

Almost disjoint families and some automorphic and injective properties of ℓ_∞/c_0

by

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Abstract. Answering questions of A. Avilés, F. Cabello Sánchez, J. Castillo, M. González and Y. Moreno we show that the following statements are independent of the usual ZFC axioms with arbitrarily large continuum: for every (some) $\omega < \kappa < 2^\omega$,

- (1) any bounded linear operator $T : c_0(\kappa) \rightarrow \ell_\infty/c_0$ extends to any superspace of $c_0(\kappa)$,
- (2) any isomorphism between any two copies of $c_0(\kappa)$ inside ℓ_∞/c_0 extends to an automorphism of ℓ_∞/c_0 .

This contrasts with Boolean, Banach-algebraic or isometric levels, where the objects known as Hausdorff gap and Luzin gap witness the failure in ZFC of the corresponding properties for the corresponding structures already at the first uncountable cardinal $\kappa = \omega_1$.

In particular, consistently, any two pairwise disjoint families in $\wp(\mathbb{N})/Fin$ of the same cardinality $\omega < \kappa < 2^\omega$ can be mapped onto each other by a linear automorphism of ℓ_∞/c_0 regardless of their different combinatorial, algebraic or topological positions in $\wp(\mathbb{N})/Fin$.

Our positive consistency results use a restricted version of Martin's axiom for a partial order that adds an infinite block diagonal matrix of an operator on ℓ_∞ which induces an operator on ℓ_∞/c_0 . The construction of its finite blocks relies on a lemma of Bourgain and Tzafriri on finite-dimensional Banach spaces.

Our negative consistency results rely on an analysis of almost disjoint families of $\mathbb{N}$, the embeddings of $c_0(\kappa)$ into ℓ_∞/c_0 they induce and their extensions to $\ell_\infty^c(\kappa)$.

1. Introduction. For the terminology and notation unexplained in the Introduction, see Section 2. Recall that a Banach space X is *injective* if every linear bounded operator from any Banach space with X as a codomain extends to any superspace of its domain. A Banach space X is *automorphic* if any isomorphism between any two of its subspaces of the same codimension extends to an automorphism of X .

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Banach spaces of the form $C(K)$ for K extremally disconnected (i.e., the Stone spaces of complete Boolean algebras, so for example $\ell_\infty(\kappa) \equiv C(\beta\kappa)$) are known to be injective but it is not known if there are injective Banach spaces which are not isomorphic to such $C(K)$ s. The Banach spaces $c_0(\kappa)$ and $\ell_2(\kappa)$ for infinite cardinals κ are known to be automorphic [26, 28] but it is not known if these are the only examples of automorphic Banach spaces (cf. Questions 6.1(1)). Several restricted versions of injective and automorphic properties are considered in the literature more or less explicitly:

DEFINITION 1.1 ([5]). Let X and Y be Banach spaces, let κ be an infinite cardinal and let $\mathcal{B}_\kappa$ denote the class of Banach spaces of density less than κ . We say that

- X is *universally separably injective* [*universally κ -injective*] (*universally Y -injective*) if for every separable space Z [for every $Z \in \mathcal{B}_\kappa$] (for every Z isomorphic to Y) and every superspace Z' of Z every bounded linear operator $T : Z \rightarrow X$ extends to an operator $\tilde{T} : Z' \rightarrow X$.
- X is *separably extensible* [*κ -extensible*] (*Y -extensible*) if for every separable subspace $Z \subseteq X$ [for every subspace $Z \subseteq X$ in $\mathcal{B}_\kappa$] (for every subspace $Z \subseteq X$ isomorphic to Y) every bounded linear operator $T : Z \rightarrow X$ extends to an operator $\tilde{T} : X \rightarrow X$.
- X is *separably automorphic* [*κ -automorphic*] (*Y -automorphic*) if for any isomorphic subspaces $Z, Z' \subseteq X$ which are separable [are in $\mathcal{B}_\kappa$] (are isomorphic to Y) of the same codimension (i.e., $\text{dens}(X/Z) = \text{dens}(X/Z')$) every isomorphism between Z and Z' extends to an automorphism of X .

For example, Johnson and Zippin [22] proved that $c_0(\kappa)$ is Y -extensible for any subspace $Y \subseteq c_0(\kappa)$ and any infinite cardinal κ , or Lindenstrauss and Rosenthal [26] proved that ℓ_∞ is 2^ω -automorphic but not automorphic, while Moreno and Plichko [28] proved that a κ -automorphic (Y -automorphic) space must be κ -extensible (Y -extensible), for any infinite cardinal κ (and any Banach space Y). Also, clearly, universal κ -injectivity (universal Y -injectivity) implies κ -extensibility (Y -extensibility), so extensibility is the weakest property of all three. A modern account and many new results on such restricted injective and automorphic properties focused on separable injectivity are presented in the monograph [5] by Avilés, Cabello Sánchez, Castillo, González and Moreno (cf. [3, 4]).

In this paper we focus on the above properties for the Banach space $\ell_\infty/c_0 \equiv C(\beta\mathbb{N} \setminus \mathbb{N})$, which is well known to be rather sensitive to additional set-theoretic assumptions [5, 10, 11, 17, 18, 23, 25, 32]. It had been an open problem whether ℓ_∞/c_0 is an injective Banach space. It was solved in the negative by Amir [2]. In fact, Rosenthal's proof of this result [31, Proposition 1.2] shows that ℓ_∞/c_0 is not universally $c_0(2^\omega)$ -injective. On the other

hand, it is shown in [5] that ℓ_∞/c_0 is universally separably injective and separably automorphic, so also separably extensible.

Fundamental phenomena of uncountable combinatorics discovered already by Hausdorff and Luzin imply that the Boolean algebra $\wp(\mathbb{N})/Fin$ corresponding to ℓ_∞/c_0 is not *Fincfin*(ω_1)-injective, nor *Fincfin*(ω_1)-extensible, nor *Fincfin*(ω_1)-automorphic ⁽¹⁾. Motivated by that, the authors of [5] proved the failure of isometric versions of injectivity of ℓ_∞/c_0 already at the first uncountable level ω_1 [5, Proposition 5.18]. The same motivation underlies the main open problem (Problem 12) discussed in [5, Section 6.4.3] whether the injective properties of ℓ_∞/c_0 as in Definition 1.1 already fail at the first uncountable level ω_1 possibly due to some similar, perhaps yet undiscovered, combinatorial phenomena or they may consistently hold, for example, up to arbitrarily large continuum. Discussing this problem in Section 6.4.3 the authors of [5] write:

the problem of injectivity-like properties of ℓ_∞/c_0 is connected at a deep level with its $c_0(\omega_1)$ -automorphic character

and they state an auxiliary open Problem 13 whether ℓ_∞/c_0 is $c_0(\omega_1)$ -automorphic. The first main result of this paper shows that this problem is undecidable by the usual axioms of ZFC. This is a consequence of the following two theorems:

THEOREM 1.2. *If $\mathfrak{a} = \omega_1$, then ℓ_∞/c_0 is neither $c_0(\omega_1)$ -automorphic nor $c_0(\omega_1)$ -extensible ⁽²⁾.*

Proof. Let $X = \overline{\text{span}}\{1_{[A_\xi]} : \xi < \kappa\}$ for some almost disjoint family $\{A_\xi : \xi \in \kappa\} \subseteq \wp(\mathbb{N})$. We will justify that X has codimension 2^ω in $C(\mathbb{N}^*)$. Choose any $A \subseteq A_1$ such that A and $A_1 \setminus A$ are infinite, and denote $Y = \{f \in C(\mathbb{N}^*) : \text{supp } f \subseteq A\}$. The quotient map from $C(\mathbb{N}^*)$ onto $C(\mathbb{N}^*)/X$ restricted to Y is an isomorphism and so $\text{dens}(C(\mathbb{N}^*)/X) \geq \text{dens}(Y) = 2^\omega$. To conclude, use Proposition 3.1 and Theorem 1.6. ■

As the hypothesis $2^\kappa > 2^\omega$ easily implies, via Rosenthal's argument, that ℓ_∞/c_0 is not universally $c_0(\kappa)$ -injective, it should be noted that the hypothesis $\mathfrak{a} = \omega_1$ of Theorem 1.2 is consistent with $2^{\omega_1} = 2^{\mathfrak{a}} = 2^\omega$ (this holds, e.g., in the Cohen model), so Theorem 1.2 also gives new information about the failure of $c_0(\kappa)$ -injectivity of ℓ_∞/c_0 as well.

⁽¹⁾ The definitions of these notions for Boolean algebras are analogous to those from Definition 1.1.

⁽²⁾ Recall that $\mathfrak{a}$ stands for the smallest infinite cardinality of a maximal family of infinite subsets of $\mathbb{N}$ whose pairwise intersections are finite (maximal almost disjoint family). It is known that $\mathfrak{a} = \omega_1$ is consistent with arbitrarily large continuum (see e.g. [16, 8]).

THEOREM 1.3. *Suppose that κ is an infinite cardinal. Assume $\mathbf{MA}_{\sigma\text{-linked}}(\kappa)$. Then ℓ_∞/c_0 is $c_0(\kappa)$ -automorphic and so $c_0(\kappa)$ -extensible.*

Proof. Use Proposition 5.7. ■

Although we have not been able to fully solve Problem 12 of [5], the above quoted passage from [5] turned out to be quite accurate, as using Theorem 1.3 and an old result of [6] we were able to conclude the consistency of a strong unknown injective property of ℓ_∞/c_0 :

THEOREM 1.4. *It is consistent with arbitrarily big continuum that for every $\kappa < 2^\omega$ every copy of $c_0(\kappa)$ in ℓ_∞/c_0 is included in a copy of $\ell_\infty(\kappa)$, and so it is consistent that ℓ_∞/c_0 is $c_0(\kappa)$ -injective.*

Proof. By the main result of [6], $\mathbf{MA}_{\sigma\text{-linked}}$ with arbitrarily big continuum is consistent with the existence of a Boolean embedding of any Boolean algebra of cardinality not exceeding 2^ω into the algebra $\wp(\mathbb{N})/Fin$; in particular, this applies to the algebras $\wp(\kappa)$ for every $\kappa < 2^\omega$ as $2^\kappa = 2^\omega$ for such κ s under $\mathbf{MA}_{\sigma\text{-centered}}(\kappa)$. Hence, in this situation, there are isometric embeddings $T_\kappa : \ell_\infty(\kappa) \rightarrow \ell_\infty/c_0$ for such κ s. Consider any copy $X \subseteq \ell_\infty/c_0$ of $c_0(\kappa)$. By Theorem 1.3 there is an automorphism $S : \ell_\infty/c_0 \rightarrow \ell_\infty/c_0$ such that $S[X] = T_\kappa[c_0(\kappa)]$. So $S^{-1}[T_\kappa[\ell_\infty(\kappa)]]$ is the required copy of $\ell_\infty(\kappa)$ containing X .

