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The finite Hilbert transform acting on L^∞

by

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Abstract. The action of the finite Hilbert transform defined on $L^\infty(-1,1)$ and taking its values in the Zygmund space $L_{\exp}(-1,1)$ is studied in detail. This is a reciprocal situation to the investigation, recently undertaken by the present authors, of the finite Hilbert transform defined on the Zygmund space $L\log L(-1,1)$ and taking its values in $L^1(-1,1)$. The fact that both $L^\infty(-1,1)$ and $L_{\exp}(-1,1)$ fail to be separable generates new features not present in [Ann. Scuola Norm. Sup. Pisa 25 (2024), 1527–1557].

1. Introduction. The *finite Hilbert transform* $T(f)$ of $f \in L^1 := L^1(-1,1)$ is the principal value integral

$$T(f)(t) := \lim_{\varepsilon\to 0^+} \frac{1}{\pi}\Big(\int_{-1}^{t-\varepsilon} + \int_{t+\varepsilon}^{1}\Big)\frac{f(x)}{x-t}\,dx,$$

which exists for a.e. $t \in (-1,1)$ and is a measurable function. Its study was motivated by problems in aerodynamics (see [CR54, R49, S54, T51, T57]) and in elasticity theory (via the study of one-dimensional singular integral operators; see [D79, G90, GK92-1, GK92-2, MP86, M67]). These applications arise via the solution of the *airfoil equation*

$$\frac{1}{\pi}\,\text{p.v.}\int_{-1}^{1}\frac{f(x)}{x-t}\,dx = g(t), \qquad \text{a.e. } t\in(-1,1), \tag{1.1}$$

where g is given and f is to be found.

Due to M. Riesz's Theorem, the finite Hilbert transform operator maps the space $L^p := L^p(-1,1)$ continuously into itself whenever $1 < p < \infty$. By the 1990s the L^p theory for the finite Hilbert transform (FHT, for short) was well established; see [K09, Ch. 11], [OE91] and [T57, §4.3]. In 1991,

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a celebrated paper of Gel'fand and Graev, [GG91], renewed an interest in the FHT by exposing its role in the inversion of the Radon transform, and hence its applications to image reconstruction in tomography; see, for example, [APS16, BKT16, KT12, SP05].

The L^p theory for the FHT on $(-1,1)$ is rather different than for the Hilbert transform on $\mathbb{R}$ or its periodic version on $\mathbb{T}$. Namely, Tricomi showed that $T\colon L^p \to L^p$ is a Fredholm operator for $1 < p < 2$ (with full range and one-dimensional kernel) and for $2 < p < \infty$ (injective and with one-codimensional kernel); see [ORS08, Example 4.21]. A deeper understanding of the FHT is obtained when considering its action on rearrangement invariant (r.i., for short) spaces on $(-1,1)$; see [COR19, COR21-1, COR21-2, COR24-1]. For example, for the FHT operator $T_X\colon X \to X$ acting on a r.i. space X with non-trivial Boyd indices, the following alternative for its fine spectrum holds:

(a) the point spectrum: $\sigma_{\rm pt}(T_X) \neq \emptyset \Leftrightarrow L^{2,\infty} \subseteq X$,
(b) the residual spectrum: $\sigma_{\rm r}(T_X) \neq \emptyset \Leftrightarrow X \subseteq L^{2,1}$,
(c) the continuous spectrum: $\sigma_{\rm c}(T_X) = \sigma(T_X) \Leftrightarrow L^{2,\infty} \not\subseteq X$ and $X \not\subseteq L^{2,1}$,

where $L^{2,1}$ and $L^{2,\infty}$ are the usual Lorentz spaces [COR21-2, Proposition 5.1].

Note that while the FHT operator maps L^p continuously into itself for $1 < p < \infty$, it does not map neither L^∞ into itself nor L^1 into itself. In the latter case, L^1 is mapped continuously into $L^0(-1,1)$, due to Kolmogorov's Theorem [BS88, Theorem III.4.9(b)]. In [COR24-2] we studied the situation when $T\colon L\log L \to L^1$, where $L\log L$ is the classical Zygmund space, which is a r.i. space close to L^1 in the sense that it contains all L^p spaces for $p > 1$.

The aim of the present paper is to study the action of the FHT on L^∞. In this case, the operator $T\colon L^\infty \to L_{\rm exp}$, denoted by T_∞, maps L^∞ continuously into $L_{\rm exp}$, where the classical Zygmund space $L_{\rm exp}$ is a r.i. space close to L^∞ in the sense that it is contained in all L^p spaces for $p < \infty$. With the objective of solving the airfoil equation (1.1), we also need to investigate the auxiliary operator $\breve{T}$ defined by

$$\breve{T}(f) := -wT\left(\frac{f}{w}\right), y$$

with $w(x) := \sqrt{1-x^2}$ for $x \in (-1,1)$, and to show that $\breve{T}\colon L^\infty \to L_{\rm exp}$ boundedly. This, together with the characterization of the range $T(L^\infty) \subsetneq L_{\rm exp}$ of T_∞, allows us to identify the inverse operator $T^{-1}\colon T(L^\infty) \to L^\infty$, and hence to solve the airfoil equation. We end the paper by showing that T_∞ is optimally defined, that is, $T\colon L^\infty \to L_{\rm exp}$ does not admit a continuous, linear, $L_{\rm exp}$-valued extension to any strictly larger r.i. domain space within L^1.

2. Preliminaries. The measure space considered in this paper is the Lebesgue measure μ on $\mathbb{R}$ restricted to the Borel σ-algebra $\mathcal{B}$ of the open

interval $(-1,1)$, which we again denote by μ. The vector space of all $\mathbb{C}$-valued, $\mathcal{B}$-simple functions is denoted by $\operatorname{sim}\mathcal{B}$. Denote by $L^0 := L^0(\mu)$ the vector space of all $\mathbb{C}$-valued measurable functions on $(-1,1)$. Measurable functions on $(-1,1)$ which coincide a.e. are identified. With respect to the a.e. pointwise order for its positive cone, L^0 is a complex vector lattice. The space L^0 is also a metrizable topological vector space for the topology of convergence in measure.

An order ideal X of L^0 is called a *Banach function space* (briefly, B.f.s.) over the measure space $((-1,1),\mathcal{B},\mu)$ if $\operatorname{sim}\mathcal{B} \subseteq X$ (equivalently, $L^\infty \subseteq X$) and if X is equipped with a lattice norm $\|\cdot\|_X$ for which it is complete. For brevity, we speak of X as a B.f.s. over $(-1,1)$. Since μ is a finite measure, necessarily $X \subseteq L^1$ (cf. [BS88, Section II.1, Definitions 1.1 and 1.3]) with a continuous inclusion (as it is a positive linear map between Banach lattices). A typical example of a B.f.s. over $(-1,1)$ is the Lebesgue space $L^p := L^p(\mu)$ for each $1 \leq p \leq \infty$.

A function f in a B.f.s. X over $(-1,1)$ has *absolutely continuous norm* if $\|f\chi_{A_n}\| \downarrow 0$ whenever $A_n \downarrow \emptyset$ in $\mathcal{B}$. The subspace X_a of X consisting of all functions which have absolutely continuous norm is an order ideal in X. That is, X_a is a closed linear subspace with the property: $f \in X_a$ and $|g| \leq |f|$ μ-a.e. implies that $g \in X_a$. A B.f.s. X has *σ-order continuous* (or *absolutely continuous*) norm if $X = X_a$, that is, for $f \in X$ given, $\|f_n\|_X \downarrow 0$ for every sequence $(f_n)_{n=1}^\infty \subseteq L^0$ satisfying $|f| \geq |f_n| \downarrow 0$ μ-a.e. For these facts and notions, see [BS88, Section I.3]. If $L^\infty \subseteq X_a$, then X_a is also a B.f.s. over $(-1,1)$. It is known that $(L^\infty)_a = \{0\}$. A B.f.s. X has the *Fatou property* if, whenever $(f_n)_{n=1}^\infty$ is a norm bounded, increasing sequence of non-negative functions in X for which $f = \sup_{n\in\mathbb{N}} f_n$ exists in L^0, then necessarily $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$.

