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Duality theory for bounded lattices: A comparative study

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Contents

Abstract

There are numerous generalizations of the celebrated Priestley duality for bounded distributive lattices to the nondistributive setting. The resulting dualities rely on an earlier foundational work of such authors as Nachbin, Birkhoff–Frink, Bruns–Lakser, Hofmann–Mislove–Stralka, and others. We undertake a detailed comparative study of the existing dualities for arbitrary bounded (nondistributive) lattices, including supplying the dual description of bounded lattice homomorphisms where it was lacking. This is achieved by working with relations instead of functions. As a result, we arrive at a landscape of categories that provide various generalizations of the category of Priestley spaces. We provide explicit descriptions of the functors yielding equivalences of these categories, together with explicit dual equivalences with the category of bounded lattices and bounded lattice homomorphisms.

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1. Introduction

There is a well-developed duality theory for distributive lattices. It was first established by Stone [Sto37] as a generalization of his celebrated duality for boolean algebras [Sto36]. The resulting spaces are known as spectral spaces, which also arise as prime spectra of commutative rings [Hoc69] (see also [DST19]). An alternative duality was established by Priestley [Pri70, Pri72] in the language of ordered Stone spaces, which became known as Priestley spaces. Cornish [Cor75] showed that the two approaches are two sides of the same coin by establishing an isomorphism between the categories Spec of spectral spaces and Pries of Priestley spaces (see Chapter 2 for details).

The above approaches heavily depend on the Prime Filter Lemma (that distinct elements of a distributive lattice can be separated by a prime filter), which is no longer available for arbitrary lattices. Over the years, various approaches have been developed to generalize Stone and Priestley dualities to arbitrary (nondistributive) lattices. The first such approach was undertaken by Urquhart [Urq78], followed by Hartung [Har92], Ploščica [Plo95], Dunn and Hartonas [HD97], Hartonas [Har97], Gehrke and van Gool [GvG14], and Jipsen and Moshier [MJ14]. More recent approaches include [Har18, CG20, BD$^+$24, Har25].

While some of these authors pointed out connections with earlier works, a comprehensive comparison of all these dualities is still lacking, and it is the goal of this article to fill in this gap. In addition, we provide an axiomatization of the category arising from the Dunn–Hartonas approach, and describe the morphisms dual to arbitrary lattice homomorphisms that was missing in the approaches of Urquhart, Hartung, Ploščica, and Gehrke–van Gool. This is done by working with relations instead of functions between the appropriate spaces, a technique that originates in modal logic and has been used to describe duals of morphisms of distributive and implicative semilattices (see [Cel03b, Cel03a, BJ11, BJ13]). In the nondistributive case it was used by Celani and González [CG20].

An important underlying theme in these considerations is Nachbin's well-known correspondence between semilattices and algebraic lattices [Nac49] (see also [BF48]), and the resulting Pontryagin-style duality for semilattices [HMS74]. This can be achieved by working either with join-semilattices and the ideal functor or with meet-semilattices and the filter functor. Since most of the above approaches follow the latter, so will we.

Every algebraic lattice is a continuous lattice, and hence carries the Scott and Lawson topologies (see, e.g., [GH$^+$03]), which are central to some of the dualities mentioned above. These topologies can be defined in the more general setting of posets, giving rise to a more general approach to various Stone-like dualities. We refer to [GH+03] as well as to [Ern91, Ern93, Ern04] and the references therein. For our purposes, working with the Scott topology yields the Jipsen–Moshier duality for semilattices, and working with the Lawson topology the BDGM duality [BD+24]. If the semilattice under consideration is a lattice, then the corresponding algebraic lattice is arithmetic (binary meets of compact elements are compact). While it is not necessary, it is convenient (and common) to restrict one's attention to bounded lattices. This results in further restriction that the top element of an arithmetic lattice is compact (equivalently, finite meets of compact elements are compact). Borrowing the terminology from pointfree topology [Joh82], we call such complete lattices coherent. Then Jipsen–Moshier duality for bounded lattices is established by working with the Scott topology (see Chapter 3), while the BDGM duality by working with the Lawson topology on coherent lattices (see Chapter 4). The equivalence of the two approaches is then a simple matter of appropriate restriction of the Cornish isomorphism between Spec and Pries.

It is interesting to note that the spaces that Hartonas works with in [Har97] are nothing other than BDGM-spaces in disguise. Thus, we derive Hartonas duality from the BDGM duality (see Chapter 5).

While in nondistributive lattices we do not have sufficient supply of prime filters, we do have sufficient supply of the filters that are meet-irreducible elements of the lattice of filters. This allows to represent arbitrary (semi)lattices as subbasic closed sets of the topology on the set of meet-irreducible filters, an approach undertaken in [CG20] (see also [Har25]). This approach is outlined in Chapter 6, where we discuss how to go back and forth between the Jipsen–Moshier and Celani–González spaces.

