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Signed trace measures on non-isotropic Riesz potentials in the unit sphere

by

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Abstract. We obtain a characterization of certain bilinear forms, or equivalently, of certain signed trace measures on non-isotropic Riesz potentials $H^s(\mathbb{S}^n)$ on the unit sphere in $\mathbb{C}^n$.

1. Introduction. It is well known that in the theory of functions of several complex variables and its relationship with harmonic analysis, a relevant model is the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, $n \geq 1$, and the corresponding study of the unit sphere $\mathbb{S}^n$, with the non-isotropic metric given by $d(\zeta, \eta) := |1-\zeta\overline{\eta}|^{1/2} = |1-\sum_{i=1}^n \zeta_i\overline{\eta_i}|^{1/2}$, $\zeta = (\zeta_1, \ldots, \zeta_n)$, $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{S}^n$. In this context, it is natural to consider spaces of regular functions of non-isotropic Sobolev type. We are interested in characterizing certain bilinear forms on these spaces, or equivalently, some trace measures on the same spaces. To be more precise, we begin with some definitions. For $0 < s < n$, the Banach space $H^s(\mathbb{S}^n)$ of non-isotropic Riesz potentials on $\mathbb{S}^n$ is the space of functions given by

$$K_s[\varphi](\zeta) := C_{n,s} \int_{\mathbb{S}^n} \frac{\varphi(\eta)}{|1-\zeta\overline{\eta}|^{n-s}}\, d\sigma(\eta), \qquad \varphi \in L^2(\mathbb{S}^n), \tag{1.1}$$

where $d\sigma$ is the normalized Lebesgue measure on $\mathbb{S}^n$. The norm in this space is given by $\|K_s[\varphi]\|_{H^s(\mathbb{S}^n)} := \|\varphi\|_{L^2(\mathbb{S}^n)}$, and $C_{n,s} = \frac{(\Gamma(\frac{n+s}{2}))^2}{(n-1)!\Gamma(s)}$ is a normalization constant, chosen so that $K_s[1] = 1$ (see for instance [9]). We observe that if s is an integer less than n, the space corresponds with a non-isotropic Sobolev space on $\mathbb{S}^n$ (see for instance [4, 12, 13]).

As we will make precise, the space $H^s(\mathbb{S}^n)$ coincides, for s small enough, with the completion of a space of regular functions ψ on $\mathbb{S}^n$ normed by

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$\|((-\Delta)^{s/2}+I)[\psi]\|_{L^2(\mathbb{S}^n)}$, where $(-\Delta)^{s/2}$ is the non-isotropic fractional Laplacian defined by

$$(-\Delta)^{s/2}[\psi](\zeta) := k_{n,s}\,\mathrm{PV}\int_{\mathbb{S}^n} \frac{\psi(\zeta)-\psi(\eta)}{|1-\zeta\overline{\eta}|^{n+s}}\,d\sigma(\eta),$$

where $k_{n,s} = \frac{4s\Gamma(n/2-s)^2}{(n-1)!\Gamma(1-2s)}$. This choice of the normalizing constant has been made, as we will see later, so that the operator $(-\Delta)^s + I$ is the inverse of the Riesz operator K_{2s}.

The main goal of this paper is the characterization of the signed trace measures for the space $H^s(\mathbb{S}^n)$, that is, the functions $\mathfrak{g}$, not necesarily non-negative, satisfying $|\int_{\mathbb{S}^n}|\varphi|^2\mathfrak{g}\,d\sigma| \le C\|\varphi\|^2_{H^s(\mathbb{S}^n)}$ for any $\varphi\in\mathcal{C}^\infty(\mathbb{S}^n)$. This problem has its origin, for $s=1$ and for $\mathbb{R}^n$, in the work of Maz'ya and Verbitsky [22]. We recall that the trace problem for non-negative functions has been thoroughly studied (see, for instance, [1] and [8]).

Our main result is the following:

Theorem 1.1. *Let $0<s<1/4$ and let $\mathfrak{g}$ be a function on $\mathbb{S}^n$, satisfying some regularity conditions that will be specified later. Then the following assertions are equivalent:*

(1) *There exists $C>0$ such that for any $\varphi\in\mathcal{C}^\infty(\mathbb{S}^n)$,*

$$\Big|\int_{\mathbb{S}^n}|\varphi|^2\mathfrak{g}\,d\sigma\Big| \le C\|\varphi\|^2_{H^s(\mathbb{S}^n)}.$$

(2) *There exists $C>0$ such that for any $\varphi,\psi\in\mathcal{C}^\infty(\mathbb{S}^n)$,*

$$\Big|\int_{\mathbb{S}^n}\varphi\psi\mathfrak{g}\,d\sigma\Big| \le C\|\varphi\|_{H^s(\mathbb{S}^n)}\|\psi\|_{H^s(\mathbb{S}^n)}.$$

(3) *The measure $d\nu := |K_s[\mathfrak{g}]|^2\,d\sigma$ is a trace measure for $H^s(\mathbb{S}^n)$, that is, there exists $C>0$ such that for any $\varphi\in\mathcal{C}^\infty(\mathbb{S}^n)$,*

$$\Big|\int_{\mathbb{S}^n}|\varphi|^2\,d\nu\Big| \le C\|\varphi\|^2_{H^s(\mathbb{S}^n)}.$$

For $n>1$, there are extra difficulties coming from the non-isotropic nature of $H^s(\mathbb{S}^n)$. The initial challenge is that the fractional Laplacian that defines $H^s(\mathbb{S}^n)$ is a non-local operator. One way to circumvent these difficulties is to consider an equivalent bilinear problem on a subspace of a weighted non-isotropic Sobolev space, stated through extensions of functions on $\mathbb{S}^n$ using a generalized non-isotropic Poisson-type operator. For the case of $\mathbb{R}^n$, the extension operator was introduced in [7]. Its extension to the upper half-plane $\mathbb{R}^{n+1}_+$ satisfies an elliptic differential equation. It is shown there that the fractional Laplacian can be expressed as the limit of certain weighted normal derivatives of the extended function. This reduces the calculation of the non-local fractional Laplacian to a limit of local operators.

We introduce a non-isotropic version of weighted non-isotropic Sobolev spaces $\mathcal{W}_s^2$ on $\mathbb{B}^n$ associated to the hermitian product of functions given by

$$(1.2)\qquad \langle v,u\rangle_{\mathcal{W}_s^2} := \int_{\mathbb{B}^n} \bigg\{ \frac{(1-|z|^2)^{1-2s}}{|z|^2} Rv\overline{Ru} + \frac{(1-|z|^2)^{-2s}}{|z|^2} \sum_{i<j} L_{i,j}v\overline{L_{i,j}u} + kv\overline{u}(1-|z|^2)^{-2s} \bigg\}\, dV(z),$$

where dV is the normalized Lebesgue measure on $\mathbb{B}^n$,

$$R = \sum_{i=1}^n z_i \frac{\partial}{\partial z_i}$$

is the complex radial derivative, $L_{ij} = \overline{z_i}\frac{\partial}{\partial z_j} - \overline{z_j}\frac{\partial}{\partial z_i}$, $1 \le i < j \le n$, are generators of the complex-tangential space, and $k > 0$ is to be chosen.

We will check that the functions in $H^s(\mathbb{S}^n)$ can be extended to functions in $\mathbb{B}^n$ with minimal norm in $\mathcal{W}_s^2$, and that these functions satisfy, for $\alpha = \beta = -\frac{n-2s}{2}$, the equation

$$(1.3)\qquad \varDelta_{\alpha,\beta} u = 0,$$

where

$$\varDelta_{\alpha,\beta} = \frac{1}{|z|^2}\bigg((1-|z|^2)R\overline{R} - \mathcal{L}_0 + \frac{n-1}{2}(R+\overline{R}) \bigg) + \alpha R + \beta \overline{R} - \alpha\beta I,$$

and

$$\mathcal{L}_0 = -\frac{1}{2}\sum_{i<j} (L_{i,j}\overline{L}_{i,j} + \overline{L}_{i,j}L_{i,j}).$$

It is proved in [3] that the Dirichlet problem associated to the operator $\varDelta_{\alpha,\beta}$, when $\operatorname{Re}(n+\alpha+\beta) > 0$, has a unique solution given by

$$u(z) = P_{\alpha,\beta}[\varphi](z) := \int_{\mathbb{S}^n} P_{\alpha,\beta}(z,\zeta)\varphi(\zeta)\, d\sigma(\zeta),$$

where

$$(1.4)\qquad P_{\alpha,\beta}(z,\zeta) = \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{(n-1)!\Gamma(n+\alpha+\beta)} \frac{(1-|z|^2)^{n+\alpha+\beta}}{(1-z\overline{\zeta})^{n+\alpha}(1-\overline{z}\zeta)^{n+\beta}}, \qquad z\in\mathbb{B}^n,\ \zeta\in\mathbb{S}^n.$$

When $\alpha = \beta = -\frac{n-2s}{2}$, we will simply write

$$P_s(z,\zeta) := P_{-\frac{n-2s}{2},-\frac{n-2s}{2}}(z,\zeta) = \frac{\Gamma(\frac{n+2s}{2})^2}{(n-1)!\Gamma(2s)} \frac{(1-|z|^2)^{2s}}{(1-z\overline{\zeta})^{n+2s}}.$$

We prove the following result:

Theorem 1.2. *Let $0 < s < 1/4$ and let φ be a regular function on $\mathbb{S}^n$ (in a sense that will be specified later).*

(1) *The following limit exists in $L^2(\mathbb{S}^n)$:*

$$\lim_{r\to 1^-}(1-r^2)^{1-2s}\frac{\partial}{\partial r}P_s[\varphi](r\zeta) = \frac{2\Gamma(1-2s)}{\Gamma(2s)}\frac{\Gamma\left(\frac{n+2s}{2}\right)^2}{\Gamma\left(\frac{n-2s}{2}\right)^2}K_{2s}^{-1}[\varphi](\zeta)$$
$$= \frac{2\Gamma(1-2s)}{\Gamma(2s)}\frac{\Gamma\left(\frac{n+2s}{2}\right)^2}{\Gamma\left(\frac{n-2s}{2}\right)^2}((-\Delta)^s + I)\varphi(\zeta).$$

(2) *There exists $C > 0$ such that for any $\varphi \in H^s(\mathbb{S}^n)$,*

$$\frac{1}{C}\|\varphi\|_{H^s(\mathbb{S}^n)} \le \|P_s[\varphi]\|_{\mathcal{W}_s^2} \le C\|\varphi\|_{H^s(\mathbb{S}^n)}.$$

The second essential component for our characterization is the use of some weighted estimates of a non-isotropic area function related to a kernel satisfying quite general conditions (see Theorem 4.1 below; this theorem could be of interest by itself). This general result will be applied to kernels that appear as composition of derivatives of the kernel that gives the solution of the Dirichlet problem (1.3), with a non-isotropic Riesz potential, that is, of the type

$$Q(z,s) = \frac{(1-|z|^2)^{-s}}{|z|}\int_{\mathbb{S}^n}\frac{\overrightarrow{\nabla}_{\mathbb{B}^n,z}P_s(z,\eta)}{|1-\zeta\overline{\eta}|^{n-s}}\,d\sigma(\eta).$$

Here $\overrightarrow{\nabla}_{\mathbb{B}^n,z}u$ is the complex vector-valued gradient of u with components $(1-|z|^2)Ru$, $(1-|z|^2)\overline{R}u$, $(1-|z|^2)^{1/2}L_{i,j}u$ and $(1-|z|^2)^{1/2}\overline{L}_{i,j}u$ for $i<j$.

A third component has been the use of the operators $K_s^{-1}K_s^{-1}K_{2s}$. In $\mathbb{R}^n$ these compositions of Riesz operators operators are just the identity operator. In the non-isotropic context, they are not the identity, but we will check that they are Calderón–Zygmund operators.

The paper is organized as follows: In Section 2, we introduce the non-isotropic Sobolev space $\mathcal{W}_s^2$ and obtain the stationary functions for their norm. In Section 3, we state the main properties of the extension non-isotropic operator P_s, and give the proof of Theorem 1.2. The four sections that follow are devoted to proving the technical results needed to obtain Theorem 1.1. More specifically, in Section 4, we obtain weighted L^2 estimates for a non-isotropic weighted area function. In Section 5, we introduce some non-isotropic vector-valued kernels, and check that their corresponding weighted area functions satisfy weighted L^2 estimates. In Section 6, we prove that the non-isotropic operator $K_s^{-1}K_s^{-1}K_{2s}$ is a Calderón–Zygmund operator. In Section 7, we recall the main properties of the non-isotropic Riesz capacities, non-negative trace measures for non-isotropic Riesz potential spaces, and Carleson measures for the extension spaces. And finally, in Section 8, we give the proof of the main Theorem 1.1.

Throughout the paper, the notation $\varphi(z) \lesssim \psi(z)$ means that there exists $C > 0$, which does not depend on z, such that $\varphi(z) \leq C\psi(z)$. We write $\varphi(z) \simeq \psi(z)$ if $\varphi(z) \lesssim \psi(z)$ and $\psi(z) \lesssim \varphi(z)$. We also recall that the letter C may denote various non-negative numerical constants, possibly different in different places.

2. The non-isotropic Sobolev space $\mathcal{W}_s^2$. If $0 < s < 1/2$, we consider the space $\mathcal{W}_s^2 := \mathcal{W}_s^2(\mathbb{B}^n)$ as the completion of functions in $\mathcal{C}_{\mathbb{C}}^\infty(\overline{\mathbb{B}^n})$ with respect to the norm

$$\|u\|_{\mathcal{W}_s^2}^2 := \langle u, u\rangle_{\mathcal{W}_s^2}, \tag{2.1}$$

where the hermitian product $\langle v, u\rangle_{\mathcal{W}_s^2}$, for $u, v \in \mathcal{C}^\infty(\overline{\mathbb{B}^n})$, is given by (1.2), with the normalizing constant given by $k = \left(\frac{n-2s}{2}\right)^2$. We observe that the choice of this constant is optional, but convenient for the forthcoming calculations.

Observe that the weights $(1-|z|^2)^{1-2s}$ and $(1-|z|^2)^{-2s}$ naturally appear when the radial derivative is considered as a derivative of weight 1 and the complex-tangential derivatives as derivatives of weight $1/2$ since $1 - 2s = 2(1 - s) - 1$ and $-2s = 2(1/2 - s) - 1$. The factor $|z|^2$ in the denominators appears as a "normalization" of the operators R and $L_{i,j}$ in a neighborhood of the origin.

2.1. An Euler–Lagrange equation. Our goal is to obtain a characterization of the stationary funcions for the functional $\|\cdot\|_{\mathcal{W}_s^2}^2$. For this purpose it will be convenient to express the corresponding Euler–Lagrange equation in terms of complex derivatives. We recall that if Ω is an open set in $\mathbb{R}^m$, the stationary functions for the functional

$$\int_\Omega \mathcal{M}(\alpha_1, \ldots, \alpha_m, u, u_1, \ldots, u_m)\, d\alpha_1 \cdots d\alpha_m,$$

where $u_i = \frac{\partial u}{\partial \alpha_i}$, $i = 1, \ldots, m$, satisfy the so-called *Euler–Lagrange equation*

$$\frac{\partial \mathcal{M}}{\partial u} - \sum_{i=1}^m \frac{\partial}{\partial \alpha_i}\frac{\partial \mathcal{M}}{\partial u_i} = 0.$$

In the complex language, if Ω is an open set in $\mathbb{C}^n$ and we consider the functional

$$\int_\Omega \mathcal{N}(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n, u, p_1, \ldots, p_n, q_1, \ldots, q_n)\, dV,$$

where $p_i = \frac{\partial u}{\partial z_i}$, $q_i = \frac{\partial u}{\partial \overline{z}_i}$, $i = 1, \ldots, n$, then using $x_i = \frac{z_i + \overline{z}_1}{2}$, $y_i = \frac{z_i - \overline{z}_1}{2i}$, $\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right)$ and $\frac{\partial}{\partial \overline{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right)$, we have the corresponding

Euler–Lagrange equation

$$\frac{\partial \mathcal{N}}{\partial u}-\sum_{i=1}^{n}\frac{\partial}{\partial z_i}\frac{\partial \mathcal{N}}{\partial p_i}-\sum_{i=1}^{n}\frac{\partial}{\partial \overline{z}_i}\frac{\partial \mathcal{N}}{\partial q_i}=0.$$

A simple calculation gives the following:

THEOREM 2.1. *Let $u\in\mathcal{C}^2_{\mathbb{R}}(\overline{\mathbb{B}^n})$. Then u is a stationary function for $\|\cdot\|^2_{\mathcal{W}^2_s}$ if and only if u is an (α,α)-harmonic function with $\alpha=-\frac{n-2s}{2}$, that is, u satisfies the PDE equation $\varDelta_{\alpha,\alpha}u=0$.* ∎

2.2. The (α,α)-harmonic functions in the language of forms. We begin by recalling some well-known notations and formulas.

PROPOSITION 2.2 ([23, Prop.16.4.4]). *Let $\omega(z)=dz_1\wedge\cdots\wedge dz_n$, $\omega_j(z)=(-1)^{j-1}dz_1\wedge\cdots\wedge\widehat{dz_j}\wedge\cdots\wedge dz_n$ and $\omega'(z)=\sum_{j=1}^n z_j\omega_j(z)$. Then $\omega(\overline{z})\wedge\omega(z)=c_n\,dV(z)$, where $c_n=(-1)^{\frac{n(n-1)}{2}}\frac{(2\pi i)^n}{n!}$ and*

$$\int_{\mathbb{S}^n} f(\eta)\omega'(\overline{\zeta})\wedge\omega(\zeta)=nc_n\int_{\mathbb{S}^n} f(\zeta)\,d\sigma(\zeta).\ ∎$$

We next obtain an equivalent formulation to the equation $\varDelta_{\alpha,\alpha}u=0$ using the language of forms. This expression will be convenient when applying Stokes' Theorem in formulas in which (α,α)-harmonic functions appear. If u is real, we introduce the forms

$$\begin{aligned}\varOmega&=\frac{(1-|z|^2)^{1-2s}}{|z|^2}\overline{R}u\,\omega'(z)\wedge\omega(\overline{z})\\&\quad+\sum_{i<j}\frac{(1-|z|^2)^{-2s}}{|z|^2}\overline{L}_{ij}u(\overline{z}_i\omega_j(z)-\overline{z}_j\omega_i(z))\wedge\omega(\overline{z}),\\ \widetilde{\varOmega}&=\frac{(1-|z|^2)^{1-2s}}{|z|^2}Ru\,\omega'(\overline{z})\wedge\omega(z)\\&\quad+\sum_{i<j}\frac{(1-|z|^2)^{-2s}}{|z|^2}L_{ij}u(z_i\omega_j(\overline{z})-z_j\omega_i(\overline{z}))\wedge\omega(z).\end{aligned}\tag{2.2}$$

A calculation gives

$$\begin{aligned}&d((-1)^n\varOmega+\widetilde{\varOmega})\\&=2\,\mathrm{Re}\biggl\{\sum_i\frac{\partial}{\partial z_i}\biggl(\frac{(1-|z|^2)^{1-2s}}{|z|^2}z_i\overline{R}u\biggr)\\&+\sum_{i<j}\biggl(\frac{\partial}{\partial z_j}\biggl(\frac{(1-|z|^2)^{-2s}}{|z|^2}\overline{z}_i\overline{L}_{ij}u\biggr)-\frac{\partial}{\partial z_i}\biggl(\frac{(1-|z|^2)^{-2s}}{|z|^2}\overline{z}_j\overline{L}_{ij}u\biggr)\biggr)\biggr\}\omega(\overline{z})\wedge\omega(z).\end{aligned}\tag{2.3}$$

Hence u is a stationary function for $\|\cdot\|_{\mathcal{W}^2_s}$ if and only if

$$d((-1)^n\Omega+\widetilde{\Omega}) = 2(1-|z|^2)^{-2s}\left(\frac{n-2s}{2}\right)^2 u\omega(\overline{z})\wedge\omega(z). \tag{2.4}$$

3. The extension operator P_s and the proof of Theorem 1.2. We recall that any function $\varphi\in L^2(\mathbb{S}^n)$ has an expansion $\varphi=\sum_{p,q}\varphi_{pq}$ in spherical harmonics, where φ_{pq} is in $H(p,q)$, the harmonic homogeneous space of total degree p in $z_1,\dots,z_n$ and total degree q in $\overline{z}_1,\dots,\overline{z}_n$ (see [23]).

Let $K_s(\cdot,\cdot)$ be the non-isotropic Riesz kernel defined by

$$K_s(\omega,\zeta) := \frac{C_{n,s}}{|1-\omega\overline{\zeta}|^{n-s}},\qquad \omega,\zeta\in\mathbb{S}^n,$$

and let K_s be the non-isotropic Riesz potential associated to the kernel, defined as in (1.1). It is then a calculation to obtain the decomposition of $K_s[\varphi]$ in spherical harmonics (see [9, (4.1)]):

$$K_s[\varphi] = \sum_{p,q}\left(\frac{\Gamma\left(\frac{n+s}{2}\right)}{\Gamma\left(\frac{n-s}{2}\right)}\right)^2\frac{\Gamma\left(p+\frac{n-s}{2}\right)\Gamma\left(q+\frac{n-s}{2}\right)}{\Gamma\left(p+\frac{n+s}{2}\right)\Gamma\left(q+\frac{n+s}{2}\right)}\varphi_{pq}. \tag{3.1}$$

We next recall some results we will use, that can be found in [3], concerning the solutions of the equation $\Delta_{\alpha,\beta}u=0$. As usual, $F(a,b;c;x)$ denotes the hypergeometric function (see [16]) given by

$$F(a,b;c;x) := \sum_{n=o}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(a)\Gamma(b)}\,\frac{\Gamma(c)}{\Gamma(c+k)}\,\frac{x^k}{k!}.$$

THEOREM 3.1. *Let $\alpha,\beta<0$.*

(1) *If u is a $\mathcal{C}^2$ function in $\mathbb{B}^n$ such that $\Delta_{\alpha,\beta}u=0$, then the spherical harmonic decomposition of $u(r\zeta)$ as a function of ζ is given by*

$$u(r\zeta)=\sum_{p,q}F_{p,q}(r^2)u_{pq}(r\zeta),\qquad z=r\zeta,\ 0<r<1,\ \zeta\in\mathbb{S}^n,$$

where $F_{p,q}(x) := F(p-\alpha,q-\beta;p+q+n;x)$, $u_{pq}\in H(p,q)$, and the series converges uniformly and absolutely on compact sets in $\mathbb{B}^n$.