The *associate space* X' of X is the vector sublattice of L^0 defined by $X' := \{g \in L^0 : fg \in L^1 \text{ for every } f \in X\}$ equipped with the lattice norm $\|g\|_{X'} := \sup\{|\int_{-1}^1 fg\,d\mu| : f \in X \text{ with } \|f\|_X \leq 1\}$. It turns out that X' is a closed linear subspace of the dual Banach space X^* of X. Moreover, X' is also a B.f.s. over $(-1,1)$ which always has the Fatou property. If X has σ-order continuous norm, then $X' = X^*$; see, for example, [LT79, p. 29].

The decreasing rearrangement $f^*\colon [0,2] \to [0,\infty]$ of a function $f \in L^0$ is the right continuous inverse of its distribution function: $\lambda \mapsto \mu(\{t \in (-1,1) : |f(t)| > \lambda\})$ for $\lambda \geq 0$. A *rearrangement invariant* space X over $(-1,1)$ is a B.f.s. with the Fatou property such that, if $g^* \leq f^*$ with $f \in X$, then $g \in X$ and $\|g\|_X \leq \|f\|_X$. In this case, its associate space X' is also a r.i. space [BS88, Proposition II.4.2].

For $\alpha \geq 0$, the Zygmund space $L^\alpha_{\exp} := L^\alpha_{\exp}(-1,1)$ consists of all measurable functions f on $(-1,1)$ for which there exists a constant $\lambda = \lambda(f) > 0$

such that

$$\int_{-1}^{1} \exp(\lambda|f(x)|)^{1/\alpha}\,dx < \infty$$

[BS88, Definition IV.6.11]. For $\alpha = 0$ the space $L^0_{\exp}$ is interpreted to be L^∞. For each $\alpha \geq 0$ the space $L^\alpha_{\exp}$ is a r.i. space with respect to the norm

$$\|f\|_{L^\alpha_{\exp}} := \sup_{0<t<2} \frac{\int_0^t f^*(s)\,ds}{t(\log(2e/t))^\alpha}, \qquad f \in L^\alpha_{\exp},$$

[BS88, Lemma IV.6.12]. A norm equivalent to the above norm in $L^\alpha_{\exp}$ is given by

$$\sup_{0<t<2} \frac{f^*(t)}{(\log(2e/t))^\alpha}, \qquad f \in L^\alpha_{\exp};$$

see [KPS82, §II.5.2, p. 115, Theorem 5.3]. The spaces $L^\alpha_{\exp}$ are non-separable, and hence their norms are not σ-order continuous. They are close to L^∞ in the sense that $L^\infty \subseteq L^\alpha_{\exp} \subseteq L^p$ for all $\alpha \geq 0$ and all $1 \leq p < \infty$; for the case $\alpha = 1$, see [BS88, Theorem IV.6.5]. Observe that $L^\alpha_{\exp} \subseteq L^\beta_{\exp}$ for all $0 \leq \alpha \leq \beta$.

For $\alpha = 1$, the Zygmund space $L_{\exp} := L^1_{\exp}$ consists of all functions having exponential integrability [BS88, Definition IV.6.1 and Lemma IV.6.2], or equivalently, all functions f on $(-1,1)$ for which there exists a constant $c = c(f) > 0$ such that

$$f^*(t) \leq c \log\left(\frac{2e}{t}\right), \qquad 0 < t < 2.$$

The r.i. space $L_{\exp}$ is a non-separable Marcinkiewicz space, and the closure of $\operatorname{sim}\mathcal{B}$ in $L_{\exp}$, denoted by $(L_{\exp})_b$, is the B.f.s. consisting of all measurable functions f on $(-1,1)$ satisfying

$$\lim_{t\to 0^+} \frac{\int_0^t f^*(s)\,ds}{t\log(2e/t)} = 0 \tag{2.1}$$

[KPS82, §II.5.2, p. 116, Lemma 5.4]; it coincides with the separable space $(L_{\exp})_a$ and does not have the Fatou property. The r.i. space $L_{\exp}$ is also an Orlicz space [BS88, Section IV.8]. Define a Young function Φ on $[0,\infty)$ by $\Phi(t) := e^t - t - 1$, which satisfies

$$\int_{-1}^{1} \Phi(|f|)\,d\mu \leq \int_{-1}^{1} e^{|f|}\,d\mu \leq 3 + 2\int_{-1}^{1} \Phi(|f|)\,d\mu, \qquad f \in L^0.$$

The previous inequality implies that the corresponding Orlicz space coincides with $L_{\exp}$ (with equivalent norms) as Φ is equivalent to the Young function

given in [BS88, Example IV.8.3(d)]. Accordingly,

$$(L_{\exp})_a = (L_{\exp})_b = \Big\{ f \in L_{\exp} : \int_{-1}^{1} e^{\lambda |f|}\, d\mu < \infty \text{ for every } \lambda \geq 0 \Big\}; \tag{2.2}$$

see [LT79, p. 120] (and [L55, Ch. 2, Sect. 3, Theorem 1] for a proof). It follows that $L^\infty \subsetneq (L_{\exp})_a \subsetneq L_{\exp}$. Indeed, direct calculations via (2.2) show that $f(x) := |\log(x)| \chi_{[0,1)}(x)$ and $g(x) := \chi_{[0,1)}(x) \log(1 + |\log(x)|)$, for $x \in (-1,1)$, belong to $L_{\exp} \setminus (L_{\exp})_a$ and $(L_{\exp})_a \setminus L^\infty$, respectively.

For $\alpha \geq 0$, the Zygmund space $L(\log L)^\alpha$ consists of all measurable functions f on $(-1,1)$ for which either one of the following two equivalent conditions holds:

$$\int_{-1}^{1} |f(x)| \big(\log(2 + |f(x)|\big)^\alpha \, dx < \infty, \qquad \int_0^2 f^*(t) \bigg(\log\bigg(\frac{2e}{t} \bigg) \bigg)^\alpha \, dt < \infty$$

[BS88, Definition IV.6.11, Lemma IV.6.12]. The space $L(\log L)^\alpha$ is r.i. with σ-order continuous norm given by

$$\|f\|_{L(\log L)^\alpha} := \int_0^2 f^*(t) \bigg(\log\bigg(\frac{2e}{t} \bigg) \bigg)^\alpha \, dt, \qquad f \in L(\log L)^\alpha.$$

For each $\alpha \geq 0$, the associate space of the space $L(\log L)^\alpha$ is $L^\alpha_{\exp}$ [KPS82, §II.5.2, p. 112, Theorem 5.2]. Since $L(\log L)^\alpha$ has σ-order continuous norm, $(L(\log L)^\alpha)^* = (L(\log L)^\alpha)' = L^\alpha_{\exp}$. The spaces $L(\log L)^\alpha$ are close to L^1 in the sense that $L^p \subseteq L(\log L)^\alpha \subseteq L^1$ for all $\alpha \geq 0$ and all $1 < p < \infty$; for the case $\alpha = 1$, see [BS88, Theorem IV.6.5]. Moreover, $L(\log L)^\beta \subseteq L(\log L)^\alpha$ for all $0 \leq \alpha \leq \beta$.

For $\alpha = 1$ the Zygmund space $L(\log L)^1$ is, of course, $L \log L$. The case when $\alpha = 0$ is interpreted as $L(\log L)^0 = L^1$.