Dunn–Hartonas work not only with the filters $\mathsf{Filt}(A)$, but also with the ideals $\mathsf{Idl}(A)$ of a lattice A, and the relation $R \subseteq \mathsf{Filt}(A) \times \mathsf{Idl}(A)$ given by $F\,R\,I$ iff $F \cap I \neq \varnothing$. They call such triples canonical L-frames and develop their duality in the language of these triples. We provide an axiomatization of such objects, and show that the resulting category is equivalent to the category of Hartonas spaces. This yields an equivalence between the Jipsen–Moshier, Bezhanishvili et al., Celani–González, Hartonas, and Dunn–Hartonas approaches (see Chapter 7).

The remaining approaches are more faithful to the Priestley approach to distributive lattices in that these authors work with not all filters and ideals, but smaller collections that are big enough to separate distinct elements of the lattice. As a consequence, axiomatizations of the corresponding spaces are more involved. We show that the Gehrke–van Gool spaces can be obtained from Dunn–Hartonas spaces (X, R, Y) by selecting appropriate closed subspaces of X and Y and restricting the relation R to those (see Chapter 8). This is done by working with distributive joins and meets, an idea that goes back to MacNeille [Mac37] (see also [BL70, HK71]), which allows us to work with an appropriate weakening of the notion of prime element. Distributive joins and meets are also known as exact [Bal84] (see also [BPP14, BPW16]). Hartung spaces can then be obtained by further restricting Gehrke–van Gool spaces to subsets X_0 of X and Y_0 of Y that are maximal in certain sense, and taking $R_0 = R \cap (X_0 \times Y_0)$ (see Chapter 9). We point out that X_0 and Y_0 are no longer closed subsets of X and Y. Because of this, instead of the subspace topology of the original topology, we work with the subspace topology of the open downset topology, which is not as well behaved (see [GvG14, pp. 447–448]).

Urquhart spaces can be obtained from Hartung spaces (X_0, R_0, Y_0) by working with the subset Z of $X_0 \times Y_0$ consisting of pairs (x, y) such that x is y-maximal and y is x-maximal, which is equipped with two natural orders $\leq_1$ and $\leq_2$ (see Chapter 10). Ploščica spaces can be thought of as an alternative presentation of Urquhart spaces, where the two quasi-orders $\leq_1$ and $\leq_2$ are replaced by a single reflexive relation R (see Chapter 11). Finally, we show that each Urquhart space $(Z, \leq_1, \leq_2)$ gives rise to the Dunn–Hartonas space (X, R, Y), thus completing the circle of correspondences (see Chapter 12).

We point out that the approaches of Urquhart, Hartung, Ploščica, and Gehrke–van Gool did not have morphisms fully developed. For example, Urquhart and Hartung developed duals of onto bounded lattice homomorphisms, while Gehrke–van Gool did the same for the more general class of admissible homomorphisms (see [GvG14, Def. 3.19]). These duals are described by appropriate pairs of functions. This approach does not generalize to arbitrary bounded lattice homomorphisms, which could be explained by the fact that restrictions of a Dunn–Hartonas morphism $(f, g)\colon (X, R, Y) \to (X', R', Y')$ to various subsets of X, X' and Y, Y' that the other authors work with may no longer be well-defined functions. We overcome this by working with pairs of relations between various such subsets, instead of pairs of functions, thus yielding a series of functors from the appropriate categories which we show are equivalences (see Chapter 13). As a result, we arrive at the diagram of equivalences in Figure 1, which captures in a nutshell how these various dualities for lattices relate to each other. In the diagram, the double arrows represent isomorphism and the single arrows equivalence of categories. In Table 1 we describe the main categories at play.

$$\begin{tikzcd}
\mathsf{CohLat}_\sigma \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathsf{JM} \arrow[r] \arrow[d, leftrightarrow] & \mathsf{CG} \\
\mathsf{CohLat}_\lambda \arrow[r, leftrightarrow] \arrow[d] & \mathsf{BDGM} \arrow[r, leftrightarrow] & \mathsf{Hs} \\
\mathsf{DH} \arrow[d] & \mathsf{Urq} \arrow[l] \arrow[r, leftrightarrow] & \mathsf{Plo} \\
\mathsf{GvG} \arrow[r] & \mathsf{Hg} \arrow[u] &
\end{tikzcd}$$

Fig. 1. Diagram of the equivalences

Letting Lat be the category of bounded lattices and bounded lattice homomorphisms, we show that composing each of the above equivalences with the corresponding dual equivalence with Lat yields a diagram that commutes up to natural isomorphism (see

Chapter 14). We conclude the paper by demonstrating how each of these dualities restricts to the distributive case. As expected, the dualities of Celani–González, Gehrke–van Gool, Hartung, Urquhart, and Ploščica all collapse to Priestley duality. We also indicate how each of the above dualities works on a nondistributive lattice, thus showcasing the similarities and differences between all these different approaches (see Chapter 15).