(2) *If $n+\alpha>0$, $n+\beta>0$, $n+\alpha+\beta>0$ and φ is a continuous function on $\mathbb{S}^n$, then the Dirichlet problem $\Delta_{\alpha,\beta}u=0$, $u_{|\mathbb{S}^n}=\varphi$ has a unique solution given by $u=P_{\alpha,\beta}[\varphi]$, where $P_{\alpha,\beta}$ is defined in (1.4). Equivalently, if $\varphi=\sum_{p,q}\varphi_{pq}$ is the expansion of φ in spherical harmonics, then*

$$u(r\zeta)=\sum_{p,q}f_{p,q}(r^2)r^{p+q}\varphi_{pq}(\zeta),\qquad z=r\zeta,\ 0<r<1,\ \zeta\in\mathbb{S}^n,$$

where $f_{p,q}(x)=\frac{F_{p,q}(x)}{F_{p,q}(1)}$.

(3) *If* $\mathrm{Re}(n-\alpha-\beta)>0$, *then an* (α,β)*-harmonic function* u *is of the form* $u=P_{\alpha,\beta}[\varphi]$ *with* $\varphi\in L^2(\mathbb{S}^n)$ *if and only if*

$$\sup_{0<r<1}\int_{\mathbb{S}^n}|u(r\zeta)|^2\,d\sigma(\zeta)<+\infty.$$

In this case, u *has radial limit a.e. in* $\mathbb{S}^n$ *and its radial maximal function is in* $L^2(\mathbb{S}^n)$.

In particular, if $\alpha=\beta=-\frac{n-2s}{2}$, $0<s$ and $P_s:=P_{\alpha,\alpha}$, then the Dirichlet problem $\varDelta_{\alpha,\beta}u=0$, $u_{|\mathbb{S}^n}=\varphi$, has a unique solution $P_s[\varphi](r\zeta)=\sum_{p,q}f_{p,q}(r^2)r^{p+q}\varphi_{pq}(\zeta)$, where $f_{p,q}(x)$ is defined by $f_{p,q}(x)=\frac{F_{p,q}(x)}{F_{p,q}(1)}$ and $F_{p,q}(x)=F(p+\frac{n-2s}{2},q+\frac{n-2s}{2};p+q+n;x)$.

The following results give a representation of the non-isotropic fractional Laplacian, a non-local operator, in terms of a limit of a weighted radial derivative of an extension operator (see [7] for the isotropic case in $\mathbb{R}^n$).

Lemma 3.2. *Let* $0<s<1/4$. *Then for any* $\varphi_{pq}\in H(p,q)$,

$$\begin{aligned}\lim_{r\to1^-}&(1-r^2)^{1-2s}r\frac{\partial}{\partial r}P_s[\varphi_{pq}](r\zeta)\\&=\frac{2\varGamma(1-2s)}{\varGamma(2s)}\frac{\varGamma\big(p+\frac{n+2s}{2}\big)\varGamma\big(q+\frac{n+2s}{2}\big)}{\varGamma\big(p+\frac{n-2s}{2}\big)\varGamma\big(q+\frac{n-2s}{2}\big)}\varphi_{pq}(\zeta)\\&=\frac{2\varGamma(1-2s)}{\varGamma(2s)}\frac{\varGamma\big(\frac{n+2s}{2}\big)^2}{\varGamma\big(\frac{n-2s}{2}\big)^2}K_{2s}^{-1}[\varphi_{pq}](\zeta)\\&=\frac{2\varGamma(1-2s)}{\varGamma(2s)}\frac{\varGamma\big(\frac{n+2s}{2}\big)^2}{\varGamma\big(\frac{n-2s}{2}\big)^2}((-\varDelta)^s+I)[\varphi_{pq}](\zeta),\end{aligned}$$

where the convergence is uniform in $\zeta\in\mathbb{S}^n$.

Proof. Recall that $P_s[\varphi_{pq}](r\zeta)=P_{-\frac{n-2s}{2},-\frac{n-2s}{2}}[\varphi_{pq}]=\frac{F_{p,q}(r^2)}{F_{p,q}(1)}r^{pq}\varphi_{pq}(\zeta)$. Thus, both the first equality of the lemma and the uniform convergence in ζ will follow once we prove that

$$\text{(3.2)}\qquad \lim_{r\to1^-}\frac{(1-r^2)^{1-2s}}{F_{p,q}(1)}r\frac{d}{dr}(F_{p,q}(r^2)r^{p+q})=\frac{2\varGamma(1-2s)}{\varGamma(2s)}\frac{\varGamma\big(p+\frac{n+2s}{2}\big)\varGamma\big(q+\frac{n+2s}{2}\big)}{\varGamma\big(p+\frac{n-2s}{2}\big)\varGamma\big(q+\frac{n-2s}{2}\big)}.$$

We have

$$\frac{d}{dr}(F_{p,q}(r^2)r^{p+q})=2F'_{p,q}(r^2)r^{p+q+1}+(p+q)r^{p+q-1}F_{p,q}(r^2).$$

Using the facts that by [16, p. 58],

$$F'_{p,q}(x) = \frac{\big(p+\frac{n-2s}{2}\big)\big(q+\frac{n-2s}{2}\big)}{p+q+n} \times F\left(p+\frac{n-2s}{2}+1, q+\frac{n-2s}{2}+1; p+q+n+1; x\right),$$

and that by [16, p. 64], for $|x|<1$ and $\operatorname{Re}(c-a-b)>0$,

$$F(a,b;c;x) = (1-x)^{c-a-b}F(c-a,c-b;c;x),$$

we see that for $p+q>0$,

$$\begin{aligned}\frac{d}{dr}\frac{F_{p,q}(r^2)}{F_{p,q}(1)}r^{p+q} &= \frac{2\big(p+\frac{n-2s}{2}\big)\big(q+\frac{n-2s}{2}\big)(1-r^2)^{2s-1}}{F_{p,q}(1)(p+q+n)} \\ &\quad\times F\left(q+\frac{n+2s}{2}, p+\frac{n+2s}{2}; p+q+n+1; r^2\right)r^{p+q} \\ &\quad+\frac{(p+q)r^{p+q-1}}{F_{p,q}(1)}F_{p,q}(r^2) \\ &=: A_{p,q}(r^2)+B_{p,q}(r^2).\end{aligned}$$

It is proved in [16, p. 6], that if $\operatorname{Re} c > \operatorname{Re} b > 0$ and $\operatorname{Re}(c-a-b)>0$, then $F(a,b;c;r^2)$ is a non-decreasing function of r and its limit as $r\to 1^-$ is $F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$. In particular, since $p+q+n-\big(q+\frac{n-2s}{2}\big) - \big(p+\frac{n-2s}{2}\big) = 2s>0$, we have

$$\lim_{r\to 1^-} F_{p,q}(r^2) = F_{p,q}(1) = \frac{\Gamma(p+q+n)\Gamma(2s)}{\Gamma\big(q+\frac{n+2s}{2}\big)\Gamma\big(p+\frac{n+2s}{2}\big)},$$

and consequently $\lim_{r\to 1^-}(1-r^2)^{1-2s}rB_{p,q}(r^2)=0$. Next, since

$$p+q+n+1-\left(q+\frac{n+2s}{2}\right)-\left(p+\frac{n+2s}{2}\right) = 1-2s>0,$$

we have

$$\begin{aligned}&\lim_{r\to 1^-}(1-r^2)^{1-2s}rA_{p,q}(r^2) \\ &\quad= \frac{2\big(p+\frac{n-2s}{2}\big)\big(q+\frac{n-2s}{2}\big)}{(p+q+n)F_{p,q}(1)}\,\frac{\Gamma(p+q+n+1)\Gamma(1-2s)}{\Gamma\big(p+\frac{n-2s}{2}+1\big)\Gamma\big(q+\frac{n-2s}{2}+1\big)} \\ &\quad= \frac{2\Gamma(1-2s)}{\Gamma(2s)}\,\frac{\Gamma\big(p+\frac{n+2s}{2}\big)\Gamma\big(q+\frac{n+2s}{2}\big)}{\Gamma\big(p+\frac{n-2s}{2}\big)\Gamma\big(q+\frac{n-2s}{2}\big)}.\end{aligned} \tag{3.3}$$

Thus, we have proved (3.2), and hence the first equality of the lemma. The second equality follows directly from (3.1).

So we are left to prove the last equality. Observe that by (1.4),

$$P_s[\varphi_{pq}](r\zeta) = \frac{\Gamma\left(\frac{n+2s}{2}\right)}{(n-1)!\Gamma(2s)} \int_{\mathbb{S}^n} \frac{(1-r^2)^{2s}\varphi_{pq}(\eta)}{|1-r\zeta\overline{\eta}|^{n+2s}}\, d\sigma(\eta).$$

The first part of the lemma for $p = q = 0$ gives

$$\lim_{r\to 1^-} (1-r^2)^{1-2s} \frac{\partial}{\partial r} P_s[1](r\zeta) = \frac{2\Gamma(1-2s)}{\Gamma(2s)} \frac{\Gamma\left(\frac{n+2s}{2}\right)^2}{\Gamma\left(\frac{n-2s}{2}\right)^2}.$$

Then

$$\begin{aligned}
(3.4)\qquad & \lim_{r\to 1^-} (1-r^2)^{1-2s} \frac{\partial}{\partial r} P_s[\varphi_{pq}](r\zeta) \\
&= \lim_{r\to 1^-} (1-r^2)^{1-2s} \frac{\partial}{\partial r} \int_{\mathbb{S}^n} P_s(r\zeta,\eta)(\varphi_{pq}(\eta)-\varphi_{pq}(\zeta))\, d\sigma(\eta) \\
&\quad + \frac{2\Gamma(1-2s)}{\Gamma(2s)} \frac{\Gamma\left(\frac{n+2s}{2}\right)^2}{\Gamma\left(\frac{n-2s}{2}\right)^2} \varphi_{pq}(\zeta).
\end{aligned}$$

We have

$$\begin{aligned}
(3.5)\qquad & (1-r^2)^{1-2s} \frac{\partial}{\partial r} \frac{(1-r^2)^{2s}}{(1-r\zeta\overline{\eta})^{\frac{n}{2}+s}(1-r\eta\overline{\zeta})^{\frac{n}{2}+s}} \\
&= \frac{-4rs}{|1-r\zeta\overline{\eta}|^{n+2s}} - \left(\frac{n}{2}+s\right)(1-r^2)\frac{\zeta\overline{\eta}+\eta\overline{\zeta}-2r|\eta\overline{\zeta}|^2}{|1-r\zeta\overline{\eta}|^{n+2s+2}}.
\end{aligned}$$

Since φ_{pq} is a polynomial, it satisfies a non-isotropic $\frac{1}{2}$-Lipschitz condition, that is, $|\varphi_{pq}(\zeta)-\varphi_{pq}(\eta)| \lesssim |1-\zeta\overline{\eta}|^{1/2}$. We have

$$\left|(1-r^2)^{1-2s} \frac{\partial}{\partial r} P_s(r\zeta,\eta)(\varphi_{pq}(\zeta)-\varphi_{pq}(\eta))\right| \lesssim \frac{1}{|1-\zeta\overline{\eta}|^{n+2s-1/2}},$$

which gives an η-integrable function, since $s < 1/4$. Thus the Dominated Convergence Theorem gives

$$\begin{aligned}
& \lim_{r\to 1^-} (1-r^2)^{1-2s} \frac{\partial}{\partial r} \int_{\mathbb{S}^n} P_s(r\zeta,\eta)(\varphi_{pq}(\eta)-\varphi_{pq}(\zeta))\, d\sigma(\eta) \\
&\qquad = \frac{\Gamma\left(\frac{n}{2}+s\right)^2(-4s)}{(n-1)!\Gamma(2s)} \int_{\mathbb{S}^n} \frac{\varphi_{pq}(\eta)-\varphi_{pq}(\zeta)}{|1-\zeta\overline{\eta}|^{n+2s}}\, d\sigma(\eta) \\
&\qquad = \frac{2\Gamma(1-2s)}{\Gamma(2s)} \frac{\Gamma\left(\frac{n+2s}{2}\right)^2}{\Gamma\left(\frac{n-2s}{2}\right)^2} (-\Delta)^s[\varphi_{pq}](\zeta),
\end{aligned}$$

as required.

Observe that the constant in the definition of the non-isotropic fractional Laplacian has been chosen so that $K_{2s}^{-1}[\varphi_{pq}] = ((-\Delta)^s + I)[\varphi_{pq}]$. ∎

THEOREM 3.3. *Let $0 < s < 1/4$ and let $\varphi \in L^2(\mathbb{S}^n)$ with development $\varphi = \sum_{p,q} \varphi_{pq}$ in harmonic homogeneous polynomials satisfying $\sum_{p,q}(p+q)\varphi_{pq} \in L^2(\mathbb{S}^n)$. Then $\varphi \in K_{2s}[L^2]$ and*

$$\lim_{r\to 1^-} (1-r^2)^{1-2s} r\frac{\partial}{\partial r} P_s[\varphi](r\zeta) = \frac{2\Gamma(1-2s)}{\Gamma(2s)} \frac{\Gamma\big(\frac{n+2s}{2}\big)^2}{\Gamma\big(\frac{n-2s}{2}\big)^2} K_{2s}^{-1}[\varphi](\zeta),$$

with convergence in $L^2(\mathbb{S}^n)$.

Proof. We begin by observing that $K_{2s}^{-1}[\varphi] \in L^2[\mathbb{S}^n]$. Indeed, by Stirling's formula and (3.1), it is enough to show that $\sum_{p,q} p^{2s}q^{2s}\varphi_{pq} \in L^2(\mathbb{S}^n)$. This is immediate since $s < 1/4$, and hence $p^{2s}q^{2s} \leq p+q$, and by hypothesis $\sum_{p,q}(p+q)\varphi_{pq} \in L^2(\mathbb{S}^n)$.

We now prove that $r\frac{\partial}{\partial r}P_s[\varphi](r\zeta) = \sum_{p,q} r\frac{\partial}{\partial r}P_s[\varphi_{pq}](r\zeta)$ by showing that this last series converges uniformly on compacts in $\mathbb{B}^n$. Indeed, it is proved in [3] that any function v such that $\Delta_{\alpha,\beta}v = 0$ satisfies $\Delta_{\alpha,\beta-1}(Rv-\beta v) = 0$ and $\Delta_{\alpha-1,\beta}(\overline{R}v-\alpha v) = 0$. In particular, from $r\frac{\partial}{\partial r} = R+\overline{R}$, we deduce that

$$r\frac{\partial}{\partial r}v = (Rv-\beta v) + (\overline{R}v-\alpha v) + (\alpha+\beta)v.$$

Next, assertion (3) in Theorem 3.1 shows that $P_s[\varphi]$ is $\big(-\frac{n-2s}{2}, -\frac{n-2s}{2}\big)$-harmonic. This fact, together with the above formula and assertion (1), implies that the series $\sum_{p,q} r\frac{\partial}{\partial r}P_s[\varphi_{pq}](r\zeta)$ converges uniformly on compact subsets of $\mathbb{B}^n$.

Following the notations in Lemma 3.2, we have, for any p, q,

$$\begin{aligned}(1-r^2)^{1-2s} r\frac{\partial}{\partial r}&P_s[\varphi_{pq}](r\zeta)\\ &= (1-r^2)^{1-2s} rA_{p,q}(r^2)\varphi_{pq}(\zeta) + (1-r^2)^{1-2s} rB_{p,q}(r^2)\varphi_{pq}(\zeta).\end{aligned}$$

Let us show that

$$\lim_{r\to 1^-} \Bigg\| \sum_{p,q} (1-r^2)^{1-2s} A_{p,q}(r^2) r\varphi_{pq}(\cdot) - \frac{2\Gamma(1-2s)\Gamma\big(\frac{n+2s}{2}\big)^2}{\Gamma(2s)\Gamma\big(\frac{n-2s}{2}\big)^2} K_{2s}^{-1}[\varphi](\cdot)\Bigg\|_{L^2(\mathbb{S}^n)} = 0.$$

From the expression of K_{2s} we see that in L^2,

$$\frac{2\Gamma(1-2s)\Gamma\big(\frac{n+2s}{2}\big)^2}{\Gamma(2s)\Gamma\big(\frac{n-2s}{2}\big)^2} K_{2s}^{-1}[\varphi] = \sum_{p,q} c_{pq}\varphi_{pq},$$

where

$$c_{pq} := \frac{2\Gamma(1-2s)}{\Gamma(2s)} \frac{\Gamma\big(p+\frac{n+2s}{2}\big)\Gamma\big(q+\frac{n+2s}{2}\big)}{\Gamma\big(p+\frac{n-2s}{2}\big)\Gamma\big(p+\frac{n-2s}{2}\big)}.$$

We then have, for any $\varepsilon > 0$ and for a fixed N large enough,

$$\begin{aligned}
&\Big\| \sum_{p,q} (1-r^2)^{1-2s} r A_{p,q}(r^2)\varphi_{pq}(\cdot) - \sum_{p,q} c_{pq}\varphi_{pq}(\cdot)\Big\|_{L^2(\mathbb{S}^n)} \\
&\quad \le \Big\| \sum_{p+q\le N} ((1-r^2)^{1-2s} r A_{p,q}(r^2) - c_{pq})\varphi_{pq}(\cdot)\Big\|_{L^2(\mathbb{S}^n)} \\
&\qquad + \Big\| \sum_{p+q>N} (1-r^2)^{1-2s} r A_{p,q}(r^2)\varphi_{pq}(\cdot)\Big\|_{L^2(\mathbb{S}^n)} + \Big\| \sum_{p+q>N} c_{pq}\varphi_{pq}(\cdot)\Big\|_{L^2(\mathbb{S}^n)} \\
&\quad \le \Big\| \sum_{p+q\le N} ((1-r^2)^{1-2s} r A_{p,q}(r^2) - c_{pq})\varphi_{pq}(\cdot)\Big\|_{L^2(\mathbb{S}^n)} \\
&\qquad + 2\Big\| \sum_{p+q>N} c_{pq}\varphi_{pq}(\cdot)\Big\|_{L^2(\mathbb{S}^n)} < \varepsilon
\end{aligned}$$

provided $1-r$ is small enough. See (3.3) and apply the fact that

$$(1-r^2)^{1-2s} r A_{p,q}(r^2) \le c_{pq}.$$

To prove that $\lim_{r\to 1^+} \|\sum_{p,q}(1-r^2)^{1-2s} r B_{p,q}(r^2)\varphi_{pq}(\cdot)\|_{L^2(\mathbb{S}^n)} = 0$, we argue in a similar way, using the facts that $\sum_{p,q}(p+q)\varphi_{pq} \in L^2(\mathbb{S}^n)$ and that by definition $B_{p,q}(r^2) = \frac{(p+q)r^{p+q-1}}{F_{p,q}(1)} F_{p,q}(r^2)$. Altogether, this gives the desired convergence in L^2. ∎

REMARK 3.4. The hypothesis on φ in the above theorem is fulfilled if the function is regular enough. For instance, that is the case if φ satisfies $\overline{L_{ij}}L_{ij}\varphi \in L^2$, $1 \le i < j \le n$. Indeed, in [2, Lemma 3.9] it is proved that if $\varphi_{pq} \in H(p,q)$, then $\sum_{i<j} \overline{L_{ij}}L_{ij}\varphi_{pq} = -2p(q+n-1)\varphi_{pq}$. Thus, taking into account that the projection from L^2 to $H(p,q)$ is given in terms of a kernel in the two variables z, ζ, which is a harmonic polynomial in ζ (see [23]), together with the integration by parts formula $\int_{\mathbb{S}^n} f L_{ij}(g)\, d\sigma = -\int_{\mathbb{S}^n} L_{ij}(f)\, g\, d\sigma$, shows that the development in harmonic polynomials of the function $\sum_{i<j}\overline{L_{ij}}L_{ij}\varphi$ is $-\sum_{p,q} 2p(q+n-1)\varphi_{pq}$. This gives the convergence in L^2 of $\sum_{p,q}(p+q)\varphi_{pq}$.

In the following theorem and its corollary we give sufficient conditions on the regularity of φ which lead us to express the non-isotropic fractional Laplacian as $\lim_{r\to 1^-}(1-r^2)^{1-2s} r\frac{\partial}{\partial r} P_s[\varphi](r\zeta)$, and consequently its relationship to the inverse operator of the non-isotropic Riesz kernel K_{2s}.

THEOREM 3.5. *Let $0 < s < 1/4$, $\varepsilon > 0$ and let φ be a function on $\mathbb{S}^n$ satisfying a Lipschitz condition $|\varphi(\zeta) - \varphi(\eta)| \lesssim |1-\zeta\overline{\eta}|^{2s+\varepsilon}$ for any $\zeta, \eta \in \mathbb{S}^n$, or more generally, $|\varphi(\zeta) - \varphi(\eta)| \lesssim |1-\zeta\overline{\eta}|^{2s}|\psi(\zeta,\eta)|$, where the function*

$\zeta \mapsto \int_{\mathbb{S}^n} \frac{|\psi(\zeta,\eta)|}{|1-\zeta\overline{\eta}|^n}\, d\sigma(\eta)$ *exists and is bounded. Then, for any* $\zeta \in \mathbb{S}^n$,

$$\lim_{r\to 1^-} (1-r^2)^{1-2s} r \frac{\partial}{\partial r} P_s[\varphi](r\zeta) = \frac{2\Gamma(1-2s)}{\Gamma(2s)} \frac{\Gamma\left(\frac{n+2s}{2}\right)^2}{\Gamma\left(\frac{n-2s}{2}\right)^2} ((-\Delta)^s + I)[\varphi](\zeta).$$

Moreover, the convergence is also in $L^2(\mathbb{S}^n)$.

