

On the heat semigroup approach to the geometric Forward-Reverse Brascamp–Lieb inequality

by

YE ZHANG

Abstract. We provide a new proof of the geometric Forward-Reverse Brascamp–Lieb inequality, using the approach of the heat semigroup, or the heat flow. Furthermore, we characterize all the Forward-Reverse Brascamp–Lieb data such that the initial relation can be preserved by some heat flow.

1. Introduction. Recall that the Forward-Reverse Brascamp–Lieb inequality, studied recently in [LC⁺18, CL21], unifies both the Brascamp–Lieb inequality and Barthe’s inequality (which is also called the Reverse Brascamp–Lieb inequality). As a result, it generalizes several famous inequalities, including Hölder’s inequality, Young’s inequality, Loomis–Whitney inequality and Prékopa–Leindler inequality.

1.1. Notation. Throughout this paper, by a *Euclidean space* V , we mean a finite-dimensional Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_V$ and the Lebesgue measure (more precisely, the measure induced by the Riemannian volume form if we regard V as a Riemannian manifold). On the direct sum of two Euclidean spaces $V_1 \oplus V_2$, the inner product is defined by

$$\langle v_1 + v_2, w_1 + w_2 \rangle_{V_1 \oplus V_2} := \langle v_1, w_1 \rangle_{V_1} + \langle v_2, w_2 \rangle_{V_2}, \quad \forall v_1, w_1 \in V_1, v_2, w_2 \in V_2.$$

With this inner product on $V_1 \oplus V_2$ the subspaces V_1 and V_2 are orthogonal to each other. We use $|\cdot|_V$ to denote the norm induced by the inner product $\langle \cdot, \cdot \rangle_V$. Given a Euclidean space V and a linear map A from V to itself, we use $\text{Tr}(A)$ and $\det(A)$ to denote its *trace* and *determinant* respectively. Moreover, for a linear map between two Euclidean spaces $A : V_1 \rightarrow V_2$, we denote the *adjoint* of A by A^* , that is, A^* is a linear map from V_2 to V_1

2020 *Mathematics Subject Classification*: Primary 26D15; Secondary 35K05, 39B72.

Key words and phrases: Barthe’s inequality, Brascamp–Lieb inequality, Forward-Reverse Brascamp–Lieb inequality, Gaussian extremizer, heat semigroup.

Received 24 March 2025; revised 22 April 2026.

Published online 6 October 2026.

satisfying

$$\langle Av_1, v_2 \rangle_{V_2} = \langle v_1, A^* v_2 \rangle_{V_1}, \quad \forall v_1 \in V_1, v_2 \in V_2.$$

We use $\mathcal{S}(V)$ to denote the set of self-adjoint linear maps on V , that is, the set of linear maps A from V to itself such that $A = A^*$. The notation $\mathcal{S}^+(V)$ (resp. $\overline{\mathcal{S}^+}(V)$) is reserved for the set of self-adjoint positive definite (resp. semidefinite) linear maps on V , that is, $A \in \mathcal{S}^+(V)$ (resp. $A \in \overline{\mathcal{S}^+}(V)$) if and only if

$$A \in \mathcal{S}(V) \quad \text{and} \quad \langle Av, v \rangle_V > 0 \quad (\text{resp. } \langle Av, v \rangle_V \geq 0), \quad \forall v \in V \setminus \{0\}.$$

For $A \in \overline{\mathcal{S}^+}(V)$, we use $\sqrt{A}$ to denote its *square root*, that is, the unique element $B \in \overline{\mathcal{S}^+}(V)$ such that $B^2 = A$. For $A, B \in \mathcal{S}(V)$, we write $A \geq B$ if $A - B \in \overline{\mathcal{S}^+}(V)$. A *centred Gaussian function* on a Euclidean space V is a function of the form

$$x \mapsto \exp(-\langle Ax, x \rangle_V), \quad \forall x \in V, A \in \mathcal{S}^+(V).$$

Furthermore, for a given Euclidean space V , we denote the *Laplacian* and *gradient* by Δ_V and ∇_V respectively. The *Hessian matrix* of a C^2 function f is denoted by $\nabla_V^2 f$. Recall that $\Delta_V f = \text{Tr}(\nabla_V^2 f)$. Also recall that $\Delta_V = \text{div}_V(\nabla_V)$, where div_V is the *divergence operator*. More generally, for $A \in \mathcal{S}^+(V)$, we use $\Delta_{V,A}$ to denote the operator

$$\Delta_{V,A} f := \text{div}_V(A \nabla_V f), \quad \forall f \in C^2.$$

In the following we use $(e^{t\Delta_{V,A}})_{t>0}$ to denote the *heat semigroup* with infinitesimal generator $\Delta_{V,A}$ on $L^1(V)$. In fact, for $f \in L^1(V)$, we have

$$(1.1) \quad (e^{t\Delta_{V,A}} f)(x) = \int_V f(y) p_t^{V,A}(x-y) dy = \int_V f(x-y) p_t^{V,A}(y) dy,$$

where the *heat kernel* of $(e^{t\Delta_{V,A}})_{t>0}$ is given by the explicit formula

$$(1.2) \quad p_t^{V,A}(x) = \frac{1}{(4\pi t)^{\frac{\dim(V)}{2}} \sqrt{\det(A)}} e^{-\frac{\langle A^{-1}x, x \rangle_V}{4t}}, \quad x \in V.$$

When $A = \text{id}_V$, the identity matrix on V , we just write $(e^{t\Delta_V})_{t>0}$ and p_t^V for simplicity. From the expression above, we know that $u(x, t) := (e^{t\Delta_{V,A}} f)(x)$ is a smooth function on $V \times (0, +\infty)$ if $f \in L^1(V)$. If in addition we have $f \geq 0$ with $\int_V f > 0$, the function u is also positive. Note that by formula (1.1), the heat semigroup can also be defined on the space of non-negative measurable functions. For example, we have $e^{t\Delta_{V,A}} 1 = 1$. However, in this case, $e^{t\Delta_{V,A}} f$ may take the value $+\infty$ at some point for general f . For more details on the heat semigroup, we refer to [Gri09, §2.7] (see also [Gri09, Theorem 7.19] for the smoothness result).

Given Euclidean spaces $(E_i)_{1 \leq i \leq k}$ and $(E^j)_{1 \leq j \leq m}$, we define $E_0 := \bigoplus_{i=1}^k E_i$, and $E^0 := \bigoplus_{j=1}^m E^j$. We use π_{E_i} and π_{E^j} to denote the orthog-

onal projection of E_0 onto E_i and of E^0 onto E^j respectively. In this paper we always regard elements in Euclidean spaces as column vectors. In this spirit throughout this paper we identify the spaces $E_0 = \bigoplus_{i=1}^k E_i$ and $E^0 = \bigoplus_{j=1}^m E^j$ with $\prod_{i=1}^k E_i$ and $\prod_{j=1}^m E^j$ via the following maps:

$$(1.3) \quad \sum_{i=1}^k v_i \mapsto \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}, \quad \sum_{j=1}^m v^j \mapsto \begin{pmatrix} v^1 \\ \vdots \\ v^m \end{pmatrix}$$

for all $v_i \in E_i$, $v^j \in E^j$, $1 \leq i \leq k$, $1 \leq j \leq m$.

Let $\mathbf{Q}$ be a linear map from E_0 to E^0 , and $\mathbf{c} := (c_i)_{1 \leq i \leq k}$, $\mathbf{d} := (d_j)_{1 \leq j \leq m}$ be collections of positive numbers such that

$$(1.4) \quad \sum_{i=1}^k c_i \dim(E_i) = \sum_{j=1}^m d_j \dim(E^j).$$

Such a triple $(\mathbf{c}, \mathbf{d}, \mathbf{Q})$ is called a *Forward-Reverse Brascamp–Lieb datum*.