Table 1. Table of the categories

Category	Objects	Location
CohLat_σ	Coherent lattices with Scott topology	Def. 2.17(1)
CohLat_λ	Coherent lattices with Lawson topology	Def. 2.17(2)
JM	Jipsen–Moshier spaces for lattices	Def. 3.10
BDGM	BDGM-spaces	Def. 4.8
Hs	Hartonas spaces for lattices	Def. 5.10
CG	Celani–González spaces	Def. 6.3
DH	Dunn–Hartonas spaces	Def. 7.7
GvG	Gehrke–van Gool spaces	Def. 8.14
Hg	Hartung spaces	Def. 9.11
Urq	Urquhart spaces	Def. 10.9
Plo	Ploščica spaces	Def. 11.2

2. Preliminaries

In this preliminary chapter we recall the Cornish isomorphism [Cor75] between Spec and Pries, the Nachbin theorem [Nac49] connecting semilattices with algebraic lattices, and the resulting Pontryagin-style duality between semilattices and algebraic lattices [HMS74]. We also specialize this to an equivalence between bounded lattices and what we term coherent lattices (the term that we borrow from pointfree topology [Joh82, p. 65]).

For a topological space X, let $\mathcal{O}(X)$ be the frame of opens of X. We recall [Joh82, Sec. II.3.4] that X is *coherent* if the compact opens form a bounded sublattice of $\mathcal{O}(X)$ that is a basis for X. Observe that each coherent space is compact. A nonempty closed subset C of X is *irreducible* if it cannot be written as the union of two proper closed subsets. The space X is *sober* if each closed irreducible subset is the closure of a unique point. Observe that each sober space is T_0. If X is coherent and sober, then X is called a *spectral space* [Hoc69, p. 43] (see also [DST19, Def. 1.1.5] or [Joh82, p. 65]; in the latter these spaces are called coherent). A map $f\colon X \to Y$ between spectral spaces is a *spectral map* provided the pullback of a compact open is compact open. Note that each spectral map is continuous. Let Spec be the category of spectral spaces and spectral maps.

A compact ordered space X is a *Priestley space* if it satisfies the Priestley separation axiom: If $x \not\leq y$, then there is a clopen upset U containing x and missing y. Let Pries be the category of Priestley spaces and continuous order-preserving maps.

THEOREM 2.1 (Cornish's isomorphism [Cor75]). Spec *is isomorphic to* Pries.

REMARK 2.2. We recall the functors establishing this isomorphism. If (X, τ) is a spectral space, then $(X, \pi, \leq)$ is a Priestley space, where π is the patch topology and $\leq$ is the specialization order of τ. Conversely, if $(X, \pi, \leq)$ is a Priestley space, then (X, τ) is a spectral space, where τ is the topology of open upsets. In addition, a map between spectral spaces is spectral iff it is continuous and order-preserving with respect to the corresponding Priestley spaces.

We next recall Nachbin's well-known result [Nac49] that algebraic frames are exactly the ideal lattices of join-semilattices. By a *join-semilattice* we mean a poset in which all finite joins exist. In particular, each join-semilattice has a least element 0.

An element k of a complete lattice L is *compact* if $k \leq \bigvee S$ implies $k \leq \bigvee T$ for some finite $T \subseteq S$. Let $K(L)$ be the set of compact elements of L. Then it is easy to see that $K(L)$ is a join-semilattice, where the order on $K(L)$ is the restriction of the order on L. The following definitions are well known (see, e.g., [GH$^+$03, Def. 1-4.2], [Joh82, p. 252]).

Definition 2.3. We call a complete lattice L

(1) a *compact lattice* if $1 \in K(L)$;
(2) an *algebraic lattice* if $K(L)$ join-generates L;
(3) an *arithmetic lattice* if L is algebraic and $K(L)$ is closed under binary meets;
(4) a *coherent lattice* if L is a compact arithmetic lattice.

Remark 2.4. It is immediate from the definition that an algebraic lattice L is coherent iff $K(L)$ is a bounded sublattice of L.

For a join-semilattice A, let $\mathsf{Idl}(A)$ be the lattice of ideals of A.

Theorem 2.5 (Nachbin's theorem [Nac49]). *Let L be a complete lattice.*

(1) *L is algebraic iff $L \cong \mathsf{Idl}(A)$ for a join-semilattice A.*
(2) *L is arithmetic iff $L \cong \mathsf{Idl}(A)$ for a lattice A.*
(3) *L is coherent iff $L \cong \mathsf{Idl}(A)$ for a bounded lattice A.*

Nachbin's theorem has an obvious reformulation for the filter lattices of meet-semilattices (by which we mean posets in which all finite meets exist). Following the standard practice in duality theory for lattices, we will have a blanket assumption that all lattices are bounded. We will thus concentrate on the following two results, where $\mathsf{Filt}(A)$ is the lattice of filters of a meet-semilattice A:

- L is an algebraic lattice iff $L \cong \mathsf{Filt}(A)$ for a meet-semilattice A;
- L is a coherent lattice iff $L \cong \mathsf{Filt}(A)$ for a bounded lattice A.

There are several natural topologies on $\mathsf{Filt}(A)$. To recall them, we introduce the following standard notation:

Notation 2.6. Let X be a poset.

(1) For $S \subseteq X$, we write ${\uparrow}S$ for the upset and ${\downarrow}S$ for the downset generated by S. Then S is an upset if $S = {\uparrow}S$ and a downset if $S = {\downarrow}S$.
(2) For $S, T \subseteq X$ we write $-S$ for set-theoretic complement and $S - T$ for set difference.

Definition 2.7 ([GH$^+$03, Def. II-1.3]). Let L be an algebraic lattice. An upset U of L is *Scott open* if $\bigvee S \in U$ implies $S \cap U \neq \varnothing$ for each directed $S \subseteq L$.