Proof. The proof in the case that φ satisfies a $2s+\varepsilon$-Lipschitz condition is the same as given in the second part of Lemma 3.2, if we consider the function φ instead of the function φ_{pq} and replace the $2s+\varepsilon$-Lipschitz condition by the $\frac{1}{2}$-Lipschitz condition.

On the other hand, if φ satisfies

$$|\varphi(\zeta) - \varphi(\eta)| \lesssim |1-\zeta\overline{\eta}|^{2s} |\psi(\zeta,\eta)|,$$

where the function $\zeta \mapsto \int_{\mathbb{S}^n} \frac{|\psi(\zeta,\eta)|}{|1-\zeta\overline{\eta}|^n}\, d\sigma(\eta)$ is bounded, then it is enough to replace in the above proof the estimate $|\varphi(\zeta)-\varphi(\eta)| \lesssim |1-\zeta\overline{\eta}|^{2s+\varepsilon}$ by $|\varphi(\zeta)-\varphi(\eta)| \lesssim |1-\zeta\overline{\eta}|^{2s}|\psi(\zeta,\eta)|$ and end the proof in an analogous way.

The convergence in L^2 follows by applying the Dominated Convergence Theorem iteratively. ∎

REMARK 3.6. Observe that the $\frac{1}{2}$-Lipschitz condition is fulfilled if the complex-tangential derivatives of φ are bounded (see the details in [6]).

COROLLARY 3.7. *Let* φ *be a function satisfying the conditions of Theorem* 3.5. *Then*

(1) $\varphi \in H^{2s}(\mathbb{S}^n)$.
(2) $\varphi = K_{2s}[((-\Delta)^s + I)[\varphi]]$.

Proof. Since φ satisfies the conditions of Theorem 3.5, we have

$$\frac{2\Gamma(1-2s)\Gamma\left(\frac{n+2s}{2}\right)^2}{\Gamma(2s)\Gamma\left(\frac{n-2s}{2}\right)^2}((-\Delta)^s + I)[\varphi](\zeta) = \lim_{r\to 1^-}(1-r^2)^{1-2s}\frac{\partial}{\partial r}P_s[\varphi](r\zeta),$$

with convergence in $L^2(\mathbb{S}^n)$. It is enough to check that if π_{pq} is the orthogonal projection from $L^2(\mathbb{S}^n)$ to $H(p,q)$, then

$$\varphi_{pq} = \pi_{pq}[\varphi] = \pi_{pq}\big[K_{2s}[((-\Delta)^s + I)[\varphi]]\big]. \tag{3.6}$$

Indeed, by Theorem 3.5,

$$\begin{aligned}
&\pi_{pq}[((-\Delta)^s + I)[\varphi]] \\
&\qquad = \frac{\Gamma(2s)\Gamma\left(\frac{n-2s}{2}\right)^2}{2\Gamma(1-2s)\Gamma\left(\frac{n+2s}{2}\right)^2}\pi_{pq}\left[\lim_{r\to 1^-}(1-r^2)^{1-2s}\frac{\partial}{\partial r}P_s[\varphi](r\zeta)\right].
\end{aligned}$$

The continuity in $L^2(\mathbb{S}^n)$ of the function $\varphi \mapsto (1-r^2)^{1-2s}\frac{\partial}{\partial r}P_s[\varphi]$ for each $r<1$ gives

$$\pi_{pq}\left[(1-r^2)^{1-2s}\frac{\partial}{\partial r}P_s[\varphi]\right] = (1-r^2)^{1-2s}\frac{\partial}{\partial r}P_s[\varphi_{pq}].$$

Letting $r\to 1^-$ and using Lemma 3.2, we deduce that the above limit is

$$\frac{2\Gamma(1-2s)}{\Gamma(2s)}\,\frac{\Gamma\big(p+\frac{n+2s}{2}\big)\Gamma\big(q+\frac{n+2s}{2}\big)}{\Gamma\big(p+\frac{n-2s}{2}\big)\Gamma\big(q+\frac{n-2s}{2}\big)}\varphi_{pq}.$$

Finally, we obtain (3.6) using the fact that by (3.1),

$$K_{2s}[\varphi_{pq}] = \left(\frac{\Gamma\big(\frac{n+2s}{2}\big)}{\Gamma\big(\frac{n-2s}{2}\big)}\right)^2\frac{\Gamma\big(p+\frac{n-2s}{2}\big)\Gamma\big(q+\frac{n-2s}{2}\big)}{\Gamma\big(p+\frac{n+2s}{2}\big)\Gamma\big(q+\frac{n+2s}{2}\big)}\varphi_{pq}. \ \blacksquare$$

3.1. Minimality of the norm in $\mathcal{W}_s^2$ of the extensions $P_s[\varphi]$

PROPOSITION 3.8. *Let $0<s<1/4$ and let φ be a real-valued function on $\mathbb{B}^n$.*

(1) *If φ satisfies the conditions of Theorem 3.3 and $v\in\mathcal{C}^2_{\mathbb{R}}(\overline{\mathbb{B}^n})$, then*

$$\operatorname{Re}\langle v, P_s[\varphi]\rangle_{\mathcal{W}_s^2} = C_{n,s}\int_{\mathbb{S}^n} vK_{2s}^{-1}[\varphi]\,d\sigma,$$

where $C_{n,s}$ is a positive constant.

(2) *If φ satisfies the conditions of Theorem 3.5 and $v\in\mathcal{C}^2_{\mathbb{R}}(\overline{\mathbb{B}^n})$, then*

$$\operatorname{Re}\langle v, P_s[\varphi]\rangle_{\mathcal{W}_s^2} = C_{n,s}\int_{\mathbb{S}^n} v((-\Delta)^s+I)[\varphi]\,d\sigma = C_{n,s}\int_{\mathbb{S}^n} vK_{2s}^{-1}[\varphi]\,d\sigma.$$

Proof. In case (1), we consider the form $v((-1)^n\Omega+\widetilde{\Omega})$, where Ω and $\widetilde{\Omega}$ are as in (2.2), with $u=P_s[\varphi]$. Then we have

$$\begin{aligned} d\big(v((-1)^n\Omega+\widetilde{\Omega})\big) &= 2\operatorname{Re}\bigg\{\frac{(1-|z|^2)^{1-2s}}{|z|^2}Rv\overline{R}u \\ &\quad+\sum_{i<j}\frac{(1-|z|^2)^{-2s}}{|z|^2}L_{i,j}v\overline{L}_{i,j}u+(1-|z|^2)^{-2s}\left(\frac{n-2s}{2}\right)^2 vu\bigg\}\omega(\overline{z})\wedge\omega(z). \end{aligned}$$

Stokes' Theorem applied to $B_r=B(0,r)$ gives, for any $0<r<1$,

$$\int_{B_r} d\big(v((-1)^n\Omega+\widetilde{\Omega})\big) = \int_{\mathbb{S}^n_r} v((-1)^n\Omega+\widetilde{\Omega}).$$

Since the restrictions to the spheres $\mathbb{S}^n_r$ of the summands in Ω that include terms of the form $(\overline{z}_i\omega_j(z)-\overline{z}_j\omega_i(z))$ are zero, using the facts that the limit in Theorem 3.3 is in $L^2(\mathbb{S}^n)$ and that $R+\overline{R}=r\frac{d}{dr}$, we deduce that

$$\begin{aligned}\lim_{r\to 1^+}\int_{\mathbb{S}^n_r}\frac{(1-r^2)^{1-2s}}{r^2}v((-1)^n\overline{R}u\,\omega'(z)\wedge\omega(\overline{z})+Ru\,\omega'(\overline{z})\wedge\omega(z))\\ =nc_n\lim_{r\to 1^+}\frac{(1-r^2)^{1-2s}}{r^2}\int_{\mathbb{S}^n_r}v(\overline{R}u+Ru)\,d\sigma\\ =nc_n\frac{2\Gamma(1-2s)\big(\Gamma\big(\frac{n+2s}{2}\big)\big)^2}{\Gamma(2s)\big(\Gamma\big(\frac{n-2s}{2}\big)\big)^2}\int_{\mathbb{S}^n}vK_{2s}^{-1}[\varphi]\,d\sigma,\end{aligned}$$

where c_n is as in Proposition 2.2. We obtain the desired formula, using

$$\int_{\mathbb{B}^n}d(v((-1)^n\Omega+\widetilde{\Omega}))=2c_n\operatorname{Re}\langle v,u\rangle_{\mathcal{W}^2_s}.$$

Case (2) is proved analogously, taking into account Corollary 3.7. ∎

Corollary 3.9. *Let $u=P_s[\varphi]$, where φ is a real-valued function on $\mathbb{B}^n$ that satisfies the conditions of Theorem* 3.3 (*or Theorem* 3.5), *and let v_1,v_2 be $\mathcal{C}^2$ real-valued functions on $\overline{\mathbb{B}}^n$ such that $v_1=v_2$ on $\mathbb{S}^n$. Then $\operatorname{Re}\langle v_1,u\rangle_{\mathcal{W}^2_s}=\operatorname{Re}\langle v_2,u\rangle_{\mathcal{W}^2_s}$.*

The next corollary gives the proof of assertion (2) in Theorem 1.2.

Corollary 3.10. *Let $\varphi\in H^s(\mathbb{S}^n)$. Then $\|P_s[\varphi]\|_{\mathcal{W}^2_s}\simeq\|\varphi\|_{H^s(\mathbb{S}^n)}$.*

Proof. We first observe that, by density, it is enough to consider $\mathcal{C}^\infty$ real-valued functions in $H^s(\mathbb{S}^n)$. By Proposition 3.8, if $u=P_s[\varphi]$, we have

$$\|u\|^2_{\mathcal{W}^2_s}\simeq\int_{\mathbb{S}^n}\varphi K_{2s}^{-1}[\varphi]\,d\sigma.$$

Next, we observe that if $\varphi=\sum_{p,q}\varphi_{p,q}$, then

$$\int_{\mathbb{S}^n}\varphi K_{2s}^{-1}[\varphi]\,d\sigma=\frac{\Gamma\big(\frac{n-2s}{2}\big)^2}{\Gamma\big(\frac{n+2s}{2}\big)^2}\sum_{p,q}\frac{\Gamma\big(p+\frac{n-2s}{2}\big)\Gamma\big(q+\frac{n-2s}{2}\big)}{\Gamma\big(p+\frac{n+2s}{2}\big)\Gamma\big(q+\frac{n+2s}{2}\big)}\|\varphi_{p,q}\|^2_{L^2(\mathbb{S}^n)},$$

whereas

$$\int_{\mathbb{S}^n}|K_s^{-1}[\varphi]|^2\,d\sigma=\frac{\Gamma\big(\frac{n-s}{2}\big)^4}{\Gamma\big(\frac{n+s}{2}\big)^4}\sum_{p,q}\frac{\Gamma\big(p+\frac{n-s}{2}\big)^2\Gamma\big(q+\frac{n-s}{2}\big)^2}{\Gamma\big(p+\frac{n+s}{2}\big)^2\Gamma\big(q+\frac{n+s}{2}\big)^2}\|\varphi_{p,q}\|^2_{L^2(\mathbb{S}^n)}.$$

But by Stirling's formula, there exists $C>0$ such that

$$\frac{1}{C}\le\frac{\Gamma\big(p+\frac{n-2s}{2}\big)}{\Gamma\big(p+\frac{n+2s}{2}\big)}\frac{\Gamma\big(p+\frac{n+s}{2}\big)^2}{\Gamma\big(p+\frac{n-s}{2}\big)^2}\le C,\qquad p\ge 1,$$

with similar estimates for the expression for q. Thus,

$$\|P_s[\varphi]\|^2_{\mathcal{W}^2_s}\simeq\|\varphi\|^2_{H^s(\mathbb{S}^n)}.\ ∎$$

The next corollary shows that among all the possible extensions of a function φ to $\mathcal{W}_s$, the function $P_s[\varphi]$ has minimal norm.

Corollary 3.11. *Let φ be a real-valued function on the unit sphere satisfying the conditions of Theorem* 3.3 (*or Theorem* 3.5) *and u a $\mathcal{C}^2$ function on $\mathcal{W}_s^2$ such that $u_{|\mathbb{S}^n} = \varphi$. Then $\|P_s[\varphi]\|_{\mathcal{W}_s^2} \le \|u\|_{\mathcal{W}_s^2}$.*

Proof. As $\operatorname{Re}\langle u, P_s[\varphi]\rangle_{\mathcal{W}_s^2} = \langle P_s[\varphi], P_s[\varphi]\rangle_{\mathcal{W}_s^2}$ by Corollary 3.9, we have

$$\begin{aligned}\|u\|^2_{\mathcal{W}_s^2} = \langle u, u\rangle_{\mathcal{W}_s^2} &= \langle u - P_s[\varphi], u - P_s[\varphi]\rangle_{\mathcal{W}_s^2}\\ &\quad + 2\operatorname{Re}\langle u, P_s[\varphi]\rangle_{\mathcal{W}_s^2} - \langle P_s[\varphi], P_s[\varphi]\rangle_{\mathcal{W}_s^2}\\ &= \langle u - P_s[\varphi], u - P_s[\varphi]\rangle_{\mathcal{W}_s^2} + \langle P_s[\varphi], P_s[\varphi]\rangle_{\mathcal{W}_s^2}\\ &= \|u - P_s[\varphi]\|^2_{\mathcal{W}_s^2} + \|P_s[\varphi]\|^2_{\mathcal{W}_s^2},\end{aligned}$$

and in particular $\|P_s[\varphi]\|^2_{\mathcal{W}_s^2} \le \|u\|^2_{\mathcal{W}_s^2}$. ■

3.2. A remark on $\|\cdot\|^2_{\mathcal{W}_s^2}$. We next obtain a useful technical result that we will need later; it shows that in $\|\cdot\|^2_{\mathcal{W}_s^2}$ we can replace $(1-|z|^2)^{-2s}|u|^2$ by $(1-|z|^2)^{1-2s}|u|^2$ and obtain an equivalent expression.

Proposition 3.12. *Let $0 < s < 1/2$ and let $\varphi \in \mathcal{C}^\infty(\overline{\mathbb{B}}^n)$. Then, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\begin{aligned}(3.7)\qquad &\int_{\mathbb{B}^n} |\varphi|^2(z)(1-|z|^2)^{-2s}\,dV(z)\\ &\le C_\varepsilon \int_{\mathbb{B}^n} |\varphi|^2(z)(1-|z|^2)^{1-2s}\,dV(z) + \varepsilon \int_{\mathbb{B}^n} |R\varphi|^2(z)\frac{(1-|z|^2)^{1-2s}}{|z|^2}\,dV(z).\end{aligned}$$

In particular,

$$\begin{aligned}\|\varphi\|_{\mathcal{W}_s^2} \simeq \int_{\mathbb{B}^n} \bigg\{ &\frac{(1-|z|^2)^{1-2s}}{|z|^2}|R\varphi|^2\\ &+ \frac{(1-|z|^2)^{-2s}}{|z|^2}\sum_{i<j}|L_{i,j}\varphi|^2 + |\varphi|^2(1-|z|^2)^{1-2s}\bigg\}\,dV(z).\end{aligned}$$

Proof. It is enough to consider φ real-valued. Set

$$\Omega = (1-|z|^2)^{1-2s}\varphi^2\omega'(\overline{z}) \wedge \omega(z).$$

Stokes' Theorem shows that if c_n is as in Proposition 2.2, then

$$\begin{aligned}0 = \int_{\mathbb{S}^n} \Omega\,d\sigma = \int_{\mathbb{B}^n} d\Omega = &\int_{\mathbb{B}^n} (1-2s)(1-|z|^2)^{1-2s}\varphi^2(z)\omega(\overline{z}) \wedge \omega(z)\\ &- \int_{\mathbb{B}^n} (1-2s)(1-|z|^2)^{-2s}\varphi^2(z)\omega(\overline{z}) \wedge \omega(z)\\ &+ \int_{\mathbb{B}^n} (1-|z|^2)^{1-2s}2\varphi(z)\overline{R}(z)\varphi(z)\omega(\overline{z}) \wedge \omega(z)\\ &+ n\int_{\mathbb{B}^n} (1-|z|^2)^{1-2s}\varphi^2(z)\omega(\overline{z}) \wedge \omega(z).\end{aligned}$$

Hence,

$$c_n \int_{\mathbb{B}^n} (1-2s)(1-|z|^2)^{-2s}\varphi^2(z)\,dV(z)$$
$$= c_n \int_{\mathbb{B}^n} (n+1-2s)(1-|z|^2)^{1-2s}\varphi^2(z)\,dV(z)$$
$$+ c_n \int_{\mathbb{B}^n} (1-|z|^2)^{1-2s} 2\varphi \overline{R}\varphi(z)\,dV(z),$$

which, using Hölder's inequality, gives (3.7). ∎

Observe that Theorems 3.3 and 3.5 and their corollaries, prove assertion (1) in Theorem 1.2. This, together with Corollary 3.10, ends the proof of Theorem 1.2.

We finish the section by obtaining an equivalence of the boundedness of a bilinear form on $H^s(\mathbb{S}^n)$ in terms of the extensions P_s. These results will be used in Section 8, where a characterization of these bilinear forms is obtained.

THEOREM 3.13. *Let* $0 < s < 1/4$ *and let* $\mathfrak{g}$ *be a real-valued function in* $L^2(\mathbb{S}^n)$ *such that* $K_{2s}[\mathfrak{g}]$ *satisfies the conditions of either Theorem* 3.3 *or Theorem* 3.5. *The following assertions are equivalent:*

(1) $|\int_{\mathbb{S}^n} \varphi\psi\mathfrak{g}\,d\sigma| \lesssim \|\varphi\|_{H^s(\mathbb{S}^n)}\|\psi\|_{H^s(\mathbb{S}^n)}$, $\varphi,\psi \in \mathcal{C}^\infty_{\mathbb{R}}(\mathbb{S}^n)$;
(2) $|\mathrm{Re}\langle P_s[\varphi\psi], P_s[K_{2s}[\mathfrak{g}]]\rangle_{\mathcal{W}^2_s}| \lesssim \|\varphi\|_{H^s(\mathbb{S}^n)}\|\psi\|_{H^s(\mathbb{S}^n)}$, $\varphi,\psi \in \mathcal{C}^\infty_{\mathbb{R}}(\mathbb{S}^n)$;
(3) $|\mathrm{Re}\langle P_s[\varphi]P_s[\psi], P_s[K_{2s}[\mathfrak{g}]]\rangle_{\mathcal{W}^2_s}| \lesssim \|\varphi\|_{H^s(\mathbb{S}^n)}\|\psi\|_{H^s(\mathbb{S}^n)}$, $\varphi,\psi \in \mathcal{C}^\infty_{\mathbb{R}}(\mathbb{S}^n)$.

Proof. In Proposition 3.8, we consider $u = P_s[K_{2s}[\mathfrak{g}]]$ and $v = P_s[\varphi\psi]$. We then have

$$\mathrm{Re}\langle P_s[\varphi\psi], P_s[K_{2s}[\mathfrak{g}]]\rangle_{\mathcal{W}^2_s} = C_{n,s}\int_{\mathbb{S}^n} \varphi\psi K_{2s}^{-1}[K_{2s}[\mathfrak{g}]]\,d\sigma = C_{n,s}\int_{\mathbb{S}^n} \varphi\psi\mathfrak{g}\,d\sigma,$$

which gives (1)⇔(2). The equivalence (2)⇔(3) is a consequence of Corollary 3.9, since $P_s[\varphi\psi]$ and $P_s[\varphi]P_s[\psi]$ have the same restriction to $\mathbb{S}^n$. ∎

4. Weighted estimates for a non-isotropic weighted area function. We consider a vector-valued kernel $\mathbf{K} : \mathbb{B}^n \times \mathbb{S}^n \to \mathbb{C}^N$ which for every $\varphi \in L^2(\mathbb{S}^n)$ defines a function $\mathbf{K}[\varphi] : \mathbb{B}^n \to \mathbb{C}^N$ given by $\mathbf{K}[\varphi](z) = \int_{\mathbb{S}^n} \mathbf{K}(z,\zeta)\varphi(\zeta)\,d\sigma(\zeta)$. The *area function* associated to $\mathbf{K}$ is the function on $\mathbb{S}^n$ defined by

$$G_{\mathbf{K}}[\varphi](\zeta) := \left(\int_{D(\zeta)} |\mathbf{K}[\varphi](z)|^2 \frac{dV(z)}{(1-|z|^2)^{n+1}}\right)^{1/2}, \tag{4.1}$$

where $D(\zeta) = \{z \in \mathbb{B}^n : |1 - z\overline{\zeta}| < \frac{\alpha}{2}(1-|z|^2)\}$, $\alpha > 1$, is the admissible region with vertex ζ.

A weight w is in the A_p *class* in $\mathbb{S}^n$ with the non-isotropic metric, $1 < p < +\infty$, if there exists $C > 0$ such that for any ball $B \subset \mathbb{S}^n$, $B = B(\zeta, r) = \{\eta \in \mathbb{S}^n : |1 - \zeta\overline{\eta}|^{1/2} < r\}$, we have

$$\left(\frac{1}{\sigma(B)} \int_B w\, d\sigma\right)\left(\frac{1}{\sigma(B)} \int_B w^{\frac{-1}{p-1}}\, d\sigma\right)^{p-1} \le C.$$

THEOREM 4.1. *Let $\mathbf{K} : \mathbb{B}^n \times \mathbb{S}^n \to \mathbb{C}^N$ be a vector-valued kernel satisfying the following conditions:*

(1) $\|G_{\mathbf{K}}[\varphi]\|_{L^2(\mathbb{S}^n)} \le C_1 \|\varphi\|_{L^2(\mathbb{S}^n)}$ *for any* $\varphi \in L^2(\mathbb{S}^n)$.

(2) *There exists* $\varepsilon > 0$ *such that* $|\mathbf{K}(z,\zeta)| \lesssim \frac{(1-|z|^2)^\varepsilon}{|1-z\overline{\zeta}|^{n+\varepsilon}}$ *for any* $z \in \mathbb{B}^n$, $\zeta \in \mathbb{S}^n$.