For $A_i \in \overline{\mathcal{S}^+}(E_i)$, $i = 1, \dots, k$ (resp. $A^j \in \overline{\mathcal{S}^+}(E^j)$, $j = 1, \dots, m$), we let $\Pi(A_1, \dots, A_k)$ (resp. $\Pi(A^1, \dots, A^m)$) denote the set of $A_0 \in \overline{\mathcal{S}^+}(E_0)$ (resp. $A^0 \in \overline{\mathcal{S}^+}(E^0)$) satisfying

$$\pi_{E_i} A_0 \pi_{E_i}^* = A_i, \quad \forall 1 \leq i \leq k \quad (\text{resp. } \pi_{E^j} A^0 \pi_{E^j}^* = A^j, \quad \forall 1 \leq j \leq m).$$

In other words, under the identification (1.3), A_0 and A^0 have the block forms

$$A_0 = \begin{pmatrix} A_1 & * & \dots & * \\ * & A_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & A_k \end{pmatrix}, \quad A^0 = \begin{pmatrix} A^1 & * & \dots & * \\ * & A^2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & A^m \end{pmatrix}.$$

Finally, for $\mathbf{c} := (c_i)_{1 \leq i \leq k}$, $\mathbf{d} := (d_j)_{1 \leq j \leq m}$, we define

$$\Lambda_{\mathbf{c}} := \sum_{i=1}^k c_i \pi_{E_i}^* \text{id}_{E_i} \pi_{E_i} \quad \text{and} \quad \Lambda_{\mathbf{d}} := \sum_{j=1}^m d_j \pi_{E^j}^* \text{id}_{E^j} \pi_{E^j}.$$

Note that with the identification (1.3), $\Lambda_{\mathbf{c}}$ and $\Lambda_{\mathbf{d}}$ have the block forms

$$\Lambda_{\mathbf{c}} = \begin{pmatrix} c_1 \text{id}_{E_1} & 0 & \dots & 0 \\ 0 & c_2 \text{id}_{E_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_k \text{id}_{E_k} \end{pmatrix},$$

$$\Lambda_{\mathbf{d}} = \begin{pmatrix} d_1 \text{id}_{E^1} & 0 & \dots & 0 \\ 0 & d_2 \text{id}_{E^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_m \text{id}_{E^m} \end{pmatrix}.$$

1.2. Forward-Reverse Brascamp–Lieb inequality. In our setting, given a Forward-Reverse Brascamp–Lieb datum $(\mathbf{c}, \mathbf{d}, \mathbf{Q})$, the *Forward-Reverse Brascamp–Lieb inequality* is the following:

$$(1.5) \quad \prod_{i=1}^k \left(\int_{E_i} f_i \right)^{c_i} \leq C(\mathbf{c}, \mathbf{d}, \mathbf{Q}) \prod_{j=1}^m \left(\int_{E^j} g_j \right)^{d_j}$$

for all non-negative measurable functions $(f_i)_{1 \leq i \leq k}, (g_j)_{1 \leq j \leq m}$ satisfying

$$(1.6) \quad \prod_{i=1}^k f_i^{c_i}(\pi_{E_i} x) \leq \prod_{j=1}^m g_j^{d_j}(\pi_{E^j} \mathbf{Q}x), \quad \forall x \in E_0.$$

Here the *Forward-Reverse Brascamp–Lieb constant* $C(\mathbf{c}, \mathbf{d}, \mathbf{Q})$ is the smallest constant for which (1.5) holds and it could be $+\infty$. Measurable functions $(f_i)_{1 \leq i \leq k}, (g_j)_{1 \leq j \leq m}$ are called *extremizers* if (1.5) holds with equality (with positive, finite value on both sides). Moreover, if $(f_i)_{1 \leq i \leq k}, (g_j)_{1 \leq j \leq m}$ are extremizers and all the functions $(f_i)_{1 \leq i \leq k}, (g_j)_{1 \leq j \leq m}$ are centred Gaussian functions, we call them *Gaussian extremizers*.

REMARK 1.1. Actually, (1.4) is a necessary condition to obtain $C(\mathbf{c}, \mathbf{d}, \mathbf{Q}) < +\infty$, which can be seen by a scaling argument.

REMARK 1.2. Here we follow the notation in [LC⁺18], which is a little bit different from the one in [CL21]. As mentioned in [CL21, Remark 1.5], the extra scaling in [CL21] is due to the duality relation there. Since we will not consider the duality relation here, we can absorb the scaling into the map. Furthermore, if we write

$$(1.7) \quad \mathbf{Q}_{ji} := \pi_{E^j} \mathbf{Q} \pi_{E_i}^*, \quad \forall 1 \leq i \leq k, 1 \leq j \leq m,$$

which is a linear map from E_i to E^j , we have $\mathbf{Q}_{ji} = c_i B_{ij}$ in [CL21, (2)]. Note that with the identification (1.3), we can write

$$(1.8) \quad \mathbf{Q} = \begin{pmatrix} \mathbf{Q}_{11} & \dots & \mathbf{Q}_{1k} \\ \vdots & \ddots & \vdots \\ \mathbf{Q}_{m1} & \dots & \mathbf{Q}_{mk} \end{pmatrix} = \begin{pmatrix} \mathbf{Q}_1 \\ \vdots \\ \mathbf{Q}_m \end{pmatrix},$$

where

$$(1.9) \quad \mathbf{Q}_j := \sum_{i=1}^k \mathbf{Q}_{ji} \pi_{E_i} = \pi_{E^j} \mathbf{Q}, \quad \forall 1 \leq j \leq m.$$

REMARK 1.3. If $k = 1$ and $\mathbf{c} = (1)$, the Forward-Reverse Brascamp–Lieb inequality reduces to the (classical) Brascamp–Lieb inequality. On the other hand, if $m = 1$ and $\mathbf{d} = (1)$, it reduces to Barthe’s inequality, or the Reverse Brascamp–Lieb inequality. In such cases we simply say *Brascamp–Lieb constant* (resp. *datum*) and *Reverse Brascamp–Lieb constant* (resp. *datum*) instead of Forward-Reverse Brascamp–Lieb constant (resp. datum).

The (classical) Brascamp–Lieb inequality was first formulated in [BL76] to find the best constants in Young’s inequality. By Lieb’s Theorem (see [Lie90, Theorem 6.2]), the Brascamp–Lieb constant can be obtained by only considering centred Gaussian functions. In other words, the Brascamp–Lieb constant is exhausted by centred Gaussian functions. Then in [BC⁺08] the authors gave a sufficient and necessary condition for the finiteness of the Brascamp–Lieb constant, as well as for the existence of the extremizers. Finally, all the extremizers are characterized in [Val08].

On the other hand, the Reverse Brascamp–Lieb inequality was formulated in [Bar98b], where a counterpart of Lieb’s Theorem is also proved. However, the extremizers are only characterized in special cases in [BKX23].

For the case of the Forward-Reverse Brascamp–Lieb inequality, a generalization of Lieb’s Theorem is proved in [LC⁺18, CL21]. A sufficient and necessary condition for the finiteness of the Forward-Reverse Brascamp–Lieb constant is also provided in [CL21].

1.3. Geometric Forward-Reverse Brascamp–Lieb datum. By a clever induction argument (cf. [CL21], see also [BC⁺08] for the case of Brascamp–Lieb inequality), it turns out that the study of general Forward-Reverse Brascamp–Lieb inequalities essentially reduced to the geometric ones. A Forward-Reverse Brascamp–Lieb datum is called *geometric* if

$$(1.10) \quad \mathbf{Q}^* \Lambda_{\mathbf{d}} \mathbf{Q} \leq \Lambda_{\mathbf{c}}$$

and there exists a $\Sigma \in \Pi(\text{id}_{E_1}, \dots, \text{id}_{E_k})$ with $\mathbf{Q} \Sigma \mathbf{Q}^* \in \Pi(\text{id}_{E^1}, \dots, \text{id}_{E^m})$. By Remark 1.2, our definition here is the same as the one in [CL21, Definition 1.21] but we write it in a more compact way by using matrices. This will help us do the computation in the proof of our main theorem.