The Scott open subsets of L form a topology, known as the *Scott topology* and denoted by $\sigma(L)$. In what follows we will freely use the following well-known facts about the Scott topology (see [GH$^+$03, Secs. II-1, II-2]).

THEOREM 2.8. *Let L be an algebraic lattice.*

(1) *The specialization order of the Scott topology on L is the existing order of L.*
(2) *Let $a \in L$. Then $\uparrow a$ is Scott open iff $a \in K(L)$.*
(3) *$\{\uparrow k : k \in K(L)\}$ is a basis for $\sigma(L)$.*
(4) *The Scott topology on L is a spectral topology.*
(5) *Let $f \colon L \to M$ be a map between algebraic lattices. Then f is Scott-continuous iff f preserves directed joins.*

We next recall the Lawson topology (see [GH$^+$03, Sec. III-1]).

DEFINITION 2.9. Let L be an algebraic lattice.

(1) The *lower topology* $\omega(L)$ on L is generated by the subbasis $\{-\uparrow x : x \in L\}$.
(2) The *Lawson topology* $\lambda(L)$ is the join of the Scott and lower topologies.

We will freely use the following well-known facts about the Lawson topology (see [GH$^+$03, Secs. III-1, V-5]). A *meet-semilattice homomorphism* between two meet-semilattices is a map $f \colon L \to M$ that preserves finite meets (and hence $f(1) = 1$).

THEOREM 2.10. *Let L be an algebraic lattice.*

(1) *The Lawson topology on L is the patch topology of the Scott topology.*
(2) *An upset of L is Lawson open iff it is Scott open.*
(3) *Let $f \colon L \to M$ be a meet-semilattice homomorphism between algebraic lattices. Then f is Lawson-continuous iff f is Scott-continuous and preserves arbitrary meets.*

REMARK 2.11.

(1) Putting together Theorems 2.1, 2.8(4), and 2.10(1), for each algebraic lattice L, we find that $(L, \sigma(L))$ is a spectral space and $(L, \lambda(L), \leq)$ is the corresponding Priestley space.
(2) In [GH$^+$03], meet-semilattices are not assumed to have a top, and hence meet-semilattice homomorphisms need not preserve the top. Under our definition of meet-semilattice homomorphisms, [GH$^+$03, Thm. III-1.8] takes the stronger form of Theorem 2.10(3) (namely, preserving nonempty meets becomes equivalent to preserving arbitrary meets).

We next define the categories of algebraic lattices with the Scott topology and algebraic lattices with the Lawson topology to which the Cornish isomorphism restricts.

DEFINITION 2.12.

(1) Let AlgLat_σ be the category of algebraic lattices with the Scott topology and Scott-continuous maps that preserve arbitrary meets.
(2) Let AlgLat_λ be the category of algebraic lattices with the Lawson topology and meet-semilattice homomorphisms that are Lawson-continuous.

Remark 2.13. Clearly isomorphisms in both AlgLat_σ and AlgLat_λ are order-isomorphisms. Let $f\colon L \to M$ be an AlgLat_σ-morphism. We show that f is spectral. Let $U \subseteq M$ be compact open. It is enough to show that $f^{-1}(U)$ is compact. By Theorem 2.8(3), it suffices to assume that $U = {\uparrow} k$ for some $k \in K(M)$. Set $x = \bigwedge f^{-1}(U)$, so $f^{-1}(U) \subseteq {\uparrow} x$. Because f preserves arbitrary meets, $f(x) = \bigwedge\{f(z) : z \in f^{-1}(U)\}$. This shows that $k \le f(x)$. Therefore, $x \in f^{-1}(U)$, and hence $f^{-1}(U) = {\uparrow} x$. Thus, $f^{-1}(U)$ is compact since each principal upset is compact. Consequently, AlgLat_σ is a (nonfull) subcategory of Spec. Similarly, AlgLat_λ is a (nonfull) subcategory of Pries.

As an immediate consequence of the above we obtain:

Theorem 2.14. *The Cornish isomorphism between* Spec *and* Pries *restricts to an isomorphism between* AlgLat_σ *and* AlgLat_λ.

Remark 2.15. Since the Scott and Lawson topologies are intrinsic to algebraic lattices, each of AlgLat_σ and AlgLat_λ is also isomorphic to the category AL, considered in [GH+03, Def. IV-1.13], whose objects are algebraic lattices and whose morphisms are maps preserving arbitrary meets and directed joins.