To prove $c_0(\kappa)$ -injectivity, let $R : c_0(\kappa) \rightarrow \ell_\infty/c_0$ be any bounded operator and let Z be any superspace of $c_0(\kappa)$. Let $R' : T_\kappa[c_0(\kappa)] \rightarrow \ell_\infty/c_0$ be such that $R = R' \circ T_\kappa|_{c_0(\kappa)}$. The operator R' can be extended to $R'' : T_\kappa[\ell_\infty(\kappa)] \rightarrow \ell_\infty/c_0$ since ℓ_∞/c_0 is $c_0(\kappa)$ -automorphic by Theorem 1.3 and so $c_0(\kappa)$ -extensible by [28]. On the other hand, by the injectivity of $\ell_\infty(\kappa)$, there is an operator $U : Z \rightarrow T_\kappa[\ell_\infty(\kappa)]$ extending $T_\kappa|_{c_0(\kappa)}$. So $R'' \circ U$ is the desired extension of R . ■

The above results should be seen in the Boolean context, where analogous properties are false in ZFC already for $\kappa = \omega_1$. This is witnessed for example by the Luzin family (see e.g. [20, 16]), which does not admit any separation into two uncountable parts. Our results imply, for example, that it is consistent that if one considers a Luzin family $\{A_\xi : \xi < \omega_1\}$ and any other almost disjoint family $\{B_\xi : \xi < \omega_1\}$ which admits a separation into two uncountable parts, then there is an automorphism S of ℓ_∞/c_0 such that $S(1_{[A_\xi]}) = 1_{[B_\xi]}$ for every $\xi < \omega_1$, while there cannot be a Boolean automorphism of $\wp(\mathbb{N})/Fin$ which sends $[A_\xi]$ to $[B_\xi]$.

A similar question which makes sense in the Boolean context is whether all or no nontrivial automorphisms of a fixed subalgebra of $\wp(\mathbb{N})/Fin$ like $Fincofin(\omega_1)$ extend to an automorphism of $\wp(\mathbb{N})/Fin$. This is investigated in [7, 19]; however, the results for $Fincofin(\omega_1)$ are negative in ZFC as well [7, Proposition 4.3].

The structure of the paper is as follows. In the next section we present notation and terminology and discuss some preliminaries. In Section 3 we deal with isometric embeddings of $\ell_\infty^c(\kappa)$ into ℓ_∞/c_0 for uncountable κ . We prove that there is in ZFC an almost disjoint family $\mathcal{B} = \{B_\xi : \xi < \omega_1\}$ such that there is an isometric embedding $S : \ell_\infty^c(\omega_1) \rightarrow C(\mathbb{N}^*)$ satisfying $S(1_{\{\xi\}}) = 1_{[B_\xi]}$ for every $\xi < \omega_1$ (Proposition 3.1). Moreover we obtain:

THEOREM 1.5. *The following conditions are equivalent ⁽³⁾:*

- (1) $\mathfrak{b} = \omega_1$,
- (2) *there is an almost disjoint family $\mathcal{A} = \{A_\xi : \xi < \omega_1\}$ such that there is no isometric embedding $S : \ell_\infty^c(\omega_1) \rightarrow C(\mathbb{N}^*)$ satisfying*

$$S(1_{\{\xi\}}) = 1_{[A_\xi]} \quad \text{for every } \xi < \omega_1.$$

Proof. Use Propositions 3.5 and 3.8. ■

In the fourth section we consider obstructions to isomorphic embeddings of $\ell_\infty^c(\kappa)$ into ℓ_∞/c_0 for uncountable κ . We prove:

THEOREM 1.6. *There is an almost disjoint family $\mathcal{A} = \{A_\xi : \xi < \mathfrak{a}\}$ such that there is no linear operator $S : \ell_\infty^c(\mathfrak{a}) \rightarrow C(\mathbb{N}^*)$ satisfying*

$$S(1_{\{\xi\}}) = 1_{[A_\xi]} \quad \text{for every } \xi < \mathfrak{a}.$$

In particular, ℓ_∞/c_0 is not universally $c_0(\mathfrak{a})$ -injective.

Proof. Use Proposition 4.3. ■

The results of Sections 3 and 4, in particular, give the proof of Theorem 1.2. In Section 5 we prove results needed to conclude Theorem 1.3. The automorphism of ℓ_∞/c_0 constructed in the proof of Theorem 1.3 is obtained from an automorphism of ℓ_∞ which preserves c_0 , i.e., can be lifted to ℓ_∞ . Such automorphisms of ℓ_∞ are represented by appropriate infinite matrices (see Section 2.1). We use a restricted version of Martin's axiom to build such a matrix as a block diagonal matrix with finite blocks. The construction of the finite blocks is based on a lemma of Bourgain and Tzafriri (see Section 2.3) concerning extensions of isomorphisms from small subspaces of finite-dimensional Banach spaces.

It should be noted that already the paper [27] of Marciszewski and Plebanek includes the use of Martin's axiom to construct a linear operator on ℓ_∞ which sends a given almost disjoint family of cardinality $\kappa < 2^\omega$ to any other one of the same cardinality (which was later exploited in [14]). However, it seems that these operators may not preserve c_0 and anyway the results of [27] do not apply to arbitrary isomorphic copies of $c_0(\kappa)$.

⁽³⁾ Recall that the *bounding number* $\mathfrak{b}$ is the minimal cardinality of a family of functions in $\mathbb{N}^{\mathbb{N}}$ which is unbounded in the order of eventual pointwise domination. It is known that $\omega < \mathfrak{b} \leq \mathfrak{a} \leq 2^\omega$ (see [8, 16]).

In particular, given any two Johnson–Lindenstrauss spaces ⁽⁴⁾ of the same density our method of proving Theorem 1.3 gives an explicit $\mathbb{N} \times \mathbb{N}$ matrix which sends one of them onto the other. In fact, if we face two almost disjoint families of cardinality $\kappa < 2^\omega$ instead of arbitrary copies of $c_0(\kappa)$ our method of constructing the matrix can be carried out without need of the lemma of Bourgain and Tzafriri.

The last section contains some questions.

2. Preliminaries

2.1. General terminology and notation. If A is a set, we denote by 1_A its characteristic function defined on an implicitly given domain. If f is a function, $f|A$ denotes the restriction of f to A and $\text{ran}(f)$ denotes the range of f . By a *simple function*, we mean a function that assumes finitely many values.

Let $\wp(X)$ denote the powerset of a set X . We will sometimes consider $\wp(X)$ as a Boolean algebra. $\text{Fincofin}(X)$ is the Boolean subalgebra of all finite and cofinite subsets of X , while Fin stands for the ideal of all finite subsets of $\mathbb{N}$. The symbols $\vee, \wedge, -$ denote Boolean operations.

All Banach spaces that appear in this paper are considered over $\mathbb{R}$. The norms we consider are mainly supremum norms. Another norm we use will be the quotient norm on ℓ_∞/c_0 . It should be clear from the context which norm is used. The symbols $c_0(\Gamma), \ell_\infty(\Gamma), \ell_1(\Gamma)$ have the standard meaning for any set Γ , and Γ is omitted if it is $\mathbb{N}$. By $\text{supp } f$ we mean the support of the function f and c_{00} stands for the linear subspace of c_0 of all finitely supported sequences. The subspace of $\ell_\infty(\kappa)$ consisting of countably supported elements is denoted by $\ell_\infty^c(\kappa)$. We sometimes identify the Banach spaces $c_0(A), \ell_\infty(A), \ell_\infty^c(A), \ell_1(A)$ with the subspaces of $c_0(\Gamma), \ell_\infty(\Gamma), \ell_\infty^c(\Gamma), \ell_1(\Gamma)$ consisting of all elements whose supports are included in $A \subseteq \Gamma$, and elements f defined on a subset $A \subseteq \Gamma$ with $f \cup \{0\}^{\mathbb{N} \setminus A}$. To keep the notation consistent with the literature, in the case of a finite set A the symbol ℓ_∞^A stands for the finite-dimensional space $\mathbb{R}^A$ with the maximum norm.

By an *operator* on a Banach space we always mean a bounded linear operator. We say that an operator T is *bounded below* by $r \in \mathbb{R}_{>0}$ if $r\|x\| \leq \|T(x)\|$ for each x in the domain of T . The algebra of operators on a Banach space X is denoted by $L(X)$. The identity on X is denoted by I_X . The symbol $\equiv$ denotes the relation of linear isometry between Banach spaces. For isomorphic Banach spaces X, Y we will also consider their *Banach–Mazur distance* $d(X, Y)$, that is,

⁽⁴⁾ Following the terminology of [29] or [13]; it is example 2 of [21], i.e., the closed subspace of ℓ_∞ generated by c_0 and the characteristic functions of an uncountable almost disjoint family. This space is also known as the Nakamura–Kakutani space [15, 2.2.10].

$$\inf \{ \|T\| \|T^{-1}\| : T \text{ is an isomorphism from } X \text{ onto } Y \}.$$

For $I \subseteq \mathbb{N}$ we will consider $I \times I$ matrices, which are formally functions from $I \times I$ into $\mathbb{R}$ and may represent certain bounded linear operators on $\ell_\infty(I)$ (and/or $c_0(I)$). For this, the rows of the matrix must be in ℓ_1 and the ℓ_1 -norms of the rows need to be bounded (and, for operators on $c_0(I)$, the entries in each column should converge to 0) (cf. [23, Section 2.1]). The norm of a matrix will always be the operator norm induced by the supremum norms in the domain and the codomain. It should be clear that the operator norm of an operator on $c_0(I)$ or on $\ell_\infty(I)$ given by a matrix M is the supremum of the ℓ_1 -norms of the rows of M . We use the usual conventions concerning multiplication of matrices. An $I \times I$ matrix M is called *block diagonal* if there is a partition $\{J_i : i \in A\}$ of I into finite sets such that $M(j, j') = 0$ whenever j, j' belong to different parts of the partition. The $J_i \times J_i$ matrices obtained from M are called *blocks* of M . Moreover, $M_n(\mathbb{Q})$ denotes the $n \times n$ matrices with rational entries.

We write $A \subseteq^* A'$ if $A \setminus A'$ is finite and $A =^* A'$ if $A \subseteq^* A'$ and $A' \subseteq^* A$. We denote by $[A]$ the equivalence class of a set $A \subseteq \mathbb{N}$ with respect to $=^*$. Similarly, for two functions $f, g \in \ell_\infty$ we write $f =^* g$ if $f = g + h$ for some $h \in c_0$ and we denote by $[f]$ the equivalence class of a function f with respect to this relation. We consider the quotient structures, the Boolean algebra $\wp(\mathbb{N})/Fin$ and the Banach space ℓ_∞/c_0 (with the usual quotient norm). Recall that, by Stone duality, we canonically identify any Boolean algebra $\mathcal{A}$ with the Boolean algebra of all clopen subsets of its Stone space $K_{\mathcal{A}}$. In particular, $\wp(\mathbb{N})/Fin$ is identified with the Boolean algebra of all clopen subsets of $\mathbb{N}^*$ and there is an isometry between ℓ_∞/c_0 and the Banach space $C(\mathbb{N}^*)$ of all continuous functions on $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ which sends $[1_A]$ to $1_{[A]}$ for $A \subseteq \mathbb{N}$.

A family $\mathcal{A}$ of subsets of $\mathbb{N}$ is *almost disjoint* if any $A, A' \in \mathcal{A}$ with $A \neq A'$ satisfy $A \cap A' =^* \emptyset$. Whenever we consider almost disjoint families, we always assume that they are infinite and consist of infinite subsets of $\mathbb{N}$. If $\mathcal{A}, \mathcal{B}$ are two families of subsets of $\mathbb{N}$, we say that $\mathcal{A}$ is *separated* from $\mathcal{B}$ if there is $C \subseteq \mathbb{N}$ such that $A \subseteq^* C$ and $B \cap C =^* \emptyset$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Such a set C is called a *separation* of $\mathcal{A}$ and $\mathcal{B}$. Note that under the identification mentioned above, if $\mathcal{A}$ is an almost disjoint family, then $\{[A] : A \in \mathcal{A}\}$ is a pairwise disjoint family of clopen subsets of $\mathbb{N}^*$ and so the closure of the span of $\{1_{[A]} : A \in \mathcal{A}\}$ is an isometric copy of $c_0(|\mathcal{A}|)$ in $C(\mathbb{N}^*)$.