(3) *There exist* $\varepsilon, \delta > 0$ *such that for any* $z \in \mathbb{B}^n$ *and* $\alpha_1, \alpha_2 \in \mathbb{S}^n$ *such that* $|1 - \alpha_1\overline{\alpha}_2| \le \delta |1 - z\overline{\alpha}_1|$,
$$|\mathbf{K}(z,\alpha_1) - \mathbf{K}(z,\alpha_2)| \lesssim \frac{(1-|z|^2)^\varepsilon |1-\alpha_1\overline{\alpha}_2|^\varepsilon}{|1-z\overline{\alpha}_1|^{n+2\varepsilon}}.$$

(4) *There exist* $\varepsilon, \delta > 0$ *such that for any* $\zeta \in \mathbb{S}^n$ *and* $\alpha_1, \alpha_2 \in \mathbb{S}^n$ *with* $|1 - \alpha_1\overline{\alpha}_2| \le \delta |1 - \alpha_1\overline{\zeta}|$,
$$|\mathbf{K}(r\alpha_1,\zeta) - \mathbf{K}(r\alpha_2,\zeta)| \lesssim \frac{(1-r^2)^\varepsilon |1-\alpha_1\overline{\alpha}_2|^\varepsilon}{|1-r\alpha_1\overline{\zeta}|^{n+2\varepsilon}}.$$

Then, for any $\omega \in A_p$,

$$\|G_{\mathbf{K}}[\varphi]\|_{L^p(\omega)} \lesssim \|\varphi\|_{L^p(\omega)}. \tag{4.2}$$

Before we prove the theorem, we recall the dyadic decomposition of a quasi-metric space given in [11] (see also [17, 18]). We state the decomposition for $\mathbb{S}^n$ and the quasi-metric $\rho(\zeta,\eta) = |1 - \zeta\overline{\eta}|$.

PROPOSITION 4.2. *Given a fixed parameter* $0 < \delta < 1$ *small enough and a fixed point* $x_0 \in \mathbb{S}^n$, *there exists a finite collection of families of sets,* $\mathcal{D}^j$, $j = 1, \dots, M$, *called* adjacent dyadic systems, *such that each* $\mathcal{D}^j$ *is a family of Borel sets* Q^k_α, $k \in \mathbb{Z}$, $\alpha \in I_k$, *called* dyadic cubes, *which are associated with points* ζ^k_α, *which we will call the* center points *of the cubes* Q^k_α, *having the following properties:*

(1) $\mathbb{S}^n = \bigcup_{\alpha \in I_k} Q^k_\alpha$ *(disjoint union) for each* $k \in \mathbb{Z}$.

(2) *If* $k < l$, *then either* $Q^l_\beta \cap Q^k_\alpha = \emptyset$ *or* $Q^l_\beta \subset Q^k_\alpha$.

(3) *There exist* $c_1, C_1 > 0$ *such that* $B(\zeta^k_\alpha, c_1\delta^k) \subset Q^k_\alpha \subset B(\zeta^k_\alpha, C_1\delta^k) =: B(Q^k_\alpha)$.

(4) *If* $k \le l$ *and* $Q^l_\beta \subset Q^k_\alpha$, *then* $B(Q^l_\beta) \subset B(Q^k_\alpha)$.

(5) *For any* $k \in \mathbb{Z}$, *there exists* α *such that* $x_0 = \zeta^k_\alpha$, *the center point of* Q^k_α.

(6) *There exists* $C > 0$, *only depending on* δ, *such that, for any non-isotropic ball* $B(\zeta, r) \subset \mathbb{S}^n$ *with* $\delta^{k+3} < r \le \delta^{k+2}$, *there exist* j *and* $Q^k_\alpha \in \mathcal{D}^j$ *such that* $B(\zeta, r) \subset Q^k_\alpha$ *and* $\operatorname{diam} Q^k_\alpha \le Cr$.

The family $\mathcal{D} = \bigcup_{j=1}^{M} \mathcal{D}^j$ is called a *dyadic decomposition* of $\mathbb{S}^n$ and we say that the set Q_α^k is a *dyadic cube of generation k centered at ζ_α^k with radius* $l(Q_\alpha^k) = \delta^k$.

We next state a lemma that follows from well-known techniques of splitting functions, which come from a method stated by A. P. Calderón and A. Zygmund, and that shows that the area function satisfies a weak L^1 estimate (see, for instance, [10]):

LEMMA 4.3. *There exists $C > 0$ such that for any $\lambda > 0$ and $\varphi \in L^1(\mathbb{S}^n)$,*

$$\sigma(\{\eta : G_{\mathbf{K}}[\varphi](\eta) > \lambda\}) \lesssim \frac{\|\varphi\|_{L^1(d\sigma)}}{\lambda}.$$

The proof of the theorem is based on the ideas in [14], where a weighted L^p estimate is obtained for a Littlewood–Paley square function associated to a function $\varphi \in \mathcal{S}$ with zero integral. We also use some results of [19]. We recall that the *Fefferman–Stein sharp function* $\varphi^\sharp$ is given by

$$\varphi^\sharp(\zeta) = \sup_{\zeta \in Q \in \mathcal{D}} \inf_c \frac{1}{\sigma(Q)} \int_Q |\varphi(\eta) - c| \, d\sigma(\eta).$$

If $0 < \delta < 1$, we write $\varphi_\delta^\sharp(\zeta) = \sup_{\zeta \in Q \in \mathcal{D}} \inf_c \big(\frac{1}{\sigma(Q)} \int_Q |\varphi(\eta) - c|^\delta \, d\sigma(\eta)\big)^{1/\delta}$. Applying [19, Theorem 3.1 and estimate (4.1)] , we have the following:

THEOREM 4.4 ([19]). *Let $1 < p < \infty$. For any $\omega \in A_p(\mathbb{S}^n)$, $\delta < 1$ and for any integrable function φ on $\mathbb{S}^n$, we have*

$$\|M[\varphi]\|_{L^p(\omega)} \lesssim \||\varphi|_\delta^\sharp\|_{L^p(\omega)},$$

where $M[\varphi]$ is the non-isotropic Hardy–Littlewood maximal function and the constants depend only on p, n and the weight ω. In particular,

$$\|\varphi\|_{L^p(\omega)} \lesssim \||\varphi|_\delta^\sharp\|_{L^p(\omega)}. \tag{4.3}$$

Proof of Theorem 4.1. We claim that it is enough to show that for $0 < \delta < 1$,

$$G_{\mathbf{K}}[\varphi]_\delta^\sharp \lesssim M[\varphi]. \tag{4.4}$$

Indeed, by (4.3) and (4.4), we see that if $\omega \in A_p(\mathbb{S}^n)$, then

$$\|G_{\mathbf{K}}[\varphi]\|_{L^p(\omega)} \lesssim \|G_{\mathbf{K}}[\varphi]_\delta^\sharp\|_{L^p(\omega)} \lesssim \|M[\varphi]\|_{L^p(\omega)} \lesssim \|\varphi\|_{L^p(\omega)},$$

which gives the desired estimate (4.2).

So, we are left to prove the claim. It is enough to show that for any $\zeta_0 \in \mathbb{S}^n$ and any cube $Q \in \mathcal{D}$ with $\zeta_0 \in Q$, there exists c_Q such that

$$\left(\frac{1}{\sigma(Q)} \int_Q |G_{\mathbf{K}}[\varphi](\zeta) - c_Q|^\delta \, d\sigma\right)^{1/\delta} \lesssim M[\varphi](\zeta_0).$$

We write $\varphi=\varphi_1+\varphi_2$, where $\varphi_1=\varphi\chi_{Q^*}$, and

$$Q^*=\lambda B(Q), \quad \text{where} \quad \lambda>1.$$

We then have

$$\begin{aligned}&\left(\frac{1}{\sigma(Q)}\int\limits_Q |G_{\mathbf{K}}[\varphi](\zeta)-c_Q|^\delta\,d\sigma\right)^{1/\delta}\\ &\lesssim\left(\frac{1}{\sigma(Q)}\int\limits_Q G_{\mathbf{K}}[\varphi_1]^\delta(\zeta)\,d\sigma\right)^{1/\delta}+\left(\frac{1}{\sigma(Q)}\int\limits_Q |G_{\mathbf{K}}[\varphi_2](\zeta)-c_Q|^\delta\,d\sigma\right)^{1/\delta}=\mathrm{I}+\mathrm{II}.\end{aligned}$$

We use the weak $(1,1)$ estimate of $G_{\mathbf{K}}$ given in Lemma 4.3 and Kolmogorov's Lemma to obtain

$$\frac{1}{\sigma(Q)}\int\limits_Q G_{\mathbf{K}}[\varphi_1]^\delta(\zeta)\,d\sigma\lesssim\frac{1}{\sigma(Q)}\sigma(Q)^{1-\delta}\Big(\int\limits_{Q^*}|\varphi_1|\,d\sigma\Big)^\delta\lesssim(M[\varphi](\zeta))^\delta,$$

which gives the desired estimate for I.

Let $\mathcal{U}=\mathcal{U}(n)$ be the group of unitary operators on the Hilbert space $\mathbb{C}^n$, that is, the group of linear operators U that preserve inner products: $\langle Uz,Uw\rangle=\langle z,w\rangle$ for $z,w\in\mathbb{C}^n$. Observe that if $U(\zeta)=\eta$, then $U(D(\zeta))=D(\eta)$.

In order to obtain the estimate for II for some chosen c_Q we begin by noting that since $\delta\le 1$, we have

$$\left(\frac{1}{\sigma(Q)}\int\limits_Q |G_{\mathbf{K}}[\varphi_2](\zeta)-c_Q|^\delta\,d\sigma\right)^{1/\delta}\le\frac{1}{\sigma(Q)}\int\limits_Q|G_{\mathbf{K}}[\varphi_2](\zeta)-c_Q|\,d\sigma(\zeta).$$

We choose

$$c_Q=\frac{1}{\sigma(Q)}\int\limits_{\eta\in Q}\Bigg(\int\limits_{D_\alpha(\eta)}|\mathbf{K}[\varphi_2](z)|^2\,\frac{dV(z)}{(1-|z|^2)^{n+1}}\Bigg)^{1/2}d\sigma(\eta).$$

We then have

$$\begin{aligned}|G_{\mathbf{K}}[\varphi_2](\zeta)-c_Q|=\Bigg|&\Bigg(\int\limits_{D_\alpha(\zeta)}\frac{|\mathbf{K}[\varphi_2](z)|^2}{(1-|z|^2)^{n+1}}\,dV(z)\Bigg)^{1/2}\\ &-\frac{1}{\sigma(Q)}\int\limits_{\eta\in Q}\Bigg(\int\limits_{D_\alpha(\eta)}\frac{|\mathbf{K}[\varphi_2](z)|^2}{(1-|z|^2)^{n+1}}\,dV(z)\Bigg)^{1/2}d\sigma(\eta)\Bigg|\end{aligned}$$

We now choose $U_\eta\in\mathcal{U}$ satisfying $U_\eta(\eta)=\zeta$ and $\|\mathrm{Id}-U_\eta\|\lesssim l(Q)$. Then the above can be bounded by

$$\frac{1}{\sigma(Q)}\int_{\eta\in Q}\Bigg[\bigg(\int_{D_\alpha(\zeta)}\frac{|\mathbf{K}[\varphi_2](z)|^2}{(1-|z|^2)^{n+1}}\,dV(z)\bigg)^{1/2} - \bigg(\int_{D_\alpha(\zeta)}\frac{|\mathbf{K}[\varphi_2](U_\eta z)|^2}{(1-|z|^2)^{n+1}}\,dV(z)\bigg)^{1/2}\Bigg]\,d\sigma(\eta)$$

$$\leq \frac{1}{\sigma(Q)}\int_{\eta\in Q}\bigg(\int_{D_\alpha(\zeta)}\frac{|\mathbf{K}[\varphi_2](z)-\mathbf{K}[\varphi_2](U_\eta z)|^2}{(1-|z|^2)^{n+1}}\,dV(z)\bigg)^{1/2}\,d\sigma(\eta),$$

Hence, applying Minkowski's inequality, we find that

$$\int_{\zeta\in Q}|G_{\mathbf{K}}[\varphi_2](\zeta)-c_Q|\,d\sigma(\zeta)$$

$$= \int_{\zeta\in Q}\frac{1}{\sigma(Q)}\int_{\eta\in Q}\bigg(\int_{z\in D_\alpha(\zeta)}\frac{|\mathbf{K}[\varphi_2](z)-\mathbf{K}[\varphi_2](U_\eta z)|^2}{(1-|z|^2)^{n+1}}\,dV(z)\bigg)^{1/2}\,d\sigma(\eta)\,d\sigma(\zeta)$$

$$\leq \frac{1}{\sigma(Q)}\int_Q\int_Q\int_{\mathbb{S}^n\setminus Q^*}\bigg(\int_{z\in D_\alpha(\zeta)}\frac{|\mathbf{K}(z,\omega)-\mathbf{K}(U_\eta z,\omega)|\,dV(z)}{(1-|z|^2)^{n+1}}\bigg)^{1/2} \times \varphi_2(\omega)\,d\sigma(\omega)\,d\sigma(\eta)\,d\sigma(\zeta).$$

Using the hypothesis (4) on the kernel $\mathbf{K}$, observing that $\|U_\eta z\| = \|z\|$, since $U_\eta\in\mathcal{U}$, and that $\|\mathrm{Id}-U_\eta\|\lesssim l(Q)$, we deduce that if $z=|z|z_0$ and $U_\eta z=|z|U_\eta z_0$ with $z_0\in\mathcal{S}^n$, then

$$|1-z_0\overline{U\eta z_0}|\leq|z_0-U_\eta z_0|\leq\|\mathrm{Id}-U_\eta\|\lesssim l(Q)<\delta|1-z_0\overline{\omega}|,$$

due to the fact that $\omega\in\mathbb{S}^n\setminus Q^*$. Hence,

$$\int_{z\in D_\alpha(\zeta)}|\mathbf{K}(z,\omega)-\mathbf{K}(U_\eta z,\omega)|^2\,\frac{dV(z)}{(1-|z|^2)^{n+1}}$$

$$\lesssim \int_{z\in D_\alpha(\zeta)}\bigg(\frac{(1-|z|^2)^\varepsilon|1-z_0\overline{U_\eta z_0}|^\varepsilon}{|1-z\overline{\omega}|^{n+2\varepsilon}}\bigg)^2\,\frac{dV(z)}{(1-|z|^2)^{n+1}}$$

$$\lesssim \int_{z\in D_\alpha(\zeta)}\frac{l(Q)^{2\varepsilon}}{|1-z\overline{\omega}|^{2n+4\varepsilon}}\,\frac{dV(z)}{(1-|z|^2)^{n+1-2\varepsilon}}.$$

Next, observe that if $z\in D_\alpha(\zeta)$ and $\omega\in\mathbb{S}^n\setminus Q^*$, then $|1-\zeta\overline{\omega}|\lesssim|1-z\overline{\omega}|$. We break the above integral over $z\in D_\alpha(\zeta)$ in two pieces, II_1 and II_2 corresponding to the integral over the sets $(1-|z|^2)\leq|1-\zeta\overline{\omega}|$ and $(1-|z|^2)\geq|1-\zeta\overline{\omega}|$, respectively, and estimate them separately. For II_1, the estimates

$|1-\zeta\overline{\omega}| \lesssim |1-z\overline{\omega}|$ and $1-|z| \leq |1-z\overline{\omega}|$ and polar coordinates give

$$\begin{aligned}
\mathrm{II}_1 &\lesssim \int\limits_{1-|z|^2\leq|1-\zeta\overline{\omega}|} \frac{l(Q)^{2\varepsilon}}{|1-z\overline{\omega}|^{2n+4\varepsilon}} \frac{dV(z)}{(1-|z|^2)^{n+1-2\varepsilon}} \\
&\leq \frac{l(Q)^{2\varepsilon}}{|1-\zeta\overline{\omega}|^{2n+4\varepsilon}} \int\limits_{1-|z|^2\leq|1-\zeta\overline{\omega}|} \frac{dV(z)}{(1-|z|^2)^{n+1-2\varepsilon}} \\
&\lesssim \frac{l(Q)^{2\varepsilon}}{|1-\zeta\overline{\omega}|^{2n+4\varepsilon}} \int\limits_0^{|1-\zeta\overline{\omega}|} \frac{dr}{(1-r)^{1-2\varepsilon}} \lesssim \frac{l(Q)^{2\varepsilon}}{|1-\zeta\overline{\omega}|^{2n+2\varepsilon}}.
\end{aligned}$$

For II_2, we have

$$\mathrm{II}_2 \lesssim l(Q)^{2\varepsilon} \int\limits_{1-|z|^2\geq|1-\zeta\overline{\omega}|} \int\limits_{|1-\zeta\overline{\omega}|}^{1} \frac{dr}{(1-r^2)^{2n+1+2\varepsilon}} \lesssim \frac{l(Q)^{2\varepsilon}}{|1-\zeta\overline{\omega}|^{2n+2\varepsilon}}.$$

Finally, we deduce that

$$\begin{aligned}
&\frac{1}{\sigma(Q)} \int\limits_{\zeta\in Q} |G_{\mathbf{K}}[\varphi_2](\zeta) - c_Q| \, d\sigma(\zeta) \\
&\quad\lesssim \frac{1}{\sigma(Q)} \int\limits_{\zeta\in Q} \frac{1}{\sigma(Q)} \int\limits_{\eta\in Q} \int\limits_{\omega\in\mathbb{S}^n\setminus Q^*} \frac{l(Q)^{2\varepsilon}}{|1-\zeta\overline{\omega}|^{2n+2\varepsilon}} \varphi_2(\omega)\, d\sigma(\omega)\, d\sigma(\eta)\, d\sigma(\zeta) \\
&\quad\lesssim \sum_k \frac{l(Q)^{\varepsilon}}{(2^k l(Q))^{n+\varepsilon}} \int\limits_{2^k B_k} \varphi_2(\omega)\, d\sigma(\omega) \lesssim \sum_k \frac{1}{(2^k)^{\varepsilon}} M[\varphi_2](\zeta_0).
\end{aligned}$$

This ends the proof of the claim, and hence of Theorem 4.1.

5. Estimates for the kernels K and J. In this section we will check that when we consider the functions (or their derivatives) that appear when we extend the non-isotropic potentials in H^s through the kernel P_s, the associated kernels satisfy the conditions in Theorem 4.1. These technical results will be essential in the proof of Theorem 1.1.

More precisely, we consider the non-isotropic vector-valued kernels

$$\mathbf{K}(z,\zeta) = \frac{(1-|z|^2)^{-s}}{|z|} \int\limits_{\mathbb{S}^n} \frac{\overrightarrow{\nabla}_{\mathbb{B}^n,z} P_s(z,\eta)}{|1-\zeta\overline{\eta}|^{n-s}}\, d\sigma(\eta), \qquad z\in\mathbb{B}^n, \tag{5.1}$$

$$\mathbf{J}(z,\zeta) = (1-|z|^2)^{1/2-s} \int\limits_{\mathbb{S}^n} \frac{P_s(z,\eta)}{|1-\zeta\overline{\eta}|^{n-s}}\, d\sigma(\eta), \qquad z\in\mathbb{B}^n, \tag{5.2}$$

where $\overrightarrow{\nabla}_{\mathbb{B}^n,z} u$ is the complex vector-valued gradient of u with components $(1-|z|^2)R_z u$, $(1-|z|^2)\overline{R}_z u$, $(1-|z|^2)^{1/2} L_{i,j,z} u$ and $(1-|z|^2)^{1/2}\overline{L}_{i,j,z} u$, for $i<j$. In what follows, we will skip the subindex z if there is no confusion.

Lemma 5.1 ([3]). *Let $0 < s < 1/2$. Then:*

(1) $|\overrightarrow{\nabla}_{\mathbb{B}^n,z} P_s(z,\eta)| \lesssim \frac{(1-|z|^2)^{2s}}{|1-z\overline{\eta}|^{n+2s}}|z|$.

(2) *The vector-valued function* $\varphi_s(z) := \int_{\mathbb{B}^n} \overrightarrow{\nabla}_{\mathbb{B}^n,z} P_s(z,\eta)d\sigma(\eta)$ *is radial and satisfies* $|\varphi_s(z)| \lesssim (1-|z|^2)^{2s}|z|$, $z \in \mathbb{B}^n$.

We state without proof the following three simple technical lemmas.

Lemma 5.2. *Assume that* $0 < \alpha < 1$ *and* $a, b > 0$. *Then*

$$\left|\frac{1}{a^\alpha} - \frac{1}{b^\alpha}\right| \lesssim \frac{|a-b|(a^\alpha + b^\alpha)}{a^\alpha b^\alpha (a+b)}.$$

Moreover, for any $\alpha > 0$ *and* $0 < a < b$,

$$\frac{1}{a^\alpha} - \frac{1}{b^\alpha} \lesssim \frac{b-a}{a^\alpha b} \lesssim \frac{b^{1/2} - a^{1/2}}{a^\alpha b^{1/2}}. \tag{5.3}$$

Lemma 5.3. *Let* $0 < s, t$ *and* $s + t < n$. *Then*

$$\int_{\mathbb{S}^n} \frac{1}{|1-r\zeta\overline{\eta}|^{n-s}} \frac{1}{|1-\alpha\overline{\eta}|^{n-t}}\, d\sigma(\eta) \simeq \frac{1}{|1-r\zeta\overline{\alpha}|^{n-s-t}}, \qquad 0<r<1,\ \zeta,\alpha\in\mathbb{S}^n.$$

Lemma 5.4. *Let* $\alpha_1, \alpha_2, \eta \in \mathbb{S}^n$ *such that* $|1-\alpha_1\overline{\eta}| < |1-\alpha_2\overline{\eta}|$. *Then, for any* $0 < r < 1$,

$$\left|\frac{1}{1-r\alpha_1\overline{\eta}} - \frac{1}{1-r\alpha_2\overline{\eta}}\right| \lesssim \frac{r|1-\alpha_1\overline{\alpha_2}|^{1/2}}{|1-r\alpha_1\overline{\eta}|\,|1-r\alpha_2\overline{\eta}|^{1/2}}.$$

Theorem 5.5. *Let* $0 < s < 1/4$.*Then the vector-valued kernels* $\mathbf{K}$ *and* $\mathbf{J}$ *defined in* (5.1) *and* (5.2) *respectively satisfy the hypotheses in Theorem* 4.1. *Consequently, for any* $\omega \in A^2(\mathbb{S}^n)$, *we have* $\|G_{\mathbf{K}}[\varphi]\|_{L^2(\omega)} \lesssim \|\varphi\|_{L^2(\omega)}$ *and* $\|G_{\mathbf{J}}[\varphi]\|_{L^2(\omega)} \lesssim \|\varphi\|_{L^2(\omega)}$.