The *lower* and *upper Boyd indices* of a r.i. space X are two indices, $\underline{\alpha}_X$ and $\overline{\alpha}_X$, which measure the effect of dilations in X and satisfy $0 \leq \underline{\alpha}_X \leq \overline{\alpha}_X \leq 1$ [BS88, Definition III.5.12]. The boundedness of the finite Hilbert transform $T\colon X \to X$ is equivalent to X having non-trivial Boyd indices, that is, $0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1$; see [KPS82, §II.8.6, pp. 170–171].

For additional details concerning the Zygmund spaces $L(\log L)^\alpha$ and $L^\alpha_{\exp}$ we refer to [BR80, Ch. III], for example.

3. The finite Hilbert transform on L^∞. In this section we establish some basic properties of the finite Hilbert transform when it acts on L^∞ and takes its values in $L_{\exp}$.

The following result, a combination of Theorem 3.1 and Corollary 3.3 in [COR24-2], will be repeatedly used.

PROPOSITION 3.1. *Let $f \in L^1$ and $g \in L\log L$ satisfy $fT(g\chi_A) \in L^1$ for every set $A \in \mathcal{B}$. Then $gT(f) \in L^1$ and the following Parseval formula is valid:*

$$\int_{-1}^{1} fT(g) = -\int_{-1}^{1} gT(f). \tag{3.1}$$

In particular, if $f \in L^\infty$ and $g \in L\log L$, then the above assumptions are satisfied, and hence (3.1) *is valid.*

PROPOSITION 3.2. *The finite Hilbert transform $T\colon L^\alpha_{\exp} \to L^{\alpha+1}_{\exp}$ is a bounded operator for $\alpha \geq 0$. In particular, $T\colon L^\infty \to L_{\exp}$ is a bounded operator.*

Proof. It is known that $T\colon L(\log L)^{\alpha+1} \to L(\log L)^\alpha$ is a bounded operator [COR24-2, Proposition 4.4]. Accordingly, its adjoint operator T^* maps $L^\alpha_{\exp} \to L^{\alpha+1}_{\exp}$ boundedly.

Given $f \in L^\alpha_{\exp} \subseteq L^1$ and $g \in L(L\log L)^{\alpha+1} \subseteq L\log L$, it follows from Proposition 3.1 that $gT(f) \in L^1$ and

$$\int_{-1}^{1} gT(f) = -\int_{-1}^{1} fT(g) = -\int_{-1}^{1} gT^*(f).$$

Thus, $T(f) \in L^{\alpha+1}_{\exp}$. It follows that $T^* = -T$. Consequently, $T\colon L^\alpha_{\exp} \to L^{\alpha+1}_{\exp}$ boundedly.

For $\alpha = 0$, we have $L^0_{\exp} = L^\infty$ and $L^1_{\exp} = L_{\exp}$ and so $T\colon L^\infty \to L_{\exp}$ boundedly. ∎

When convenient, the operator $T\colon L^\infty \to L_{\exp}$ will also be denoted by T_∞, where the subscript indicates briefly that the domain space is L^∞.

REMARK 3.3. It is shown in [COR24-2, Proposition 4.15] that $T\colon L\log L \to L^1$ is not compact. According to Schauder's Theorem also its adjoint operator $T\colon L^\infty \to L_{\exp}$ (cf. the proof of Proposition 3.2) is not compact. A similar argument, via Gantmacher's Theorem, shows that $T\colon L^\infty \to L_{\exp}$ is not a weakly compact operator (since this is the case for $T\colon L\log L \to L^1$ [COR24-1, Theorem 4.7]).

Moreover, the operator $T\colon L^\infty \to L_{\exp}$ is not order bounded. For, otherwise, there exists $0 \leq f \in L_{\exp}$ such that $|T(\chi_A)| \leq f$ for all $A \in \mathcal{B}$. Since $L_{\exp} \subseteq L^2$, say, also $f \in L^2$, and so $\{T(\chi_A) : A \in \mathcal{B}\}$ is an order bounded subset of L^2. This is a contradiction to [COR20, Proposition 3.2(vii)], with $X = L^2$ there.

A central role is played by the bounded, continuous function

$$w(x) := \sqrt{1-x^2}, \quad x \in (-1,1). \tag{3.2}$$

PROPOSITION 3.4. *The finite Hilbert transform $T\colon L^\infty \to L_{\exp}$ is injective.*

Proof. Theorem 3.4 in [COR24-2] implies, for $f \in L\log L$, that $T(f) = 0$ if and only if $f = c/w$ for some $c \in \mathbb{C}$. Since $1/w \notin L^\infty$, it follows that $T\colon L^\infty \to L_{\exp}$ is injective. ∎

PROPOSITION 3.5. *The smallest r.i. space over $(-1,1)$ which contains $T(L^\infty)$ is $L_{\exp}$.*

Proof. Let X be a r.i. space over $(-1,1)$ such that $T(L^\infty) \subseteq X$. Then $T(\chi_{(-1,1)}) \in X$. From the definition of T it follows that $T(\chi_{(-1,1)})(x) = \frac{1}{\pi}\log\left(\frac{1-x}{1+x}\right)$ for $x \in (-1,1)$. Direct calculation also shows that

$$\left(\log\left(\frac{1-x}{1+x}\right)\right)^*(t) = \log\left(\frac{4-t}{t}\right), \qquad 0 < t < 2.$$

Consider the function $F(x) := 1 + \left|\log\left(\frac{1-x}{1+x}\right)\right|$ for $x \in (-1,1)$, which belongs to X (as the constant function $1 \in L^\infty$ belongs to X). Then

$$F^*(t) := 1 + \left(\log\left(\frac{1-x}{1+x}\right)\right)^*(t) \geq \log\left(\frac{2e}{t}\right), \qquad 0 < t < 2.$$

Let $f \in L_{\exp}$. Then there exists a positive constant M_f such that $f^*(t) \leq M_f \log(2e/t)$ for $0 < t < 2$. It follows that $f^*(t) \leq M_f F^*(t)$ for $0 < t < 2$. Since X is r.i. and $F \in X$, this implies that $f \in X$ [BS88, Theorem II.4.6]. So, $L_{\exp} \subseteq X$. ∎

REMARK 3.6. In the proof of Proposition 3.5 it was noted that $h := \pi T(\chi_{(-1,1)})$ satisfies $h^*(s) \geq \log(2e/s) - 1$ for $s \in (-1,1)$, that is,

$$\frac{\int_0^t h^*(s)\,ds}{t\log(2e/t)} \geq \frac{\int_0^t (\log(2e/s) - 1)\,ds}{t\log(2e/t)}, \qquad 0 < t < 2.$$

Direct calculation shows that $\lim_{t\to 0^+} \frac{\int_0^t (\log(2e/s)-1)\,ds}{t\log(2e/t)} = 1 \neq 0$ and so $h \notin (L_{\exp})_a$; see (2.1). Accordingly, $T(L^\infty) \not\subseteq (L_{\exp})_a$. Since $(L_{\exp})_a$ does not have the Fatou property, it is not a r.i. space and so the previous observation does not follow from Proposition 3.5.

4. The auxiliary operator $\breve{T}$ on L^∞. Let $f \in L^1$ satisfy $f/w \in L^1$. Kolmogorov's Theorem ensures that $T(f/w) \in L^0$ and so the function $\breve{T}(f) \in L^0$, given pointwise a.e. in $(-1,1)$ by

$$\breve{T}(f) := -w\,T\left(\frac{f}{w}\right), \tag{4.1}$$

is well defined. For $f \in L^1$, the function $\widehat{T}(f) \in L^0$, given pointwise a.e. in $(-1,1)$ by

$$\widehat{T}(f) := -\frac{1}{w}T(wf), \tag{4.2}$$

is also well defined.

The operator $f \mapsto \widehat{T}(f)$ arose when solving the airfoil equation (1.1) (i.e., for the inversion of T) for $T\colon L^p \to L^p$ with $1 < p < 2$, and, more generally, for $T\colon X \to X$ with X a r.i. space satisfying $1/2 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1$. The operator $f \mapsto \breve{T}(f)$ arose when determining the solution of the airfoil equation for $T\colon L^p \to L^p$ with $2 < p < \infty$ and for $T\colon X \to X$ with X a r.i. space satisfying $0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1/2$. We refer to [COR19] and the references therein.