A partial order $\mathbb{P}$ is said to be σ -linked if $\mathbb{P} = \bigcup \mathbb{P}_n$, where for each $n \in \mathbb{N}$ the set $\mathbb{P}_n$ is *linked*, i.e., any two elements of $\mathbb{P}_n$ are compatible in $\mathbb{P}$. $\text{MA}_{\sigma\text{-linked}}(\kappa)$ stands for a restricted version of Martin's axiom which states that given a σ -linked partial order $\mathbb{P}$ and a family $\mathcal{D}$ of cardinality κ of dense subsets of $\mathbb{P}$ there is a filter $\mathbb{G} \subseteq \mathbb{P}$ such that $\mathbb{G} \cap \mathbb{D} \neq \emptyset$ for every $\mathbb{D} \in \mathcal{D}$.

For more terminology concerning Martin's axiom, see [24, 33]. For further terminology and notation, the reader may consult the books [1, 5, 24].

2.2. The Banach space ℓ_∞/c_0

DEFINITION 2.1. $\pi : \ell_\infty \rightarrow \ell_\infty/c_0$ is the quotient map.

DEFINITION 2.2. Let a be a finite set, $\{x_\xi : \xi \in a\} \subseteq X$, $\{y_\xi : \xi \in a\} \subseteq Y$ be two indexed sets of vectors in Banach spaces X, Y respectively and let $0 < p, q < \infty$ be real numbers. We write $\langle x_\xi : \xi \in a \rangle \stackrel{p,q}{\sim} \langle y_\xi : \xi \in a \rangle$ if for any sequence $t = (t_\xi)_{\xi \in a} \in \ell_\infty^a$ we have

$$p^{-1} \left\| \sum_{\xi \in a} t_\xi y_\xi \right\|_Y \leq \left\| \sum_{\xi \in a} t_\xi x_\xi \right\|_X \leq q \left\| \sum_{\xi \in a} t_\xi y_\xi \right\|_Y.$$

We record a few simple but useful observations.

LEMMA 2.3. Denote $X' = \text{span} \{x_\xi : \xi \in a\}$ and $Y' = \text{span} \{y_\xi : \xi \in a\}$. Then the following are equivalent:

- (1) $\langle x_\xi : \xi \in a \rangle \stackrel{p,q}{\sim} \langle y_\xi : \xi \in a \rangle$,
- (2) there is an isomorphism $V : X' \rightarrow Y'$ such that $V(x_\xi) = y_\xi$ which is bounded from above by p and from below by q^{-1} ,
- (3) there is an isomorphism $V : X' \rightarrow Y'$ such that $V(x_\xi) = y_\xi$ with $\|V\| \leq p$ and $\|V^{-1}\| \leq q$.

LEMMA 2.4. The following properties hold:

- (1) $\langle x_\xi : \xi \in a \rangle \stackrel{p,q}{\sim} \langle y_\xi : \xi \in a \rangle$ if and only if $\langle y_\xi : \xi \in a \rangle \stackrel{q,p}{\sim} \langle x_\xi : \xi \in a \rangle$,
- (2) if $\langle x_\xi : \xi \in a \rangle \stackrel{p,q}{\sim} \langle y_\xi : \xi \in a \rangle$ and $\langle y_\xi : \xi \in a \rangle \stackrel{p',q'}{\sim} \langle z_\xi : \xi \in a \rangle$, then $\langle x_\xi : \xi \in a \rangle \stackrel{pp',qq'}{\sim} \langle z_\xi : \xi \in a \rangle$,
- (3) if $\langle x_\xi : \xi \in a \rangle \stackrel{p,q}{\sim} \langle y_\xi : \xi \in a \rangle$ and $a' \subseteq a$, then $\langle x_\xi : \xi \in a' \rangle \stackrel{p,q}{\sim} \langle y_\xi : \xi \in a' \rangle$.

LEMMA 2.5. Suppose that a is a finite set and $\{y_\xi : \xi \in a\} \subseteq \ell_\infty$ is such that $\{\pi(y_\xi) : \xi \in a\}$ is a linearly independent set. Let $u > 1$. There exists $N \in \mathbb{N}$ such that $\langle y_\xi | [n, \infty) : \xi \in a \rangle \stackrel{1,u}{\sim} \langle \pi(y_\xi) : \xi \in a \rangle$ for all $n > N$.

Proof. Define $f_n(t) = \sum_{\xi \in a} t_\xi y_\xi | [n, \infty)$ for $n \in \mathbb{N}$ and $t \in \ell_\infty^a$. We always have $\|f_n(t)\| \geq \|\pi(f_0(t))\|$. For $n \in \mathbb{N}$ let

$$E_n = \{t \in S_{\ell_\infty^a} : u^{-1} \cdot \|f_n(t)\| < \|\pi(f_0(t))\|\}.$$

By the continuity of π , f_n and the corresponding norms, the sets E_n are open. For every $t \in S_{\ell_\infty^a}$ we have $\|f_n(t)\| \searrow \|\pi(f_0(t))\|$, by the very definition of f_n s and the norms, and $\|\pi(f_0(a))\| > 0$, by the fact that $\{\pi(y_\xi) : \xi \in a\}$ is linearly independent. Consequently, $E_n \subseteq E_m$ whenever $n < m$ and for every $t \in S_{\ell_\infty^a}$ there is $n \in \mathbb{N}$ such that $t \in E_n$. Since $S_{\ell_\infty^a}$ is compact, there is $N \in \mathbb{N}$ such that $S_{\ell_\infty^a} = \bigcup_{n \leq N} E_n = E_N$. Then $u^{-1} \cdot \|f_n(t)\| < \|\pi(f_0(t))\|$

whenever $n \geq N$ for all $t \in S_{\ell_\infty^a}$, so also, by linearity, for all $t \in \ell_\infty^a$, which gives us the conclusion. ■

LEMMA 2.6. *Suppose that a is a finite set and $\{y_\xi : \xi \in a\} \subseteq \ell_\infty(A)$ for some subset $A \subseteq \mathbb{N}^*$. Let $u > 1$. Then there is $N \in \mathbb{N}$ such that $\langle y_\xi : \xi \in a \rangle \stackrel{1,u}{\sim} \langle y_\xi|(A \cap n) : \xi \in a \rangle$ for every $n > N$.*

Proof. Denote $Y = \text{span}\{y_\xi : \xi \in a\}$, $Y_n = \text{span}\{y_\xi|(A \cap n) : \xi \in a\}$ and define $P_n : Y \rightarrow Y_n$ by $P(y) = y|(A \cap n)$. Clearly, P_n s are bounded from above by 1. Let

$$E_n = \{y \in S_Y : u^{-1} \cdot \|y\| < \|y|(A \cap n)\|\}.$$

For every $y \in S_Y$ we have $\|y|(A \cap n)\| \nearrow \|y\| > 0$, so $y \in E_n$ for some $n \in \mathbb{N}$. Since S_Y is compact and E_n s are open, there is $N \in \mathbb{N}$ such that $S_Y = \bigcup_{n \leq N} E_n = E_N$. Then for $n \geq N$ the operator P_n is bounded below by u^{-1} on S_Y , so, by linearity, on the whole space Y as well. By Lemma 2.3, this finishes the proof. ■

2.3. Around a theorem of Bourgain and Tzafriri

LEMMA 2.7 ([9, Theorem 5]). *For every $K \geq 1$, there exist constants $c = c(K) > 0$ and $C = C(K) < \infty$ such that, whenever*

$$\ell_p^n = Y_1 \oplus Z_1 = Y_2 \oplus Z_2$$

for some integer n and some $1 \leq p \leq \infty$, and

- (i) $\dim(Y_1) = \dim(Y_2) = h$ and $d(Y_1, Y_2) \leq K$,
- (ii) $\|P_{Y_i}\|, \|P_{Z_i}\| \leq K$ for $i = 1, 2$ ⁽⁵⁾,
- (iii) $h \leq c\sqrt{n}$,

then $d(Z_1, Z_2) \leq C$.

LEMMA 2.8. *For every $\rho > 1$ there are $0 < c_1(\rho), c_2(\rho) < \infty$ such that whenever Y_1, Y_2 are subspaces of ℓ_∞^n for some $n \in \mathbb{N}$ and $T : Y_1 \rightarrow Y_2$ is an isomorphism with*

- (1) $\dim(Y_1) = \dim(Y_2) = h \leq c_1(\rho)\sqrt{n}$,
- (2) $d(Y_i, \ell_\infty^h) \leq \rho$ for $i = 1, 2$,
- (3) $\|T\|, \|T^{-1}\| \leq \rho$,

then there is an automorphism $\tilde{T}$ of ℓ_∞^n extending T such that $\|\tilde{T}\|, \|\tilde{T}^{-1}\| < c_2(\rho)$.

Moreover, if Y_1, Y_2 are generated by vectors $\{v_k : k < h\}$ and $\{w_k : k < h\}$, respectively, all of whose coordinates are rational, and $T(v_k) = w_k$ for all $k < h$, then we can choose $\tilde{T}$ as above whose matrix has all entries rational.

⁽⁵⁾ P_{Y_i}, P_{Z_i} are projections determined by the decompositions $\ell_p^n = Y_i \oplus Z_i$ for $i = 1, 2$.

Proof. Let $S_i : Y_i \rightarrow \ell_\infty^h$ for $i = 1, 2$ be isomorphisms with $\|S_i\| \cdot \|S_i^{-1}\| \leq \rho$. Let $\phi_j^i = S_i^*(e_j^*)$ for $1 \leq j \leq h$ and $i = 1, 2$. So $\|\phi_j^i\| \leq \|S_i\|$. Using the Hahn–Banach theorem, extend each ϕ_j^i to $\psi_j^i \in (\ell_\infty^n)^*$ preserving the norm. Define $P_{Y_i} : \ell_\infty^n \rightarrow Y_i$ by

$$P_{Y_i}(f) = S_i^{-1}(\psi_1^i(f), \dots, \psi_h^i(f)) \quad \text{for } f \in \ell_\infty^n, i = 1, 2.$$

It is clear that P_{Y_i} is a projection from ℓ_∞^n onto Y_i of norm not exceeding ρ , so $P_{Z_i} = I - P_{Y_i}$ is a projection onto $\ker(P_{Y_i}) = Z_i$ of norm not exceeding $1 + \rho$. By (3), $d(Y_1, Y_2) \leq \rho^2$.