We present in Sections 5.1–5.4 the proof of Theorem 5.5 for $\mathbf{K}$. The estimates for $\mathbf{J}$ are obtained analogously, taking into account that

$$\left|\int_{\mathbb{S}^n} P_s(z,\eta)\, d\sigma(\eta)\right| \lesssim 1.$$

5.1. Proof of (1) in Theorem 4.1 for K. The estimate $\|G_{\mathbf{K}}[\varphi]\|_{L^2(\mathbb{S}^n)} \lesssim \|\varphi\|_{L^2(\mathbb{S}^n)}$ is an immediate consequence of Fubini's Theorem and Corollary 3.10.

5.2. Proof of (2) in Theorem 4.1 for K. We observe that if to prove the estimate for $\mathbf{K}$ given by (5.1), we had used directly $|\overrightarrow{\nabla}_{\mathbb{B}^n,z} P_s(z,\eta)| \lesssim \frac{(1-|z|^2)^{2s}}{|1-z\overline{\eta}|^{n+2s}}$, we would have obtained an integral which does not converge as $|z| \to 1$. Instead, we will add and subtract the term $\frac{1}{|1-z\overline{\zeta}|^{n-s}}$ before we enter the modulus inside the integral. This argument will be used repeatedly in what follows. We write

$$|K(z,\zeta)| \le \frac{(1-|z|^2)^{-s}}{|z|} \int\limits_{\mathbb{S}^n} |\overrightarrow{\nabla}_{\mathbb{B}^n,z} P_s(z,\eta)| \left| \frac{1}{|1-\zeta\overline{\eta}|^{n-s}} - \frac{1}{|1-z\overline{\zeta}|^{n-s}} \right| d\sigma(\eta)$$
$$+ \frac{(1-|z|^2)^{-s}}{|z|\,|1-z\overline{\zeta}|^{n-s}} \left| \int\limits_{\mathbb{S}^n} \overrightarrow{\nabla}_{\mathbb{B}^n,z} P_s(z,\eta)\, d\sigma(\eta) \right| = \mathcal{A} + \mathcal{B}.$$

To estimate $\mathcal{A}$, we use the fact that, by Lemma 5.1, $|\overrightarrow{\nabla}_{\mathbb{B}^n,z} P_s(z,\eta)| \lesssim \frac{(1-|z|^2)^{2s}}{|1-z\overline{\zeta}|^{n+2s}}|z|$, and we decompose the domain of integration into $S_1 := \{\eta \in \mathbb{S}^n : |1-\zeta\overline{\eta}| > |1-z\overline{\zeta}|\}$ and $S_2 := \{\eta \in \mathbb{S}^n : |1-\zeta\overline{\eta}| \le |1-z\overline{\zeta}|\}$; we denote by $\mathcal{A}_1$ (respectively $\mathcal{A}_2$) the corresponding integrals. Using (5.3), Lemma 5.3 and $s < 1/4$, we find that

$$\mathcal{A}_1 \lesssim (1-|z|^2)^{-s} \int\limits_{S_1} \frac{(1-|z|^2)^{2s}}{|1-z\overline{\eta}|^{n+2s}} \frac{|1-z\overline{\eta}|^{1/2}}{|1-z\overline{\zeta}|^{n-s}|1-\zeta\overline{\eta}|^{1/2}}\, d\sigma(\eta)$$
$$\lesssim \frac{1}{|1-z\overline{\zeta}|^{n-s}} \int\limits_{S_1} \frac{(1-|z|^2)^{s}}{|1-z\overline{\eta}|^{n+2s-1/2}} \frac{1}{|1-\zeta\overline{\eta}|^{1/2}}\, d\sigma(\eta) \lesssim \frac{(1-|z|^2)^{s}}{|1-z\overline{\zeta}|^{n+s}}.$$

Similarly, using again (5.3) and Lemma 5.3, we prove that $\mathcal{A}_2 \lesssim \frac{(1-|z|^2)^s}{|1-z\overline{\zeta}|^{n+s}}$.

Finally, by Lemma 5.1(2), we have

$$\mathcal{B} \lesssim \frac{(1-|z|^2)^{-s}(1-|z|^2)^{2s}}{|1-z\overline{\zeta}|^{n-s}} \lesssim \frac{(1-|z|^2)^{s}}{|1-z\overline{\zeta}|^{n-s}}.$$

5.3. Proof of (3) in Theorem 4.1 for K. We will show that there exist $\varepsilon, \delta > 0$ such that for any $z \in \mathbb{B}^n$ and $\alpha_1, \alpha_2 \in \mathbb{S}^n$ with $|1-\alpha_1\overline{\alpha}_2| \le \delta|1-z\overline{\alpha}_1|$,

$$|\mathbf{K}(z,\alpha_1) - \mathbf{K}(z,\alpha_2)| \lesssim \frac{(1-|z|^2)^{\varepsilon}|1-\alpha_1\overline{\alpha}_2|^{\varepsilon}}{|1-z\overline{\alpha}_1|^{n+2\varepsilon}}.$$

Let $z = rz_0 \in \mathbb{B}^n$, where $z_0 = z/|z|$. We first observe that by choosing δ small enough, and $\alpha_1, \alpha_2 \in \mathbb{S}^n$ such that $|1-\alpha_1\overline{\alpha}_2| \le \delta|1-z\overline{\alpha_1}|$, we obtain $|1-z\overline{\alpha}_1| \simeq |1-z\overline{\alpha}_2|$.

Since $|1-z\overline{\alpha_1}| \simeq (1-r) + |1-z_0\overline{\alpha_1}|$, we consider the following two cases, with k to be chosen later:

Case 1: $1-r > k|1-z_0\overline{\alpha_1}|$.

Case 2: $1-r \le k|1-z_0\overline{\alpha_1}|$.

In Case 1, we have $|1-z\overline{\alpha_1}| \simeq 1-r$. We assume, in addition, that the points in the region of integration satisfy $|1-\alpha_1\overline{\eta}| < |1-\alpha_2\overline{\eta}|$ (the other case is proved analogously). Then, using Lemma 5.2, we find that

$$\left| \frac{1}{|1-\alpha_1\overline{\eta}|^{n-s}} - \frac{1}{|1-\alpha_2\overline{\eta}|^{n-s}} \right| \lesssim \frac{|1-\alpha_1\overline{\alpha}_2|^{1/2}}{|1-\alpha_1\overline{\eta}|^{n-s}|1-\alpha_2\overline{\eta}|^{1/2}}.$$

Thus, by Lemmas 5.1–5.3 and the fact that we are in Case 1, we get

$$\begin{aligned}
|\mathbf{K}(z,\alpha_1)-\mathbf{K}(z,\alpha_2)| &\lesssim \int_{\mathbb{S}^n} \frac{(1-|z|^2)^s}{|1-z\overline{\eta}|^{n+2s}} \frac{|1-\alpha_1\overline{\alpha_2}|^{1/2}}{|1-\alpha_1\overline{\eta}|^{n-s}|1-\alpha_2\overline{\eta}|^{1/2}}\,d\sigma(\eta)\\
&\lesssim \frac{|1-\alpha_1\overline{\alpha_2}|^{1/2}}{(1-|z|^2)^{n+s}} \int_{\mathbb{S}^n} \frac{d\sigma(\eta)}{|1-\alpha_1\overline{\eta}|^{n-s}|1-\alpha_2\overline{\eta}|^{1/2}}\\
&\simeq \frac{|1-\alpha_1\overline{\alpha_2}|^{1/2}}{(1-|z|^2)^{n+s}|1-\alpha_1\overline{\alpha_2}|^{1/2-s}} = \frac{|1-\alpha_1\overline{\alpha_2}|^{s}}{(1-|z|^2)^{n+s}}\\
&\simeq (1-|z|^2)^s \frac{|1-\alpha_1\overline{\alpha_2}|^{s}}{|1-z\overline{\alpha_1}|^{n+2s}},
\end{aligned}$$

which is the estimate that we wanted to prove.

In Case 2, we see that $|1-z_0\overline{\alpha_1}| \simeq |1-z\overline{\alpha_1}| \simeq |1-z\overline{\alpha_2}|$. We have

$$\begin{aligned}
&|\mathbf{K}(z,\alpha_1)-\mathbf{K}(z,\alpha_2)|\\
&\quad\lesssim \frac{(1-|z|^2)^{-s}}{|z|}\left|\int_{\mathbb{S}^n} \overrightarrow{\nabla}_{\mathbb{B}^n,z} P_s(z,\eta)\left(\frac{1}{|1-\alpha_1\overline{\eta}|^{n-s}}\right.\right.\\
&\qquad\qquad\left.\left. - \frac{1}{|1-z_0\overline{\alpha_1}|^{n-s}} - \frac{1}{|1-\alpha_2\overline{\eta}|^{n-s}} + \frac{1}{|1-z_0\overline{\alpha_2}|^{n-s}}\right) d\sigma(\eta)\right|\\
&\qquad + \frac{(1-|z|^2)^{-s}}{|z|}\left|\int_{\mathbb{S}^n} \overrightarrow{\nabla}_{\mathbb{B}^n,\,z} P_s(z,\eta)\,d\sigma(\eta)\right|\left|\frac{1}{|1-z_0\overline{\alpha_1}|^{n-s}} - \frac{1}{|1-z_0\overline{\alpha_2}|^{n-s}}\right|\\
&\quad = \mathcal{A}_1+\mathcal{A}_2.
\end{aligned}$$

Using Lemmas 5.1(2) and 5.2, we deduce that $\mathcal{A}_2 \lesssim (1-|z|^2)^s \frac{|1-\alpha_1\overline{\alpha_2}|^s}{|1-z\overline{\alpha_1}|^{n+2s}}$.

Now, we deal with the term $\mathcal{A}_1$. We decompose the integral into $\mathcal{A}_{1,1}$ and $\mathcal{A}_{1,2}$, corresponding to the integrals over the sets $B_1 = \mathcal{S}^n \setminus B(z_0, \delta|1-z\overline{\alpha_1}|)$ and $B_2 = B(z_0,\delta|1-z\overline{\alpha_1}|)$. Then

$$\begin{aligned}
\mathcal{A}_{1,1} \lesssim{}& \frac{(1-|z|^2)^{-s}}{|z|}\int_{B_1} |\overrightarrow{\nabla}_{\mathbb{B}^n,\,z} P_s(z,\eta)|\left|\frac{1}{|1-\alpha_1\overline{z_0}|^{n-s}} - \frac{1}{|1-\alpha_2\overline{z_0}|^{n-s}}\right| d\sigma(\eta)\\
&+ \frac{(1-|z|^2)^{-s}}{|z|}\int_{B_1} |\overrightarrow{\nabla}_{\mathbb{B}^n,\,z} P_s(z,\eta)|\left|\frac{1}{|1-\alpha_1\overline{\eta}|^{n-s}} - \frac{1}{|1-\alpha_2\overline{\eta}|^{n-s}}\right| d\sigma(\eta).
\end{aligned}$$

By Lemmas 5.1(1), 5.2 and 5.3, and using the fact that if $\eta \in B_1$, then $|1-z_0\overline{\eta}| \geq \delta|1-z_0\overline{\alpha_1}|$, we deduce that $\mathcal{A}_{1,1} \lesssim \frac{(1-|z|^2)^s|1-\alpha_1\overline{\alpha_2}|^{1/2}}{|1-z\overline{\alpha_1}|^{n+s+1/2}}$.

So we are left to estimate $\mathcal{A}_{1,2}$, that is, $\eta \in B_2 = B(z_0,\delta|1-z\overline{\alpha_1}|)$. Let $\alpha(u),\beta(v)$, $u,v \in [0,1]$, be two complex-tangential curves such that $\alpha(1)=\alpha_1$, $\alpha(0)=\alpha_2$, $\operatorname{Long}(\alpha) \leq 2|1-\alpha_1\overline{\alpha_2}|^{1/2}$, and $\beta(0)=z_0$, $\beta(1)=\eta$, $\operatorname{Long}(\beta) \leq 2|1-z\overline{\eta}|^{1/2}$ (a proof of the existence of these curves can be found, for instance, in [6]). As a consequence, for any u,v we have $|1-\alpha(u)\overline{\beta(v)}| \simeq$

$|1-z_0\overline{\alpha_1}|$. We then find that

$$\left|\left(\frac{1}{|1-\alpha_1\overline{\eta}|^{n-s}}-\frac{1}{|1-\alpha_2\overline{\eta}|^{n-s}}\right)-\left(\frac{1}{|1-\alpha_1\overline{z_0}|^{n-s}}-\frac{1}{|1-\alpha_2\overline{z_0}|^{n-s}}\right)\right|$$
$$=\left|\int_0^1\frac{d}{du}\left(\frac{1}{|1-\alpha(u)\overline{\eta}|^{n-s}}-\frac{1}{|1-\alpha(u)\overline{\eta}|^{n-s}}\right)du\right|$$
$$=\left|\int_0^1\frac{d}{dv}\frac{d}{du}\int_0^1\frac{1}{|1-\alpha(u)\overline{\beta(v)}|^{n-s}}\,du\,dv\right|\lesssim\frac{|1-\alpha_1\overline{\alpha_2}|^{1/2}|1-z_0\overline{\eta}|^{1/2}}{|1-z_0\overline{\alpha_1}|^{n+1-s}}.$$

Hence, since $|1-z_0\overline{\eta}|\le\delta|1-z\overline{\alpha_1}|$ for $\eta\in B_2$, we have

$$\mathcal{A}_{1,2}\lesssim\int_{B_1}\frac{(1-|z|^2)^s}{|1-z_0\overline{\eta}|^{n+2s}}\,\frac{|1-\alpha_1\overline{\alpha_2}|^{1/2}|1-z_0\overline{\eta}|^{1/2}}{|1-z_0\overline{\alpha_1}|^{n+1-s}}\,d\sigma(\eta)$$
$$\lesssim\frac{(1-|z|^2)^s|1-\alpha_1\overline{\alpha_2}|^{1/2}}{|1-z_0\overline{\alpha_1}|^{n+s+1/2}}.$$

5.4. Proof of (4) in Theorem 4.1 for K. We have to prove that there exist $\varepsilon,\delta>0$ such that for any $z\in\mathbb{B}^n$ and $\alpha_1,\alpha_2\in\mathbb{S}^n$ with $|1-\alpha_1\overline{\alpha}_2|\le\delta|1-\alpha_1\overline{\zeta}|$,

$$|\mathbf{K}(r\alpha_1,\zeta)-\mathbf{K}(r\alpha_2,\zeta)|\lesssim\frac{(1-r^2)^{\varepsilon}|1-\alpha_1\overline{\alpha}_2|^{\varepsilon}}{|1-r\zeta\overline{\alpha_1}|^{n+2\varepsilon}}.$$

We fix $\alpha_1,\alpha_2,\zeta\in\mathbb{S}^n$ such that $|1-\alpha_1\overline{\alpha_2}|<\delta|1-\alpha_1\overline{\zeta}|$. Then we find that $|1-\alpha_1\overline{\zeta}|\simeq|1-\alpha_2\overline{\zeta}|$. Without loss of generality, in the region of integration, we will assume that $|1-\alpha_1\overline{\eta}|\le|1-\alpha_2\overline{\eta}|$ and we will denote this set by $\mathbb{S}^n_1$.

With an abuse of notation we write

$$\overrightarrow{\nabla}_{\mathbb{B}^n}\frac{(1-r^2)^{2s}}{|1-r\alpha_1\overline{\eta}|^{n+2s}}=\overrightarrow{\nabla}_{\mathbb{B}^n,z}\frac{(1-|z|^2)^{2s}}{|1-z\overline{\eta}|^{n+2s}}_{|z=r\alpha_1}.$$

Since by Lemma 5.1(2), the function $\varphi_s(z):=\int_{\mathbb{B}^n}\overrightarrow{\nabla}_{\mathbb{B}^n}P_s(z,\zeta)\,d\sigma(\zeta)$ is radial, we obtain

(5.4) $\quad|\mathbf{K}(r\alpha_1,\zeta)-\mathbf{K}(r\alpha_2,\zeta)|$

$$=\frac{(1-r^2)^{-s}}{r}\left|\int_{\mathbb{S}^n}\left(\overrightarrow{\nabla}_{\mathbb{B}^n}\frac{(1-r^2)^{2s}}{|1-r\alpha_1\overline{\eta}|^{n+2s}}-\overrightarrow{\nabla}_{\mathbb{B}^n}\frac{(1-r^2)^{2s}}{|1-r\alpha_2\overline{\eta}|^{n+2s}}\right)\frac{d\sigma(\eta)}{|1-\eta\overline{\zeta}|^{n-s}}\right|$$
$$=\frac{(1-r^2)^{-s}}{r}\left|\int_{\mathbb{S}^n}\left(\overrightarrow{\nabla}_{\mathbb{B}^n}\frac{(1-r^2)^{2s}}{|1-r\alpha_1\overline{\eta}|^{n+2s}}-\overrightarrow{\nabla}_{\mathbb{B}^n}\frac{(1-r^2)^{2s}}{|1-r\alpha_2\overline{\eta}|^{n+2s}}\right)\right.$$
$$\left.\times\left(\frac{1}{|1-\eta\overline{\zeta}|^{n-s}}-\frac{1}{|1-r\alpha_1\overline{\zeta}|^{n-s}}\right)d\sigma(\eta)\right|.$$

We need to obtain a suitable estimate for

$$\left|\overrightarrow{\nabla}_{\mathbb{B}^n}\frac{(1-r^2)^{2s}}{|1-r\alpha_1\overline{\eta}|^{n+2s}}-\overrightarrow{\nabla}_{\mathbb{B}^n}\frac{(1-r^2)^{2s}}{|1-r\alpha_2\overline{\eta}|^{n+2s}}\right|. \tag{5.5}$$

The proof is straightforward, by evaluating carefully the different derivatives involved in (5.5). For the radial derivative R_z (the argument for $\overline{R}_z$ is completely analogous), we obtain, using Lemmas 5.2 and 5.4,

$$\begin{aligned}(1-r^2)&\left|R_z\left(\frac{(1-|z|^2)^{2s}}{|1-z\overline{\eta}|^{n+2s}}\right)_{|z=r\alpha_1}-R_z\left(\frac{(1-|z|^2)^{2s}}{|1-z\overline{\eta}|^{n+2s}}\right)_{|z=r\alpha_2}\right|\\ &\lesssim (1-r^2)^{2s}r\frac{|1-\alpha_1\overline{\alpha_2}|^{1/2}}{|1-r\alpha_1\overline{\eta}|^{n+2s}|1-r\alpha_2\overline{\eta}|^{1/2}}.\end{aligned}$$

A careful calculation gives

$$\begin{aligned}(1-r^2)^{1/2}&\left|L_{ij}\left(\frac{(1-|z|^2)^{2s}}{|1-z\overline{\eta}|^{n+2s}}\right)_{|z=r\alpha_1}-L_{ij}\left(\frac{(1-|z|^2)^{2s}}{|1-z\overline{\eta}|^{n+2s}}\right)_{|z=r\alpha_2}\right|\\ &\lesssim r(1-r^2)^{2s}\frac{|1-\alpha_1\overline{\alpha_2}|^{1/2}}{|1-r\alpha_1\overline{\eta}|^{n+2s}|1-r\alpha_2\overline{\eta}|^{1/2}}.\end{aligned}$$

Thus, we have obtained for (5.5) the estimate

$$\begin{aligned}&\left|\overrightarrow{\nabla}_{\mathbb{B}^n}\frac{(1-r^2)^{2s}}{|1-r\alpha_1\overline{\eta}|^{n+2s}}-\overrightarrow{\nabla}_{\mathbb{B}^n}\frac{(1-r^2)^{2s}}{|1-r\alpha_2\overline{\eta}|^{n+2s}}\right|\\ &\qquad\lesssim r(1-r^2)^{2s}\frac{|1-\alpha_1\overline{\alpha_2}|^{1/2}}{|1-r\alpha_1\overline{\eta}|^{n+2s}|1-r\alpha_2\overline{\eta}|^{1/2}}.\end{aligned}$$

On the other hand, using again Lemma 5.2, we get

$$\begin{aligned}&\left|\frac{1}{|1-\eta\overline{\zeta}|^{n-s}}-\frac{1}{|1-r\alpha_1\overline{\zeta}|^{n-s}}\right|\\ &\qquad\lesssim\frac{|1-r\alpha_1\overline{\eta}|^{1/2}}{|1-\eta\overline{\zeta}|^{n-s}|1-r\alpha_1\overline{\zeta}|^{1/2}}+\frac{|1-r\alpha_1\overline{\eta}|^{1/2}}{|1-\eta\overline{\zeta}|^{1/2}|1-r\alpha_1\overline{\zeta}|^{n-s}}.\end{aligned}$$

We can now finally proceed to bound (5.4). If we choose $0<\varepsilon<1/2-2s$, and use $|1-\alpha_1\overline{\alpha_2}|\lesssim|1-\alpha_2\overline{\eta}|$, we conclude, by the above estimates, that

$$\begin{aligned}&|\mathbf{K}(r\alpha_1,\zeta)-\mathbf{K}(r\alpha_2,\zeta)|\\ &\quad\lesssim\int_{\mathbb{S}^n}\frac{(1-r^2)^s|1-\alpha_1\overline{\alpha_2}|^{1/2}}{|1-r\alpha_1\overline{\eta}|^{n+2s}|1-r\alpha_2\overline{\eta}|^{1/2}}\\ &\qquad\times\left(\frac{|1-r\alpha_1\overline{\eta}|^{1/2}}{|1-\eta\overline{\zeta}|^{n-s}|1-r\alpha_1\overline{\zeta}|^{1/2}}+\frac{|1-r\alpha_1\overline{\eta}|^{1/2}}{|1-\eta\overline{\zeta}|^{1/2}|1-r\alpha_1\overline{\zeta}|^{n-s}}\right)d\sigma(\eta)\end{aligned}$$

$$\lesssim \int_{\mathbb{S}^n} \frac{(1-r^2)^s|1-\alpha_1\overline{\alpha_2}|^\varepsilon}{|1-r\alpha_1\overline{\eta}|^{n+2s-1/2+\varepsilon}}$$
$$\times \left(\frac{1}{|1-\eta\overline{\zeta}|^{n-s}|1-r\alpha_1\overline{\zeta}|^{1/2}} + \frac{1}{|1-\eta\overline{\zeta}|^{1/2}|1-r\alpha_1\overline{\zeta}|^{n-s}}\right) d\sigma(\eta)$$
$$\lesssim \frac{(1-r^2)^s|1-\alpha_1\overline{\alpha_2}|^\varepsilon}{|1-r\alpha_1\overline{\zeta}|^{n+s+\varepsilon}}.$$

This ends the proof of Theorem 5.5.