In the case $k = 1$ and $\mathbf{c} = (1)$, the only choice of Σ is $\text{id}_{E_1} (= \text{id}_{E_0})$. As a result, the geometric conditions reduce to

$$\mathbf{Q}_j \mathbf{Q}_j^* = \text{id}_{E^j}, \quad \forall 1 \leq j \leq m, \quad \sum_{j=1}^m d_j \mathbf{Q}_j^* \mathbf{Q}_j \leq \text{id}_{E_0},$$

where $\mathbf{Q}_j$ is defined by (1.9). By (1.4), the traces of the left-hand side and the right-hand side of the last inequality coincide, which implies equality,

that is,

$$\sum_{j=1}^m d_j \mathbf{Q}_j^* \mathbf{Q}_j = \text{id}_{E_0}.$$

Thus, the geometric conditions here reduce to the ones of the geometric Brascamp–Lieb inequality (cf. [BC⁺08, Definition 2.1]).

Given two Forward-Reverse Brascamp–Lieb data $(\mathbf{c}, \mathbf{d}, \mathbf{Q})$, $(\mathbf{c}', \mathbf{d}', \mathbf{Q}')$, we say they are *equivalent* if $\mathbf{c} = \mathbf{c}'$, $\mathbf{d} = \mathbf{d}'$ and there exist invertible linear maps $C_i : E_i \rightarrow E_i$, $1 \leq i \leq k$, and $D_j : E^j \rightarrow E^j$, $1 \leq j \leq m$, such that

$$\mathbf{Q}'_{ji} = D_j^{-1} \mathbf{Q}_{ji} C_i^{-1}, \quad \forall 1 \leq i \leq k, 1 \leq j \leq m,$$

where $\mathbf{Q}_{ji}$ is defined in (1.7).

1.4. Main result. In the literature, there are several methods of proving the Brascamp–Lieb inequality and the Reverse Brascamp–Lieb inequality: (1) by the rearrangement inequality [BL76, BLL74]; (2) by optimal transport [Bar98b, Bar98a]; (3) by a probabilistic argument [Bor00, Leh14]; (4) by the heat semigroup [BCE04, BC⁺11, BH09, BC⁺08, Bor00, CCE09, CLL04]. Regarding the Forward-Reverse Brascamp–Lieb inequality, the proof given in [CL21] is by a probabilistic argument developing the ones in [Bor00, Leh14]. The target of the present paper is to give a new proof of the geometric Forward-Reverse Brascamp–Lieb inequality by using the heat semigroup. Here is our main result: if the Forward-Reverse Brascamp–Lieb datum $(\mathbf{c}, \mathbf{d}, \mathbf{Q})$ is geometric, then the relation (1.6) can be preserved by the heat flow.

THEOREM 1.4. *Given a geometric Forward-Reverse Brascamp–Lieb datum $(\mathbf{c}, \mathbf{d}, \mathbf{Q})$, if all the functions $(f_i)_{1 \leq i \leq k}, (g_j)_{1 \leq j \leq m}$ are non-negative and integrable, and (1.6) holds, that is,*

$$\prod_{i=1}^k f_i^{c_i}(\pi_{E_i} x) \leq \prod_{j=1}^m g_j^{d_j}(\pi_{E^j} \mathbf{Q} x), \quad \forall x \in E_0,$$

then

$$(1.11) \quad \prod_{i=1}^k (e^{t\Delta_{E_i}} f_i)^{c_i}(\pi_{E_i} x) \leq \prod_{j=1}^m (e^{t\Delta_{E^j}} g_j)^{d_j}(\pi_{E^j} \mathbf{Q} x), \quad \forall x \in E_0, t > 0.$$

REMARK 1.5. Recall that for the geometric Brascamp–Lieb inequality, the heat semigroup approach is a little different from what we stated above. To be more precise, given a geometric Forward-Reverse Brascamp–Lieb datum with $k = 1$ and $\mathbf{c} = (1)$ (see Section 1.3 for details), instead of the preservation property under the heat flow, the usual heat semigroup ap-

proach showed that the following function is increasing in time t :

$$Q(t) := \int_{E_0} \prod_{j=1}^m (e^{t\Delta_{E^j}} g_j)^{d_j}(\mathbf{Q}_j x) dx, \quad t \geq 0.$$

In fact, our Theorem 1.4 implies $Q'(t) \geq 0$: Fix a time $t \geq 0$ and a positive number s . Define the function $f_{1,t}$ on $E_1 = E_0$ by

$$f_{1,t}(x) := \prod_{j=1}^m (e^{t\Delta_{E^j}} g_j)^{d_j}(\mathbf{Q}_j x), \quad x \in E_1 = E_0.$$

It follows from Theorem 1.4 that

$$(1.12) \quad (e^{s\Delta_{E_1}} f_{1,t})(x) \leq \prod_{j=1}^m (e^{(t+s)\Delta_{E^j}} g_j)^{d_j}(\mathbf{Q}_j x), \quad \forall x \in E_1 = E_0.$$

Taking the integral in (1.12), since the heat semigroup preserves the L^1 norm, we obtain

$$\int_{E_0} f_{1,t}(x) dx \leq \int_{E_0} \prod_{j=1}^m (e^{(t+s)\Delta_{E^j}} g_j)^{d_j}(\mathbf{Q}_j x) dx,$$

which is exactly $Q(t) \leq Q(t+s)$. Note that in [BB19, §2.5], the authors proved that $f_{1,t}$ satisfies $\partial_t f_{1,t} \geq \Delta_{E_1} f_{1,t}$, which also implies (1.12) by the maximum principle for heat equations (the argument is quite similar to the one in the proof of Theorem 1.4 so we omit the details).

The proof of Theorem 1.4 relies on the maximum principle for parabolic equations inspired by [BCE04, BH09]. See Lemma 2.1 below for more details. We remark that in [ILS25], a similar idea also works for the proof of the Borell–Brascamp–Lieb inequality, which is another generalization of the Prékopa–Leindler inequality.

However, to use such a maximum principle, we need some regularity of the initial data. Since we should take (1.6) into account when we do the approximation, it turns out that, instead of continuity, the semicontinuity is the suitable regularity for our purpose. And general L^1 functions can be approximated by upper/lower semicontinuous functions from below/above, which is guaranteed by the famous Vitali–Carathéodory Theorem. See Theorem 2.6 below for more details.

As a corollary, from the long-time behavior of the heat equation (see Lemma 2.9 below), we obtain the Forward-Reverse Brascamp–Lieb constant in this geometric setting.

COROLLARY 1.6. *If $(\mathbf{c}, \mathbf{d}, \mathbf{Q})$ is a geometric Forward-Reverse Brascamp–Lieb datum, then $C(\mathbf{c}, \mathbf{d}, \mathbf{Q}) = 1$.*

Furthermore, we can show that the heat semigroup approach only works for a Forward-Reverse Brascamp–Lieb datum which is essentially geometric.

In other words, the heat flow preservation also gives a characterization of the data which are equivalent to geometric ones. To be more precise, the following theorem holds.

THEOREM 1.7. *Let $(\mathbf{c}, \mathbf{d}, \mathbf{Q})$ be a Forward-Reverse Brascamp–Lieb datum. Then the following are equivalent:*

- (i) *The datum $(\mathbf{c}, \mathbf{d}, \mathbf{Q})$ is equivalent to a geometric one.*
- (ii) *There exist $A_i \in \mathcal{S}^+(E_i)$, $1 \leq i \leq k$, and $A^j \in \mathcal{S}^+(E^j)$, $1 \leq j \leq m$ such that if all the functions $(f_i)_{1 \leq i \leq k}$, $(g_j)_{1 \leq j \leq m}$ are non-negative and integrable, and (1.6) holds, that is,*

$$\prod_{i=1}^k f_i^{c_i}(\pi_{E_i} x) \leq \prod_{j=1}^m g_j^{d_j}(\pi_{E^j} \mathbf{Q} x), \quad \forall x \in E_0,$$

then

$$(1.13) \quad \prod_{i=1}^k (e^{t\Delta_{E_i, A_i}} f_i)^{c_i}(\pi_{E_i} x) \leq \prod_{j=1}^m (e^{t\Delta_{E^j, A^j}} g_j)^{d_j}(\pi_{E^j} \mathbf{Q} x),$$

$\forall x \in E_0, t > 0.$

- (iii) *There exist Gaussian extremizers.*

The proofs of Theorem 1.4 and Theorem 1.7 will be provided in Section 2 and Section 3 respectively.