So we are in a position to apply Lemma 2.7 for $K = \rho^2 + 1$. The conclusion is that there are $c(\rho^2 + 1), C(\rho^2 + 1)$ such that if $h \leq c(\rho^2 + 1)\sqrt{n}$, then $d(Z_1, Z_2) \leq C(\rho^2 + 1)$. So there is an isomorphism $Q : Z_1 \rightarrow Z_2$ such that $\|Q\| \cdot \|Q^{-1}\| \leq C(\rho^2 + 1)$. Let $R = \sqrt{\|Q^{-1}\|/\|Q\|}Q$. Then

$$\begin{aligned} R^{-1} &= \sqrt{\|Q\|/\|Q^{-1}\|}Q^{-1}, \\ \|R\| &= \|R^{-1}\| = \sqrt{\|Q\| \cdot \|Q^{-1}\|} \leq \sqrt{C(\rho^2 + 1)}. \end{aligned}$$

Now define $\tilde{T} : \ell_\infty^n \rightarrow \ell_\infty^n$ by

$$\tilde{T} = TP_{Y_1} + RP_{Z_1}.$$

It is clear that $\tilde{T}|_{Y_1} = T$. Also

$$\begin{aligned} \tilde{T}(T^{-1}P_{Y_2} + R^{-1}P_{Z_2}) &= (TP_{Y_1} + RP_{Z_1})(T^{-1}P_{Y_2} + R^{-1}P_{Z_2}) \\ &= TT^{-1}P_{Y_2} + 0 + 0 + RR^{-1}P_{Z_2} = P_{Y_2} + P_{Z_2} = I. \end{aligned}$$

We obtain $(T^{-1}P_{Y_2} + R^{-1}P_{Z_2})\tilde{T} = I$ in an analogous way. So

$$\tilde{T}^{-1} = T^{-1}P_{Y_2} + R^{-1}P_{Z_2}.$$

As a result, we have

$$\|\tilde{T}\|, \|\tilde{T}^{-1}\| \leq \rho^2 + \sqrt{C(\rho^2 + 1)} \cdot (\rho + 1).$$

So $c_2(\rho) = \rho^2 + \sqrt{C(\rho^2 + 1)} \cdot (\rho + 1) + 1$ and $c_1(\rho) = c(\rho^2 + 1)$ work.

To prove the last part of the lemma, choose vectors $\{v_k : h \leq k < n\}$ with rational coordinates such that $\mathcal{B} = \{v_k : k < n\}$ is a basis for ℓ_∞^n . The set

$$\mathcal{V} = \{U \in L(\ell_\infty^n) : U \text{ invertible}; \|U\|, \|U^{-1}\| < c_2(\rho)\}$$

is open in the norm topology on $L(\ell_\infty^n)$ and the operator $U \xrightarrow{f} \mathcal{M}_{\mathcal{B}}^{\text{st}}(U)$, which assigns to every $U \in L(\ell_\infty^n)$ the matrix of U with respect to the basis $\mathcal{B}$ and the standard basis, is a continuous linear isomorphism, where the space of matrices is also considered with the norm topology. As a result, $f[\mathcal{V}]$ is also open. Moreover, for every $k < h$, $Uv_k = w_k$ if and only if $\mathcal{M}_{\mathcal{B}}^{\text{st}}(U)(i, k) = w_k(i)$ for all $i < n$. Choose any $\tilde{T}$ from the conclusion of the first part of the lemma. The matrix $\mathcal{M}_{\mathcal{B}}^{\text{st}}(\tilde{T})$ has rational entries in the first h columns and $\mathcal{M}_{\mathcal{B}}^{\text{st}}(\tilde{T}) \in f[\mathcal{V}]$, so we can find $A \in f[\mathcal{V}]$ with all

entries rational and $w_k(i) = \mathcal{M}_B^{\text{st}}(\tilde{T})(i, k) = A(i, k)$ for $k < h, i < n$. By the assumption that $\{v_k : k < n\}$ have rational coordinates, $\mathcal{M}_B^{\text{st}}(\text{Id}) \in M_n(\mathbb{Q})$. Since in the standard algorithm for inverting a matrix the entries of $\mathcal{M}_{\text{st}}^B(\text{Id}) = \mathcal{M}_B^{\text{st}}(\text{Id})^{-1}$ are obtained from the entries of $\mathcal{M}_B^{\text{st}}(\text{Id})$ by a number of multiplications, additions and divisions of these entries (for instance, using Gaussian elimination), we know that also $\mathcal{M}_{\text{st}}^B(\text{Id}) \in M_n(\mathbb{Q})$. If $\tilde{U} = f^{-1}(A)$, then

$$\mathcal{M}_{\text{st}}^{\text{st}}(\tilde{U}) = \mathcal{M}_{\text{st}}^B(\text{Id}) \cdot \mathcal{M}_B^{\text{st}}(\tilde{U}) \in M_n(\mathbb{Q}). \blacksquare$$

3. On isometric embeddings of $\ell_\infty^c(\omega_1)$ into $C(\mathbb{N}^*)$

PROPOSITION 3.1. *There exists an almost disjoint family $\{A_\xi : \xi < \omega_1\} \subseteq \mathcal{P}(\mathbb{N})$ and an isometric embedding $S : \ell_\infty^c(\omega_1) \rightarrow C(\mathbb{N}^*)$ such that $S(1_{\{\xi\}}) = 1_{[A_\xi]}$ for every $\xi < \omega_1$.*

DEFINITION 3.2. Let $\{s_\alpha : \alpha < \omega_1\}$ be a family of functions. We call it a *coherent family* if for each $\alpha < \omega_1$,

- (i) $s_\alpha : \omega \cdot \alpha \rightarrow \omega$ is injective,
- (ii) $s_\alpha|_{\omega \cdot \beta} = s_\beta$ for every $\beta < \alpha$.

A construction of such a family is given for instance in [24, Chapter II, Theorem 5.9] (actually, our family is a subfamily of the family constructed there, consisting only of functions enumerated with limit ordinals). The following lemma ensures that having such functions we are able to find an almost disjoint family with desired properties.

LEMMA 3.3. *Suppose that $\{s_\alpha : \alpha < \omega_1\}$ is a coherent family. Let*

$$A_\xi := s_{\xi+1}[[\omega \cdot \xi, \omega \cdot \xi + \omega]].$$

Then $\{A_\xi : \xi < \omega_1\}$ is an almost disjoint family and there is an isometric embedding $S : \ell_\infty^c(\omega_1) \rightarrow C(\mathbb{N}^)$ such that $S(1_{\{\xi\}}) = 1_{[A_\xi]}$ for every $\xi < \omega_1$.*

Proof. For $\xi < \omega_1$ let us denote $\xi^+ = \xi + 1$ and $I_\xi = [\omega \cdot \xi, \omega \cdot \xi^+)$. Under this notation, $A_\xi = s_{\xi^+}[I_\xi]$. Since $\{s_\alpha : \alpha < \omega_1\}$ is a coherent family, for $\xi > \eta$ we have

$A_\xi \cap A_\eta = s_{\xi^+}[I_\xi] \cap s_{\eta^+}[I_\eta] = s_{\xi^+}[I_\xi] \cap s_{\xi^+}[I_\eta] = s_{\xi^+}[I_\xi \cap I_\eta] = s_{\xi^+}[\emptyset] = \emptyset$, so the family $\{A_\xi : \xi < \omega_1\}$ is indeed almost disjoint. Let $\mathcal{A}$ be the Boolean algebra of all countable and cocountable subsets of ω_1 and let

$$U_X = \bigcup_{\xi \in X} I_\xi$$

for $X \subseteq \omega_1$. Now note that to prove the lemma it is enough to construct a Boolean monomorphism $h : \mathcal{A} \rightarrow \wp(\mathbb{N})/\text{Fin}$ such that $h(\{\xi\}) = [A_\xi]$ for every $\xi < \omega_1$. This is because such a monomorphism induces an isometry $T_h : C(K) \rightarrow C(\mathbb{N}^*)$ satisfying $T_h(1_X) = 1_{h(X)}$ for $X \subseteq \omega_1$ countable or

cocountable, where K is the Stone space of $\mathcal{A}$. The isometry is determined on the dense subspace of continuous simple functions on K . Note that, under the standard identification of elements of $\mathcal{A}$ with clopen subsets of K , $\ell_\infty^c(\omega_1)$ is the closed subspace of $C(K)$ generated by the characteristic functions 1_X for X countable. Since

$$T_h(1_{\{\xi\}}) = 1_{h(\{\xi\})} = 1_{[A_\xi]}$$

for every $\xi < \omega_1$, $S = T_h|_{\ell_\infty^c(\omega_1)}$ would work for the second part of the lemma.

From now on we focus on constructing h as above. For a countable $X \subseteq \omega_1$ define $\alpha(X) = \min \{\alpha < \omega_1 : U_X \subseteq U_{\omega \cdot \alpha}\}$. Define $h : \mathcal{A} \rightarrow \wp(\mathbb{N})/Fin$ by

$$h(X) = \begin{cases} [s_{\alpha(X)}[U_X]] & \text{if } X \text{ is countable,} \\ [\mathbb{N} \setminus s_{\alpha(\omega_1 \setminus X)}[U_{\omega_1 \setminus X}]] & \text{if } X \text{ is cocountable.} \end{cases}$$

The definition clearly implies that $h(-A) = -h(A)$ for any $A \in \mathcal{A}$. For countable $X, Y \subseteq \omega_1$ and a Boolean operation $\diamond \in \{\vee, \wedge, -\}$ we have

$$\begin{aligned} s_{\alpha(X)}[U_X] \diamond s_{\alpha(Y)}[U_Y] &=^* s_{\alpha(X \cup Y)}[U_X] \diamond s_{\alpha(X \cup Y)}[U_Y] \\ &= s_{\alpha(X \cup Y)}[U_X \diamond U_Y] = s_{\alpha(X \cup Y)}[U_{X \diamond Y}] =^* s_{\alpha(X \diamond Y)}[U_{X \diamond Y}] \end{aligned}$$

since s_α s are injective and coherent by Definition 3.2. Thus $h(A \wedge B) = h(A) \wedge h(B)$, $h(A \vee B) = h(A) \vee h(B)$ and $h(A \setminus B) = h(A) - h(B)$ for countable $A, B \in \mathcal{A}$. From that, taking complements, we can conclude that the same holds for arbitrary $A, B \in \mathcal{A}$ and that h defined above is a Boolean homomorphism. Moreover, if $X \neq \emptyset$, then U_X is infinite, so $h(X) = [s_{\alpha(X)}[U_X]] \neq \emptyset$. Thus, h is a monomorphism. ■

PROPOSITION 3.4. *Let $\{A_\xi : \xi < \kappa\}$ be an almost disjoint family. If there exists an isometric embedding $S : \ell_\infty^c(\kappa) \rightarrow C(\mathbb{N}^*)$ such that $S(1_{\{\xi\}}) = 1_{[A_\xi]}$ for every $\xi < \kappa$, then for every $F \in [\kappa]^{\leq \omega}$ there is $V_F \subseteq \mathbb{N}$ such that $A_\xi \subseteq^* V_F$ for $\xi \in F$ and $A_\xi \cap V_F =^* \emptyset$ for $\xi \notin F$.*