Now suppose that L, M are coherent lattices and $f\colon L \to M$ is an AlgLat_σ-morphism. Since f preserves arbitrary meets, it has a left adjoint $\ell\colon M \to L$. By [GH+03, Cor. IV-1.12], $\ell[K(M)] \subseteq K(L)$. In addition, we have:

Lemma 2.16. *Let $f\colon L \to M$ be an AlgLat_σ-morphism with left adjoint ℓ. The following are equivalent:*

(1) *ℓ preserves finite meets.*
(2) *ℓ preserves finite meets of compact elements.*
(3) *f^{-1} preserves finite joins of principal filters.*
(4) *f^{-1} preserves finite joins of principal filters of compact elements.*

Proof. (1)⇔(2) One direction is obvious. For the other, let $S \subseteq L$ be finite. Because ℓ is order-preserving, $\ell(\bigwedge S) \le \bigwedge\{\ell(x) : x \in S\}$. To see the equality, let $x \in S$ and $k \in K(M)$ with $k \le \ell(x)$. Since ℓ preserves arbitrary joins, $\ell(x) = \bigvee\{\ell(m) : m \in {\downarrow} x \cap K(L)\}$. As k is compact and this join is directed, there is $m \in K(L)$ with $m \le x$ and $k \le \ell(m)$. Now, if $k \in K(M)$ with $k \le \bigwedge\{\ell(x) : x \in S\}$, then for each $x \in S$ there is $m_x \in K(L)$ with $m_x \le x$ and $k \le \ell(m_x)$. Since ℓ preserves finite meets of compact elements,

$$k \le \ell\left(\bigwedge\{m_x : x \in S\}\right) \le \ell\left(\bigwedge S\right).$$

This yields the reverse inequality, and hence ℓ preserves all finite meets.

(1)⇔(3) Since $f^{-1}({\uparrow} y) = {\uparrow}\ell(y)$ for each $y \in M$, for $S \subseteq M$ finite we have

$$f^{-1}\left(\bigvee\{{\uparrow} y : y \in S\}\right) = f^{-1}\left({\uparrow}\bigwedge S\right) = {\uparrow}\ell\left(\bigwedge S\right)$$

and

$$\bigvee\{f^{-1}({\uparrow} y) : y \in S\} = \bigvee\{{\uparrow}\ell(y) : y \in S\} = {\uparrow}\bigwedge\{\ell(y) : y \in S\}.$$

From this we see that f^{-1} preserves finite joins of principal filters iff ℓ preserves finite meets. Therefore, (1) and (3) are equivalent.

(2)⇔(4) This follows from the same argument as for (1)⇔(3). ∎

Definition 2.17.

(1) Let CohLat_σ be the category of coherent lattices and AlgLat_σ-morphisms whose left adjoint preserves finite meets.
(2) Let CohLat_λ be the category of coherent lattices and AlgLat_λ-morphisms whose left adjoint preserves finite meets.

Clearly CohLat_σ is a (nonfull) subcategory of AlgLat_σ and CohLat_λ is a (nonfull) subcategory of AlgLat_λ. As an immediate consequence we obtain:

Theorem 2.18. *The isomorphism of Theorem* 2.14 *restricts to an isomorphism between* CohLat_σ *and* CohLat_λ*.*

We recall from the introduction that Lat is the category of bounded lattices and bounded lattice homomorphisms. Let also SLat be the category of meet-semilattices and meet-semilattice homomorphisms.

Theorem 2.19.

(1) *The categories* AlgLat_σ *and* AlgLat_λ *are each dually equivalent to* SLat*.*
(2) *The categories* CohLat_σ *and* CohLat_λ *are each dually equivalent to* Lat*.*

Sketch of proof. (1) In [HMS74, Thm. I.3.9] (see also [GH+03, Thm. IV-1.16]), Nachbin's theorem was generalized to a duality between SLat and the category of Stone semilattices. Since the latter is (isomorphic to) AlgLat_λ, the result follows by applying Theorem 2.14.

(2) The dual equivalence of AlgLat_λ and SLat restricts to a dual equivalence of CohLat_λ and Lat (see, e.g., [HMS74, Prop. 3.19]). Now apply Theorem 2.18. ∎

Remark 2.20.

(1) Theorem 2.19(1) together with Remark 2.15 implies that SLat is also dually equivalent to AL (see [Ern91, Cor. 5.9(3)], [GH+03, Thm. IV.1.16]).
(2) The functors establishing the dual equivalences of Theorem 2.19 are described as follows. One functor sends a semilattice or lattice A to $\mathsf{Filt}(A)$ and a morphism $\alpha\colon A \to B$ to $\alpha^{-1}\colon \mathsf{Filt}(B) \to \mathsf{Filt}(A)$. The other sends an algebraic or coherent lattice L to $(K(L), \geq)$ and a morphism $f\colon L \to M$ to the restriction of its left adjoint.

3. The Jipsen–Moshier approach

In [MJ14], Jipsen and Moshier defined two categories of topological spaces and showed that they are dually equivalent to SLat and Lat. In this chapter we show that these two categories of spaces are isomorphic to AlgLat_σ and CohLat_σ, respectively.

For a topological space X, we write $\leq$ for the specialization order on X. A nonempty subset F of X is a *filter* if for each $x, y \in F$, there is $z \in F$ with $z \leq x, y$. Observe that when X is a meet-semilattice, this notion is equivalent to the standard notion of filter. Let $\mathsf{OF}(X)$ be the open filters and $\mathsf{KOF}(X)$ the compact open filters of X. A subset A of X is *F-saturated* if it is an intersection of open filters. For $A \subseteq X$, set

$$\mathsf{fsat}(A) = \bigcap \{U \in \mathsf{OF}(X) : A \subseteq U\}.$$

Then fsat is a closure operator on X, and so its fixpoints form the complete lattice $\mathsf{FSat}(X)$, where meet is given by intersection and join by $\bigvee \mathcal{S} = \mathsf{fsat}(\bigcup \mathcal{S})$ (see, e.g., [DP02, Prop. 7.2(ii)]).