2. Proof of Theorem 1.4. In this section we give the proof of our main theorem, Theorem 1.4. We split the proof into four parts. In Section 2.1 we prove the key ingredient of the proof: the maximum principle for parabolic equations. Then in Section 2.2 we establish three lemmas to deal with the corresponding three assumptions of Lemma 2.1. The proof under an extra assumption on the regularity of initial data is provided in Section 2.3. Finally, in Section 2.4 we end the proof with an approximation process by using the Vitali–Carathéodory Theorem.

2.1. Maximum principle for parabolic equations. In this section we prove a slightly improved version of [BH09, Lemma 1], which is closely related to the maximum principle for parabolic equations. To be more precise, we lower the regularity required for the initial datum at time 0. Although the proof is similar, for the sake of completeness, we include the proof here.

LEMMA 2.1. *Let V be a Euclidean space and $\mathbf{C} \in C^2(V \times (0, +\infty))$. Assume furthermore that:*

(1) for any point $(v_*, t_*) \in V \times (0, +\infty)$ such that

$$(2.1) \quad \begin{cases} \mathbf{C}(v_*, t_*) \leq 0, \\ \nabla_V \mathbf{C}(v_*, t_*) = 0, \\ \nabla_V^2 \mathbf{C}(v_*, t_*) \geq 0, \end{cases}$$

we have $\partial_t \mathbf{C}(v_*, t_*) \geq 0$;

(2) for every $T > 0$, we have

$$(2.2) \quad \liminf_{|v|_V \rightarrow +\infty} \left(\inf_{t \in (0, T]} \mathbf{C}(v, t) \right) \geq 0;$$

(3) for every $v_0 \in V$, we have

$$(2.3) \quad \liminf_{v \rightarrow v_0, t \rightarrow 0^+} \mathbf{C}(v, t) \geq 0,$$

where all the “lim inf”s can take the value $+\infty$.

Then

$$\mathbf{C}(v, t) \geq 0, \quad \forall (v, t) \in V \times (0, +\infty).$$

Proof. Assume on the contrary that there is a point $(v_0, t_0) \in V \times (0, +\infty)$ such that

$$\mathbf{C}(v_0, t_0) = -2\delta < 0.$$

We can pick an $\epsilon > 0$ such that $\mathbf{C}(v_0, t_0) + \epsilon t_0 < -\delta < 0$. Also pick a $T \in [t_0, +\infty)$. Now we consider the following function on $V \times (0, T]$:

$$\mathbf{C}_\epsilon(v, t) := \mathbf{C}(v, t) + \epsilon t.$$

From the construction above we know that

$$\inf_{(v, t) \in V \times (0, T]} \mathbf{C}_\epsilon(v, t) \leq \mathbf{C}_\epsilon(v_0, t_0) < -\delta < 0.$$

By definition, there is a sequence $\{(v_n, t_n)\}_{n \geq 1} \subset V \times (0, T]$ such that

$$\mathbf{C}_\epsilon(v_n, t_n) \rightarrow \inf_{(v, t) \in V \times (0, T]} \mathbf{C}_\epsilon(v, t) \quad \text{as } n \rightarrow +\infty.$$

It follows from (2.2) that there exists an $R > 0$ such that if $|v|_V > R$, then

$$\inf_{t \in (0, T]} \mathbf{C}_\epsilon(v, t) \geq \inf_{t \in (0, T]} \mathbf{C}(v, t) \geq -\delta,$$

which implies $|v_n|_V \leq R$ for n large enough. After passing to some subsequence, we can assume that $(v_n, t_n) \rightarrow (v_*, t_*) \in V \times [0, T]$. We deduce that $t_* > 0$, since otherwise

$$\begin{aligned} \liminf_{v \rightarrow v_*, t \rightarrow 0^+} \mathbf{C}(v, t) &\leq \liminf_{n \rightarrow +\infty} \mathbf{C}(v_n, t_n) \\ &\leq \lim_{n \rightarrow +\infty} \mathbf{C}_\epsilon(v_n, t_n) = \inf_{(v, t) \in V \times (0, T]} \mathbf{C}_\epsilon(v, t) < 0, \end{aligned}$$

which contradicts (2.3). Now by continuity of $\mathbf{C}$ we have

$$\mathbf{C}(v_*, t_*) \leq \mathbf{C}_\epsilon(v_*, t_*) = \inf_{(v,t) \in V \times (0,T]} \mathbf{C}_\epsilon(v, t) < 0.$$

This implies that at the point (v_*, t_*) , we have (2.1) and $\partial_t \mathbf{C}(v_*, t_*) + \epsilon \leq 0$. But (2.1) implies $\partial_t \mathbf{C}(v_*, t_*) \geq 0$ by the first assumption and this leads to a contradiction. ■

2.2. Some preparation lemmas. Towards the proof of Theorem 1.4, we make some preparations to deal with the three assumptions in Lemma 2.1.

LEMMA 2.2. *Assume $(\mathbf{c}, \mathbf{d}, \mathbf{Q})$ is a geometric Forward-Reverse Brascamp-Lieb datum.*

(1) *We have*

$$(2.4) \quad \sum_{i=1}^k \frac{1}{c_i} \left| \sum_{j=1}^m d_j \mathbf{Q}_{ji}^* v^j \right|_{E_i}^2 \leq \sum_{j=1}^m d_j |v^j|_{E^j}^2, \quad \forall v^j \in E^j, 1 \leq j \leq m.$$

(2) *If $X_i \in \mathcal{S}(E_i)$, $1 \leq i \leq k$ and $Y_j \in \mathcal{S}(E^j)$, $1 \leq j \leq m$, satisfy*

$$(2.5) \quad \sum_{i=1}^k c_i \pi_{E_i}^* X_i \pi_{E_i} \leq \mathbf{Q}^* \left(\sum_{j=1}^m d_j \pi_{E^j}^* Y_j \pi_{E^j} \right) \mathbf{Q},$$

then

$$(2.6) \quad \sum_{i=1}^k c_i \operatorname{Tr}(X_i) \leq \sum_{j=1}^m d_j \operatorname{Tr}(Y_j).$$

Proof. With the identification (1.3), if we use the language of matrices, then to prove (2.4) is equivalent to proving that

$$\Lambda_{\mathbf{d}} \mathbf{Q} \Lambda_{\mathbf{c}}^{-1} \mathbf{Q}^* \Lambda_{\mathbf{d}} \leq \Lambda_{\mathbf{d}},$$

which is equivalent to $A^* A \leq \operatorname{id}_{E^0}$, where $A := \Lambda_{\mathbf{c}}^{-1/2} \mathbf{Q}^* \Lambda_{\mathbf{d}}^{1/2}$. On the other hand, from (1.10) we have $AA^* \leq \operatorname{id}_{E_0}$, which is equivalent to saying that $\|A^*\|_{E_0 \rightarrow E^0} \leq 1$, and thus $\|A\|_{E^0 \rightarrow E_0} \leq 1$, which implies $A^* A \leq \operatorname{id}_{E^0}$. This ends the proof of the first part. For the second part, in the same spirit we write (2.5) as

$$\begin{pmatrix} c_1 X_1 & 0 & \dots & 0 \\ 0 & c_2 X_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_k X_k \end{pmatrix} \leq \mathbf{Q}^* \begin{pmatrix} d_1 Y_1 & 0 & \dots & 0 \\ 0 & d_2 Y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_m Y_m \end{pmatrix} \mathbf{Q}.$$