Proof. Fix $F \in [\kappa]^{\leq \omega}$. Suppose that $S(1_F)(x) < 1$ for some $x \in [A_\xi]$ with $\xi \in F$. Then

$$1 \geq S(2 \cdot 1_{\{\xi\}} - 1_F)(x) = 2 \cdot 1_{[A_\xi]}(x) - S(1_F)(x) > 2 - 1 = 1,$$

a contradiction. Similarly, if $S(1_F)(x) > 0$ for some $x \in [A_\xi]$ with $\xi \notin F$, then

$$1 \geq S(1_{\{\xi\}} + 1_F)(x) = 1_{[A_\xi]}(x) + S(1_F)(x) > 1,$$

which again gives a contradiction. We conclude that $[A_\xi] \subseteq \{x \in \mathbb{N}^* : S(1_F)(x) = 1\}$ for $\xi \in F$ and $[A_\xi] \cap \{x \in \mathbb{N}^* : S(1_F)(x) > 0\} = \emptyset$ for $\xi \notin F$. By the compactness of $S(1_F)^{-1}[\{1\}]$, there is a finite family $\{U_i\}_{i=1}^n \subseteq \mathcal{P}(\mathbb{N})$

such that

$$\{x \in \mathbb{N}^* : S(1_F)(x) = 1\} \subseteq \bigcup_{i=1}^n [U_i] \subseteq \{x \in \mathbb{N}^* : S(1_F)(x) > 0\}.$$

The set $V_F = \bigcup_{i=1}^n U_i$ has the desired property. ■

PROPOSITION 3.5. *There is an almost disjoint family $\mathcal{A} = \{A_\xi : \xi < \mathfrak{b}\}$ such that there is no isometric embedding $S : \ell_\infty^c(\mathfrak{b}) \rightarrow C(\mathbb{N}^*)$ satisfying*

$$S(1_{\{\xi\}}) = 1_{[A_\xi]} \quad \text{for every } \xi < \mathfrak{b}.$$

Proof. By [16, Theorem 3.3] (case $\mathfrak{b}_5$) there exists an almost disjoint family $\mathcal{A} = \mathcal{B} \cup \mathcal{C} \subseteq [\omega]^\omega$ such that $|\mathcal{C}| = \omega$, $|\mathcal{B}| = \mathfrak{b}$, and the families $\mathcal{B}$ and $\mathcal{C}$ cannot be separated. Then by Proposition 3.4 for any enumeration $\{A_\xi : \xi < \mathfrak{b}\}$ of $\mathcal{A}$ there is no isometric embedding $S : \ell_\infty^c(\mathfrak{b}) \rightarrow C(\mathbb{N}^*)$ satisfying $S(1_{\{\xi\}}) = 1_{[A_\xi]}$ for every $\xi < \mathfrak{b}$. ■

LEMMA 3.6. *Suppose that A, B are countably infinite sets, $C \subseteq \mathbb{N}$ is infinite, and*

- (1) $A \subseteq B$, $B \setminus A$ is infinite,
- (2) $f : A \rightarrow C$ is an injection,
- (3) $g : B \rightarrow \mathbb{N}$ is an injection,
- (4) $\text{ran}(g) \subseteq^* C$,
- (5) $C \setminus \text{ran}(g)$ is infinite,
- (6) $f =^* g|A$,
- (7) $F \subseteq C$ is finite.

Then there is an injection $h : B \rightarrow C$ such that $f \subseteq h$, $h =^ g$ and $F \subseteq \text{ran}(h)$.*

Proof. Let $D_1 = \{x \in B \setminus A : g(x) \in \mathbb{N} \setminus C\}$, so by (2), $\text{ran}(f \cup (g|E)) \subseteq C$ for any $E \subseteq B \setminus A$ disjoint from D_1 . By (3) and (4), D_1 is finite.

Let $D_2 = \{x \in B \setminus A : \exists y \in A, g(x) = f(y)\}$, so by (2) and (3), $f \cup (g|E)$ is an injection for any $E \subseteq B \setminus A$ disjoint from D_2 . By (6), D_2 is finite.

Let $D_3 \supseteq D_1 \cup D_2$ be any subset of $B \setminus A$ of finite cardinality not smaller than $|F|$. It exists by (1). And let $E = (B \setminus A) \setminus D_3$. Finally, let $h' = f \cup (g|E)$.

By the above remarks, $h' : A \cup E \rightarrow C$ is injective and $h' =^* (g|(A \cup E))$ by (6). Now define $h'' : D_3 \rightarrow C \setminus \text{ran}(h')$ by putting any distinct values on distinct elements of D_3 as long as $\text{ran}(h'')$ includes $F \setminus \text{ran}(h')$. This can be done as $|D_3| \geq |F|$ and since $C \setminus \text{ran}(h')$ is infinite by (5) and (6). Now $h = h' \cup h''$ works. ■

LEMMA 3.7. *Let $\{A_\xi : \xi < \omega_1\}$ be an almost disjoint family and suppose that there is an almost increasing sequence $\{V_\alpha\}_{\alpha < \omega_1} \subseteq \mathcal{P}(\mathbb{N})$ such that $A_\xi \subseteq^* V_\alpha$ for $\xi < \alpha$ and $A_\xi \cap V_\alpha =^* \emptyset$ for $\omega_1 > \xi \geq \alpha$. Then there is a*

coherent family $\{s_\alpha : \alpha < \omega_1\}$ such that

$$A_\xi =^* s_{\xi+1} [[\omega \cdot \xi, \omega \cdot \xi + \omega]].$$

Consequently, by Lemma 3.3, there is an isometric embedding $S : \ell_\infty^c(\omega_1) \rightarrow C(\mathbb{N}^*)$ such that $S(1_{\{\xi\}}) = 1_{[A_\xi]}$ for every $\xi < \omega_1$.

Proof. As in the proof of Lemma 3.1, we denote $\xi^+ = \xi + 1$ and $I_\xi = [\omega \cdot \xi, \omega \cdot \xi^+)$. For any $\alpha = \beta + k$ where $\beta < \omega_1$ is a limit ordinal and $k \in \omega$ consider

$$(i) \quad W_\alpha = V_\beta \cup \bigcup_{i < k} A_{\beta+i}.$$

Note that the assumptions of the lemma concerning $\{V_\alpha\}_{\alpha < \omega_1}$ imply the same properties for $\{W_\alpha\}_{\alpha < \omega_1}$, i.e., it is an almost increasing sequence such that W_α is a separation of $\{A_\xi : \xi < \alpha\}$ and $\{A_\xi : \alpha \leq \xi < \omega_1\}$ for every $\alpha < \omega_1$. It follows that it is enough to find $s_\alpha : \omega \cdot \alpha \rightarrow \omega$ such that, for any $\alpha < \omega_1$,

- (ii) s_α is an injection,
- (iii) $s_\alpha|_{\omega \cdot \beta} =^* s_\beta$ for every $\beta < \alpha$,
- (iv) $\text{ran}(s_\alpha) = W_\alpha$.

Then by (i)–(iv) and the separation properties of $\{W_\alpha\}_{\alpha < \omega_1}$,

$$s_{\xi^+}[I_\xi] = \text{ran}(s_{\xi^+}) \setminus s_{\xi^+}[\omega \cdot \xi] =^* \text{ran}(s_{\xi^+}) \setminus \text{ran}(s_\xi) = W_{\xi^+} \setminus W_\xi =^* A_\xi.$$

To construct s_ξ we proceed in the standard way by induction on $\xi < \omega_1$. In the successor case, define s_{ξ^+} as an extension of s_ξ by any bijection from I_ξ onto $A_\xi \setminus W_\xi$. There is such a bijection as both sets are infinite and countable by (i) and by the properties of the sequence of W_ξ s.

Now suppose that ξ is a limit ordinal. To construct s_ξ satisfying (ii)–(iv) fix a strictly increasing sequence $\{\xi_n\}_{n \in \mathbb{N}}$ of ordinals such that $\xi_0 = 0$ and $\sup_{n \in \mathbb{N}} \xi_n = \xi$. Let $\sigma : \mathbb{N} \rightarrow W_\xi$ be a bijection. Recursively over $n \geq 1$ we define a sequence $(t_n)_{n \in \mathbb{N}}$ such that

- (a) $t_n : \omega \cdot \xi_n \rightarrow W_\xi$ is an injection,
- (b) $t_n =^* s_{\xi_n}$,
- (c) $t_k \subseteq t_n$ for $k < n$,
- (d) $\sigma[n] \subseteq \text{ran}(t_n)$.

Put $t_0 = \emptyset$. Having already defined $(t_k)_{k < n}$ satisfying (a)–(d) we apply Lemma 3.6 to $A = \omega \cdot \xi_{n-1}$, $f = t_{n-1}$, $B = \omega \cdot \xi_n$, $g = s_{\xi_n}$, $C = W_\xi$, $F = \sigma[n]$. To do so, we check that hypotheses (1)–(7) of this lemma are satisfied: hypotheses (1) and (7) follow directly from the definitions of A, B, F, σ ; (2) follows from the inductive hypothesis (a); (3) from the inductive hypothesis (ii) for $\xi_n < \xi$; (4) and (5) from (i) and the inductive hypothesis (iv) for

$\xi_n < \xi$; and (6) from the inductive hypothesis (iii) for $\xi_n < \xi$ and the inductive hypothesis (b). So Lemma 3.6 gives us t_n as in (a)–(d). This completes the recursive construction of $(t_n)_{n \in \mathbb{N}}$.

Finally, we set $s_\xi := \bigcup_{n \in \mathbb{N}} t_n$ and check that it satisfies (ii)–(iv). Being an increasing union of injections (by (a) and (c)) with the images contained in W_ξ , s_ξ is an injection with the image in W_ξ itself, so (ii) and one of the inclusions of (iv) hold. Condition (d) ensures that $\text{ran}(s_\xi) = \bigcup_n \text{ran}(t_n) \supseteq \bigcup_n \sigma[n] = \text{ran}(\sigma) = W_\xi$, so the other inclusion of (iv) holds as well. For any $\alpha < \xi$ there exists $n \in \mathbb{N}$ such that $\xi_n > \alpha$ and $s_\xi|_{\omega \cdot \alpha} = t_n|_{\omega \cdot \alpha} =^* s_{\xi_n}|_{\omega \cdot \alpha} =^* s_\alpha$, so s_ξ satisfies (iii), as required. ■

PROPOSITION 3.8. *If $\mathfrak{b} > \omega_1$, then for every almost disjoint family $\mathcal{A} = \{A_\xi : \xi < \omega_1\}$ there is an isometric embedding $S : \ell_\infty^c(\omega_1) \rightarrow C(\mathbb{N}^*)$ satisfying*

$$S(1_{\{\xi\}}) = 1_{[A_\xi]} \quad \text{for every } \xi < \omega_1.$$

Proof. Assume $\mathfrak{b} > \omega_1$ and fix an almost disjoint family $\mathcal{A} = \{A_\xi : \xi < \omega_1\}$. We will construct an almost increasing sequence $\{V_\alpha\}_{\alpha < \omega_1}$ of subsets of $\mathbb{N}$ such that

- (a) $A_\xi \subseteq^* V_\alpha$ for $\xi < \alpha < \omega_1$,
- (b) $A_\xi \cap V_\alpha =^* \emptyset$ for $\omega_1 > \xi \geq \alpha$.

This is done by induction on $\alpha < \omega_1$. For the successor case, we put $V_{\alpha+1} = V_\alpha \cup A_\alpha$. For the limit case, we note that the inductive hypotheses (a), (b) imply that $B \cap C =^* \emptyset$ for any $B \in \mathcal{B}$ and any $C \in \mathcal{C}$, where

$$\mathcal{B} = \{A_\xi : \xi < \alpha\} \cup \{V_\xi : \xi < \alpha\}, \quad \mathcal{C} = \{A_\xi : \omega_1 > \xi \geq \alpha\};$$

moreover, $\mathcal{B}$ is countable and $|\mathcal{C}| = \omega_1 < \mathfrak{b}$. By [16, Theorem 3.3] (case $\mathfrak{b}_7$) this implies the existence of a separation $V_\alpha \subseteq \mathbb{N}$ such that the appropriate cases of (a) and (b) hold.