6. $K_s^{-1}K_s^{-1}K_{2s}$ is a Calderón–Zygmund operator. We begin the section by recalling a fundamental result (see for instance [15, Thm. 7.11] for the version in $\mathbb{R}^n$, [5] for the homogeneous version, and the references therein).

THEOREM 6.1. *Assume T is a* Calderón–Zygmund kernel *on $\mathbb{S}^n \times \mathbb{S}^n \setminus \{\zeta = \eta\}$, that is, it satisfies the following assumptions:*

(1) *T defines a bounded operator from $L^2(\mathbb{S}^n)$ to $L^2(\mathbb{S}^n)$.*
(2) $|T(\zeta,\eta)| \lesssim \frac{1}{|1-\zeta\overline{\eta}|^n}$.
(3) *There exist $\varepsilon, \delta > 0$ such that if $|1-\zeta_1\overline{\zeta_2}| < \varepsilon|1-\zeta_1\overline{\eta}|$, then*
$$|T(\zeta_1,\eta) - T(\zeta_2,\eta)| \lesssim \left(\frac{|1-\zeta_1\overline{\zeta_2}|}{|1-\zeta_1\overline{\eta}|}\right)^\delta \frac{1}{|1-\zeta_1\overline{\eta}|^n}.$$
(4) *There exist $\varepsilon, \delta > 0$ such that if $|1-\eta_1\overline{\eta_2}| < \varepsilon|1-\zeta\overline{\eta_1}|$, then*
$$|T(\zeta,\eta_1) - T(\zeta,\eta_2)| \lesssim \left(\frac{|1-\eta_1\overline{\eta_2}|}{|1-\zeta\overline{\eta_1}|}\right)^\delta \frac{1}{|1-\zeta\overline{\eta_1}|^n}.$$

Then, for any weight w in the Muckenhoupt class $A^2(\mathbb{S}^n)$, the operator T is bounded from $L^2(w)$ to $L^2(w)$.

Observe that the non-isotropic operator $K_s^{-1}K_s^{-1}K_{2s}$ is not the identity. Using the above result, we will prove that it is a Calderón–Zygmund operator:

THEOREM 6.2. *Let $0 < s < 1/4$ and let $T = K_s^{-1}K_s^{-1}K_{2s}$. Then T is a Calderón–Zygmund operator, and in particular, for any weight w in the Muckenhoupt class $A^2(\mathbb{S}^n)$, T is bounded from $L^2(w)$ to $L^2(w)$.*

We first prove the following proposition:

PROPOSITION 6.3. *Let $0 < s < 1/4$. Then the operator $K_s^{-1}K_{2s}$ on $L^2(\mathbb{S}^n)$ is defined by a kernel $\phi(\zeta,\eta)$ that satisfies the following estimates:*

(1) $|\phi(\zeta,\eta)| \lesssim \frac{1}{|1-\zeta\overline{\eta}|^{n-s}}$.
(2) *There exists $\varepsilon > 0$ such that if $|1-\eta_1\overline{\eta_2}| < \varepsilon|1-\zeta\overline{\eta_1}|$, then*
$$|\phi(\zeta,\eta_1) - \phi(\zeta,\eta_2)| \lesssim \frac{|1-\eta_1\overline{\eta_2}|^{2s}}{|1-\zeta\overline{\eta_1}|^{n-s+2s}}.$$

(3) *There exist $\varepsilon > 0$ such that if $|1-\zeta_1\overline{\zeta_2}| < \varepsilon|1-\zeta_1\overline{\eta}|$, then*

$$|\phi(\zeta_1,\eta)-\phi(\zeta_2,\eta)| \lesssim \frac{|1-\zeta_1\overline{\zeta_2}|^{1/2-s}}{|1-\zeta_1\overline{\eta}|^{n-s+1/2-s}}.$$

Proof. We first observe that $K_s^{-1}[K_{2s}]$ is an operator from $L^2(\mathbb{S}^n)$ to itself, since it is enough to consider the expression of both operators K_s^{-1} and K_{2s} on harmonic polynomials (see (3.1)).

We also observe that by Corollary 3.7, on regular functions, $K_s^{-1}[\psi] = ((-\Delta)^{s/2}+I)\psi$. We will then infer that on regular functions, the operator $K_s^{-1}[K_{2s}]$ is defined by the kernel

$$\begin{aligned}(6.1)\quad &\phi(\zeta,\eta)\\ &:= \int_{\mathbb{S}^n} \frac{1}{|1-\zeta\overline{\omega}|^{n+s}}\left(\frac{1}{|1-\omega\overline{\eta}|^{n-2s}}-\frac{1}{|1-\zeta\overline{\eta}|^{n-2s}}\right)d\sigma(\omega)+\frac{k}{|1-\zeta\overline{\eta}|^{n-2s}}.\end{aligned}$$

We begin by proving assertion (1). Since the second summand on the right of 6.1 has a better estimate than the one in (1), we just have to deal with the first summand. By Lemma 5.2,

$$\left|\int_{\mathbb{S}^n} \frac{1}{|1-\zeta\overline{\omega}|^{n+s}}\left(\frac{1}{|1-\omega\overline{\eta}|^{n-2s}}-\frac{1}{|1-\zeta\overline{\eta}|^{n-2s}}\right)d\sigma(\omega)\right| \lesssim \frac{1}{|1-\zeta\overline{\eta}|^{n-s}}.$$

We next prove (2). There exists $C>0$ such that

(6.2)

$$\begin{aligned}|\phi(\zeta,\eta_1)-\phi(\zeta,\eta_2)| \lesssim \int_{\mathbb{S}^n} \frac{1}{|1-\zeta\overline{\omega}|^{n+s}}\bigg|&\frac{1}{|1-\omega\overline{\eta_1}|^{n-2s}}-\frac{1}{|1-\zeta\overline{\eta_1}|^{n-2s}}\\ &-\left(\frac{1}{|1-\omega\overline{\eta_2}|^{n-2s}}-\frac{1}{|1-\zeta\overline{\eta_2}|^{n-2s}}\right)\bigg|\,d\sigma(\omega)\\ &+C\left|\frac{1}{|1-\zeta\overline{\eta_1}|^{n-2s}}-\frac{1}{|1-\zeta\overline{\eta_2}|^{n-2s}}\right|.\end{aligned}$$

The estimate for the second summand is immediate, since by Lemma 5.2,

$$\left|\frac{1}{|1-\zeta\overline{\eta_1}|^{n-2s}}-\frac{1}{|1-\zeta\overline{\eta_2}|^{n-2s}}\right| \lesssim \frac{|1-\eta_1\overline{\eta_2}|^{1/2}}{|1-\zeta\overline{\eta_1}|^{n-2s+1/2}}+\frac{|1-\eta_1\overline{\eta_2}|^{1/2}}{|1-\zeta\overline{\eta_2}|^{n-2s+1/2}},$$

which is an estimate of the type (2).

We next deal with the first summand. We decompose the unit sphere into two regions and estimate each of the corresponding integrals separately:

- $B(\zeta, \frac{|1-\zeta\overline{\eta_1}|}{k})$, $k>0$ large enough to be chosen later on,
- $B(\zeta, \frac{|1-\zeta\overline{\eta_1}|}{k})^c$.

We begin with the estimate of the integral over $B(\zeta, \frac{|1-\zeta\overline{\eta_1}|}{k})$, which we denote by $\mathcal{A}_1$.

We choose $\varepsilon > 0$ small enough and $k > 0$ large enough so that the points ζ and $\omega \in B(\zeta, |1-\zeta\overline{\eta_1}|/k)$ and η_1, η_2 can be joined by two complex-tangential curves $\alpha(t)$ and $\beta(u)$ of lengths bounded, up to a constant, by $|1-\zeta\overline{\omega}|^{1/2}$ and $|1-\eta_1\overline{\eta_2}|^{1/2}$, respectively (see [6]), and in addition, for any t, u, we have $|1-\alpha(t)\overline{\beta(u)}| \simeq |1-\zeta\overline{\eta_1}|$. Then, for $\omega \in B(\zeta, |1-\zeta\overline{\eta_1}|/k)$,

$$(6.3)\qquad \left|\frac{1}{|1-\omega\overline{\eta_1}|^{n-2s}} - \frac{1}{|1-\zeta\overline{\eta_1}|^{n-2s}} - \left(\frac{1}{|1-\omega\overline{\eta_2}|^{n-2s}} - \frac{1}{|1-\zeta\overline{\eta_2}|^{n-2s}}\right)\right|$$
$$= \left|\int_0^1 \frac{d}{du}\left(\frac{1}{|1-\omega\overline{\beta(u)}|^{n-2s}} - \frac{1}{|1-\zeta\overline{\beta(u)}|^{n-2s}}\right) du\right|$$
$$= \left|\int_0^1\int_0^1 \frac{d}{du}\frac{d}{ds}\frac{1}{|1-\alpha(t)\overline{\beta(u)}|^{n-2s}}\, dt\, du\right| \lesssim \frac{|1-\zeta\overline{\omega}|^{1/2}\,|1-\eta_1\overline{\eta_2}|^{1/2}}{|1-\zeta\overline{\eta_1}|^{n-2s+1}}.$$

Plugging the above estimate into the integral over $B(\zeta, |1-\zeta\overline{\eta_1}|/k)$, we get

$$\mathcal{A}_1 \lesssim \int_{B(\zeta, \frac{|1-\zeta\overline{\eta_1}|}{k})} \frac{|1-\eta_1\overline{\eta_2}|^{1/2}}{|1-\zeta\overline{\omega}|^{n+s-1/2}|1-\zeta\overline{\eta_1}|^{n-2s+1}}\, d\sigma(\omega)$$
$$\lesssim \frac{|1-\eta_1\overline{\eta_2}|^{1/2}}{|1-\zeta\overline{\eta_1}|^{n-s+1/2}}.$$

The estimate of the integral over $B(\zeta, |1-\zeta\overline{\eta_1}|/k)^c$ follows from Lemma 5.2.

Next, we prove (3). We assume that $|1-\zeta_1\overline{\zeta_2}| < \varepsilon|1-\zeta_1\overline{\eta}|$. Arguing as in the previous statement, using Corollary 3.7, we have to estimate

$$(6.4)\qquad \int_{\mathbb{S}^n} \left\{\frac{1}{|1-\zeta_1\overline{\omega}|^{n+s}}\left(\frac{1}{|1-\omega\overline{\eta}|^{n-2s}} - \frac{1}{|1-\zeta_1\overline{\eta}|^{n-2s}}\right)\right.$$
$$\left. - \frac{1}{|1-\zeta_2\overline{\omega}|^{n+s}}\left(\frac{1}{|1-\omega\overline{\eta}|^{n-2s}} - \frac{1}{|1-\zeta_2\overline{\eta}|^{n-2s}}\right)\right\} d\sigma(\omega)$$
$$+ C\left(\frac{|1-\zeta_1\overline{\zeta_2}|^{1/2}}{|1-\zeta_1\overline{\eta}|^{n-2s+1/2}} + \frac{|1-\zeta_1\overline{\zeta_2}|^{1/2}}{|1-\zeta_2\overline{\eta}|^{n-2s+1/2}}\right).$$

It is clear that the two last summands on the right satisfy the required estimate. So it remains to estimate the first one. We consider the region of integration $\{\omega : |1-\zeta_1\overline{\omega}| < |1-\zeta_2\overline{\omega}|\}$, the argument being analogous for the complementary region.

Integrating over the ball defined by $B(\zeta_1, \varepsilon_1|1-\zeta_1\overline{\eta}|)$, $B(\eta, \varepsilon_1|1-\zeta_1\overline{\eta}|)$ and $\Omega = \mathbb{S}^n \setminus (B(\zeta_1, \varepsilon_1|1-\zeta_1\overline{\eta}|) \cup B(\eta, \varepsilon_1|1-\zeta_1\overline{\eta}|))$, we conclude that the first summand in (6.4) is bounded by $\frac{|1-\zeta_1\overline{\zeta_2}|^{1/2}}{|1-\zeta_1\overline{\eta}|^{n-s+1/2}}$. ∎

The following technical result will be used later on to finish the proof of assertion (4) of Theorem 6.1.

LEMMA 6.4. *Let $0 < s < 1/4$ and $\zeta, \omega, \eta_1, \eta_2 \in \mathbb{S}^n$. Then there exists $\varepsilon > 0$ such that if $|1-\omega\overline{\zeta}| < \varepsilon|1-\omega\overline{\eta_1}|$ and $|1-\eta_1\overline{\eta_2}| < \varepsilon|1-\omega\overline{\eta_1}|$, then*

$$|\phi(\omega,\eta_1)-\phi(\zeta,\eta_1)-(\phi(\omega,\eta_2)-\phi(\zeta,\eta_2))| \lesssim \frac{|1-\zeta\overline{\omega}|^{1/2-s}|1-\eta_1\overline{\eta_2}|^{2s}}{|1-\zeta\overline{\eta_1}|^{n+1/2}}.$$

Proof. We have, by Lemma 5.3,

$$\begin{aligned}\phi(\zeta,\eta) &= \int_{\mathbb{S}^n} \frac{1}{|1-\zeta\overline{\xi}|^{n+s}}\left(\frac{1}{|1-\xi\overline{\eta}|^{n-2s}} - \frac{1}{|1-\zeta\overline{\eta}|^{n-2s}}\right) d\sigma(\xi) + \frac{k}{|1-\zeta\overline{\eta}|^{n-2s}}\\ &=: \phi_1(\zeta,\eta) + \frac{k}{|1-\zeta\overline{\eta}|^{n-2s}}.\end{aligned}$$

We begin by observing that the part of the estimate corresponding to the term $\frac{k}{|1-\zeta\overline{\eta}|^{n-2s}}$ follows as in (6.3), by joining ζ and ω and η_1 and η_2 by complex-tangential curves with respective lengths $|1-\zeta\overline{\omega}|^{1/2}$ and $|1-\eta_1\overline{\eta_2}|^{1/2}$ and observing that the complex-tangential derivatives are bounded by $\frac{1}{|1-\zeta\overline{\eta_1}|^{n-2s+1}}$. Consequently, these terms are bounded by

$$\frac{|1-\zeta\overline{\omega}|^{1/2}|1-\eta_1\overline{\eta_2}|^{1/2}}{|1-\zeta\overline{\eta_1}|^{n-2s+1}},$$

which is better than the desired estimates.

Hence, the lemma will follow once we obtain the desired estimates for ϕ_1 in place of ϕ. If we denote by ξ the integration variable in the definition of ϕ_1, we will assume in addition that we are in the region $\Omega = \{\xi : |1-\xi\overline{\omega}| < |1-\xi\overline{\zeta}|,\ |1-\xi\overline{\eta_1}| < |1-\xi\overline{\eta_2}|\}$. The other cases can be proved analogously.

As in the previous cases, for some ε_1 small enough, we split the unit sphere into several regions:

$$\begin{aligned}S_1 &= B(\zeta, \varepsilon_1|1-\zeta\overline{\eta_1}|) \cap \Omega,\\ S_2 &= B(\eta_1, \varepsilon_1|1-\zeta\overline{\eta_1}|) \cap \Omega,\\ S_3 &= (\mathbb{S}^n \setminus (S_1 \cup S_2)) \cap \Omega.\end{aligned}$$

We observe that

$$\begin{aligned}&\phi_1(\omega,\eta_1)-\phi_1(\zeta,\eta_1)-(\phi_1(\omega,\eta_2)-\phi_1(\zeta,\eta_2))\\ &\quad= \left(\frac{1}{|1-\omega\overline{\xi}|^{n+s}} - \frac{1}{|1-\zeta\overline{\xi}|^{n+s}}\right)\\ &\qquad\times\left(\frac{1}{|1-\xi\overline{\eta_1}|^{n-2s}} - \frac{1}{|1-\xi\overline{\eta_2}|^{n-2s}} - \frac{1}{|1-\omega\overline{\eta_1}|^{n-2s}} + \frac{1}{|1-\omega\overline{\eta_2}|^{n-2s}}\right)\\ &\qquad- \frac{1}{|1-\zeta\overline{\xi}|^{n+s}}\\ &\qquad\times\left(\frac{1}{|1-\omega\overline{\eta_1}|^{n-2s}} - \frac{1}{|1-\omega\overline{\eta_2}|^{n-2s}} - \frac{1}{|1-\zeta\overline{\eta_1}|^{n-2s}} + \frac{1}{|1-\zeta\overline{\eta_2}|^{n-2s}}\right)\\ &\quad= A+B.\end{aligned}$$

The argument used in (6.3) implies that

$$\int_{S_1} |A|\, d\sigma(\xi) \lesssim \frac{|1-\omega\overline{\zeta}|^{1/2-s}|1-\eta_1\overline{\eta_2}|^{1/2}}{|1-\zeta\overline{\eta_1}|^{n-2s+1}} \lesssim \frac{|1-\omega\overline{\zeta}|^{1/2-s}|1-\eta_1\overline{\eta_2}|^{2s}}{|1-\zeta\overline{\eta_1}|^{n+1/2}}.$$

Furthermore, using $|1-\zeta\overline{\omega}| \lesssim |1-\zeta\overline{\xi}|$, we have

$$\int_{S_1} |B|\, d\sigma(\xi) \lesssim \frac{|1-\omega\overline{\zeta}|^{1/2-s}|1-\eta_1\overline{\eta_2}|^{2s}}{|1-\zeta\overline{\eta_1}|^{n+1/2}}.$$

We follow with the estimates over S_2:

$$\int_{S_2} |A|\, d\sigma(\xi) \lesssim \frac{|1-\omega\overline{\zeta}|^{1/2}|1-\eta_1\overline{\eta_2}|^{2s}}{|1-\zeta\overline{\eta_1}|^{n+s+1/2}} + \frac{|1-\omega\overline{\zeta}|^{1/2}|1-\eta_1\overline{\eta_2}|^{1/2}}{|1-\zeta\overline{\eta_1}|^{n-s+1}}.$$

In the estimate of the integral of B over S_2, we again use the argument in (6.3) to obtain

$$\begin{aligned}\int_{S_2} |B|\, d\sigma(\xi) &\lesssim \int_{S_2} \frac{1}{|1-\zeta\overline{\eta_1}|^{n+s}} \frac{|1-\omega\overline{\zeta}|^{1/2}|1-\eta_1\overline{\eta_2}|^{1/2}}{|1-\zeta\overline{\eta_1}|^{n-2s+1}}\, d\sigma(\xi) \\ &\lesssim \frac{|1-\omega\overline{\zeta}|^{1/2}|1-\eta_1\overline{\eta_2}|^{1/2}}{|1-\zeta\overline{\eta_1}|^{n-s+1}}.\end{aligned}$$

A straightforward calculation also gives

$$\int_{S_3} (|A|+|B|)\, d\sigma(\xi) \lesssim \frac{|1-\omega\overline{\zeta}|^{1/2}|1-\eta_1\overline{\eta_2}|^{1/2}}{|1-\zeta\overline{\eta_1}|^{n-s+1}}.$$

This ends the proof of Lemma 6.4. ∎

6.1. Proof of Theorem 6.2. We recall that we are assuming that $0 < s < 1/4$ and we want to prove that $T = K_s^{-1}K_s^{-1}K_{2s}$ is a Calderón–Zygmund operator; this will be deduced by checking that T satisfies the hypotheses in Theorem 6.1.

The fact that assertion (1) in Theorem 6.1 holds is a consequence of the developments of K_{2s} and K_s^{-1} in harmonic homogeneous polynomials and the expression of the operator T in terms of its development (3.1) in harmonic polynomials.

Let us check (2) in Theorem 6.1. From Corollary 3.7 we see that the kernel associated to $K_s^{-1}K_s^{-1}K_{2s}$ is $T(\zeta,\eta) = T_1(\zeta,\eta) + C\phi(\zeta,\eta)$, where

$$T_1(\zeta,\eta) = \int_{\mathbb{S}^n} \frac{1}{|1-\zeta\overline{\omega}|^{n+s}}(\phi(\omega,\eta)-\phi(\zeta,\eta))\, d\sigma(\omega),$$

and ϕ is as in Proposition 6.3. We will follow the methods used in that proposition, replacing $\frac{1}{|1-\omega\overline{\eta}|^{n-2s}}$ by $\phi(\omega,\eta)$ and taking into account the estimates obtained for the kernel ϕ. Using the proposition we deduce that

$|\phi(\zeta,\eta)| \lesssim \frac{1}{|1-\zeta\overline{\eta}|^{n-s}} \lesssim \frac{1}{|1-\zeta\overline{\eta}|^{n}}$. Next, the estimate $|T_1(\zeta,\eta)| \lesssim \frac{1}{|1-\zeta\overline{\eta}|^{n}}$ follows easily by splitting the integral into two parts, over $|1-\zeta\overline{\omega}| < \varepsilon|1-\zeta\overline{\eta}|$ and over the complement.