This implies

$$\begin{aligned} \sqrt{\Sigma} \begin{pmatrix} c_1 X_1 & 0 & \dots & 0 \\ 0 & c_2 X_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_k X_k \end{pmatrix} \sqrt{\Sigma} \\ \leq \sqrt{\Sigma} \mathbf{Q}^* \begin{pmatrix} d_1 Y_1 & 0 & \dots & 0 \\ 0 & d_2 Y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_m Y_m \end{pmatrix} \mathbf{Q} \sqrt{\Sigma}, \end{aligned}$$

where Σ is the matrix appearing in the definition of geometric Forward-Reverse Brascamp–Lieb datum. Taking the trace on both sides and using the property that $\text{Tr}(\mathbf{A}\mathbf{B}) = \text{Tr}(\mathbf{B}\mathbf{A})$ for suitable linear maps $\mathbf{A}, \mathbf{B}$, we obtain

$$\begin{aligned} \text{Tr} \left(\Sigma \begin{pmatrix} c_1 X_1 & 0 & \dots & 0 \\ 0 & c_2 X_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_k X_k \end{pmatrix} \right) \\ \leq \text{Tr} \left(\mathbf{Q} \Sigma \mathbf{Q}^* \begin{pmatrix} d_1 Y_1 & 0 & \dots & 0 \\ 0 & d_2 Y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_m Y_m \end{pmatrix} \right). \end{aligned}$$

Since $(\mathbf{c}, \mathbf{d}, \mathbf{Q})$ is geometric, the matrices have the following block forms:

$$\begin{aligned} \Sigma \begin{pmatrix} c_1 X_1 & 0 & \dots & 0 \\ 0 & c_2 X_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_k X_k \end{pmatrix} &= \begin{pmatrix} c_1 X_1 & * & \dots & * \\ * & c_2 X_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & c_k X_k \end{pmatrix}, \\ \mathbf{Q} \Sigma \mathbf{Q}^* \begin{pmatrix} d_1 Y_1 & 0 & \dots & 0 \\ 0 & d_2 Y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_m Y_m \end{pmatrix} &= \begin{pmatrix} d_1 Y_1 & * & \dots & * \\ * & d_2 Y_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & d_m Y_m \end{pmatrix}. \end{aligned}$$

So the trace inequality yields (2.6). ■

LEMMA 2.3. *Assume f is a bounded function with compact support on some Euclidean space V . Then*

$$\lim_{|x|_V \rightarrow +\infty} \sup_{t \in (0, +\infty)} |(e^{t\Delta_V} f)(x)| = 0.$$

Proof. Assume $|f| \leq M$ and that the support of f is contained in $K := \{v : |v|_V \leq R\}$. Then combining (1.1) with (1.2), we obtain

$$|(e^{t\Delta_V} f)(x)| \leq \int_V |f(y)| p_t^V(x - y) dy \leq \frac{M}{(4\pi t)^{\frac{\dim(V)}{2}}} \int_K e^{-\frac{|x-y|_V^2}{4t}} dy.$$

Note that when $|x|_V \geq 2R$ and $y \in K$, we have $|x - y|_V \geq |x|_V - |y|_V \geq |x|_V - R \geq \frac{|x|_V}{2}$. Thus the last term can be bounded by

$$\frac{C(M, R, \dim(V))}{t^{\frac{\dim(V)}{2}}} e^{-\frac{|x|_V^2}{16t}} \leq \frac{C'(M, R, \dim(V))}{|x|_V^{\dim(V)}} \rightarrow 0$$

as $|x|_V \rightarrow +\infty$. Since the last term is independent of t , we can take the supremum over $t > 0$. ■

LEMMA 2.4. *Assume f and g are integrable functions on some Euclidean space V .*

- (1) *If f is upper semicontinuous and bounded from above, then for every $x_0 \in V$, we have*

$$\limsup_{x \rightarrow x_0, t \rightarrow 0^+} (e^{t\Delta_V} f)(x) \leq f(x_0).$$

- (2) *If g is lower semicontinuous and bounded from below, then for every $x_0 \in V$, we have*

$$\liminf_{x \rightarrow x_0, t \rightarrow 0^+} (e^{t\Delta_V} g)(x) \geq g(x_0).$$

Proof. It suffices to prove the first assertion since the proof for the second one is quite similar. In this proof we use $B(x, r)$ to denote the ball in V centred at x with radius r . Now fix a point $x_0 \in V$. For every $\epsilon > 0$, by the upper semicontinuity of f at x_0 , there exists a $\delta > 0$ such that

$$f(z) \leq f(x_0) + \frac{\epsilon}{2}, \quad \forall z \in B(x_0, 2\delta).$$

Then from (1.1), we have

$$\begin{aligned} (2.7) \quad (e^{t\Delta_V} f)(x) &= \int_V f(x - y) p_t^V(y) dy \\ &= \int_{B(0, \delta)} f(x - y) p_t^V(y) dy + \int_{B(0, \delta)^c} f(x - y) p_t^V(y) dy. \end{aligned}$$

For the first term, if $x \in B(x_0, \delta)$ and $y \in B(0, \delta)$, we have $x - y \in B(x_0, 2\delta)$ and thus

$$\begin{aligned}
(2.8) \quad \int_{B(0,\delta)} f(x-y)p_t^V(y) dy &\leq \left(f(x_0) + \frac{\epsilon}{2}\right) \int_{B(0,\delta)} p_t^V(y) dy \\
&= \left(f(x_0) + \frac{\epsilon}{2}\right) - \left(f(x_0) + \frac{\epsilon}{2}\right) \int_{B(0,\delta)^c} p_t^V(y) dy.
\end{aligned}$$

Now for the second terms in (2.7) and (2.8), by (1.2) and a change of variables, we have

$$\begin{aligned}
&\left| \left(f(x_0) + \frac{\epsilon}{2}\right) \int_{B(0,\delta)^c} p_t^V(y) dy \right| \\
&\leq \frac{\left|f(x_0) + \frac{\epsilon}{2}\right|}{(4\pi t)^{\frac{\dim(V)}{2}}} \int_{B(0,\delta)^c} e^{-\frac{|y|_V^2}{4t}} dy = \left|f(x_0) + \frac{\epsilon}{2}\right| \int_{B(0,\delta/\sqrt{4\pi t})^c} e^{-\pi|z|_V^2} dz, \\
&\int_{B(0,\delta)^c} f(x-y)p_t^V(y) dy \\
&\leq \frac{M}{(4\pi t)^{\frac{\dim(V)}{2}}} \int_{B(0,\delta)^c} e^{-\frac{|y|_V^2}{4t}} dy = M \int_{B(0,\delta/\sqrt{4\pi t})^c} e^{-\pi|z|_V^2} dz,
\end{aligned}$$

where $M > 0$ is an upper bound of f . The last terms of the two estimates above can be made sufficiently small as $t \rightarrow 0^+$ by the dominated convergence theorem. So there exists an $\eta > 0$ such that when $t \in (0, \eta)$, we have

$$-\left(f(x_0) + \frac{\epsilon}{2}\right) \int_{B(0,\delta)^c} p_t^V(y) dy + \int_{B(0,\delta)^c} f(x-y)p_t^V(y) dy \leq \frac{\epsilon}{2}.$$

Combining the estimates of all the terms, we conclude that for every $\epsilon > 0$, there exist $\delta, \eta > 0$ such that when $x \in B(x_0, \delta)$ and $t \in (0, \eta)$, we obtain

$$(e^{t\Delta_V} f)(x) \leq f(x_0) + \epsilon. \blacksquare$$

2.3. The proof under suitable regularity. The target of this subsection is to prove the following proposition, which is a special case of Theorem 1.4.