In this section we investigate the operators $\breve{T}$ and $\widehat{T}$ acting between (possibly) different pairs of r.i. spaces X and Y, with no restriction on their Boyd indices.

It is well known that positive operators between B.f.s.'s are (automatically) continuous. The finite Hilbert transform is far from being positive. However, a related result is available.

LEMMA 4.1. *Let X, Y be B.f.s.'s over $(-1,1)$.*

(i) *If $T(X) \subseteq Y$, then $T\colon X \to Y$ is continuous.*
(ii) *If $\widehat{T}(X) \subseteq Y$, then $\widehat{T}\colon X \to Y$ is continuous.*
(iii) *If $\breve{T}(X) \subseteq Y$, then $\breve{T}\colon X \to Y$ is continuous.*

Proof. (i) Let $f_n \to 0$ in X and $T(f_n) \to g$ in Y. Since $X \subseteq L^1$ continuously, also $f_n \to 0$ in L^1. Kolmogorov's Theorem implies that $T(f_{n_j}) \to 0$ a.e. for some subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$. Also $T(f_{n_j}) \to g$ in L^1 (as $Y \subseteq L^1$ continuously) and so $T(f_{n_{j_k}}) \to g$ a.e. for some subsequence $\{f_{n_{j_k}}\}_{k=1}^{\infty}$ of $\{f_{n_j}\}_{j=1}^{\infty}$. Since $T(f_{n_{j_k}}) \to 0$ a.e., we can conclude that $g = 0$. By the Closed Graph Theorem, $T\colon X \to Y$ is continuous.

(ii) Let $f_n \to 0$ in X and $\widehat{T}(f_n) \to g$ in Y. Since $|wf_n| \leq |f_n|$ for $n \in \mathbb{N}$, also $(wf_n) \to 0$ in X. Hence, $(wf_n) \to 0$ in L^1. Kolmogorov's Theorem implies that $T(wf_{n_j}) \to 0$ a.e. for some subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$. Then also $(-1/w)T(wf_{n_j}) = \widehat{T}(f_{n_j}) \to 0$ a.e. Moreover, $\widehat{T}(f_{n_j}) \to g$ in L^1 and so $\widehat{T}(f_{n_{j_k}}) \to g$ a.e. for some subsequence $\{f_{n_{j_k}}\}_{k=1}^{\infty}$ of $\{f_{n_j}\}_{j=1}^{\infty}$. Since $(-1/w)T(wf_{n_{j_k}}) = \widehat{T}(f_{n_{j_k}}) \to 0$ a.e., we can conclude that $g = 0$. By the Closed Graph Theorem, $\widehat{T}\colon X \to Y$ is continuous.

(iii) Let $f_n \to 0$ in X and $\breve{T}(f_n) \to g$ in Y. We point out that $\breve{T}$ defined on X means that $f/w \in L^1$ for every $f \in X$ and that $\breve{T}(f) \in Y$. Accordingly, the positive linear operator $f \in X \mapsto f/w \in L^1$ is well defined, and hence it is continuous. So, $f_n/w \to 0$ in L^1. Kolmogorov's Theorem

implies that $T(f_{n_j}/w) \to 0$ a.e. for some subsequence $\{f_{n_j}\}_{j=1}^\infty$ of $\{f_n\}_{n=1}^\infty$. Then also $-wT(f_{n_j}/w) = \breve{T}(f_{n_j}) \to 0$ a.e. Moreover, $\breve{T}(f_{n_j}) \to g$ in L^1 and so $\breve{T}(f_{n_{j_k}}) \to g$ a.e. for some subsequence $\{f_{n_{j_k}}\}_{k=1}^\infty$ of $\{f_{n_j}\}_{j=1}^\infty$. Since $-wT(f_{n_{j_k}}/w) = \breve{T}(f_{n_{j_k}}) \to 0$ a.e., we can conclude that $g = 0$. By the Closed Graph Theorem, $\breve{T}\colon X \to Y$ is continuous. ∎

The following result makes the relation between $\breve{T}$ and $\widehat{T}$ precise.

THEOREM 4.2. *Let X, Y be r.i. spaces such that $X \subseteq L\log L$ and $L^{2,\infty} \subseteq Y$. Suppose that $\widehat{T}\colon X \to Y$ boundedly. Then*

$$\breve{T}\colon Y' \to X'$$

boundedly. Moreover, for all $f \in Y'$ and $g \in X$,

$$\int_{-1}^{1} \widehat{T}(g) f = -\int_{-1}^{1} g\breve{T}(f). \tag{4.3}$$

If in addition both X and Y are separable, then $(\widehat{T})^* = -\breve{T}$.

Proof. Let $f \in Y'$. Observe that $f/w \in L^1$ because $f \in Y' \subseteq (L^{2,\infty})' = L^{2,1}$ and $1/w \in L^{2,\infty}$. Hence, $\breve{T}(f) = -wT(f/w) \in L^0$. Let $g \in X$. Given $A \in \mathcal{B}$, note that $\widehat{T}(g\chi_A) \in Y$, and so Hölder's inequality yields

$$\frac{f}{w} T(wg\chi_A) = -f\widehat{T}(g\chi_A) \in L^1.$$

This allows us to apply Proposition 3.1 (with $f/w \in L^1$ in place of f and $gw \in L\log L$ in place of g) to conclude that $-g\widetilde{T}(f) = gwT(f/w) \in L^1$ and

$$\int_{-1}^{1} g\breve{T}(f) = -\int_{-1}^{1} gw\, T\left(\frac{f}{w}\right) = \int_{-1}^{1} \frac{f}{w} T(gw) = -\int_{-1}^{1} f\widehat{T}(g).$$

This verifies (4.3) and, because $f\widehat{T}(g) \in L^1$ for every $g \in X$, also shows that $\breve{T}(f) \in X'$. Hence, $\breve{T}(Y') \subseteq X'$ and so Lemma 4.1 implies that $\breve{T}\colon Y' \to X'$ is bounded.

When X and Y are both separable (equivalently, have σ-order continuous norms [BS88, Corollary I.5.6]), we have $X^* = X'$ and $Y^* = Y'$. The identity $(\widehat{T})^* = -\breve{T}$ is then clear from (4.3). ∎

COROLLARY 4.3. *The operator $\breve{T}\colon L^\alpha_{\exp} \to L^{\alpha+1}_{\exp}$ is bounded for each $\alpha \geq 0$. In particular, $\breve{T}\colon L^\infty \to L_{\exp}$ is bounded.*

Proof. We can apply Theorem 4.2 to $X = L(\log L)^{\alpha+1}$ and $Y = L(\log L)^\alpha$ since $\widehat{T}\colon L(\log L)^{\alpha+1} \to L(\log L)^\alpha$ boundedly [COR24-2, Proposition 4.5], with $X \subseteq L\log L$ and $L^{2,\infty} \subseteq Y$. ∎

Remark 4.4.

(i) Since $\widehat{T}_p := \widehat{T}\colon L^p \to L^p$ boundedly for $1 < p < 2$ [OE91, Lemma 2.3], it follows from Theorem 4.2 that $\check{T}_{p'} := \check{T}\colon L^{p'} \to L^{p'}$ satisfies $(\widehat{T}_p)^* = -\check{T}_{p'}$ with $2 < p' < \infty$; compare this with [OE91, Proposition 2.6].

(ii) For a r.i. space X with $1/2 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$, it is known that $\widehat{T}\colon X \to X$ boundedly [COR19, Theorem 3.2]. If X is also separable, then Theorem 4.2 implies that $\check{T}\colon X' \to X'$ boundedly (denoted by $\check{T}_{X'}$) and $(\widehat{T}_X)^* = -\check{T}_{X'}$ with $0 < \underline{\alpha}_{X'} \le \overline{\alpha}_{X'} < 1/2$; compare with [COR19, Theorem 3.3].