We next examine assertion (3) in Theorem 6.1. We have

$$\begin{aligned}(6.5)\quad & |T(\zeta_1,\eta)-T(\zeta_2,\eta)| \\ &= \int_{\mathbb{S}^n} \frac{1}{|1-\zeta_1\overline{\omega}|^{n+s}}(\phi(\omega,\eta)-\phi(\zeta_1,\eta))\,d\sigma(\omega) \\ &\quad - \int_{\mathbb{S}^n} \frac{1}{|1-\zeta_2\overline{\omega}|^{n+s}}(\phi(\omega,\eta)-\phi(\zeta_2,\eta))\,d\sigma(\omega) + C(\phi(\zeta_1,\eta)-\phi(\zeta_2,\eta)).\end{aligned}$$

Using Proposition 6.3(3), we find that the last summand on the right is bounded, up to a constant, by $\frac{|1-\zeta_1\overline{\zeta_2}|^{1/2-s}}{|1-\zeta_1\overline{\eta}|^{n-s+1/2-s}}$, which is better than the required estimate. So we estimate the first two terms. We consider the integrands over $\Omega = \{\omega : |1-\zeta_1\overline{\omega}| < |1-\zeta_2\overline{\omega}|\}$ (the other situation is analogous). We split the integrals into three regions:

- $S_1 = B(\eta, \varepsilon_1|1-\eta\overline{\zeta_1}|)\cap\Omega$ for some ε_1 small enough,
- $S_2 = B(\zeta_1, \varepsilon_1|1-\eta\overline{\zeta_1}|)\cap\Omega$,
- $S_3 = \Omega\setminus(S_1\cup S_2)$.

We begin with the integrals in (6.5) over S_1. Rewriting the integrand and using Proposition 6.3(1, 3), we have

$$\begin{aligned}\Bigg| \int_{S_1} \bigg(\frac{1}{|1-\zeta_1\overline{\omega}|^{n+s}} - \frac{1}{|1-\zeta_2\overline{\omega}|^{n+s}}\bigg)(\phi(\omega,\eta)-\phi(\zeta_1,\eta))\,d\sigma(\omega) \\ + \int_{S_1} \frac{1}{|1-\zeta_2\overline{\omega}|^{n+s}}(-\phi(\zeta_1,\eta)+\phi(\zeta_2,\eta))\,d\sigma(\omega)\Bigg| \\ \lesssim \frac{|1-\zeta_1\overline{\zeta_2}|^{1/2-s}}{|1-\zeta_1\overline{\eta}|^{n+1/2-s}}.\end{aligned}$$

We next estimate the integrals in (6.5) over S_2. Since we are assuming that $|1-\zeta_1\overline{\omega}| < |1-\zeta_2\overline{\omega}|$, we have $|1-\zeta_1\overline{\zeta_2}| \lesssim |1-\zeta_2\overline{\omega}|$. Hence, an easy calculation gives

$$\begin{aligned}\Bigg| \int_{S_2} \bigg(\frac{1}{|1-\zeta_1\overline{\omega}|^{n+s}} - \frac{1}{|1-\zeta_2\overline{\omega}|^{n+s}}\bigg)(\phi(\omega,\eta)-\phi(\zeta_1,\eta))\,d\sigma(\omega) \\ + \int_{S_2} \frac{1}{|1-\zeta_2\overline{\omega}|^{n+s}}(-\phi(\zeta_1,\eta)+\phi(\zeta_2,\eta))\,d\sigma(\omega)\Bigg| \\ \lesssim \frac{|1-\zeta_1\overline{\zeta_2}|^{1/2-2s}}{|1-\zeta_1\overline{\eta}|^{n-2s+1/2}}.\end{aligned}$$

Next, we prove the estimate of the integrals in (6.5) over S_3. We use the same decomposition as in S_1 and we obtain (by Proposition 6.3(1, 3))

$$\int_{S_3} \left\{ \frac{|1-\zeta_1\overline{\zeta_2}|^{1/2}}{|1-\zeta_1\overline{\omega}|^{n+s+1/2}} \left(\frac{1}{|1-\omega\overline{\eta}|^{n-s}} + \frac{1}{|1-\zeta_1\overline{\eta}|^{n-s}} \right) + \frac{1}{|1-\zeta_2\overline{\omega}|^{n+s}} \frac{|1-\zeta_1\overline{\zeta_2}|^{1/2}}{|1-\zeta_1\overline{\eta}|^{n-s+1/2}} \right\} d\sigma(\omega)$$
$$\lesssim \frac{|1-\zeta_1\overline{\zeta_2}|^{1/2}}{|1-\zeta_1\overline{\eta}|^{s+1/2}} \frac{1}{|1-\zeta_1\overline{\eta}|^{n-s}} + \frac{1}{|1-\zeta_1\overline{\eta}|^{s}} \frac{|1-\zeta_1\overline{\zeta_2}|^{1/2}}{|1-\zeta_1\overline{\eta}|^{n-s+1/2}} \lesssim \frac{|1-\zeta_1\overline{\zeta_2}|^{1/2}}{|1-\zeta_1\overline{\eta}|^{n+1/2}}.$$

We now verify assertion (4) in Theorem 6.1. We assume that $|1-\eta_1\overline{\eta_2}|$ is much smaller than $|1-\zeta\overline{\eta_1}|$ and we will prove that there exists $\delta > 0$ with

$$|T(\zeta,\eta_1) - T(\zeta,\eta_2)| \lesssim \frac{|1-\eta_1\overline{\eta_2}|^{\delta}}{|1-\zeta\overline{\eta_1}|^{n+\delta}}.$$

We write

$$\begin{aligned} &T(\zeta,\eta_1) - T(\zeta,\eta_2) \\ &\quad = \int_{\mathbb{S}^n} \frac{1}{|1-\zeta\overline{\omega}|^{n+s}} \big(\phi(\omega,\eta_1) - \phi(\zeta,\eta_1) - (\phi(\omega,\eta_2) - \phi(\zeta,\eta_2))\big)\, d\sigma(\omega) \\ &\quad\quad + C(\phi(\zeta,\eta_1) - \phi(\zeta,\eta_2)) \end{aligned}$$

By Proposition 6.3(2), the second summand satisfies the desired estimate, so we estimate the first. We proceed by decomposing the sphere into three sets:

- $S_1' = B(\zeta, \varepsilon|1-\zeta\overline{\eta_1}|)$ with $\varepsilon > 0$ small enough,
- $S_2' = B(\eta_1, \varepsilon|1-\eta_1\overline{\eta_2}|)$,
- $S_3' = \mathbb{S}^n \setminus (S_1' \cup S_2')$.

Lemma 6.4 shows that the integral over S_1' is bounded by $\frac{|1-\eta_1\overline{\eta_2}|^{2s}}{|1-\zeta\overline{\eta_1}|^{n+2s}}$.

For the integral over S_2', since we do not have the singularity on ζ, we estimate separately each of the summands, applying Proposition 6.3(1)(2), and we obtain

$$\int_{S_2} \frac{1}{|1-\zeta\overline{\omega}|^{n+s}} (|\phi(\omega,\eta_1) - \phi(\omega,\eta_2)| + |\phi(\zeta,\eta_1) - \phi(\zeta,\eta_2)|)\, d\sigma(\omega) \lesssim \frac{|1-\eta_1\overline{\eta_2}|^{s}}{|1-\zeta\overline{\eta_1}|^{n+s}}.$$

Finally, again by Proposition 6.3(2), the corresponding integral over S_3' is also bounded by $\frac{|1-\eta_1\overline{\eta_2}|^{s}}{|1-\zeta\overline{\eta_1}|^{n+s}}$.

7. Capacities, trace measures for $H^s(\mathbb{S}^n)$ and Carleson measures for $P_s[H^s(\mathbb{S}^n)]$

DEFINITION 7.1. Let $E \subset \mathbb{S}^n$ be a closed set in the unit sphere. The *nonisotropic Riesz capacity* of E is defined by

$$\mathrm{Cap}_s(E) := \inf\{\|f\|_2^2 : K_s(|f|) \geq 1 \text{ on } E\}.$$

We list some properties of the equilibrium measure for a compact set in $\mathbb{S}^n$, which will be used below and that are essentially due to O. Frostman (see [1, Theorem 2.2.7]):

THEOREM 7.2. *Given a closed set $E \subset \mathbb{S}^n$, there exists a positive capacitary measure ν_E on $\mathbb{S}^n$ such that:*

(1) *ν_E is supported on E and $\nu_E(E) = \mathrm{Cap}_s(E)$,*
(2) *$q_E := K_s K_s(\nu_E) \geq 1$ a.e. on E,*
(3) *$q_E \in H^s$ and $\|q_E\|^2_{H^s(\mathbb{S}^n)} \lesssim \mathrm{Cap}_s(E)$,*
(4) *there is a constant $C > 0$ independent of E such that for any $\zeta \in \mathbb{S}^n$, $q_E(\zeta) \leq C$.*

REMARK 7.3. We observe that by Lemma 5.3, if we consider $p_E := K_{2s}(\nu_E)$ instead of $q_E = K_s K_s[\nu_E]$, we have $p_E \simeq q_E$. The proof of the non-isotropic version of a result by I. E. Verbitsky (see [20, Chapter 2, Lemma 3.6]) shows that if $1 < \alpha < \frac{n}{n-2s}$, then both weights p_E^α and q_E^α are in the Muckenhoupt class A_1, that is, for any $\eta \in \mathbb{S}^n$ and any $\delta > 0$,

$$\frac{1}{\delta^n} \int_{B(\eta,\delta)} p_E^\alpha \, d\sigma \lesssim p_E^\alpha(\eta).$$

7.1. Regularization in $\mathbb{S}^n$. Although there is no natural group structure in $\mathbb{S}^n$, the following regularization procedure can be considered in the unit sphere (see [4]). For any $\ell \geq 1$, we consider a non-negative $\mathcal{C}^\infty$ function φ_ℓ, defined in the unit disc $\mathbb{D}$, radial, supported in $|z| \leq 1/\ell$ and satisfying $\int_{\mathbb{S}^n} \varphi_\ell(1 - \zeta\overline{\eta})\, d\sigma(\eta) = 1$. If $\psi \in \mathcal{C}(\mathbb{S}^n)$, we define its regularized function ψ_ℓ by

$$\psi_\ell(\zeta) := \int_{\mathbb{S}^n} \varphi_\ell(1 - \zeta\overline{\eta})\psi(\eta)\, d\sigma(\eta).$$

We observe that ψ_ℓ converges uniformly to ψ. If $\psi \in L^2(\mathbb{S}^n)$, then similar arguments to the classical case in $\mathbb{R}^n$ show that ψ_ℓ converges to ψ in $L^2(\mathbb{S}^n)$. If ν is a finite measure on $\mathbb{S}^n$, we define, as usual, its regularization ν_ℓ. By

$$\nu_\ell(\zeta) := \int_{\mathbb{S}^n} \varphi_\ell(1 - \zeta\overline{\eta})\, d\nu(\eta).$$

It is also clear that ν_ℓ converges weakly to ν.

For simplicity of notation, if ν_E is the capacitary measure of a closed set E, we write $\nu_{E,\ell} := (\nu_E)_\ell$ for the regularization of ν_E, and also $p_{E,\ell} := (p_E)_\ell$ for the regularization of the function p_E.

In [4, Lemma 1] it is shown that if g and h are measurable bounded functions defined on the unit ball in $\mathbb{C}^1$ and if $\zeta, \eta \in \mathbb{S}^n$, then

$$\int_{\mathbb{S}^n} g(\langle \zeta, \omega\rangle) h(\langle \omega, \eta\rangle)\, d\sigma(\omega) = \int_{\mathbb{S}^n} h(\langle \zeta, \omega\rangle) g(\langle \omega, \eta\rangle)\, d\sigma(\omega). \tag{7.1}$$

In particular, applying (7.1) to $g_r(z) = \frac{1}{|1-rz|^{n-2s}}$ for any $0 < r < 1$, and to $h(z) = \varphi_\ell(z)$, and letting $r \to 1^+$, we deduce by the Monotone Convergence Theorem that

$$\begin{aligned}\int_{\mathbb{S}^n} \frac{1}{|1-\zeta\overline{\eta}|^{n-2s}} \int_{\mathbb{S}^n} \varphi_\ell(1-\eta\overline{\omega})\, d\nu(\omega)\, d\sigma(\eta) \\ = \int_{\mathbb{S}^n} \frac{1}{|1-\eta\overline{\omega}|^{n-2s}} \int_{\mathbb{S}^n} \varphi_\ell(1-\zeta\overline{\eta})\, d\nu(\omega)\, d\sigma(\eta).\end{aligned}$$

Also, if ν_E is as above, we obtain $p_{E,\ell} = K_{2s}[\nu_{E,\ell}]$.

Lemma 7.4. *Let $E \subset \mathbb{S}^n$ be a compact set and let ν_E be the extremal measure. For $\ell \geq 1$, let $\nu_{E,\ell}$ be the regularized measure defined as before. Then, for any $\ell \geq 1$,*

(1) $\nu_{E,\ell}(\mathbb{S}^n) \simeq \mathrm{Cap}_s(E)$,
(2) *there exists $M > 0$ such that $p_{E,\ell} = K_{2s}[\nu_{E,\ell}] \leq M$.*

Proof. (1) is an immediate consequence of Fubini's Theorem and Theorem 7.2(1); and (2) is an immediate consequence of the boundedness of p_E by Theorem 7.2(4) and Remark 7.3. ∎

Proposition 7.5. *If $E \subset \mathbb{S}^n$ is a closed set and $1/2 < \alpha$, then*

$$\|p^\alpha_{E,\ell}\|^2_{H^s} \lesssim \mathrm{Cap}_s(E).$$

Proof. Since $p^\alpha_{E,\ell} \in \mathcal{C}^\infty(\mathbb{S}^n)$, from Corollary 3.11, Proposition 3.8 and Lemma 7.4 we deduce that

$$\begin{aligned}\|p^\alpha_{E,\ell}\|^2_{H^s_2} &\lesssim \|P_s[p_{E,\ell}]^\alpha\|^2_{\mathcal{W}^2_s} = \langle P_s[p_{E,\ell}]^\alpha, P_s[p_{E,\ell}]^\alpha\rangle_{\mathcal{W}^2_s} \\ &\simeq \langle P_s[p_{E,\ell}]^{2\alpha-1}, P_s[p_{E,\ell}]\rangle_{\mathcal{W}^2_s} \\ &\simeq \int_{\mathbb{S}^n} p^{2\alpha-1}_{E,\ell} K^{-1}_{2s}[p_{E,\ell}]\, d\sigma \simeq \int_{\mathbb{S}^n} p^{2\alpha-1}_{E,\ell}\, d\nu_{E,\ell} \lesssim \mathrm{Cap}_s(E). \ ∎\end{aligned}$$

We recall the characterization of the positive trace measures for $H^s(\mathbb{S}^n)$ (see for instance [1] for a proof in the isotropic case).

Proposition 7.6. *Let $0 < s < 1/2$ and let μ be a positive Borel measure on $\mathbb{S}^n$. Then μ is a trace measure for $H^s(\mathbb{S}^n)$, that is, $\int_{\mathbb{S}^n} |f|^2\, d\mu \lesssim \|f\|^2_{H^s(\mathbb{S}^n)}$*

for every $f \in H^s(\mathbb{S}^n)$, if and only if there exists $C_\mu > 0$ such that for any compact set $E \subset \mathbb{S}^n$, $\mu(E) \le C_\mu \mathrm{Cap}_s(E)$.

DEFINITION 7.7. Let $E \subset \mathbb{S}^n$. Then the *tent* $T(E)$ over E is defined by $T(E) = \mathbb{B}^n \setminus \bigcup_{\xi \notin E} D(\xi)$, where $D(\xi)$ is the admissible region introduced in Section 4.

REMARK 7.8. If we choose $0 < t < n$ and we add and subtract the term $\frac{1}{|1-z\overline{\eta}|^{n-t}}$ in the integrand defining the function $P_s[\frac{1}{|1-\zeta\,\overline{\cdot}|^{n-t}}]$, and we use the fact that $P_s[1]$ is a bounded function, then Lemmas 5.2 and 5.3 give

$$\int_{\mathbb{S}^n} \frac{(1-|z|^2)^{2s}}{|1-z\overline{\zeta}|^{n+2s}} \frac{d\sigma(\zeta)}{|1-\zeta\overline{\eta}|^{n-t}} \lesssim \frac{1}{|1-z\overline{\eta}|^{n-t}}.$$

Consequently, if for every closed set E in $\mathbb{S}^n$ we know that $\mu(T(E)) \lesssim \mathrm{Cap}_s(E)$, then μ is a Carleson measure for $P_s[K_s(L^2(\mathbb{S}^n))]$. Equivalently, $\int_{\mathbb{S}^n} |P_s[\varphi]|^2\, d\mu \lesssim \|\varphi\|^2_{K_s(L^2(\mathbb{S}^n))}$ for every $\varphi \in K_s(L^2(\mathbb{S}^n))$ (see [9] and the references therein).

In the proof of the main theorem in Section 8, we also need the following result on trace measures which is due to I. E. Verbitsky.

PROPOSITION 7.9 ([21]). *Let g be an integrable function on $\mathbb{S}^n$ such that $|g|^p\, d\sigma$ is a trace measure for $K_s[L^p(\mathbb{S}^n)]$. Let h be a measurable function on $\mathbb{S}^n$ for which there exists $C > 0$ such that for any weight w in A_1,*

$$\int_{\mathbb{S}^n} |h|^p w \le C \int_{\mathbb{S}^n} |g|^p w. \tag{7.2}$$

Then $|h|^p\, d\sigma$ is a trace measure for $K_s[L^p]$.

In particular, applying Theorem 6.2, we deduce the following:

COROLLARY 7.10. *If $|g|^2\, d\sigma$ is a trace measure for $K_s[L^2(\mathbb{S}^n)]$, then so also is the measure $|K_s^{-1} K_s^{-1} K_{2s}[g]|^2\, d\sigma$.*

8. Proof of Theorem 1.1. The theorem is a consequence of the following result:

THEOREM 8.1. *Let $0 < s < 1/4$ and let $\mathfrak{g}$ be a function in $L^2[\mathbb{S}^n]$ such that $K_{2s}[\mathfrak{g}]$ satisfies the conditions of either Theorem 3.3 or Theorem 3.5. Then the following assertions are equivalent:*

(1) *For any $\varphi \in \mathcal{C}^\infty(\mathbb{S}^n)$,*

$$\Big| \int_{\mathbb{S}^n} |\varphi|^2 \mathfrak{g}\, d\sigma \Big| \lesssim \|\varphi\|^2_{H^s(\mathbb{S}^n)}.$$

(2) *The measures $|RP_s[K_{2s}[\mathfrak{g}]]|^2 \frac{(1-|z|^2)^{1-2s}}{|z|^2}\, dV$, $|T_{ij}P_s[K_{2s}[\mathfrak{g}]]|^2 \frac{(1-|z|^2)^{-2s}}{|z|^2}\, dV$ and $|P_s[K_{2s}[\mathfrak{g}]]|^2 (1-|z|^2)^{-2s}\, dV$ are Carleson measures for the space $P_s[H^s(\mathbb{S}^n)]$.*

(3) *The measure $d\nu := |K_s[\mathfrak{g}]|^2\, d\sigma$ is a trace measure for $H^s(\mathbb{S}^n)$.*

8.1. Proof of (2)⇒(1) in Theorem 8.1. In fact, we will show that $\mathfrak{g}$ satisfies assertion (3) in Theorem 3.13, and consequently (1) in this theorem is fulfilled. We have

- $R(P_s[\varphi]P_s[\psi]) = RP_s[\varphi]P_s[\psi] + P_s[\varphi]RP_s[\psi]$,
- $T_{i,j}(P_s[\varphi]P_s[\psi]) = T_{i,j}P_s[\varphi]P_s[\psi] + P_s[\varphi]T_{i,j}P_s[\psi]$, $i < j$.

By Hölder's inequality, the hypothesis that $\frac{(1-|z|^2)^{1-2s}}{|z|^2}|RP_s[K_{2s}[\mathfrak{g}]]|^2\,dV(z)$ is a Carleson measure for $P_s(H^s)$, and Corollary 3.10, we deduce that

$$\begin{aligned}
&\left|\int_{\mathbb{B}^n} \frac{(1-|z|^2)^{1-2s}}{|z|^2} RP_s[\varphi]P_s[\psi]\overline{RP_s[K_{2s}[\mathfrak{g}]]}\,dV(z)\right|^2 \\
&\qquad \le \left\{\int_{\mathbb{B}^n} \frac{(1-|z|^2)^{1-2s}}{|z|^2}|RP_s[\varphi]|^2\,dV(z)\right\}^{1/2} \\
&\qquad\quad \times \left\{\int_{\mathbb{B}^n} \frac{(1-|z|^2)^{1-2s}}{|z|^2}|RP_s[K_{2s}[\mathfrak{g}]]|^2|P_s[\psi]|^2\,dV(z)\right\}^{1/2} \\
&\qquad \lesssim \|\varphi\|_{H^s}\|\psi\|_{H^s}.
\end{aligned}$$

Analogous estimates are obtained for the integral involving $P_s[\varphi]RP_s[\psi]$.

The same argument holds for the complex-tangential derivative, by using instead the fact that $\frac{(1-|z|^2)^{-2s}}{|z|^2}|T_{i,j}P_s[K_{2s}[\mathfrak{g}]]|^2\,dV(z)$ is a Carleson measure for $P_s(H^s)$.