Now use Lemma 3.7 to conclude the proof. ■

4. Obstructions to isomorphic embeddings of $\ell_\infty^c(\kappa)$ into ℓ_∞/c_0

LEMMA 4.1. *Suppose that $S : \ell_\infty^c(\kappa) \rightarrow C(\mathbb{N}^*)$ is a linear operator such that $S(1_{\{\xi\}}) = 1_{[A_\xi]}$ for every $\xi < \kappa$, where κ is an uncountable cardinal and $\{A_\xi : \xi < \kappa\}$ is an almost disjoint family. Let*

$$r = \inf_{C \in [\kappa]^{\leq \omega}} \sup_{f, x} \left\{ |S(f)(x)| : f \in \ell_\infty^c(\kappa \setminus C), \|f\| = 1, x \in \bigcup_{\xi \in \kappa \setminus C} [A_\xi] \right\},$$

$$R = \inf_{C \in [\kappa]^{\leq \omega}} \sup_{f, x} \left\{ |S(f)(x)| : f \in \ell_\infty^c(\kappa \setminus C), \|f\| = 1, x \notin \bigcup_{\xi < \kappa} [A_\xi] \right\}.$$

Then $R \geq r$.

Proof. Fix $C \in [\kappa]^{<\omega}$. By the definition of r , whenever we have a finite family $\mathcal{F}$ of elements of $\ell_\infty^c(\kappa)$ and $\varepsilon > 0$, we can find $g \in \ell_\infty^c(\kappa \setminus (C \cup \bigcup_{f \in \mathcal{F}} \text{supp } f))$ of norm 1 such that $|S(g)(x)| > r - \varepsilon$ for some $x \in [A_\xi]$ where $\xi \in \kappa \setminus (C \cup \bigcup_{f \in \mathcal{F}} \text{supp } f)$. Therefore, fixing $\varepsilon > 0$ we choose a sequence $\{f_n : n \in \mathbb{N}\}$ of disjointly supported elements of norm 1 of $\ell_\infty^c(\kappa \setminus C)$ such that for every $n \in \mathbb{N}$ there is a different ordinal ξ_n in $\kappa \setminus C$ and $x_n \in [A_{\xi_n}]$ with

$$(*) \quad |S(f_n)(x_n)| > r - \varepsilon.$$

For any $x \in \mathbb{N}^*$ let $\epsilon_k = \text{sgn}(f_k(x))$. Note that

$$\begin{aligned} \sum_{k \in \mathbb{N}} |S(f_k)(x)| &= \sup_{E \in [\mathbb{N}]^{<\omega}} \sum_{k \in E} |S(f_k)(x)| = \sup_{E \in [\mathbb{N}]^{<\omega}} S\left(\sum_{k \in E} \epsilon_k f_k\right)(x) \\ &\leq \sup_{E \in [\mathbb{N}]^{<\omega}} \|S\| \cdot \left\| \sum_{k \in E} \epsilon_k f_k \right\| \leq \|S\|. \end{aligned}$$

Applying Rosenthal's [31, Lemma 1.1] (for $\Lambda = \Gamma = \mathbb{N}$, $E_n = \{n\}$ and $\mu_n(E) = \sum_{k \in E} |S(f_k)(x_n)|$) and passing to an infinite subset of $\mathbb{N}$ we may assume that for every $n \in \mathbb{N}$ we have

$$(**) \quad \sum_{k \in \mathbb{N} \setminus \{n\}} |S(f_k)(x_n)| < \varepsilon.$$

Let $J : \ell_\infty \rightarrow \ell_\infty^c(\kappa)$ and $R : C(\mathbb{N}^*) \rightarrow \ell_\infty$ be given by the formulas

$$J((a_n)_{n \in \mathbb{N}}) = \sum_{n \in \mathbb{N}} a_n f_n$$

for $(a_n)_{n \in \mathbb{N}} \in \ell_\infty$, where the infinite sum is defined coordinatewise, and

$$R(g) = (g(x_n))_{n \in \mathbb{N}}$$

for $g \in C(\mathbb{N}^*)$. Let $T, T' : \ell_\infty \rightarrow \ell_\infty$ be given by $T = R \circ S \circ J$ and

$$T'((a_k)_{k \in \mathbb{N}}) = \left(\sum_{k \in \mathbb{N}} a_k S(f_k)(x_n) \right)_{n \in \mathbb{N}}.$$

By (**), T' is well defined and bounded. Then $T|_{c_0} = T'|_{c_0}$ because $T(1_{\{k\}}) = (S(f_k)(x_n))_{n \in \mathbb{N}} = T'(1_{\{k\}})$ for each $k \in \mathbb{N}$. So by [1, Theorem 2.5.4] there exists an infinite subset $M \subseteq \mathbb{N}$ such that T and T' coincide on $\ell_\infty(M)$.

Let $f = J(1_M) = \sum_{n \in M} f_n$, where the infinite sum is defined coordinatewise. Observe that $S(f)(x_n) = T(1_M)(n) = T'(1_M)(n) = \sum_{k \in M} S(f_k)(x_n)$ for $n \in M$. So by (*) and (**) for every $n \in M$ we have

$$(***) \quad |S(f)(x_n)| \geq |S(f_n)(x_n)| - \sum_{k \in M \setminus \{n\}} |S(f_k)(x_n)| > r - 2\varepsilon.$$

Let x be any accumulation point of $\{x_n : n \in M\}$ in $\mathbb{N}^*$. Clearly $|S(f)(x)| \geq r - 2\varepsilon$ by (***). As x_n s are contained in pairwise disjoint sets $[A_{\xi_n}]$ and

each $[A_\xi]$ is open, we conclude that $x \notin \bigcup_{\xi < \kappa} [A_\xi]$. Also, $f \in \ell_\infty^c(\kappa \setminus C)$ and $\|f\| = 1$. Since C and $\varepsilon > 0$ were arbitrary, we get $R \geq r$. ■

LEMMA 4.2. *Let $\mathcal{A}$ be a maximal almost disjoint family. Then for every subset $X \subseteq \mathbb{N}$ such that X is not almost covered by finitely many elements of $\mathcal{A}$ there exist $\mathfrak{a}$ many $A \in \mathcal{A}$ such that $|A \cap X| = \omega$.*

Proof. Towards a contradiction, suppose that $X \subseteq \mathbb{N}$ is not almost covered by finitely many elements of $\mathcal{A}$, but $|\{A \in \mathcal{A} : |A \cap X| = \omega\}| < \mathfrak{a}$. Let $\mathcal{B} = \{A \in \mathcal{A} : |A \cap X| = \omega\}$. Then there exists $C \subseteq X$ such that $\{A \cap X : A \in \mathcal{B}\} \cup \{C\}$ is an almost disjoint family of subsets of X . But this means that also the family $\mathcal{A} \cup \{C\}$ is almost disjoint, which contradicts the assumption that $\mathcal{A}$ is maximal. ■

PROPOSITION 4.3. *Suppose κ is an uncountable cardinal and $\mathcal{A} = \{A_\xi : \xi < \kappa\}$ is a maximal almost disjoint family ⁽⁶⁾ of subsets of $\mathbb{N}$. Let T be an isometric embedding $T : c_0(\kappa) \rightarrow C(\mathbb{N}^*)$ satisfying $T(1_{\{\xi\}}) = 1_{[A_\xi]}$ for every $\xi < \kappa$. Then no linear operator $S : \ell_\infty^c(\kappa) \rightarrow C(\mathbb{N}^*)$ extends T .*

Proof. Let $T : c_0(\kappa) \rightarrow C(\mathbb{N}^*)$ be such that $T(1_{\{\xi\}}) = 1_{[A_\xi]}$ for each $\xi < \kappa$. Suppose $S : \ell_\infty^c(\kappa) \rightarrow C(\mathbb{N}^*)$ is an operator which extends T . Let R and r be as in Lemma 4.1. Fix $C \in [\kappa]^{\leq \omega}$. Using the definition of R find $f \in \ell_\infty^c(\kappa \setminus C)$ with $\|f\| = 1$ such that

$$|S(f)(x)| > R - 1/2$$

for some $x \notin \bigcup_{\xi < \kappa} [A_\xi]$. By the continuity of $S(f)$ there is an infinite $B \subseteq \mathbb{N}$ such that $x \in [B]$ and $|S(f)(y)| > R - 1/2$ for every $y \in [B]$. As $x \notin \bigcup_{\xi < \kappa} [A_\xi]$, we conclude that B is not almost covered by finitely many elements of $\mathcal{A}$, and so by Lemma 4.2, B has infinite intersections with uncountably many elements of $\mathcal{A}$. Let $\xi \in \kappa \setminus C$ be such that ξ is not in the support of f and $A_\xi \cap B \neq \emptyset$. Then $\|f \pm 1_\xi\| = 1$ but

$$|S(f \pm 1_\xi)(y)| = |S(f)(y) \pm 1_{[A_\xi]}(y)| = |S(f)(y)| + 1 > R - 1/2 + 1 > R + 1/2$$

for every $y \in [A_\xi] \cap [B]$. Since C was arbitrary, this shows that $r \geq R + 1/2$, which contradicts Lemma 4.1. ■

5. Extending isomorphisms of isomorphic copies of $c_0(\kappa)$ to automorphisms under $\text{MA}_{\sigma\text{-linked}}(\kappa)$. In this section we consider an uncountable cardinal $\kappa < \mathfrak{c}$, assume $\text{MA}_{\sigma\text{-linked}}(\kappa)$ and fix two isomorphic copies X_1, X_2 of $c_0(\kappa)$ inside ℓ_∞/c_0 and an isomorphism $U : X_1 \rightarrow X_2$. The aim of this section is to show that under this hypothesis there is an automorphism $\tilde{U}$ of ℓ_∞/c_0 which extends U .

In the first part of this section we introduce terminology and notation concerning objects related to X_1, X_2 and U , which will be used in the entire

⁽⁶⁾ Note that consistently κ could have countable cofinality by the result of [12].

section. Recall the terminology of Definitions 2.1 and 2.2. Fix an isomorphism $T : c_0(\kappa) \rightarrow X_1$. Then $S = U \circ T$ is an isomorphism $c_0(\kappa) \rightarrow X_2$.

In order to construct an extension of U , we choose $\mathcal{F} = \{f_\xi : \xi < \kappa\}$ and $\mathcal{G} = \{g_\xi : \xi < \kappa\}$, two indexed subsets of ℓ_∞ , such that $\pi(f_\xi) = T(1_{\{\xi\}})$ and $\pi(g_\xi) = S(1_{\{\xi\}}) = U(\pi(f_\xi))$. We will additionally assume that all the coordinates of all f_ξ s and g_ξ s are rational numbers and $\|f_\xi\| = \|\pi(f_\xi)\|$, $\|g_\xi\| = \|\pi(g_\xi)\|$. This can be easily achieved, as a perturbation of any $f \in \ell_\infty$ by an element of c_0 does not affect the value of $\pi(f)$. Observe that for any finite $a \subseteq \kappa$ and $\rho \geq \|T\|, \|T^{-1}\|, \|S\|, \|S^{-1}\|$ the operators T, S witness that

$$\langle \pi(f_\xi) : \xi \in a \rangle \stackrel{\rho, \rho}{\sim} \langle 1_{\{\xi\}} : \xi \in a \rangle \stackrel{\rho, \rho}{\sim} \langle \pi(g_\xi) : \xi \in a \rangle,$$

where the $1_{\{\xi\}}$ s are considered as elements of $c_0(\kappa)$.