PROPOSITION 2.5. *Let $(\mathbf{c}, \mathbf{d}, \mathbf{Q})$ be a geometric Forward-Reverse Brascamp-Lieb datum. Assume that the functions $(f_i)_{1 \leq i \leq k}$ are non-negative, upper semicontinuous, with compact support, and the functions $(g_j)_{1 \leq j \leq m}$ are non-negative, lower semicontinuous. If (1.6) holds, that is,*

$$\prod_{i=1}^k f_i^{c_i}(\pi_{E_i} x) \leq \prod_{j=1}^m g_j^{d_j}(\pi_{E^j} \mathbf{Q} x), \quad \forall x \in E_0,$$

then

$$(2.9) \quad \prod_{i=1}^k (e^{t\Delta_{E_i}} f_i)^{c_i}(\pi_{E_i} x) \leq \prod_{j=1}^m (e^{t\Delta_{E^j}} g_j)^{d_j}(\pi_{E^j} \mathbf{Q} x), \quad \forall x \in E_0, t > 0.$$

Proof. For any $\delta > 0$, we define $u_i := \log(e^{t\Delta_{E_i}} f_i)$, $1 \leq i \leq k$, $g_j^\delta := g_j + \delta$, $v_j^\delta := \log(e^{t\Delta_{E^j}} g_j^\delta)$, $1 \leq j \leq m$. Since $e^{t\Delta_{E^j}} g_j^\delta = e^{t\Delta_{E^j}} g_j + \delta$, by the definition we know that all these functions are smooth and

$$(2.10) \quad \partial_t u_i = \Delta_{E_i} u_i + |\nabla_{E_i} u_i|_{E_i}^2, \quad \forall 1 \leq i \leq k,$$

$$(2.11) \quad \partial_t v_j^\delta = \Delta_{E^j} v_j^\delta + |\nabla_{E^j} v_j^\delta|_{E^j}^2, \quad \forall 1 \leq j \leq m.$$

Now on the space $E_0 \times (0, +\infty)$ we construct

$$\mathbf{C}(x, t) := \sum_{j=1}^m d_j v_j^\delta(\pi_{E^j} \mathbf{Q}x, t) - \sum_{i=1}^k c_i u_i(\pi_{E_i} x, t).$$

We show that this $\mathbf{C}$ satisfies all the three assumptions of Lemma 2.1.

Assume first that at the point (x_*, t_*) we have $\nabla_{E_0} \mathbf{C}(x_*, t_*) = 0$ and $\nabla_{E_0}^2 \mathbf{C}(x_*, t_*) \geq 0$. From $\nabla_{E_0} \mathbf{C}(x_*, t_*) = 0$ we obtain

$$(2.12) \quad c_i \nabla_{E_i} u_i(\pi_{E_i} x_*, t_*) = \sum_{j=1}^m d_j \mathbf{Q}_{ji}^* \nabla_{E^j} v_j^\delta(\pi_{E^j} \mathbf{Q}x_*, t_*), \quad \forall 1 \leq i \leq k.$$

Using (2.4) with $v^j = \nabla_{E^j} v_j^\delta(\pi_{E^j} \mathbf{Q}x_*, t_*)$, $1 \leq j \leq m$, we have

$$(2.13) \quad \sum_{i=1}^k c_i |\nabla_{E_i} u_i(\pi_{E_i} x_*, t_*)|_{E_i}^2 \leq \sum_{j=1}^m d_j |\nabla_{E^j} v_j^\delta(\pi_{E^j} \mathbf{Q}x_*, t_*)|_{E^j}^2.$$

From $\nabla_{E_0}^2 \mathbf{C}(x_*, t_*) \geq 0$ we know that (2.5) holds with

$$X_i = \nabla_{E_i}^2 u_i(\pi_{E_i} x_*, t_*), \quad 1 \leq i \leq k, \quad Y_j = \nabla_{E^j}^2 v_j^\delta(\pi_{E^j} \mathbf{Q}x_*, t_*), \quad 1 \leq j \leq m.$$

It follows from (2.6) that

$$(2.14) \quad \sum_{i=1}^k c_i \Delta_{E_i} u_i(\pi_{E_i} x_*, t_*) \leq \sum_{j=1}^m d_j \Delta_{E^j} v_j^\delta(\pi_{E^j} \mathbf{Q}x_*, t_*).$$

Combining (2.10)–(2.11) with (2.13)–(2.14), we have

$$\begin{aligned} \partial_t \mathbf{C}(x_*, t_*) &= \sum_{j=1}^m d_j \partial_t v_j^\delta(\pi_{E^j} \mathbf{Q}x_*, t_*) - \sum_{i=1}^k c_i \partial_t u_i(\pi_{E_i} x_*, t_*) \\ &= \sum_{j=1}^m d_j (\Delta_{E^j} v_j^\delta(\pi_{E^j} \mathbf{Q}x_*, t_*) + |\nabla_{E^j} v_j^\delta(\pi_{E^j} \mathbf{Q}x_*, t_*)|_{E^j}^2) \\ &\quad - \sum_{i=1}^k c_i (\Delta_{E_i} u_i(\pi_{E_i} x_*, t_*) + |\nabla_{E_i} u_i(\pi_{E_i} x_*, t_*)|_{E_i}^2) \geq 0, \end{aligned}$$

which shows that the first assumption of Lemma 2.1 holds.

Now Lemma 2.3 implies

$$\liminf_{|x|_{E_0} \rightarrow +\infty} \sup_{t \in (0, +\infty)} \sum_{i=1}^k c_i u_i(\pi_{E_i} x, t) \rightarrow -\infty.$$

Together with the fact that $(v_j^\delta)_{1 \leq j \leq m}$ are bounded from below by construction, it is easy to show that the second assumption of Lemma 2.1 holds.

Finally, for the third assumption of Lemma 2.1, by Lemma 2.4, for any $x_0 \in E_0$ we have

$$\liminf_{x \rightarrow x_0, t \rightarrow 0^+} \mathbf{C}(x, t) \geq \sum_{j=1}^m d_j \log(g_j^\delta)(\pi_{E^j} \mathbf{Q} x_0) - \sum_{i=1}^k c_i \log(f_i)(\pi_{E_i} x_0) \geq 0.$$

Thus Lemma 2.1 applies to the function $\mathbf{C}$ here and we obtain

$$\sum_{j=1}^m d_j v_j^\delta(\pi_{E^j} \mathbf{Q} x, t) - \sum_{i=1}^k c_i u_i(\pi_{E_i} x, t) \geq 0.$$

Since $\delta > 0$ is arbitrary, letting $\delta \rightarrow 0^+$, we prove the proposition. ■

2.4. Approximation process. In this subsection we remove the regularity assumptions on the functions $(f_i)_{1 \leq i \leq k}$, $(g_j)_{1 \leq j \leq m}$ in Proposition 2.5 by using the famous Vitali–Carathéodory Theorem, which states that any integrable function can be approximated from above by lower semicontinuous functions and from below by upper semicontinuous functions. We refer to [Rud87, §2.25] for the proof.

THEOREM 2.6 (Vitali–Carathéodory Theorem). *Given a Euclidean space V and a function $f \in L^1(V)$, for every $\epsilon > 0$, there exist two functions u and v such that $u \leq f \leq v$, u is upper semicontinuous and bounded from above, v is lower semicontinuous and bounded from below, and*

$$\int_V (v - u) \leq \epsilon.$$

Furthermore, if $f \geq 0$, we can pick u in such a way that $u \geq 0$ and u has compact support.

REMARK 2.7. The last part of Theorem 2.6 is not stated in [Rud87, §2.25] but it can be seen from the proof there.