(iii) A proof of Corollary 4.3 avoiding the use of duality is also possible. This requires showing, for some constant $C > 0$ and some $2 < p_0 < \infty$, that $\|\check{T}\|_{L^p \to L^p} \le Cp$ for all $p_0 < p < \infty$, and then applying the classical result of Zygmund on extrapolation; see [Z59, Theorem XII.(4·41) on p. 119 of Volume II]. Obtaining the uniform estimate needed for $\|\check{T}\|_{L^p \to L^p}$ follows, via analogy, the steps established in the corresponding study of the operator $\widehat{T}$, in particular, by adapting [COR24-2, Lemmas 4.6, 4.7 and 6.1].

5. Inversion of the FHT on L^∞. In this section we study the action of T and $\check{T}$ on L^∞. In particular, the range $T(L^\infty)$ of T in $L_{\exp}$ is characterized, which is then used to solve the airfoil equation.

Compare the following result with [OE91, Proposition 2.6] and [COR19, Theorem 3.3].

Theorem 5.1.

(i) *The operator $T_\infty\colon L^\infty \to L_{\exp}$ is injective.*

(ii) *Let $f \in L^\infty$. Then $T(f) \in L_{\exp}$ and $\check{T}T(f) = f$.*

(iii) *The operator $Q\colon L^\infty \to L^\infty$ given by*

$$Q(f) := \left(\frac{1}{\pi}\int_{-1}^{1}\frac{f}{w}\right)\chi_{(-1,1)}, \qquad f \in L^\infty, \tag{5.1}$$

is a bounded, rank-1, projection of L^∞ onto $\operatorname{span}(\chi_{(-1,1)}) \subseteq L^\infty$ satisfying $\|Q\|_{L^\infty \to L^\infty} = 1$. Furthermore,

$$T\check{T}(f) = f - Q(f), \qquad f \in L^\infty.$$

(iv) *For each $f \in L^\infty$,*

$$\int_{-1}^{1}\frac{T(f)}{w} = 0.$$

Proof. Proposition 3.4 establishes (i). Let $f \in L^\infty$ and fix any $2 < p < \infty$, in which case $f \in L^p$. It follows from [COR19, Theorem 3.3] for $X = L^p$

that (ii), (iii) and (iv) hold. That $\|Q\|_{L^\infty \to L^\infty} = 1$ follows from (5.1) and $\int_{-1}^{1}(1/w) = \pi$. ∎

REMARK 5.2. The operator $Q_{L_{\exp}}\colon L_{\exp} \to L_{\exp}$ given by

$$Q_{L_{\exp}}(g) := \left(\frac{1}{\pi} \int_{-1}^{1} \frac{g}{w}\right)\chi_{(-1,1)}, \qquad g \in L_{\exp},$$

is also a bounded, rank-1 projection of $L_{\exp}$ onto $\operatorname{span}(\chi_{(-1,1)}) \subseteq L_{\exp}$. Indeed, since $1/w \in L\log L$, Hölder's inequality shows that $g/w \in L^1$ and

$$\left|Q_{L_{\exp}}(g)\right| \leq \frac{1}{\pi}\left\|\frac{1}{w}\right\|_{L\log L} \|g\|_{L_{\exp}} \chi_{(-1,1)}, \qquad g \in L_{\exp},$$

with $\chi_{(-1,1)} \in L^\infty \subseteq L_{\exp}$. The previous pointwise inequality on $(-1,1)$ implies that $Q_{L_{\exp}}(g) \in L^\infty \subseteq L_{\exp}$ and that $Q_{L_{\exp}}$ is bounded with operator norm

$$\left\|Q_{L_{\exp}}\right\| \leq \frac{1}{\pi}\left\|\frac{1}{w}\right\|_{L\log L} \|\chi_{(-1,1)}\|_{L_{\exp}} = \frac{1}{\pi}\left\|\frac{1}{w}\right\|_{L\log L}.$$

Clearly the range $Q_{L_{\exp}}(L_{\exp})$ equals $\operatorname{span}(\chi_{(-1,1)}) \subseteq L^\infty \subseteq L_{\exp}$.

Suppose now that a function $g_0 \in L_{\exp}$ satisfies $\breve{T}(g_0) \in L^\infty$, in which case $T\breve{T}(g_0) \in L_{\exp}$. To prove that the identity

$$T\breve{T}(g_0) = g_0 - Q_{L_{\exp}}(g_0) \tag{5.2}$$

holds in $L_{\exp}$, fix any $2 < p < \infty$, in which case $L_{\exp} \subseteq L^p$. Moreover, the identity

$$T\breve{T}(g_0) = g_0 - Q(g_0) \tag{5.3}$$

holds in L^p [OE91, Proposition 2.6], where $Q(f) := \left(\frac{1}{\pi}\int_{-1}^{1}\frac{f}{w}\right)\chi_{(-1,1)} \in L^p$ for $f \in L^p$. But both sides of (5.3) belong to $L_{\exp}$ and the proof of (5.2) is complete.

Recall that the function $1/w$ belongs to $L\log L = (L_{\exp})'$. So, it defines a continuous linear functional $\varphi_{1/w}\colon L_{\exp} \to \mathbb{C}$ by

$$\varphi_{1/w}(g) := \left\langle g, \frac{1}{w}\right\rangle = \int_{-1}^{1} \frac{g}{w}, \qquad g \in L_{\exp},$$

which satisfies $\|\varphi_{1/w}\|_{L_{\exp}^*} \leq \left\|\frac{1}{w}\right\|_{L\log L}$. Observe that

$$\operatorname{Ker}(\varphi_{1/w}) = \left\{g \in L_{\exp} : \int_{-1}^{1} \frac{g}{w} = 0\right\} = \operatorname{Ker}(Q_{L_{\exp}})$$

is a proper, closed, infinite-dimensional subspace of $L_{\exp}$. Indeed, if $g \in L_{\exp}$ is any odd function, then $\int_{-1}^{1}(g/w) = 0$ and so $g \in \operatorname{Ker}(\varphi_{1/w})$.

Theorem 5.1(iv) implies that $T(L^\infty) \subseteq \mathrm{Ker}(\varphi_{1/w})$. In particular, T_∞ is not surjective.

An important consequence of Theorem 5.1 is the following characterization of the range space of T_∞.

COROLLARY 5.3. *A function $g \in L_{\exp}$ belongs to the range space $T(L^\infty) \subseteq L_{\exp}$ if and only if it satisfies both $\breve{T}(g) \in L^\infty$ and $\varphi_{1/w}(g) = 0$. That is,*

$$T(L^\infty) = \{g \in \mathrm{Ker}(\varphi_{1/w}) : \breve{T}(g) \in L^\infty\} = \{g \in \mathrm{Ker}(Q_{L_{\exp}}) : \breve{T}(g) \in L^\infty\}.$$

Proof. Let $g \in L_{\exp}$ satisfy $\breve{T}(g) \in L^\infty$ and $\int_{-1}^{1}(g/w) = 0$, that is, $Q_{L_{\exp}}(g) = 0$. Remark 5.2 implies $T(\breve{T}(g)) = g$. Accordingly, $g \in T(L^\infty)$.