We will construct an isomorphism $\bar{U} : \ell_\infty \rightarrow \ell_\infty$ such that $\bar{U}[c_0] = c_0$ and $\bar{U}(f_\xi) - g_\xi \in c_0$ for each $\xi < \kappa$. Then we can define $\tilde{U}$ by

$$\tilde{U}(\pi(f)) = \pi(\bar{U}(f)).$$

The condition $U[c_0] = c_0$ ensures that $\tilde{U}$ is well defined and injective. The surjectivity of $\bar{U}$ implies the surjectivity of $\tilde{U}$. The boundedness of $\bar{U}$ provides the boundedness of $\tilde{U}$. Since $\pi[\mathcal{F}]$ is linearly dense in X_1 and $\tilde{U}$ coincides with U on $\pi[\mathcal{F}]$, $\tilde{U}$ is indeed the desired extension of U . The above terminology and notation will be used in the rest of this section.

DEFINITION 5.1. Suppose that κ is an uncountable cardinal, $X_1, X_2 \subseteq \ell_\infty/c_0$ two isomorphic copies of $c_0(\kappa)$, $T, S : c_0(\kappa) \rightarrow \ell_\infty/c_0$ and $\mathcal{F}, \mathcal{G} \subseteq \ell_\infty/c_0$ as above. Let $\rho = \max \{\|T\|, \|T^{-1}\|, \|S\|, \|S^{-1}\|\}$.

We define $\mathbb{P} = \mathbb{P}_{\mathcal{F}, \mathcal{G}} = \langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ as follows. The set $\mathbb{P}$ consists of triples of the form $p = (n_p, M_p, a_p)$ where

- (a) $n_p \in \mathbb{N}$,
- (b) M_p is an $n_p \times n_p$ matrix with rational entries such that

$$\|v\|/c_2(8\rho^2) \leq \|M_p v\| \leq c_2(8\rho^2)\|v\|$$

for any vector $v \in \ell_\infty^{n_p}$ (c_2 as in Lemma 2.8),

- (c) $a_p \in [\kappa]^{<\omega}$ such that for $n \geq n_p$,

$$\begin{aligned} \langle f_\xi[n, \infty) : \xi \in a_p \rangle &\stackrel{1,2}{\sim} \langle \pi(f_\xi) : \xi \in a_p \rangle, \\ \langle g_\xi : \xi \in a_p \rangle &\stackrel{2,1}{\sim} \langle g_\xi[n, \infty) : \xi \in a_p \rangle. \end{aligned}$$

For $p, q \in \mathbb{P}$ we write $p \leq_{\mathbb{P}} q$ if

- (i) $n_p \geq n_q$,
- (ii) the matrix M_p is of the form

$$\begin{bmatrix} M_q & 0 \\ 0 & M_{qp} \end{bmatrix}$$

where M_{qp} is an $[n_q, n_p) \times [n_q, n_p)$ matrix,

- (iii) $a_p \supseteq a_q$,
- (iv) $M_{qp}(f_\xi|[n_q, n_p]) = g_\xi|[n_q, n_p]$ for every $\xi \in a_q$.

LEMMA 5.2. $\mathbb{P}_{\mathcal{F}, \mathcal{G}} = \langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ as above is a partial order.

LEMMA 5.3. Suppose that κ is an uncountable cardinal, $X_1, X_2 \subseteq \ell_\infty/c_0$ two isomorphic copies of $c_0(\kappa)$, $T, S : c_0(\kappa) \rightarrow \ell_\infty/c_0$, $\rho \in \mathbb{R}$ and $\mathcal{F}, \mathcal{G} \subseteq \ell_\infty/c_0$ as above. Let $\mathbb{P} = \mathbb{P}_{\mathcal{F}, \mathcal{G}}$. Let $p = (n, H, a_p) \in \mathbb{P}$ and $q = (n, H, a_q) \in \mathbb{P}$, and let $N \in \mathbb{N}$. Then there exists $r = (n_r, M_r, a_r) \in \mathbb{P}$ such that $r \leq p, q$ and $n_r \geq N$.

Proof. Let $a_r = a_p \cup a_q$. By Lemmas 2.5 and 2.6 there is $n_r \in \mathbb{N}$ with $n, N < n_r$, $|a_r| \leq c_1(8\rho^2)\sqrt{n_r - n}$ (the constant c_1 is as in Lemma 2.8) such that

$$(1) \quad \begin{aligned} \langle f_\xi|[k, \infty) : \xi \in a_r \rangle &\stackrel{1,2}{\sim} \langle \pi(f_\xi) : \xi \in a_r \rangle, \\ \langle \pi(g_\xi) : \xi \in a_r \rangle &\stackrel{2,1}{\sim} \langle g_\xi|[k, \infty) : \xi \in a_r \rangle \end{aligned}$$

for $k \geq n_r$, and

$$(2) \quad \begin{aligned} \langle f_\xi|[n, n_r) : \xi \in a_r \rangle &\stackrel{2,1}{\sim} \langle f_\xi|[n, \infty) : \xi \in a_r \rangle, \\ \langle g_\xi|[n, \infty) : \xi \in a_r \rangle &\stackrel{1,2}{\sim} \langle g_\xi|[n, n_r) : \xi \in a_r \rangle. \end{aligned}$$

To complete the definition of r we need to find M_r . By Lemma 2.4, condition (c) for p, q and the lower and upper bounds of S, T imply that for $a \subseteq a_p$ or $a \subseteq a_q$,

$$\langle f_\xi|[n, \infty) : \xi \in a \rangle \stackrel{\rho, 2\rho}{\sim} \langle 1_{\{\xi\}} : \xi \in a \rangle \stackrel{2\rho, \rho}{\sim} \langle g_\xi|[n, \infty) : \xi \in a \rangle,$$

where the $1_{\{\xi\}}$ s are elements of $c_0(\kappa)$. As a result, for any real-valued sequence $(t_\xi)_{\xi \in a_r}$,

$$\begin{aligned} \left\| \sum_{\xi \in a_r} t_\xi f_\xi|[n, \infty) \right\| &\leq \left\| \sum_{\xi \in a_p} t_\xi f_\xi|[n, \infty) \right\| + \left\| \sum_{\xi \in a_q \setminus a_p} t_\xi f_\xi|[n, \infty) \right\| \\ &\leq 2\rho \left\| \sum_{\xi \in a_p} t_\xi 1_\xi \right\| + 2\rho \left\| \sum_{\xi \in a_q \setminus a_p} t_\xi 1_\xi \right\| \leq 4\rho \max_{\xi \in a_r} |t_\xi| = 4\rho \left\| \sum_{\xi \in a_r} t_\xi 1_\xi \right\|, \end{aligned}$$

and, on the other hand,

$$\left\| \sum_{\xi \in a_r} t_\xi 1_\xi \right\| \leq \rho \left\| \sum_{\xi \in a_r} t_\xi \pi(f_\xi) \right\| \leq \rho \left\| \sum_{\xi \in a_r} t_\xi f_\xi|[n, \infty) \right\|.$$

We may also write analogous inequalities for g_ξ s instead of f_ξ s. Consequently,

$$(3) \quad \langle f_\xi|[n, \infty) : \xi \in a_r \rangle \stackrel{\rho, 4\rho}{\sim} \langle 1_{\{\xi\}} : \xi \in a_r \rangle \stackrel{4\rho, \rho}{\sim} \langle g_\xi|[n, \infty) : \xi \in a_r \rangle.$$

Combining (2) and (3), by Lemma 2.4 we get

$$(4) \quad \langle f_\xi|[n, n_r) : \xi \in a_r \rangle \stackrel{2\rho, 4\rho}{\sim} \langle 1_{\{\xi\}} : \xi \in a_r \rangle \stackrel{4\rho, 2\rho}{\sim} \langle g_\xi|[n, n_r) : \xi \in a_r \rangle.$$

Let $Y_{\mathcal{F}} = \text{span}\{f_{\xi}[[n, n_r)] : \xi \in a_r\}$, $Y_{\mathcal{G}} = \text{span}\{g_{\xi}[[n, n_r)] : \xi \in a_r\}$ and $V : Y_{\mathcal{F}} \rightarrow Y_{\mathcal{G}}$ be given by $V(f_{\xi}[[n, n_r)]) = g_{\xi}[[n, n_r)]$. Then $\dim(Y_{\mathcal{F}}) = \dim(Y_{\mathcal{G}}) = |a_r|$ and (4) implies that $d(Y_{\mathcal{F}}, \ell_{\infty}^{|a_r|}), d(Y_{\mathcal{G}}, \ell_{\infty}^{|a_r|}) \leq 8\rho^2$ and $\|V\|, \|V^{-1}\| \leq 8\rho^2$.

So, as $|a_r| \leq c_1(8\rho^2)\sqrt{n_r - n}$, we can apply Lemma 2.8 to conclude that there is an automorphism W of $\ell_{\infty}^{n_r \setminus n}$ which extends V such that

$$(5) \quad \|W\|, \|W^{-1}\| \leq c_2(8\rho^2)$$

and, since f_{ξ} s and g_{ξ} s have all coordinates rational, the matrix M of W has rational entries. We define

$$M_r = \begin{bmatrix} H & 0 \\ 0 & M \end{bmatrix}.$$

Now we are left with checking that $r \in \mathbb{P}$ and $r \leq p, q$. Condition (a) of Definition 5.1 is clear. Also M_r is an $n_p \times n_p$ matrix with rational entries by (b) for p, q and the choice of M . Moreover, for any $v \in \ell_{\infty}^{n_r}$,

$$M_r v = H(v[[0, n)]) \cup M(v[[n, n_r)]),$$

so $\|M_r v\| = \max\{\|H(v[[0, n)])\|, \|M(v[[n, n_r)])\|\}$. Since by (b) for p, q and by (5) we have $\|H\|, \|H^{-1}\|, \|M\|, \|M^{-1}\| \leq c_2(8\rho^2)$, we get

$$\begin{aligned} \frac{\|v\|}{c_2(8\rho^2)} &= \frac{\max\{\|v[[0, n)]\|, \|v[[n, n_r)]\|\}}{c_2(8\rho^2)} \leq \max\{\|Hv[[0, n)]\|, \|Mv[[n, n_r)]\|\} \\ &\leq c_2(8\rho^2) \cdot \max\{\|v[[0, n)]\|, \|v[[n, n_r)]\|\} = c_2(8\rho^2)\|v\|. \end{aligned}$$

So condition (b) from Definition 5.1 is satisfied for $r = (n_r, M_r, a_r)$. Also (c) of Definition 5.1 is satisfied for r by (1) and so $r \in \mathbb{P}_{\mathcal{F}, \mathcal{G}}$. To check that $r \leq p, q$ we first note that conditions (i)–(iii) of Definition 5.1 follow directly from the choices of n_r , M_r and a_r respectively. Finally, condition (iv) follows from the definition of V and the fact that W extends V . ■

LEMMA 5.4. $\mathbb{P}_{\mathcal{F}, \mathcal{G}}$ as in Definition 5.1 is σ -linked and so it satisfies the c.c.c.