Proof of Theorem 1.4. It follows from Theorem 2.6 that for each $1 \leq i \leq k$, there exists a sequence $\{u_{i,n}\}_{n \geq 1} \subset L^1(E_i)$ such that for each $n \geq 1$, $u_{i,n}$ is non-negative, upper semicontinuous, with compact support, $u_{i,n} \leq f_i$, and $u_{i,n} \rightarrow f_i$ in $L^1(E_i)$ as $n \rightarrow +\infty$. Similarly, for each $1 \leq j \leq m$, there exists a sequence $\{v_{j,n}\}_{n \geq 1} \subset L^1(E^j)$ such that for each $n \geq 1$, $v_{j,n}$ is non-negative, lower semicontinuous, $g_j \leq v_{j,n}$, and $v_{j,n} \rightarrow g_j$ in $L^1(E^j)$

as $n \rightarrow +\infty$. Now, for each $n \geq 1$, by (1.6) we have

$$\prod_{i=1}^k (u_{i,n})^{c_i} (\pi_{E_i} x) \leq \prod_{j=1}^m (v_{j,n})^{d_j} (\pi_{E_j} \mathbf{Q} x), \quad \forall x \in E_0.$$

Applying Proposition 2.5, we obtain

$$\prod_{i=1}^k (e^{t\Delta_{E_i}} u_{i,n})^{c_i} (\pi_{E_i} x) \leq \prod_{j=1}^m (e^{t\Delta_{E_j}} v_{j,n})^{d_j} (\pi_{E_j} \mathbf{Q} x), \quad \forall x \in E_0, t > 0.$$

Fix $x \in E_0$ and $t > 0$. We let $n \rightarrow +\infty$. Since all the heat kernels are bounded functions for fixed time, the L^1 convergences imply the pointwise convergences, and we prove Theorem 1.4. ■

REMARK 2.8. The main difficulty of the approximation process lies in the preservation of the relation (1.6). For example, it is easy to assume that each f_i is bounded and has compact support, but the boundedness for each g_j is harder since we need to take care of (1.6). To be more precise, even a change of values on a small set in E_0 will affect the values of $(g_j)_{1 \leq j \leq m}$ a lot. For instance, let $\mathbf{Q}$ be a rotation on $\mathbb{R}^2$. Let $L := [-\epsilon, \epsilon] \times [-M, M]$. Then with ϵ small the measure of the projection of L onto the first variable is small. However, we cannot control the measure of the projection of $\mathbf{Q}(L)$ onto the first variable for every fixed $\mathbf{Q}$. This explains why we used upper/lower semicontinuous functions instead of continuous functions.

To prove Corollary 1.6 we need the following long-time behavior of the heat equation, which is a direct result of (1.1), (1.2), and the dominated convergence theorem.

LEMMA 2.9. *Assume V is a Euclidean space, $A \in \mathcal{S}^+(V)$, and $f \in L^1(V)$. Then for every $x \in V$, we have*

$$\lim_{t \rightarrow +\infty} (4\pi t)^{\frac{\dim(V)}{2}} (e^{t\Delta_{V,A}} f)(2\sqrt{t}x) = \frac{1}{\sqrt{\det(A)}} e^{-\langle A^{-1}x, x \rangle_V} \left(\int_V f \right).$$

Proof of Corollary 1.6. Without loss of generality we can assume all functions $(f_i)_{1 \leq i \leq k}, (g_j)_{1 \leq j \leq m}$ are integrable. Then Theorem 1.4 yields

$$\prod_{i=1}^k (e^{t\Delta_{E_i}} f_i)^{c_i}(0) \leq \prod_{j=1}^m (e^{t\Delta_{E_j}} g_j)^{d_j}(0), \quad \forall t > 0.$$

Multiplying both sides by

$$\prod_{i=1}^k (4\pi t)^{\frac{c_i \dim(E_i)}{2}} = \prod_{j=1}^m (4\pi t)^{\frac{d_j \dim(E_j)}{2}},$$

where the equality is given by (1.4), and letting $t \rightarrow +\infty$, we obtain

$$\prod_{i=1}^k \left(\int_{E_i} f_i \right)^{c_i} \leq \prod_{j=1}^m \left(\int_{E^j} g_j \right)^{d_j}$$

by Lemma 2.9. This implies $C(\mathbf{c}, \mathbf{d}, \mathbf{Q}) \leq 1$. To prove the opposite direction, it suffices to notice that from (1.10) we have

$$\sum_{j=1}^m d_j |\pi_{E^j} \mathbf{Q} x|_{E^j}^2 \leq \sum_{i=1}^k c_i |\pi_{E_i} x|_{E_i}^2, \quad \forall x \in E_0,$$

or equivalently,

$$\prod_{i=1}^k (e^{-|\pi_{E_i} x|_{E_i}^2})^{c_i} \leq \prod_{j=1}^m (e^{-|\pi_{E^j} \mathbf{Q} x|_{E^j}^2})^{d_j}, \quad \forall x \in E_0.$$

As a result, the choice $f_i = e^{-|\cdot|_{E_i}^2}$, $1 \leq i \leq k$, and $g_j = e^{-|\cdot|_{E^j}^2}$, $1 \leq j \leq m$, gives $C(\mathbf{c}, \mathbf{d}, \mathbf{Q}) \geq 1$. ■

3. Proof of Theorem 1.7. (i) $\Rightarrow$ (ii). From (i) there exists a geometric Forward-Reverse Brascamp–Lieb datum $(\mathbf{c}, \mathbf{d}, \mathbf{Q}')$ such that

$$\mathbf{Q}'_{ji} = D_j^{-1} \mathbf{Q}_{ji} C_i^{-1}, \quad \forall 1 \leq i \leq k, 1 \leq j \leq m,$$

for some invertible linear maps $C_i : E_i \rightarrow E_i$, $1 \leq i \leq k$, and $D_j : E^j \rightarrow E^j$, $1 \leq j \leq m$. Now define

$$\mathbf{C} := \sum_{i=1}^k \pi_{E_i}^* C_i \pi_{E_i}, \quad \mathbf{D} := \sum_{j=1}^m \pi_{E^j}^* D_j \pi_{E^j}.$$

With these notations we have $\mathbf{Q} = \mathbf{D} \mathbf{Q}' \mathbf{C}$. In the following we use $\circ$ to denote the composition of maps. Since (1.10) holds, we have

$$\prod_{i=1}^k F_i^{c_i}(\pi_{E_i} x) \leq \prod_{j=1}^m G_j^{d_j}(\pi_{E^j} \mathbf{Q}' x), \quad \forall x \in E_0,$$

where $F_i := f_i \circ C_i^{-1}$, $1 \leq i \leq k$, $G_j := g_j \circ D_j$, $1 \leq j \leq m$. It follows from Theorem 1.4 that

$$\prod_{i=1}^k (e^{t \Delta_{E_i} F_i})^{c_i}(\pi_{E_i} x) \leq \prod_{j=1}^m (e^{t \Delta_{E^j} G_j})^{d_j}(\pi_{E^j} \mathbf{Q}' x), \quad \forall x \in E_0, t > 0.$$

By a direct calculation using (1.1) and (1.2), we obtain

$$\prod_{i=1}^k (e^{t \Delta_{E_i, A_i} f_i})^{c_i}(C_i^{-1} \pi_{E_i} x) \leq \prod_{j=1}^m (e^{t \Delta_{E^j, A^j} g_j})^{d_j}(D_j \pi_{E^j} \mathbf{Q}' x), \quad \forall x \in E_0, t > 0.$$

where $A_i := C_i^{-1}(C_i^{-1})^*$, $1 \leq i \leq k$, $A^j := D_j D_j^*$, $1 \leq j \leq m$. This implies (1.13).