Conversely, let $g \in T(L^\infty)$. Then there exists $f \in L^\infty$ such that $g = T(f) \in L_{\exp}$. From Theorem 5.1(ii) we have $\breve{T}(g) = \breve{T}T(f) = f \in L^\infty$. Moreover, Theorem 5.1(iv) implies that

$$\int_{-1}^{1} \frac{g}{w} = \int_{-1}^{1} \frac{T(f)}{w} = 0. \ \blacksquare$$

REMARK 5.4. We have seen that $T_\infty(L^\infty) \subseteq \mathrm{Ker}(\varphi_{1/w})$. This inclusion is actually *proper*. Indeed, let $X := L\log L$, $Y := L^1$ and let $S\colon X \to Y$ denote the finite Hilbert transform $T\colon L\log L \to L^1$, briefly $T_{\log}$. Then [COR24-2, Proposition 4.12(i)] implies that S is *not* surjective. Moreover, $S^* = -T_\infty$ is injective. According to [R73, (a)$\Leftrightarrow$(c) in Theorem 4.15], with S in place of T there, it follows that $T_\infty(L^\infty)$ is *not* norm-closed in $L_{\exp}$. So $T_\infty(L^\infty)$ cannot be norm-closed in the closed subspace $\mathrm{Ker}(\varphi_{1/w})$ either.

We seek necessary conditions on a function g to satisfy the condition $\breve{T}(g) \in L^\infty$ appearing in the characterization of the range space $T(L^\infty)$ in Corollary 5.3. In this regard, the following result (cf. [COR24-1, Theorem 2.3]) is relevant. A further class of functions g is given in Corollary 5.8.

PROPOSITION 5.5. *Let g satisfy a uniform λ-Hölder condition on $(-1,1)$ for $\lambda \in (0,1]$, that is, for some constant $K_g > 0$ we have*

$$|g(s) - g(t)| \le K_g |s-t|^\lambda, \qquad s,t \in (-1,1).$$

Then, with $B(\cdot,\cdot)$ denoting the Beta function,

$$w(t)\left|\left(T\left(\frac{g}{w}\right)\right)(t)\right| \le \frac{2}{\pi} K_g\, B(1/2,\lambda), \quad t \in (-1,1).$$

In particular, $\breve{T}(g) \in L^\infty$.

The range space $T(L^\infty)$ has been identified in Corollary 5.3. The next result presents various facts aimed at a better understanding of $T(L^\infty)$. For the definition of the weak-$*$ topology $\sigma(L_{\exp}, L\log L)$ on $L_{\exp} = (L\log L)^*$ see [R73, Section 3.14], for example.

PROPOSITION 5.6. *The following assertions hold for T_∞ and $\breve{T}$:*

(i) $T(L^\infty)$ *is a proper, weak-$*$ dense linear subspace of* $\operatorname{Ker}(\varphi_{1/w})$.
(ii) $\breve{T}(L^\infty) \not\subseteq L^\infty$.
(iii) $(I_{L^\infty} - Q)(L^\infty) \not\subseteq T(L^\infty) \not\subseteq L^\infty$, *with Q as defined in* (5.1).
(iv) $L^\infty \not\subseteq \breve{T}(L^\infty)$.
(v) $L^\infty \not\subseteq \breve{T}(L_{\exp}) \not\subseteq L_{\exp}$.
(vi) $L_{\exp} \not\subseteq \breve{T}(L_{\exp})$.

Proof. (i) It was shown in Remark 5.4 that $T(L^\infty) \subsetneq \operatorname{Ker}(\varphi_{1/w})$. Recall that, if X and Y are Banach spaces and $S\colon X \to Y$ is a bounded linear operator with adjoint operator $S^*\colon Y^* \to X^*$, then the weak-$*$ closure of $S^*(Y^*)$ equals $\operatorname{Ker}(S)^\perp$ [R73, Theorems 4.7 and 4.12]. We apply this when S is the operator $T_{\log}$, in which case $(T_{\log})^* = -T_\infty$ with $T_\infty\colon L^\infty \to L_{\exp}$. Accordingly, the weak-$*$ closure of $T_\infty(L^\infty)$ is equal to $\operatorname{Ker}(T_{\log})^\perp$. Since $\operatorname{Ker}(T_{\log}) = \operatorname{span}(1/w)$ [COR24-2, Theorem 3.4], it follows that the weak-$*$ closure of $T_\infty(L^\infty)$ equals $\operatorname{Ker}(\varphi_{1/w})$.

(ii) Direct computation shows, for $x \in (-1,1)$, that

$$\breve{T}(\chi_{[0,1)})(x) = -w(x)T\left(\frac{\chi_{[0,1)}}{w}\right)(x) = \frac{-1}{\pi}\log\left|\frac{(1-x)^{1/2}(1+x)^{-1/2}+1}{(1-x)^{1/2}(1+x)^{-1/2}-1}\right|;$$

see [OE91, proof of Lemma 4.3]. This function is plainly unbounded on $(-1,1)$ whereas $\chi_{[0,1)} \in L^\infty$.

(iii) That $T(L^\infty) \not\subseteq L^\infty$ is clear since $T(\chi_{(-1,1)})(x) = \frac{1}{\pi}\log\left(\frac{1-x}{1+x}\right)$ for $x \in (-1,1)$.

Assume, by way of contradiction, that $(I_{L^\infty} - Q)(L^\infty) \subseteq T(L^\infty)$. We show that this implies $\breve{T}(L^\infty) \subseteq L^\infty$. So, fix $g \in L^\infty$. Then $(I_{L^\infty} - Q)(g) = T(f)$ for some $f \in L^\infty$ and so $\breve{T}(g) = \breve{T}(I_{L^\infty} - Q)(g) + \breve{T}(Q(g))$. But $T(1/w) = 0$ implies that

$$\breve{T}(Q(g)) = \left(\frac{-1}{\pi}\int_{-1}^{1}\frac{g}{w}\right)wT\left(\frac{1}{w}\right) = 0.$$

Hence, $\breve{T}(g) = \breve{T}(I_{L^\infty} - Q)(g) = \breve{T}T(f)$, which, by Theorem 5.1(ii), equals $f \in L^\infty$. That is, $\breve{T}(L^\infty) \subseteq L^\infty$, which contradicts (ii).

(iv) Assume, by way of contradiction, that $L^\infty \subseteq \breve{T}(L^\infty)$. Then $T(L^\infty) \subseteq T\breve{T}(L^\infty) = (I_{L^\infty} - Q)(L^\infty) \subseteq L^\infty$ by Theorem 5.1(iii). Hence, $T(L^\infty) \subseteq L^\infty$, which is not true by part (iii) of the current proposition.

(v) Assuming the contrary, that is, $\breve{T}(L_{\exp}) \subseteq L_{\exp}$, we will arrive at the containment $\widehat{T}(L\log L) \subseteq L\log L$, which does not hold, as shown in [COR24-2, Proposition 4.12(iv)].

Let $f \in L\log L$ and select any $g \in L_{\exp}$. Since $1/w \in L^{2,\infty} \subseteq L^{3/2}$ and $g \in L_{\exp} \subseteq L^6$, it follows that $g/w \in L^{6/5} \subseteq L\log L$, as $1/(3/2) + 1/6 = 1/(6/5)$. By assumption, for each $A \in \mathcal{B}$, the function $\check{T}(g\chi_A)$ belongs to $L_{\exp}$ and so Hölder's inequality yields

$$(wf)T\left(\frac{g}{w}\chi_A\right) = -f\check{T}(g\chi_A) \in L^1.$$

Since $A \in \mathcal{B}$ is arbitrary with $wf \in L^1$ and $g/w \in L\log L$, Proposition 3.1 implies that $(g/w)T(wf) \in L^1$, and hence that

$$g\widehat{T}(f) = -\frac{g}{w}T(wf) \in L^1.$$

Accordingly, $g\widehat{T}(f) \in L^1$ for every $g \in L_{\exp}$, and so $\widehat{T}(f) \in (L_{\exp})' = L\log L$. Hence, $\widehat{T}(L\log L) \subseteq L\log L$, which was noted to be false.

