univalent on $\widetilde{D}_{n_k}$, it has uniform distortion on $\widetilde{Q}^{n_k}$ and by Koebe's theorem there is a constant $B > 0$ such that

$$\frac{|F'_{n_k}(\xi)|}{|F'_{n_k}(\eta)|} \le B, \qquad \forall \xi, \eta \in D_{n_k}.$$

Pulling back to the z-plane, we get

$$R_k/r_k \le B.$$

If $\ell = n_k - n_{k_0}$, then $\Psi^\ell = \mathcal{Z}_i \circ f^\ell \circ \mathcal{Z}_i^{-1}$ and Proposition 3.1 implies

$$|(\Psi^\ell)'(z)| \ge \frac{\alpha_{l_k}}{4\pi}.$$

Since

$$F_{n_k} = (\Psi^\ell \circ \mathcal{Z}^i \circ f^{n_{k_0}})^{-1},$$

its derivative satisfies

$$|F'_{n_k}(z)| \le \frac{4\pi}{A\alpha_{l_k}}.$$

Because the diameter of $\widetilde{Q}^{n_k}$ is less than $16\sqrt{2}\alpha_{k-1}$, the diameter of $F_{n_k}(\widetilde{Q}^{n_k})$ tends to 0, and therefore $R_k \to 0$.

Furthermore,

$$\begin{aligned} \frac{m(E \cap D(z, R_k))}{m(D(z, R_k))} &\ge \frac{m(E \cap D(z, R_k))}{B^2 m(D(z, r_k))} \ge \frac{m(E \cap U_k))}{B^2 m(U_k)} \\ &\ge \frac{1}{B^2} \frac{m(Y \cap \widetilde{Q}^{n_k})}{B^2 m(\widetilde{Q}^{n_k})} \ge \frac{1}{B^4} \frac{m(Y \cap \widetilde{Q}^{n_k})}{m(Q^{n_k})} \ge \frac{C}{4B^4}. \end{aligned}$$

Since this inequality says that

$$\lim_{k\to\infty} \frac{m(E \cap D(z, R_k))}{m(D(z, R_k))} \ge \frac{C}{4B^4} > 0,$$

it implies that the density of z in E is positive, contradicting our assumption that z is a density point of the complementary set $\mathbb{C} \setminus E$. Therefore E has full Lebesgue measure. ∎

With this lemma, we can now prove

Main Theorem 2. *For almost every point $z \in \mathbb{C}$,*

$$\omega(z) = P_f = \bigcup_{i=1}^{N} \{\lambda_i, f(\lambda_i), \ldots, f^{p_i-1}(\lambda_i), \infty\}.$$

Proof. Since f is not ergodic, for almost every point $z \in \mathbb{C}$,

$$\lim_{n\to\infty} d(f^n(z), P_f) = 0.$$

This implies that $\omega(z) \subseteq P_f$ for almost every point $z \in \mathbb{C}$. To prove the theorem we need to show that $\{z \in \mathbb{C} : \omega(z) \subsetneq P_f\}$ has zero Lebesgue measure. By Lemma 4.1, we only need to show that

$$\{z \in E : \omega(z) \subsetneq P_f\}$$

has zero Lebesgue measure.

In each plane $\mathcal{Z}_i$, let $S = S_{\delta_i}$ be a square in $\mathcal{G}_k^i$ in the strip Hor_k^i. Given $l \in \mathbb{N}$ and $q \in \{1, \ldots, N\}$, let

$$K_l^q = \{\mathcal{P}^{\boldsymbol{\sigma}}(S) : \boldsymbol{\sigma} = \{\delta^{i_0}, \ldots, \delta^{i_l}\},\ i_0 \neq m, \ldots, i_l \neq m\}.$$

That is, K_l^m consists of all regions L such that *none* of its successive images under the composition map Ψ^l belong to the $\mathcal{Z}_m$ plane, and whose final image is a square S' in a horizontal strip Hor_{k+l}^j of some $\mathcal{Z}_j$-plane.

Recall that for any i, each square $S' = \mathrm{Rect}_{k+l,n}^i \subset \mathrm{Hor}_{k+l}^i$ is evenly divided into N rectangles $\mathrm{Rect}_{j,k+l,n}^i$. This implies that

$$m(S' \cap \mathrm{Rect}_{j,k+l,n}^i) \leq \frac{1}{N} m(S'),$$

which in turn implies that

$$m\Big(S' \cap \bigcup_{j \neq m} \mathrm{Rect}_{j,k+l,n}^i\Big) \leq \frac{N-1}{N} m(S').$$

Fix q' and $L_{q'} = \mathcal{P}_{\boldsymbol{\iota}}^{\boldsymbol{\sigma}}(S) \in K_{q'}^l$. Koebe's distortion theorem for the map Ψ^l implies that there is a distortion factor D such that for each q,

$$\frac{m(L_{q'} \cap \bigcup_{L \in K_{l+1}^q} L)}{m(L_{q'})} \leq D \frac{m(S' \cap \bigcup_{j \neq q} \mathrm{Rect}^j)}{m(S')} \leq D \frac{N-1}{N}.$$

Since Ψ^l is univalent for all l, its image is outside a large disk in some $\mathcal{Z}^i$-plane. Thus, the distortion factor D on each square S' of side π is close to 1. Thus we can assume that $ND/(N-1) < 1$. For example

$$D = \frac{N+a}{N-a} \quad \text{for some } 0 < a < \frac{N}{2N-1}.$$

Therefore, for each q,

$$\frac{m((\bigcap_{l=0}^{\infty} \bigcup_{\mathcal{P} \in K_l^q} \mathcal{P}) \cap S)}{m(S)} = 0,$$

and so

$$m\Big(\Big(\bigcap_{l=0}^{\infty} \bigcup_{\mathcal{P} \in K_l^q} \mathcal{P}\Big) \cap S\Big) = 0.$$

Finally, since S was arbitrary, set

$$\mathcal{W}_i = \bigcup_{q=1}^{N} \bigcup_{k} \bigcup_{S\in \mathrm{Hor}_k^i} \Big(\bigcap_{l=0}^{\infty} \bigcup_{\mathcal{P}\in K_l^q} \mathcal{P}\Big).$$

Thus $m(\mathcal{W}_i) = 0$, and if $W = \bigcup_{i=1}^{N} Z_i^{-1}(\mathcal{W}_i)$, then $m(W) = 0$. This implies $\bigcup_{n=1}^{\infty} f^{-n}(W)$ also has zero Lebesgue measure.