(ii) $\Rightarrow$ (iii). Assume now (1.13) holds. We replace x with $2\sqrt{t}x$, multiply both sides by

$$\prod_{i=1}^k (4\pi t)^{\frac{c_i \dim(E_i)}{2}} = \prod_{j=1}^m (4\pi t)^{\frac{d_j \dim(E^j)}{2}},$$

as before, and let $t \rightarrow +\infty$. It follows from Lemma 2.9 that

$$(3.1) \quad \prod_{i=1}^k \left(\int_{E_i} f_i \right)^{c_i} e^{-c_i \langle A_i^{-1} \pi_{E_i} x, \pi_{E_i} x \rangle_{E_i}} \\ \leq \frac{\prod_{i=1}^k (\det(A_i))^{c_i/2}}{\prod_{j=1}^m (\det(A^j))^{d_j/2}} \prod_{j=1}^m \left(\int_{E^j} g_j \right)^{d_j} e^{-d_j \langle (A^j)^{-1} \pi_{E^j} \mathbf{Q}x, \pi_{E^j} \mathbf{Q}x \rangle_{E^j}}$$

for all $x \in E_0$. Now let $x = 0$. We obtain

$$C(\mathbf{c}, \mathbf{d}, \mathbf{Q}) \leq \frac{\prod_{i=1}^k (\det(A_i))^{c_i/2}}{\prod_{j=1}^m (\det(A^j))^{d_j/2}}.$$

Replacing x with rx ($r > 0$) in (3.1), taking the logarithm, and then dividing both sides by r^2 , we obtain

$$-\sum_{i=1}^k c_i \langle A_i^{-1} \pi_{E_i} x, \pi_{E_i} x \rangle_{E_i} \leq \frac{C'}{r^2} - \sum_{j=1}^m d_j \langle (A^j)^{-1} \pi_{E^j} \mathbf{Q}x, \pi_{E^j} \mathbf{Q}x \rangle_{E^j}.$$

Here C' is a constant independent of $x \in E_0$ and $r > 0$. Letting $r \rightarrow +\infty$ we have

$$\sum_{j=1}^m d_j \langle (A^j)^{-1} \pi_{E^j} \mathbf{Q}x, \pi_{E^j} \mathbf{Q}x \rangle_{E^j} \leq \sum_{i=1}^k c_i \langle A_i^{-1} \pi_{E_i} x, \pi_{E_i} x \rangle_{E_i},$$

or equivalently,

$$\prod_{i=1}^k (e^{-\langle A_i^{-1} \pi_{E_i} x, \pi_{E_i} x \rangle_{E_i}})^{c_i} \leq \prod_{j=1}^m (e^{-\langle (A^j)^{-1} \pi_{E^j} \mathbf{Q}x, \pi_{E^j} \mathbf{Q}x \rangle_{E^j}})^{d_j}.$$

This means that the choice

$$f_i = e^{-\langle A_i^{-1} \pi_{E_i} x, \pi_{E_i} x \rangle_{E_i}}, \quad 1 \leq i \leq k, \quad g_j = e^{-\langle (A^j)^{-1} \pi_{E^j} \mathbf{Q}x, \pi_{E^j} \mathbf{Q}x \rangle_{E^j}}, \quad 1 \leq j \leq m,$$

gives

$$C(\mathbf{c}, \mathbf{d}, \mathbf{Q}) = \frac{\prod_{i=1}^k (\det(A_i))^{c_i/2}}{\prod_{j=1}^m (\det(A^j))^{d_j/2}} = \frac{\prod_{i=1}^k (\int_{E_i} f_i)^{c_i}}{\prod_{j=1}^m (\int_{E^j} g_j)^{d_j}},$$

which is exactly (iii).

(iii) $\Rightarrow$ (i). Since we have Gaussian extremizers, the Forward-Reverse Brascamp–Lieb datum $(\mathbf{c}, \mathbf{d}, \mathbf{Q})$ is equivalent to some geometric one by [CL21, Theorem 1.23]. ■

Acknowledgements. YZ would like to thank Neal Bez for his kind invitation to Saitama University and thank Anthony Gauvan and Hiroshi Tsuji for the fruitful discussions during the visit. YZ also would like to thank the anonymous referee for many valuable suggestions and remarks.

Funding. This project has received funding from (i) the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement GEOSUB, No. 945655); (ii) the JSPS Grant-in-Aid for Early-Career Scientists (No. 24K16928).

References

- [Bar98a] F. Barthe, *Optimal Young’s inequality and its converse: a simple proof*, Geom. Funct. Anal. 8 (1998), 234–242.
- [Bar98b] F. Barthe, *On a reverse form of the Brascamp–Lieb inequality*, Invent. Math. 134 (1998), 335–361.
- [BCE04] F. Barthe and D. Cordero-Erausquin, *Inverse Brascamp–Lieb inequalities along the heat equation*, in: Geometric Aspects of Functional Analysis, Lecture Notes in Math. 1850, Springer, Berlin, 2004, 65–71.
- [BC⁺11] F. Barthe, D. Cordero-Erausquin, M. Ledoux, and B. Maurey, *Correlation and Brascamp–Lieb inequalities for Markov semigroups*, Int. Math. Res. Notices 2011, 2177–2216.
- [BH09] F. Barthe and N. Huet, *On Gaussian Brunn–Minkowski inequalities*, Studia Math. 191 (2009), 283–304.
- [BB19] J. Bennett and N. Bez, *Generating monotone quantities for the heat equation*, J. Reine Angew. Math. 756 (2019), 37–63.
- [BC⁺08] J. Bennett, A. Carbery, M. Christ, and T. Tao, *The Brascamp–Lieb inequalities: finiteness, structure and extremals*, Geom. Funct. Anal. 17 (2008), 1343–1415.
- [Bor00] Ch. Borell, *Diffusion equations and geometric inequalities*, Potential Anal. 12 (2000), 49–71.
- [BKX23] K. J. Boroczky, P. Kalantopoulos, and D. Xi, *The case of equality in geometric instances of Barthe’s reverse Brascamp–Lieb inequality*, in: Geometric Aspects of Functional Analysis, Lecture Notes in Math. 2327, Springer, Cham, 2023, 129–165.
- [BL76] H. J. Brascamp and E. H. Lieb, *Best constants in Young’s inequality, its converse, and its generalization to more than three functions*, Adv. Math. 20 (1976), 151–173.
- [BLL74] H. J. Brascamp, E. H. Lieb, and J. M. Luttinger, *A general rearrangement inequality for multiple integrals*, J. Funct. Anal. 17 (1974), 227–237.
- [CCE09] E. A. Carlen and D. Cordero-Erausquin, *Subadditivity of the entropy and its relation to Brascamp–Lieb type inequalities*, Geom. Funct. Anal. 19 (2009), 373–405.
- [CLL04] E. A. Carlen, E. H. Lieb, and M. Loss, *A sharp analog of Young’s inequality on S^N and related entropy inequalities*, J. Geom. Anal. 14 (2004), 487–520.

- [CL21] T. A. Courtade and J. Liu, *Euclidean forward-reverse Brascamp–Lieb inequalities: finiteness, structure, and extremals*, J. Geom. Anal. 31 (2021), 3300–3350.
- [Gri09] A. Grigor’yan, *Heat Kernel and Analysis on Manifolds*, AMS/IP Stud. Adv. Math. 47, Amer. Math. Soc., Providence, RI, and Int. Press, Boston, MA, 2009.
- [ILS25] K. Ishige, Q. Liu, and P. Salani, *A parabolic PDE-based approach to Borell–Brascamp–Lieb inequality*, Math. Ann. 392 (2025), 4891–4937.
- [Leh14] J. Lehec, *Short probabilistic proof of the Brascamp–Lieb and Barthe theorems*, Canad. Math. Bull. 57 (2014), 585–597.
- [Lie90] E. H. Lieb, *Gaussian kernels have only Gaussian maximizers*, Invent. Math. 102 (1990), 179–208.
- [LC⁺18] J. Liu, T. A. Courtade, P. W. Cuff, and S. Verdú, *A forward-reverse Brascamp–Lieb inequality: entropic duality and Gaussian optimality*, Entropy 20 (2018), no. 6, art. 418, 32 pp.
- [Rud87] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1987.
- [Val08] S. I. Valdimarsson, *Optimisers for the Brascamp–Lieb inequality*, Israel J. Math. 168 (2008), 253–274.

Ye Zhang

Scuola Internazionale Superiore di Studi Avanzati (SISSA)

34136 Trieste, Italy

E-mail: yezhang@sissa.it

zhangye0217@gmail.com