(vi) Assume that $L_{\exp} \subseteq \check{T}(L_{\exp})$, in which case $T(L_{\exp}) \subseteq T\check{T}(L_{\exp})$. Fix $2 < p < \infty$. As indicated in (5.3), the identity $T\check{T}(h) = h - Q(h)$ holds in L^p with each of the individual terms $h, Q(h)$ and $T\check{T}(h)$ belonging to L^p. If $g \in L_{\exp}$ then $g \in L^p$ and so $T\check{T}(g) = g - Q(g)$. From the definition of Q it is clear that $Q(g) \in L^\infty \subseteq L_{\exp}$, and so $g - Q(g) \in L_{\exp}$. Hence, $T\check{T}(g) \in L_{\exp}$. It follows that $T(L_{\exp}) \subseteq L_{\exp}$, that is, $T\colon L_{\exp} \to L_{\exp}$ boundedly (cf. Lemma 4.1). This is impossible as $L_{\exp}$ has trivial Boyd indices. ∎

Recall from (1.1) that the airfoil equation is $T(f) = g$, that is,

$$\frac{1}{\pi}\,\text{p.v.}\int_{-1}^{1}\frac{f(x)}{x-t}\,dx = g(t), \qquad \text{a.e. } t \in (-1,1).$$

Given g, an *inversion formula* is needed to solve this equation for f. Theorem 5.1 and Corollary 5.3 allow us to obtain such an inversion formula for $T_\infty\colon L^\infty \to L_{\exp}$.

THEOREM 5.7. *Let g belong to the range of $T_\infty\colon L^\infty \to L_{\exp}$. The unique solution $f \in L^\infty$ of the airfoil equation* (1.1) *is*

$$f = \check{T}(g) = -wT\left(\frac{g}{w}\right).$$

Proof. Define $f := \check{T}(g)$ and note that $f \in L^\infty$ by Corollary 5.3. Remark 5.2 implies that $T(f) = T(\check{T}(g)) = g - Q_{L_{\exp}}(g)$. But $g = T_\infty(h)$ for some $h \in L^\infty$ and so Theorem 5.1(iv) implies that

$$Q_{L_{\exp}}(g) = \left(\frac{1}{\pi}\int_{-1}^{1}\frac{T(h)}{w}\right)\chi_{(-1,1)} = 0.$$

Hence, $T(f) = g$. That is, $f = \check{T}(g)$ is the unique solution (cf. Proposition 3.4) of the airfoil equation. ∎

According to Proposition 5.6(i) we can interpret the operator T_∞ as taking its values in the proper, closed subspace $\mathrm{Ker}(\varphi_{1/w})$ of $L_{\exp}$. Theorem 5.7 implies that $T\colon L^\infty \to \mathrm{Ker}(\varphi_{1/w})$ is a vector space isomorphism onto its range $T(L^\infty) \subsetneq \mathrm{Ker}(\varphi_{1/w})$ whose inverse map is the vector space isomorphism $\breve{T}\colon T(L^\infty) \to L^\infty$.

COROLLARY 5.8. *The linear map $\breve{T}\colon T(L^\infty) \to L^\infty$ is a closed operator defined on the proper linear subspace $T(L^\infty)$ of* $\mathrm{Ker}(\varphi_{1/w})$.

Proof. Consider any sequence $(g_n)_{n=1}^\infty \subseteq T(L^\infty)$ satisfying $g_n \to g$ in $\mathrm{Ker}(\varphi_{1/w})$ and $\breve{T}(g_n) \to f$ in L^∞. Let $(h_n)_{n=1}^\infty \subseteq L^\infty$ satisfy $g_n = T(h_n)$ for each $n \in \mathbb{N}$, in which case $T(h_n) \to g$ in $\mathrm{Ker}(\varphi_{1/w})$. On the other hand, $g_n = T(h_n)$ implies, via Theorem 5.7, that $h_n = \breve{T}(g_n)$ for each $n \in \mathbb{N}$, and so $h_n \to f$ in L^∞. By the continuity of $T_\infty\colon L^\infty \to \mathrm{Ker}(\varphi_{1/w})$ it follows that $T(h_n) \to T(f)$ in $\mathrm{Ker}(\varphi_{1/w})$, and hence that $g = T(f)$ with $f \in L^\infty$. That is, $g \in T(L^\infty)$. By the uniqueness statement in Theorem 5.7 we can conclude that $f = \breve{T}(g)$. Accordingly, $\breve{T}\colon T(L^\infty) \to L^\infty$ is a closed linear operator defined in $\mathrm{Ker}(\varphi_{1/w})$. ∎

REMARK 5.9. Observe that $\mathrm{Ker}(\varphi_{1/w})$ is a *proper*, closed subspace of $L_{\exp}$ which contains $T(L^\infty)$. This does not contradict Proposition 3.5 as $\mathrm{Ker}(\varphi_{1/w})$ is not an order ideal in L^0, and hence is not a B.f.s. Indeed, $f = \chi_{[0,1)} - \chi_{(-1,0)}$ is an odd function and so $f \in \mathrm{Ker}(\varphi_{1/w})$. However, $|f| = \chi_{(-1,1)}$ satisfies $\varphi_{1/w}(|f|) = \pi$ and so $|f| \notin \mathrm{Ker}(\varphi_{1/w})$.

6. The optimal domain for T taking values in $L_{\exp}$. In this section we prove that the operator $T_\infty\colon L^\infty \to L_{\exp}$ is optimally defined. The approach used for this is first to explicitly determine the *optimal domain* of T with values in $L_{\exp}$, denoted by $[T, L_{\exp}]$, and then to show that it is isomorphic to L^∞. This is the same approach that was used to show that $T\colon X \to X$ is optimally defined whenever X is a r.i. space with non-trivial Boyd indices (cf. [COR21-1, §3]) and also used to show that $T\colon L\log L \to L^1$ is optimally defined [COR24-2, §5]. This strategy required an in-depth study of the optimal domain (cf. [COR19, Theorem 5.3], [COR21-1, Theorem]) that we will also use here.

Define the space of functions

$$[T, L_{\exp}] := \{f \in L^1 : T(h) \in L_{\exp} \text{ for all } |h| \leq |f|\}, \tag{6.1}$$

referred to as the *optimal domain* of T with values in $L_{\exp}$. Proposition 3.4 implies that the inclusion $L^\infty \subseteq [T, L_{\exp}]$ is valid. It will be shown that the opposite containment $[T, L_{\exp}] \subseteq L^\infty$ also holds.

We begin by investigating the structure and exhibiting various properties of $[T, L_{\exp}]$.

PROPOSITION 6.1.

(i) *Let $f \in L^1$. The following conditions are equivalent:*

(a) $f \in [T, L_{\exp}]$.

(b) $\sup_{|h|\le|f|} \|T(h)\|_{L_{\exp}} < \infty$.

(c) $T(f\chi_A) \in L_{\exp}$ *for every* $A \in \mathcal{B}$.

(d) $\sup_{A\in\mathcal{B}} \|T(f\chi_A)\|_{L_{\exp}} < \infty$.

(e) $T(\theta f) \in L_{\exp}$ *for every* $\theta \in L^\infty$ *with* $|\theta| = 1$ *a.e.*

(f) $\sup_{|\theta|=1} \|T(\theta f)\|_{L_{\exp}} < \infty$.

(g) $fT(g) \in L^1$ *for every* $g \in L\log L$.

(ii) *Every function* $f \in [T, L_{\exp}]$ *satisfies the inequalities*

$$\sup_{A\in\mathcal{B}} \|T(\chi_A f)\|_{L_{\exp}} \le \sup_{|\theta|=1} \|T(\theta f)\|_{L_{\exp}} \le \sup_{|h|\le|f|} \|T(h)\|_{L_{\exp}} \le 4 \sup_{A\in\mathcal{B}} \|T(\chi_A f)\|_{L_{\exp}}.$$

Proof. The proof of the equivalences (a)⇔(b)⇔(c)⇔(d)⇔(e)⇔(f) in part (i) follows by adapting the proof of [COR19, Proposition 4.1] to the case when $X = L_{\exp}$, after observing that the non-triviality of the Boyd indices of X is not used in these proofs, and by using Proposition 3.1 above in place of [COR19, Proposition 3.1].