DEFINITION 3.1.

(1) Let X be a topological space. We call X a *Jipsen–Moshier space*, or JM-space for short, if X is sober and $\mathsf{KOF}(X)$ forms a basis that is closed under finite intersections.
(2) Let $\mathsf{JM}_{\mathsf{SLat}}$ be the category of Jipsen–Moshier spaces and maps $f\colon X \to Y$ satisfying $U \in \mathsf{KOF}(Y)$ implies $f^{-1}(U) \in \mathsf{KOF}(X)$.

REMARK 3.2. JM-spaces are called HMS-spaces in [MJ14], to emphasize their reliance on Hofmann–Mislove–Stralka duality for semilattices [HMS74]. The definition of a JM-space is a strengthening of the definition of a spectral space in that the coherence condition on compact open subsets holds for the smaller collection $\mathsf{KOF}(X)$, and similarly for morphisms. Observe that $\mathsf{KOF}(X)$ is a meet-subsemilattice of $\mathsf{FSat}(X)$ and every $\mathsf{JM}_{\mathsf{SLat}}$-morphism is automatically continuous.

Because we will be viewing algebraic lattices as topological spaces, we will typically denote them by X, Y, Z. We recall the following results from [MJ14, Secs. 2, 3].

THEOREM 3.3. *Let X be a topological space and $\leq$ its specialization order.*

(1) *A filter of X with respect to $\leq$ is compact iff it is principal.*
(2) *$X \in \mathsf{JM}_{\mathsf{SLat}}$ iff $(X, \leq)$ is an algebraic lattice and the topology of X is the Scott topology.*

REMARK 3.4. We briefly comment on the proof of Theorem 3.3(2). Using Theorem 2.8, it is easy to see that each algebraic lattice with the Scott topology is a JM-space. The key then is to prove that a JM-space is an algebraic lattice and the topology is the Scott topology. This can be seen as follows. Since every JM-space is sober, directed joins exist. This together with the fact that compact open filters are principal implies that arbitrary meets exist, and hence X is a complete lattice. In addition, X is algebraic since compact open filters are precisely the principal upsets of compact elements, which also implies that the topology is the Scott topology.

Putting Theorem 2.8(2) and Theorem 3.3(1) together yields the following result, which will be used throughout.

LEMMA 3.5. *Let X be an algebraic lattice. Viewing X as a topological space with the Scott topology,* $\mathsf{KOF}(X) = \{{\uparrow}k : k \in K(X)\}$.

We next turn our attention to morphisms. The next result can be derived from [MJ14, Sec. 5]. To keep the paper self-contained, we give a short proof.

LEMMA 3.6. *Let $f\colon X \to Y$ be a map between algebraic lattices. Then f is a $\mathsf{JM}_{\mathsf{SLat}}$-morphism iff f is an AlgLat_σ-morphism.*

Proof. Suppose that f is a AlgLat_σ-morphism. Let $U \in \mathsf{KOF}(Y)$. By Remark 2.13, f is spectral, and hence $f^{-1}(U)$ is compact open. In addition, since f preserves arbitrary meets, $f^{-1}(U)$ is a filter. Thus, $f^{-1}(U) \in \mathsf{KOF}(X)$, and so f is a $\mathsf{JM}_{\mathsf{SLat}}$-morphism.

Conversely, let f be a $\mathsf{JM}_{\mathsf{SLat}}$-morphism. Then f is Scott-continuous by Remark 3.2 and Theorem 3.3(2). If U is compact open, then U is a finite union from $\mathsf{KOF}(Y)$. Therefore, $f^{-1}(U)$ is a finite union from $\mathsf{KOF}(X)$. Consequently, f is spectral. To see that f preserves arbitrary meets, first observe that f is order preserving since the orders on X and Y are the specialization orders for the Scott topology by Theorem 2.8(1). Now let $S \subseteq X$ and set $x = \bigwedge S$. Then $f(x) \leq \bigwedge f[S]$. For the reverse inequality, let $l \in K(Y)$ with $l \leq \bigwedge f[S]$. Then $l \leq f(z)$ for each $z \in S$. By Lemma 3.5, there is $k \in K(X)$ with $f^{-1}(\uparrow l) = \uparrow k$. Therefore, if $z \in S$, then $k \leq z$. This yields $k \leq x$, and so $l \leq f(k) \leq f(x)$, again since f is order preserving. Because $\bigwedge f[S]$ is the join of all such l, we get $\bigwedge f[S] \leq f(x)$, and hence $f(x) = \bigwedge f[S]$. Thus, f preserves arbitrary meets, and hence is an AlgLat_σ-morphism. ∎

THEOREM 3.7. *AlgLat_σ is isomorphic to $\mathsf{JM}_{\mathsf{SLat}}$.*

Proof. By Theorem 3.3(2), $X \in \mathsf{AlgLat}_\sigma$ iff $(X, \sigma(X)) \in \mathsf{JM}_{\mathsf{SLat}}$. Moreover, by Lemma 3.6, $f \in \mathsf{Hom}_{\mathsf{AlgLat}_\sigma}(X, Y)$ iff $f \in \mathsf{Hom}_{\mathsf{JM}_{\mathsf{SLat}}}(X, Y)$. Thus, the assignment $X \mapsto (X, \sigma(X))$ and $f \mapsto f$ defines the desired isomorphism of categories. ∎

The next result is an immediate consequence of the above and Theorem 2.19(1).