Proof. For $n \in \mathbb{N}$ and for an $n \times n$ matrix H with rational entries let

$$\mathbb{P}(n, H) = \{p \in \mathbb{P}_{\mathcal{F}, \mathcal{G}} : n = n_p, H = M_p\}.$$

It is clear that there are countably many n, H as above and that every element of $\mathbb{P}_{\mathcal{F}, \mathcal{G}}$ belongs to $\mathbb{P}(n, H)$ for some n, H as above. Lemma 5.3 implies that each $\mathbb{P}(n, H)$ is linked, i.e., any two elements of it are compatible. ■

LEMMA 5.5. For every $n \in \mathbb{N}$ the set

$$D_n = \{p \in \mathbb{P}_{\mathcal{F}, \mathcal{G}} : n_p \geq n\}$$

is dense in $\mathbb{P}_{\mathcal{F}, \mathcal{G}}$ as in Definition 5.1.

Proof. Apply Lemma 5.3 to $p = q$ and $N \geq n$. ■

LEMMA 5.6. *For every $\xi < \kappa$ the set*

$$E_\xi = \{p \in \mathbb{P}_{\mathcal{F}, \mathcal{G}} : \xi \in a_p\}$$

is dense in $\mathbb{P}_{\mathcal{F}, \mathcal{G}}$ as in Definition 5.1.

Proof. Fix $\xi \in \kappa$ and $p = (n_p, M_p, a_p)$. Since $\|f_\xi\| = \|\pi(f_\xi)\|$ and $\|g_\xi\| = \|\pi(g_\xi)\|$ by the choice of f_ξ and g_ξ at the beginning of this section, the triple $q = (n_p, M_p, \{\xi\})$ belongs to $\mathbb{P}_{\mathcal{F}, \mathcal{G}}$ because condition (c) of Definition 5.1 is satisfied for q . Apply Lemma 5.5 to find $r \leq p, q$. Then $r \in E_\xi$. ■

PROPOSITION 5.7. *Suppose that κ is an infinite cardinal. Assume $\mathbf{MA}_{\sigma\text{-linked}}(\kappa)$. Let X_1, X_2 be two isomorphic copies of $c_0(\kappa)$ inside ℓ_∞/c_0 and let $U : X_1 \rightarrow X_2$ be an isomorphism. Then there is an automorphism $\bar{U}$ of ℓ_∞/c_0 which extends U .*

Proof. As noted at the beginning of this section, under the hypothesis of the proposition we have certain isomorphic embeddings S, T of $c_0(\kappa)$ into ℓ_∞/c_0 and $f_\xi, g_\xi \in \ell_\infty$ for $\xi < \kappa$ such that $\pi(f_\xi) = T(1_{\{\xi\}})$, $\pi(g_\xi) = S(1_{\{\xi\}})$, and it is enough to construct an automorphism $\bar{U} : \ell_\infty \rightarrow \ell_\infty$ such that $\bar{U}[c_0] = c_0$ and $\bar{U}(f_\xi) - g_\xi \in c_0$ for each $\xi < \kappa$.

To do so, we consider the partial order $\mathbb{P} = \mathbb{P}_{\mathcal{F}, \mathcal{G}}$. By Lemma 5.4 and Lemmas 5.5 and 5.6 and the hypothesis of $\mathbf{MA}_{\sigma\text{-linked}}(\kappa)$ there is a filter $\mathbb{G} \subseteq \mathbb{P}$ such that $D_n \cap \mathbb{G} \neq \emptyset \neq E_\xi \cap \mathbb{G}$ for each $n \in \mathbb{N}$ and each $\xi < \kappa$.

Now define an $\mathbb{N} \times \mathbb{N}$ matrix M by putting $M(i, j) = k$ if and only if there is $p \in \mathbb{G}$ such that $M_p(i, j) = k$. Note that since $\mathbb{G}$ is a filter, given $p, q \in \mathbb{G}$ such that $i, j < n_p, n_q$, there is $r \in \mathbb{G}$ such that $r \leq p, q$ and so $M_p(i, j) = M_r(i, j) = M_q(i, j)$. By Lemma 5.5, for any pair $(i, j) \in \mathbb{N}$ there is $p \in \mathbb{G}$ such that $n_p > i, j$ and M_p is an $n_p \times n_p$ matrix. Hence, M is a well defined $\mathbb{N} \times \mathbb{N}$ matrix.

Moreover, according to Lemma 5.5, there is a decreasing sequence $\{p_k : k \in \mathbb{N}\} \subseteq \mathbb{G}$ with p_0 being the trivial condition such that $n_{p_k} \geq k$. It follows from condition (ii) of Definition 5.1 that there are $[n_{p_k}, n_{p_{k+1}}) \times [n_{p_k}, n_{p_{k+1}})$ matrices M_k such that M is block diagonal built from blocks M_k , that is, $M(i, j) = 0$ unless there is k such that $i, j \in [n_{p_k}, n_{p_{k+1}})$ and $M(i, j) = M_k(i, j)$. In particular $M1_{\{i\}}$, which is one of the columns of the matrix M , is a finitely supported vector for any $i \in \mathbb{N}$. It follows that M defines an operator $\bar{U}_{00} : c_{00} \rightarrow c_{00}$ given by $\bar{U}_{00}(f) = Mf$ for $f \in c_{00}$.

Condition (b) of Definition 5.1 implies that $\|M_k\| \leq c_2(8\rho^2)$ for each k and so $\|\bar{U}_{00}\| \leq c_2(8\rho^2)$. Consequently, $\bar{U}_{00}$ may be extended to an operator $\bar{U}_0$ on c_0 with the same norm. Moreover, condition (b) of Definition 5.1 also implies that for each $k \in \mathbb{N}$ there are $[n_{p_k}, n_{p_{k+1}}) \times [n_{p_k}, n_{p_{k+1}})$ matrices $\tilde{M}_k$ which are inverses of M_k s. The same argument as for M implies that the matrix $\tilde{M}$ formed by the matrices $\tilde{M}_k$ defines the inverse of $\bar{U}_0$ of norm not exceeding $c_2(8\rho^2)$.

Multiplication by matrices M and $\tilde{M}$ defines operators $\bar{U}$, $\bar{U}^{-1}$ on ℓ_∞ [23, Propositions 2.1 and 2.2] whose composition (from either side) is the identity. In fact, $\bar{U} = \bar{U}_0^{**}$ and similarly for the inverse [23, Proposition 2.5]. It follows that $\bar{U}$ given by the matrix M is an automorphism of ℓ_∞ with $\bar{U}[c_0] = c_0$.

Now fix $\xi < \kappa$ and applying Lemma 5.6 fix $p \in \mathbb{G}$ such that $\xi \in a_p$. We claim that $\bar{U}(f_\xi) = g_\xi + h_\xi$, where $h_\xi \in \ell_\infty(n_p)$. Indeed, since $\mathbb{G}$ is a filter, $M(i, j) = k$ if and only if there is $q \in \mathbb{G}$ such that $M_q(i, j) = k$ and $q \leq p$, so the claim follows from conditions (ii) and (iv) of Definition 5.1. ■

6. Questions. Besides Problem 12 of [5] whether ℓ_∞/c_0 can be consistently universally κ -injective for $\kappa > \omega_1$ we believe the following problems are open, natural and interesting:

QUESTIONS 6.1. *Is any of the following statements provable in ZFC (or is its negation consistent with ZFC)?*

- (1) if ℓ_∞/c_0 is 2^ω -automorphic, then $2^\omega = \omega_1$;
- (2) if ℓ_∞/c_0 is 2^ω -extensible, then $2^\omega = \omega_1$;
- (3) ℓ_∞/c_0 is not $c_0(2^\omega)$ -automorphic;
- (4) ℓ_∞/c_0 is not $c_0(2^\omega)$ -extensible;
- (5) ℓ_∞/c_0 is not $c_0(\mathfrak{a})$ -automorphic;
- (6) ℓ_∞/c_0 is not $c_0(\mathfrak{a})$ -extensible.

We note that ℓ_∞/c_0 is not automorphic (the argument is as in [28, Proposition 5.2]; we do not use it directly as the hypothesis on codimensions is missing there). To see this, let $X \subseteq \ell_\infty/c_0$ be the kernel of a surjective operator from ℓ_∞/c_0 onto $\ell_2(\mathfrak{c})$. It exists because ℓ_∞ admits such an operator by [30, Remark 2, p. 203] and ℓ_∞/c_0 admits a (necessarily complemented) copy of ℓ_∞ . Considering an isomorphism $S : \ell_\infty/c_0 \oplus \ell_\infty/c_0 \rightarrow \ell_\infty/c_0$ we obtain an isomorphic copy $Y = S[\{0\} \oplus X]$ of X which cannot be sent by an automorphism onto X as the quotient by X does not contain ℓ_∞/c_0 , while the quotient by Y does. Moreover, the codimensions of X and Y agree, that is, $\text{dens}((\ell_\infty/c_0)/X) = \text{dens}(\ell_2(\mathfrak{c})) = \mathfrak{c} = \text{dens}((\ell_\infty/c_0)/Y)$.

Taking into account Proposition 4.3, one way to solve in the negative items (5) and (6) (and so (3) and (4)) may be through generalizing Proposition 3.1 from ω_1 to $\mathfrak{a}$ by constructing an isomorphic embedding of $\ell_\infty^c(\mathfrak{a})$ into ℓ_∞/c_0 in ZFC. We note here that this may be delicate because, for example, it is proved in [11] that consistently 2^ω is arbitrarily large and $\ell_\infty(c_0(\omega_2))$ cannot be isomorphically embedded into ℓ_∞/c_0 . So in such a situation $\ell_\infty^c(\omega_2)$ simply does not embed into ℓ_∞/c_0 as $\ell_\infty(c_0(\omega_2)) \subseteq \ell_\infty^c(\omega_2)$. However, in these (Cohen) models, $\mathfrak{a} = \omega_1$.

Another observation is that in the model of Theorem 1.4 all the items above besides (1) and (2) above question hold by Proposition 4.3, as $\mathfrak{a} = 2^\omega$

and $\ell_\infty^c(\mathfrak{a})$ embeds into ℓ_∞/c_0 since the Boolean algebra of countable and cocountable subsets of 2^ω has cardinality 2^ω . Also we do not know the answer to the following:

QUESTION 6.2. *Suppose that $\kappa < \mathfrak{b}$ is an uncountable cardinal. Does every isometric embedding $T : c_0(\kappa) \rightarrow \ell_\infty/c_0$ extend to an isometric embedding $T : \ell_\infty^c(\kappa) \rightarrow \ell_\infty/c_0$?*

The positive answer would generalize Theorem 1.5. In fact, some of the questions above concerning the existence of isomorphic embeddings may be solved by providing even stronger Boolean embeddings, as we do not know the answers to the following:

QUESTIONS 6.3. *Are the following provable in ZFC?*

- (1) *The Boolean algebra of all countable and all cocountable subsets of $\mathfrak{a}$ embeds into $\wp(\mathbb{N})/\text{Fin}$.*
- (2) *Every embedding of the Boolean algebra $\text{Fincofin}(\kappa)$ into $\wp(\mathbb{N})/\text{Fin}$ can be extended to an embedding of the Boolean algebra of all countable and all cocountable subsets of κ if $\kappa < \mathfrak{b}$.*

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