Finally, using again Hölder's inequality and that by hypothesis the measure $|P_s[K_{2s}[\mathfrak{g}]]|^2(1-|z|^2)^{-2s}\,dV(z)$ is a Carleson measure for $P_s(H^s(\mathbb{S}^n))$, we also have

$$\left|\int_{\mathbb{B}^n} (1-|z|^2)^{-2s}P_s[\varphi]P_s[\psi]\overline{P_s[K_{2s}[\mathfrak{g}]]}\,dV(z)\right| \lesssim \|\varphi\|_{H^s}\|\psi\|_{H^s}.$$

8.2. Proof of (3)⇒(2) in Theorem 8.1. We observe that we may assume that $\mathfrak{g} \in L^2(\mathbb{S}^n)$ and is real-valued. We assume that $|K_s[\mathfrak{g}]|^2\,d\sigma$ is a trace measure for $K_s[L^2(\mathbb{S}^n)]$. Since $K_s^{-1}K_s^{-1}K_{2s}$ is a Calderón–Zygmund operator (Theorem 6.2), and since

$$K_s^{-1}K_s^{-1}K_{2s}K_s[\mathfrak{g}] = K_s^{-1}K_{2s}[\mathfrak{g}],$$

Proposition 7.9 implies that $|K_s^{-1}K_{2s}[\mathfrak{g}]|\,d\sigma$ is also a trace measure for $H^s[\mathbb{S}^n]$. By Remark 7.8, it is enough to show that if $E \subset \mathbb{S}^n$ is any closed set, and L is either R or $T_{i,j}$ with $i<j$ or Id, then

$$\int_{T(E)} \frac{(1-|z|^2)^{1-2s}}{|z|^2}|LP_s[K_{2s}[\mathfrak{g}]]|^2\,dV(z) \lesssim \mathrm{Cap}_s(E).$$

Let $\alpha \in \left(\frac{1}{2}, \frac{n}{2(n-2s)}\right)$. Noting first that by Remark 7.3, $p_E \gtrsim 1$ a.e. on E, and then using Fubini's Theorem, we obtain

$$
\begin{aligned}
&\int_{T(E)} |RP_s[K_{2s}[\mathfrak{g}]](z)|^2 \frac{(1-|z|^2)^{1-2s}}{|z|^2}\, dV(z) \\
&\lesssim \int_{\mathbb{B}^n} |RP_s[K_{2s}[\mathfrak{g}]](z)|^2 \frac{(1-|z|^2)^{1-2s}}{|z|^2} \frac{1}{(1-|z|^2)^n} \int_{I_z} p_E^{2\alpha}(\zeta)\, d\sigma(\zeta)\, dV(z) \\
&\lesssim \int_{\mathbb{S}^n} \int_{D(\zeta)} |RP_s[K_{2s}[\mathfrak{g}]](z)|^2 \frac{(1-|z|^2)^{1-2s-n}}{|z|^2} p_E^{2\alpha}(\zeta)\, dV(z)\, d\sigma(\zeta) \\
&= \int_{\mathbb{S}^n} \int_{D(\zeta)} \left\{ \frac{(1-|z|^2)^{-s+1}}{|z|} \left| RP_s\big[K_s[K_s^{-1}[K_{2s}[\mathfrak{g}]]]\big](z) \right| \right\}^2 \frac{dV(z)}{(1-|z|^2)^{n+1}}\, p_E^{2\alpha}\, d\sigma \\
&= \|G_{\mathbf{K}}[K_s^{-1}[K_{2s}[\mathfrak{g}]]]\|^2_{L^2(p_E^{2\alpha})},
\end{aligned}
$$

where $G_{\mathbf{K}}$ is the area function defined in (4.1) with $\mathbf{K}$ as in Theorem 5.5.

We next observe that $p_E = \lim_\ell p_{E,\ell}$ in $L^2(\mathbb{S}^n)$, and in particular, there exists a subsequence such that $p_E = \lim_i p_{E,\ell_i}$ a.e. in $\mathbb{S}^n$. Hence, since $p_E^{2\alpha} \in A_2(\mathbb{S}^n)$ (see Remark 7.3), by Theorem 5.5 the above is bounded by

$$
\begin{aligned}
&\left(\int_{\mathbb{S}^n} |K_s^{-1}[K_{2s}[\mathfrak{g}]](\zeta)|^2 p_E^{2\alpha}(\zeta)\, d\sigma(\zeta) \right)^2 \\
&\qquad\qquad \le \liminf_i \left(\int_{\mathbb{S}^n} |K_s^{-1}[K_{2s}[\mathfrak{g}]](\zeta)|^2 p_{E,\ell_i}^{2\alpha}(\zeta)\, d\sigma(\zeta) \right)^2.
\end{aligned}
$$

Since $|K_s^{-1}[K_{2s}[\mathfrak{g}]]|^2$ is a trace measure for $H^s(\mathbb{S}^n)$, the above is bounded by $\liminf_i \|p_{E,\ell_i}^\alpha\|^2_{H^s} \lesssim \mathrm{Cap}_s(E)$, where the last estimate follows from Proposition 7.5.

The proof for the complex-tangential part follows by an analogous argument.

Finally, the previous arguments applied to $P_s[K_{2s}[\mathfrak{g}]]$ in place of $RP_s[K_{2s}[\mathfrak{g}]]$ give

$$
\begin{aligned}
&\int_{T(E)} |P_s[K_{2s}[\mathfrak{g}]](z)|^2 (1-|z|^2)^{-2s}\, dV(z) \\
&\lesssim \int_{\mathbb{S}^n} \int_{D(\zeta)} \left| (1-|z|^2)^{1/2-s} P_s\big[K_s[K_s^{-1}[K_{2s}[\mathfrak{g}]]]\big](z) \right|^2 p_E^{2\alpha}(\zeta)\, \frac{dV(z)}{(1-|z|^2)^{n+1}}\, d\sigma(\zeta) \\
&= \left\| G_{\mathbf{J}}\big[K_s^{-1}[K_{2s}[\mathfrak{g}]]\big] \right\|^2_{L^2(p_E^{2\alpha})}.
\end{aligned}
$$

Since $p_E^{2\alpha} \in A_2$, Theorem 5.5 and Proposition 7.5 show that the above is bounded by

$$\Big(\int_{\mathbb{S}^n} |K_s^{-1}[K_{2s}[\mathfrak{g}]](\zeta)|^2 p_E^{2\alpha}(\zeta)\,d\sigma(\zeta)\Big)^2 \lesssim \mathrm{Cap}_s(E).$$

8.3. Proof of (1)⇒(3) in Theorem 8.1. We observe that in the proof of this implication, we again only need to assume that $\mathfrak{g}$ is in $L^2(\mathbb{S}^n)$ and is real-valued. By the hypothesis (1), we find that for any $\varphi, \psi \in \mathcal{C}_{\mathbb{R}}^\infty(\mathbb{S}^n)$,

$$\Big|\int_{\mathbb{S}^n} \varphi\psi\mathfrak{g}\,d\sigma\Big| \lesssim \|\varphi\|_{H^s(\mathbb{S}^n)}\|\psi\|_{H^s(\mathbb{S}^n)}.$$

We want to prove that for any compact set $E \subset \mathbb{S}^n$,

$$\int_E |K_s[\mathfrak{g}]|^2\,d\sigma \lesssim \mathrm{Cap}_s(E), \tag{8.1}$$

For $1 \le \ell, m \in \mathbb{N}$, we consider the test functions

$$\varphi_{\ell,m} := \frac{K_s[(\chi_E K_s[\mathfrak{g}])_\ell]}{p_{E,m}^\alpha}, \qquad \psi_m := p_{E,m}^\alpha,$$

where $p_{E,m} = K_{2s}[\nu_{E,m}]$, with ν_E the capacitary measure of E and with $\alpha \in \big(\frac{1}{2}, \frac{n}{2(n-2s)}\big)$.

We first observe that since the sequence $(\chi_E K_s[\mathfrak{g}])_\ell$ converges in $L^2(\mathbb{S}^n)$ to the function $\chi_E K_s[\mathfrak{g}]$, we have

$$\int_E |K_s[\mathfrak{g}]|^2\,d\sigma = \int_{\mathbb{S}^n} \chi_E K_s[\mathfrak{g}]K_s[\mathfrak{g}]\,d\sigma = \lim_\ell \int_{\mathbb{S}^n} (\chi_E K_s[\mathfrak{g}])_\ell K_s[\mathfrak{g}]\,d\sigma. \tag{8.2}$$

Next, by hypothesis, for any $\ell, m \ge 1$,

$$\begin{aligned}\int_{\mathbb{S}^n} (\chi_E K_s[\mathfrak{g}])_\ell K_s[\mathfrak{g}]\,d\sigma &= \int_{\mathbb{S}^n} K_s[(\chi_E K_s[\mathfrak{g}])_\ell]\mathfrak{g}\,d\sigma = \Big|\int_{\mathbb{S}^n} \varphi_{\ell,m}\psi_m\mathfrak{g}\,d\sigma\Big| \\ &\lesssim \|\varphi_{\ell,m}\|_{H^s(\mathbb{S}^n)}\|\psi_m\|_{H^s(\mathbb{S}^n)}.\end{aligned}$$

We already know by Proposition 7.5 that $\|\psi_m\|_{H^s}^2 \lesssim \mathrm{Cap}_s(E)$. We claim that

$$\|\varphi_{\ell,m}\|_{H^s}^2 = \Big\|\frac{K_s[(\chi_E K_s[\mathfrak{g}])_\ell]}{p_{E,m}^\alpha}\Big\|_{H^s}^2 \lesssim \int_{\mathbb{S}^n} (\chi_E K_s[\mathfrak{g}])_\ell K_s[\mathfrak{g}]\,d\sigma. \tag{8.3}$$

Using the claim and (8.2), we deduce that

$$\int_E K_s[\mathfrak{g}]^2\,d\sigma \lesssim \mathrm{Cap}_s(E),$$

which gives (8.1).

So, we are left to show (8.3). By Corollary 3.11 and Proposition 3.12, we see that

(8.4)
$$\begin{aligned}\|\varphi_{\ell,m}\|^2_{H^s} &= \left\| \frac{K_s[(\chi_E K_s[\mathfrak{g}])_\ell]}{p^\alpha_{E,m}} \right\|^2_{H^s} \lesssim \left\| \frac{P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]]}{P_s[p_{E,m}]^\alpha} \right\|^2_{\mathcal{W}^2_s} \\ &\lesssim \int_{\mathbb{B}^n} \Bigg\{ \frac{(1-|z|^2)^{1-2s}}{|z|^2} \left| R\frac{P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]]}{P_s[p_{E,m}]^\alpha} \right|^2 \\ &\quad + \sum_{i<j} \frac{(1-|z|^2)^{-2s}}{|z|^2} \left| L_{ij}\frac{P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]]}{P_s[p_{E,m}]^\alpha} \right|^2 \\ &\quad + k(1-|z|^2)^{1-2s} \left| \frac{P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]]}{P_s[p_{E,m}]^\alpha} \right|^2 \Bigg\} \, dV =: \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3.\end{aligned}$$

For $k = 1, 2$, we denote by $\mathcal{A}_k^{\mathrm{num}}$ (respectively $\mathcal{A}_k^{\mathrm{den}}$) the terms obtained when we differentiate with respect to R or $L_{i,j}$ the numerator (or the denominator) of $\mathcal{A}_k$.

We begin by estimating the terms in $\mathcal{A}_1^{\mathrm{num}}$. We first observe that for any $z \in \mathbb{B}^n$,

$$P_s[p_{E,m}](z) \gtrsim \frac{1}{\sigma(I_z)} \int_{I_z} p_{E,m} \, d\sigma,$$

where $I_z = \{\zeta \in \mathbb{S}^n;\ z \in D(\zeta)\}$, and $D(\zeta)$ is the admissible region corresponding to ζ. Then, applying Hölder's inequality twice, we get

$$\begin{aligned}&\mathcal{A}_1^{\mathrm{num}} \\ &\lesssim \int_{\mathbb{B}^n} \frac{(1-|z|^2)^{1-2s}}{|z|^2} |R(P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]])|^2 \left(\frac{\int_{I_z} p_{E,m}(\eta)\, d\sigma(\eta)}{(1-|z|^2)^n} \right)^{-2\alpha} dV(z) \\ &\lesssim \int_{\mathbb{B}^n} \frac{(1-|z|^2)^{1-2s}}{|z|^2} |R(P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]])|^2 \left(\frac{\int_{I_z} p^{-1}_{E,m}(\eta)\, d\sigma(\eta)}{(1-|z|^2)^n} \right)^{2\alpha} dV(z) \\ &\lesssim \int_{\mathbb{S}^n} \int_{D(\zeta)} \left(\frac{(1-|z|^2)^{1-s}}{|z|} |R(P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]])| \right)^2 \frac{dV(z)}{(1-|z|^2)^{n+1}} \frac{d\sigma(\eta)}{p^{2\alpha}_{E,m}(\eta)}.\end{aligned}$$

Since by Remark 7.3, $p^{2\alpha}_{E,m} \in A_2$ with constants independent of E and m (and hence also $p^{-2\alpha}_{E,m} \in A_2$), Theorem 5.5 gives

$$\mathcal{A}_1^{\mathrm{num}} \lesssim \int_{\mathbb{S}^n} |(\chi_E K_s[\mathfrak{g}])_\ell|^2 \frac{1}{p^{2\alpha}_{E,m}(\eta)} \, d\sigma(\eta).$$

Next, letting first $\ell \to \infty$, using the convergence of $((\chi_E K_s[\mathfrak{g}])_\ell)_\ell$ in $L^2(\mathbb{S}^n)$, we obtain

$$\lim_m \mathcal{A}_1^{\mathrm{num}} \lesssim \int_{\mathbb{S}^n} |(\chi_E K_s[\mathfrak{g}])|^2 \frac{d\sigma(\eta)}{p_{E,m}^{2\alpha}}.$$

Then since

$$p_{E,m}(\zeta) = \int_{\mathbb{S}^n} \frac{d\nu_{E,m}(\eta)}{|1-\zeta\overline{\eta}|^{n-2s}} \gtrsim \nu_{E,m}(\mathbb{S}^n) \simeq \mathrm{Cap}_s(E),$$

the Dominated Convergence Theorem shows (since $p_E \gtrsim 1$ a.e. on E) that

$$\lim_{\ell,m} \mathcal{A}_1^{\mathrm{num}} \lesssim \int_E |K_s[\mathfrak{g}]|^2(\eta)\, d\sigma(\eta).$$

We observe that an analogous argument gives the desired estimate for the terms in $\mathcal{A}_2^{\mathrm{num}}$ obtained when we calculate the complex-tangential derivatives of the numerator. So, up to now we have obtained

$$\mathcal{A}_1^{\mathrm{num}} + \mathcal{A}_2^{\mathrm{num}} \lesssim \int_E |K_s[\mathfrak{g}]|^2(\eta)\, d\sigma(\eta). \tag{8.5}$$

Now we proceed to estimate $\mathcal{A}_k^{\mathrm{den}}$, $k=1,2$. Proposition 3.8 applied to $v_{\ell m} = \frac{P_s[(K_s[\chi_E K_s[\mathfrak{g}]])_\ell]^2}{P_s[p_{E,m}]^{2\alpha+1}}$ and $u_m = P_s[p_{E,m}]$ implies that $\mathrm{Re}\,\langle v_{\ell m}, u_m\rangle_{\mathcal{W}_s^2} = C_{n,s}\int_{\mathbb{S}^n} v_{\ell,m} K_{2s}^{-1}[p_{E,m}]\, d\sigma$, where $C_{n,s}>0$. But

$$\begin{aligned}\int_{\mathbb{S}^n} v_{\ell,m} K_{2s}^{-1}[p_{E,m}]\, d\sigma &= \int_{\mathbb{S}^n} \frac{(K_s[(\chi_E K_s[\mathfrak{g}])_\ell])^2}{p_{E,m}^{2\alpha+1}} K_{2s}^{-1}[p_{E,m}] \\ &= \int_{\mathbb{S}^n} \frac{(K_s[(\chi_E K_s[\mathfrak{g}])_\ell])^2}{p_{E,m}^{2\alpha+1}}\, d\nu_{E,m} \ge 0.\end{aligned}$$

As a consequence, $\mathrm{Re}\,\langle v_{\ell,m}, u_m\rangle_{\mathcal{W}_s^2} \ge 0$, and we then have

$$\begin{aligned}0 \le -(2\alpha{+}1) &\int_{\mathbb{B}^n} \frac{(1-|z|^2)^{-2s}}{|z|^2} \frac{(P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]])^2}{(P_s[p_{E,m}])^{2\alpha+2}} \\ &\times\Big((1-|z|^2)|RP_s[p_{E,m}]|^2 + \sum_{i<j} |L_{ij}P_s[p_{E,m}]|^2\Big)\, dV(z) \\ &+2\mathrm{Re}\int_{\mathbb{B}^n} \frac{(1-|z|^2)^{1-2s}}{|z|^2} \frac{P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]]}{P_s(p_{E,m})^{2\alpha+1}} RP_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]]\overline{R}P_s[p_{E,m}]\, dV \\ &+2\mathrm{Re}\int_{\mathbb{B}^n} \frac{(1-|z|^2)^{-2s}}{|z|^2} \sum_{i<j} \frac{P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]]}{P_s[p_{E,m}]^{2\alpha+1}} \\ &\qquad\times L_{ij}P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]]\overline{L_{ij}}P_s[p_{E,m}]\, dV \\ &+\int_{\mathbb{B}^n} k(1-|z|^2)^{-2s} \frac{P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]]^2}{P_s[p_{E,m}]^{2\alpha+1}} P_s(p_{E,m})\, dV(z).\end{aligned}$$

From the above estimate and Hölder's inequality, we deduce that

$$\begin{aligned}
&\mathcal{A}_1^{\mathrm{den}} + \mathcal{A}_2^{\mathrm{den}} \\
&\quad\lesssim \bigg(\int_{\mathbb{B}^n} \frac{(1-|z|^2)^{1-2s}}{|z|^2} \frac{|RP_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]]|^2}{P_s[p_{E,m}]^{2\alpha}} \, dV(z) \bigg)^{1/2} \\
&\qquad \times \bigg(\int_{\mathbb{B}^n} \frac{(1-|z|^2)^{1-2s}}{|z|^2} \frac{|P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]]|^2}{P_s[p_{E,m}]^{2\alpha+2}} |RP_s[p_{E,m}]|^2 \, dV(z) \bigg)^{1/2} \\
&\qquad + \bigg(\int_{\mathbb{B}^n} \frac{(1-|z|^2)^{-2s}}{|z|^2} \sum_{i<j} \frac{|L_{ij} P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]]|^2}{P_s[p_{E,m}]^{2\alpha}} \, dV(z) \bigg)^{1/2} \\
&\qquad \times \bigg(\int_{\mathbb{B}^n} \frac{(1-|z|^2)^{-2s}}{|z|^2} \frac{|P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]]|^2}{P_s[p_{E,m}]^{2\alpha+2}} |L_{i,j} P_s[p_{E,m}]|^2 \, dV(z) \bigg)^{1/2} \\
&\qquad + \int_{\mathbb{B}^n} (1-|z|^2)^{-2s} \frac{|P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]]|^2}{P_s[p_{E,m}]^{2\alpha}} \, dV(z) \\
&\quad\lesssim \frac{1}{\varepsilon} (A_1^{\mathrm{num}} + A_2^{\mathrm{num}}) + \varepsilon (\mathcal{A}_1^{\mathrm{den}} + \mathcal{A}_2^{\mathrm{den}}) \\
&\qquad + \int_{\mathbb{B}^n} (1-|z|^2)^{-2s} \frac{|P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]]|^2}{P_s[p_{E,m}]^{2\alpha}} \, dV(z).
\end{aligned}$$

Hence, using (8.5), we obtain

$$\mathcal{A}_1^{\mathrm{den}} + \mathcal{A}_2^{\mathrm{den}} \lesssim \int_E K_s[\mathfrak{g}]^2(\eta) \, d\sigma(\eta) + \int_{\mathbb{B}^n} (1-|z|^2)^{-2s} \frac{|P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]]|^2}{P_s[p_{E,m}]^{2\alpha}} \, dV(z),$$

and consequently

$$\mathcal{A}_1 + \mathcal{A}_2 \lesssim \int_E K_s[\mathfrak{g}]^2(\eta) \, d\sigma(\eta) + \int_{\mathbb{B}^n} (1-|z|^2)^{-2s} \frac{|P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]]|^2}{P_s[p_{E,m}]^{2\alpha}} \, dV(z).$$

But, applying Proposition 3.12, we see that

$$\begin{aligned}
&\int_{\mathbb{B}^n} (1-|z|^2)^{-2s} \frac{|P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]]|^2}{P_s[p_{E,m}]^{2\alpha}} \, dV(z) \\
&\qquad \lesssim \frac{1}{\varepsilon} \int_{\mathbb{B}^n} (1-|z|^2)^{1-2s} \frac{|P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]]|^2}{P_s[p_{E,m}]^{2\alpha}} \, dV(z) \\
&\qquad\quad + \varepsilon \int_{\mathbb{B}^n} (1-|z|^2)^{1-2s} R\bigg(\frac{P_s[K_s[(\chi_E K_s[\mathfrak{g}])_\ell]]}{P_s[p_{E,m}]^{2\alpha}} \bigg) \, dV(z).
\end{aligned}$$

Altogether this gives $\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 \lesssim \int_E K_s[\mathfrak{g}]^2(\eta) \, d\sigma(\eta) + \mathcal{A}_3$.

The estimate for $\mathcal{A}_3$ follows by the arguments used in the estimate of $\mathcal{A}_1^n$, applying Theorem 5.5 to $G_{\mathbf{J}}$.

So, we have just proved that, using (8.4),

$$\|\varphi\|_{H^s}^2 \lesssim \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 \lesssim \int_E K_s[\mathfrak{g}]^2(\eta)\, d\sigma(\eta),$$

which finishes the proof of Theorem 8.1. ∎

We finish with a characterization of a bilinear problem deduced from Theorem 8.1.

Corollary 8.2. *Let* $0 < s < 1/4$ *and let* $\mathfrak{g}$ *be a function satisfying the conditions in Theorem* 3.3 *or* 3.5. *Then the following assertions are equivalent:*

(1) *For any* $\varphi, \psi \in \mathcal{C}^\infty(\mathbb{S}^n)$,

$$\Big|\int_{\mathbb{S}^n} ((-\Delta)^s + I)(\varphi\psi)\mathfrak{g}\, d\sigma\Big| \lesssim \|\varphi\|_{H^s(\mathbb{S}^n)}\|\psi\|_{H^s(\mathbb{S}^n)}.$$

(2) *The measure* $d\nu := |K_s[K_{2s}^{-1}[\mathfrak{g}]]|^2\, d\sigma$ *is a trace measure for* $H^s(\mathbb{S}^n)$.

Proof. Since by Corollary 3.7, $((-\Delta)^s + I)(\varphi\psi) = K_{2s}^{-1}[\varphi\psi]$, we have

$$\int_{\mathbb{S}^n} ((-\Delta)^s + I)(\varphi\psi)\mathfrak{g} = \int_{\mathbb{S}^n} \varphi\psi\mathfrak{h}\, d\sigma,$$

where $\mathfrak{h} = K_{2s}^{-1}[\mathfrak{g}]$. Hence, applying Theorem 8.1, we finish the proof. ∎

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