COROLLARY 3.8 ([MJ14, Thm. 5.2]). *$\mathsf{JM}_{\mathsf{SLat}}$ is dually equivalent to SLat.*

REMARK 3.9. The functors yielding the duality of Corollary 3.8 are essentially those described in Remark 2.20. That is, $X \in \mathsf{JM}_{\mathsf{SLat}}$ is sent to $\mathsf{KOF}(X)$, which is isomorphic to $(K(X), \geq)$, and $A \in \mathsf{SLat}$ to $\mathsf{Filt}(A)$. A $\mathsf{JM}_{\mathsf{SLat}}$-morphism $f\colon X \to X'$ is sent to the restriction of f^{-1} to $\mathsf{KOF}(X')$, and a SLat-morphism $f\colon A \to B$ to $f^{-1}\colon \mathsf{Filt}(B) \to \mathsf{Filt}(A)$.

We now consider a (nonfull) subcategory of $\mathsf{JM}_{\mathsf{SLat}}$ that is dually equivalent to Lat.

DEFINITION 3.10. Let JM be the category of Jipsen–Moshier spaces such that $\mathsf{KOF}(X)$ forms a bounded sublattice of $\mathsf{FSat}(X)$ and $\mathsf{JM}_{\mathsf{SLat}}$-morphisms f such that f^{-1} preserves finite joins of compact open filters.

Putting Theorem 3.3(2) together with [MJ14, Thm. 3.2] yields:

LEMMA 3.11. *Let $X \in \mathsf{JM}_{\mathsf{SLat}}$. Then $X \in \mathsf{JM}$ iff X is a coherent lattice and the topology of X is the Scott topology.*

REMARK 3.12. The key idea of the proof is as follows. By Lemma 3.5, $\mathsf{KOF}(X)$ is isomorphic to $(K(X), \geq)$. Thus, $\mathsf{KOF}(X)$ is a bounded sublattice of $\mathsf{FSat}(X)$ iff $K(X)$ is a bounded sublattice of X.

In addition, putting Lemmas 2.16, 3.5 and 3.6 together yields:

LEMMA 3.13. *Let $f\colon X \to Y$ be an AlgLat_σ-morphism between coherent lattices topologized with the Scott topology. Then f is a CohLat_σ-morphism iff f is a JM-morphism.*

We thus obtain:

THEOREM 3.14. *CohLat_σ is isomorphic to JM.*

Proof. By Lemma 3.11, $X \in \mathsf{CohLat}_\sigma$ iff $(X, \sigma(X)) \in \mathsf{JM}$. By Lemma 3.13, we know that $f \in \mathsf{Hom}_{\mathsf{CohLat}_\sigma}(X, Y)$ iff $f \in \mathsf{Hom}_{\mathsf{JM}}(X, Y)$. Therefore, the isomorphism between AlgLat_σ and $\mathsf{JM}_{\mathsf{SLat}}$ restricts to an isomorphism between CohLat_σ and JM. ∎

The next result is an immediate consequence of the above and Theorem 2.19(2).

Corollary 3.15 ([MJ14, Thm. 5.2]). JM *is dually equivalent to* Lat.

Remark 3.16. The functors yielding the duality of Corollary 3.15 are the restrictions of those given in Remark 3.9.

4. The Bezhanishvili–Dmitrieva–de Groot–Moraschini approach

In this chapter we consider the spaces introduced in [BD+24], which we term BDGM-spaces. The same way JM-spaces are a strengthening of spectral spaces, BDGM-spaces are a strengthening of Priestley spaces, and the Cornish isomorphism between Spec and Pries restricts to an isomorphism between the categories of JM-spaces and BDGM-spaces. As we pointed out in the previous chapter, these categories come in two flavors, depending on whether we want to obtain a dual equivalence for SLat or Lat. We thus arrive at two categories $\mathsf{BDGM}_{\mathsf{SLat}}$ and BDGM. We show that $\mathsf{BDGM}_{\mathsf{SLat}}$ is isomorphic to $\mathsf{AlgLat}_{\lambda}$, from which we deduce that $\mathsf{BDGM}_{\mathsf{SLat}}$ is isomorphic to $\mathsf{JM}_{\mathsf{SLat}}$. We then show that the above isomorphism restricts to an isomorphism between BDGM and $\mathsf{CohLat}_{\lambda}$, from which we deduce that BDGM is isomorphic to JM.

Definition 4.1.

(1) We call an ordered topological space X a *BDGM-space* if it is a compact space, a meet-semilattice, and the following strengthening of the Priestley separation axiom holds:

$$\text{If } x \not\leq y, \text{ then there is a clopen filter } F \text{ such that } x \in F \text{ and } y \notin F.$$

(2) A *BDGM-morphism* between BDGM-spaces is a continuous meet-semilattice morphism.