Applying the special case of Proposition 3.1 (i.e., when $f \in L^\infty$ and $g \in L\log L$) we can argue as in [COR19, proof of (d)⇒(g) in Proposition 4.1] to conclude that (b)⇒(g).

Finally, suppose that (g) is satisfied. Fix any $A \in \mathcal{B}$. Then $(f\chi_A)T(g) \in L^1$ for every $g \in L\log L$ (by assumption). Apply Proposition 3.1 to $f\chi_A$ in place of f to obtain $gT(f\chi_A) \in L^1$ for all $g \in L\log L$. Accordingly, $T(f\chi_A) \in (L\log L)' = L_{\exp}$, which establishes (c).

The inequalities in part (ii) can be proved exactly as in [COR19, proof of the "Moreover" part of Proposition 4.1]. ∎

PROPOSITION 6.2. *The space* $[T, L_{\exp}]$ *is a linear lattice for the a.e. pointwise order and, when equipped with the norm*

$$\|f\|_{[T,L_{\exp}]} := \sup\{\|T(h)\|_{L_{\exp}} : |h| \le |f|\}, \qquad f \in [T, L_{\exp}], \tag{6.2}$$

it is a B.f.s. over $(-1, 1)$.

Moreover, $[T, L_{\exp}]$ *is the largest B.f.s., within* L^0*, to which* $T_\infty\colon L^\infty \to L_{\exp}$ *admits an* $L_{\exp}$*-valued, continuous linear extension.*

Proof. Proposition 6.1 shows that $\|\cdot\|_{[T,L_{\exp}]}$ is finite on $[T, L_{\exp}]$. That $[T, L_{\exp}]$ is a vector space and $\|\cdot\|_{[T,L_{\exp}]}$ is a lattice seminorm follows the argument given in [COR19, p. 1850]. To show that $\|\cdot\|_{[T,L_{\exp}]}$ is actually a norm, let $f \in [T, L_{\exp}]$ satisfy $\|f\|_{[T,L_{\exp}]} = 0$. Suppose that $f \ne 0$. Choose

$A \in \mathcal{B}$ with $\mu(A) > 0$ such that $f\chi_A \in L^\infty$ and $f(x) \neq 0$ for all $x \in A$. Since $|f\chi_A| \leq |f|$, it follows from (6.2) that $\|T(f\chi_A)\|_{L_{\exp}} = 0$ and so $T(f\chi_A) = 0$. The injectivity of $T_\infty \colon L^\infty \to L_{\exp}$ implies that $f\chi_A = 0$ a.e. This contradiction establishes the claim. Completeness of $[T, L_{\exp}]$ follows the arguments given in [COR19, proof of Lemma 4.4]. The proof that $[T, L_{\exp}]$ has the Fatou property can be deduced from the arguments used to establish [COR19, Proposition 4.5].

Establishing the optimality of $[T, L_{\exp}]$ follows [COR24-1, proof of Theorem 4.6] as explicitly done in [COR20, proof of Proposition 2.7].

We point out that the proofs from [COR19] indicated above correspond to the case of $[T, X]$ when X is a r.i. space with non-trivial Boyd indices. However, they remain valid in our current setting since the non-triviality of the Boyd indices is not actually used in those proofs from [COR19]. ∎

The following remarkable inequality, namely [COR24-1, Theorem 3.11], will be needed below.

THEOREM 6.3. *Let $A \in \mathcal{B}$ satisfy $\mu(A) > 0$. Then*

$$\|T(\chi_A)\|_{L_{\exp}} > \frac{1}{\pi e^2}.$$

PROPOSITION 6.4. *The inclusion $[T, L_{\exp}] \subseteq L^\infty$ is valid.*

Proof. Let $f \in [T, L_{\exp}]$. For each $n \in \mathbb{N}$, set $A_n := \{x \in (-1, 1) : n \leq |f(x)| < n + 1\}$. Note that $n\chi_{A_n} \leq |f|$ a.e. Let $g \in L \log L$. By applying Proposition 3.1 to $n\chi_{A_n} \in L^\infty$ and $g \in L \log L$ we can deduce that both $gT(n\chi_{A_n})$ and $n\chi_{A_n}T(g)$ are integrable and satisfy

$$\int_{-1}^{1} gT(n\chi_{A_n}) = -\int_{-1}^{1} n\chi_{A_n}T(g), \qquad n \in \mathbb{N}.$$

Thus, using the equivalence (a)⇔(g) in Proposition 6.1 and the inequality $n\chi_{A_n} \leq |f|$ yields

$$\Big|\int_{-1}^{1} gT(n\chi_{A_n})\Big| \leq \int_{-1}^{1} |fT(g)| < \infty, \qquad n \in \mathbb{N}.$$

Since the previous inequality holds for arbitrary $g \in L \log L$, it follows that the set $\{nT(\chi_{A_n}) : n \in \mathbb{N}\}$ is weak-$*$ bounded in $L_{\exp}$ and hence, by the uniform boundedness principle, it is norm bounded in $L_{\exp} = (L \log L)^*$. That is, for some constant $C > 0$, we have

$$\|T(\chi_{A_n})\|_{L_{\exp}} \leq \frac{C}{n}, \qquad n \in \mathbb{N},$$

which implies $\|T(\chi_{A_n})\|_{L_{\exp}} \to 0$ for $n \to \infty$. But Theorem 6.3 implies

$$\|T(\chi_A)\|_{L_{\exp}} > \frac{1}{\pi e^2},$$

for every $A \in \mathcal{B}$ with $\mu(A) > 0$. Consequently, there exists $n_0 \in \mathbb{N}$ such that $\mu(A_n) = 0$ for all $n \geq n_0$, that is, $|f(x)| \leq n_0$ for a.e. $x \in (-1,1)$. So, $f \in L^\infty$. ∎

As announced, we now answer the question regarding the optimal extension of $T_\infty \colon L^\infty \to L_{\exp}$.

THEOREM 6.5. *The identity $[T, L_{\exp}] = L^\infty$ holds as an order and bi-continuous isomorphism between B.f.s.'s. Hence, $T_\infty \colon L^\infty \to L_{\exp}$ does not admit a continuous, linear, $L_{\exp}$-valued extension to any strictly larger B.f.s. within L^0.*

Proof. Propositions 6.2 and 6.4 show that L^∞ and $[T, L_{\exp}]$ are equal as vector spaces.

To prove that L^∞ is continuously included in $[T, L_{\exp}]$, fix $f \in L^\infty$. For each $h \in L^0$ satisfying $|h| \leq |f|$ observe that $h \in L^\infty$ with $\|h\|_{L^\infty} \leq \|f\|_{L^\infty}$. Proposition 3.2 yields $\|T(h)\|_{L_{\exp}} \leq \|T\| \cdot \|h\|_{L^\infty} \leq \|T\| \cdot \|f\|_\infty$ and so we can conclude from (6.2) that

$$\|f\|_{[T,L_{\exp}]} \leq \|T\| \cdot \|f\|_{L^\infty}, \qquad f \in L^\infty,$$

where $\|T\|$ denotes the operator norm $\|T\|_{L^\infty \to L_{\exp}}$. That is, the natural inclusion $L^\infty \subseteq [T, L_{\exp}]$ is continuous.

Since both L^∞ and $[T, L_{\exp}]$ are B.f.s.'s (cf. Proposition 6.2), it follows from the Open Mapping Theorem that the natural inclusion $L^\infty \subseteq [T, L_{\exp}]$ is an isomorphism.

Proposition 6.2 implies that $T_\infty \colon L^\infty \to L_{\exp}$ does not admit a continuous, linear, $L_{\exp}$-valued extension to any strictly larger B.f.s. within L^0. ∎

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