To complete the proof, note that

$$\{z \in E : \omega_f(z) \subsetneq P_f\} \subset \bigcup_{n=1}^{\infty} f^{-n}(W),$$

so that it has zero Lebesgue measure. ∎

Main Theorem 3. *There is no f-invariant finite measure absolutely continuous with respect to Lebesgue measure.*

Proof. Suppose there is an f-invariant absolutely continuous measure ρ. Let

$$\mathcal{W}^i(k) = Y^{\infty} \cap \{z \in \mathcal{Z}_i : |\Im z| > \alpha_k\} \quad \text{and} \quad W_0^i(k) = \mathcal{Z}_i^{-1}(\mathcal{W}^i(k)).$$

Then

$$f^{p_i+1}(W_0^i(k)) \subset \bigcup_{i=1}^{N} W_0^i(k+1) \subsetneq \bigcup_{i=1}^{N} W_0^i(k).$$

For each $i = 1, \ldots, N$ and $j = 1, \ldots, p_i$, set $W_j^i(k) = f^j(W_0^i(k))$. For each k, $W_j^i(k)$ is a bounded set containing $f^j(\lambda_i)$. Let β_k^j be the radius of the maximal disk in $W_j^i(k)$ centered at $f^j(\lambda_i)$. By Lemma 2.4, these disks form a nested sequence whose radii go to zero, so that for all j, $\beta_k^j \to 0$ as $k \to \infty$.

Let

$$\mathbf{W}(k) = \bigcup_{i=1}^{N} \bigcup_{j=0}^{p_i} W_j^i(k)$$

and let $p = \max\{p_1, \ldots, p_N\}$. Then, for all pairs $q, n \in \mathbb{N}$,

$$f^{(p+1)n+q}(\mathbf{W}(k)) \subset \mathbf{W}(k+n).$$

Since $E^{\infty} \subset \bigcup W_0^i(1) \subset \mathbf{W}(1)$, it follows from Lemma 4.1 that

$$m\Big(\mathbb{C} \setminus \bigcup_{n\in\mathbb{N}} f^{-n}(\mathbf{W}(1))\Big) = 0.$$

By the absolute continuity of ρ,

$$\rho\Big(\mathbb{C} \setminus \bigcup_{n\in\mathbb{N}} f^{-n}(\mathbf{W}(1))\Big) = 0.$$

Moreover, since ρ is also f-invariant, there is an $r>0$ such that $\rho(\mathbf{W}(1))=r$. Furthermore, since $f^{k(p+1)}(\mathbf{W}(1)) \subset \mathbf{W}(k)$ for each k, the invariance implies that $\rho(\mathbf{W}(k)) \geq \rho(\mathbf{W}(1)) = r$.

Claim. *There exists $r' > 0$ such that for each k,*

$$\rho\Big(\mathbf{W}(k) \setminus \bigcup_{i=1}^{N} W_0^i(k)\Big) \geq r'.$$

If not, for all $r' \leq r/2$, there exists a k such that

$$\rho\Big(\mathbf{W}(k) \setminus \bigcup_{i=1}^{N} W_0^i(k)\Big) < r'.$$

However, this implies

$$\begin{aligned}\rho\Big(\mathbf{W}(k) \setminus \bigcup_{i=1}^{N} W_0^i(k)\Big) &\geq \rho\Big(\bigcup_{i=1}^{N} W_1^i(k)\Big) = \rho\Big(f\Big(\bigcup_{i=1}^{N} W_0^i(k)\Big)\Big) \\ &\geq \rho\Big(\bigcup_{i=1}^{N} W_0^i(k)\Big) \geq r - r' \geq r',\end{aligned}$$

which is a contradiction. Thus the claim holds, and so for all k,

$$\rho\Big(\bigcup_{i=1}^{N}\bigcup_{j=1}^{p_i} W_i^j(k)\Big) \geq r'.$$

Furthermore, since

$$\bigcap_{k=1}^{\infty}\Big(\bigcup_{i=1}^{N}\bigcup_{j=1}^{p_i} W_i^j(k)\Big) = \bigcup_{i=1}^{N}\{\lambda_i, \dots, f^{p_i-1}(\lambda_i)\},$$

it follows that

$$\rho\Big(\bigcup_{i=1}^{N}\{\lambda_i, \dots, f^{p_i-1}(\lambda_i)\}\Big) \geq r',$$

which contradicts the absolute continuity of ρ. ∎

Acknowledgements. The authors would like to express their sincere gratitude to Weiyuan Qiu for raising the ergodicity question during a talk. The authors are also grateful to the anonymous referees for their careful reading and valuable suggestions, which have significantly improved the manuscript.

Funding. The first author was supported by the AMS–Simons Research Enhancement Grant (No. 01009877) and PSC-CUNY awards. The second author was supported by gifts from the Simons Foundation (Nos. 523341 and 942077) and PSC-CUNY awards. This material is also based upon work supported by the National Science Foundation under Grant No. 1440140, while the third author was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2022 semester.

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