(3) Let $\mathsf{BDGM}_{\mathsf{SLat}}$ be the category of BDGM-spaces and BDGM-morphisms.

Remark 4.2. It is clear from the above definition that each BDGM-space is a Priestley space, and so $\leq$ is a closed order on X. In particular, ${\uparrow}x, {\downarrow}x$ are closed for each $x \in X$. Moreover, an argument similar to that for Priestley spaces shows that

$$\{F - G : F, G \in \mathsf{CLF}(X)\}$$

is a subbasis for the topology of X, where $\mathsf{CLF}(X)$ denotes the collection of clopen filters of X.

It is trivial to see that each algebraic lattice with the Lawson topology is a BDGM-space. The following lemma shows the converse.

Lemma 4.3. *Let X be a BDGM-space.*

(1) *X is a complete lattice.*

(2) $\mathsf{CLF}(X) = \{{\uparrow}k : k \in K(X)\}$.

(3) *X is an algebraic lattice and the topology is the Lawson topology.*

Proof. (1) Since $1 \in X$, it is enough to show that $\bigwedge S$ exists in X for each nonempty $S \subseteq X$. Let T be the set of all finite meets from S. Because X is compact and each ${\downarrow}t$ is closed, $\bigcap \{{\downarrow}t : t \in T\}$ is nonempty. This intersection is the set S^l of lower bounds of S. If $x_1, \dots, x_n \in S^l$ and $t_1, \dots, t_m \in T$, then $x_i \le t_j$ for each i, j, so

$$t_1 \wedge \cdots \wedge t_m \in {\uparrow}x_1 \cap \cdots \cap {\uparrow}x_n \cap {\downarrow}t_1 \cap \cdots \cap {\downarrow}t_m.$$

Therefore, $\{{\uparrow}x : x \in S^l\} \cup \{{\downarrow}t : t \in T\}$ has the finite intersection property, and hence compactness implies that there is

$$s \in \bigcap \{{\uparrow}x : x \in S^l\} \cap \bigcap \{{\downarrow}t : t \in T\}.$$

Thus, $s = \bigwedge S$.

(2) Let $F \in \mathsf{CLF}(X)$. Then $\{F\} \cup \{{\downarrow}x : x \in F\}$ is a collection of closed sets with the finite intersection property. Therefore, there is $k \in F \cap \bigcap \{{\downarrow}x : x \in F\}$, which forces $F = {\uparrow}k$. To see that $k \in K(X)$, suppose that $S \subseteq X$ with $k \le \bigvee S =: s$. Then ${\uparrow}s \subseteq {\uparrow}k$, so $\bigcap \{{\uparrow}x : x \in S\} = {\uparrow}s \subseteq {\uparrow}k$. Since each ${\uparrow}x$ is closed and ${\uparrow}k = F$ is clopen, compactness implies that there are $x_1, \dots, x_n \in S$ with ${\uparrow}x_1 \cap \cdots \cap {\uparrow}x_n \subseteq {\uparrow}k$, and so $k \le x_1 \vee \cdots \vee x_n$. Thus, k is compact.

Conversely, let $k \in K(X)$. The separation axiom for BDGM-spaces shows that the filter ${\uparrow}k$ is an intersection of clopen filters. Therefore, by the previous paragraph, there is $S \subseteq K(X)$ with ${\uparrow}k = \bigcap \{{\uparrow}l : l \in S\}$. This implies that $k = \bigvee S$. Since k is compact, k is a finite join from S, and hence ${\uparrow}k$ is a finite intersection of clopen filters. Thus, ${\uparrow}k$ is clopen.

(3) By (1), for X to be an algebraic lattice, it is enough to show that each $x \in X$ is a join from $K(X)$. By the separation axiom for BDGM-spaces and (2), we have ${\uparrow}x = \bigcap \{{\uparrow}k : k \in K(X),\ k \le x\}$. Thus, $x = \bigvee({\downarrow}x \cap K(X))$. Since $\{{\uparrow}k - {\uparrow}l : k, l \in K(X)\}$ is a subbasis for the topology on X, the topology must be the Lawson topology. ∎

Proposition 4.4.

(1) *Let X be an ordered Stone space. Then $X \in \mathsf{BDGM}_{\mathsf{SLat}}$ iff $X \in \mathsf{AlgLat}_\lambda$.*
(2) *A map $f\colon X \to Y$ is a $\mathsf{BDGM}_{\mathsf{SLat}}$-morphism iff f is an AlgLat_λ-morphism.*

Proof. (1) As we pointed out above, if $X \in \mathsf{AlgLat}_\lambda$, then $X \in \mathsf{BDGM}_{\mathsf{SLat}}$. The converse follows from Lemma 4.3(3).

(2) By (1), $X, Y \in \mathsf{BDGM}_{\mathsf{SLat}}$ iff $X, Y \in \mathsf{AlgLat}_\lambda$. Furthermore, the topology on a BDGM-space is the Lawson topology by Lemma 4.3(3). Thus, f is a $\mathsf{BDGM}_{\mathsf{SLat}}$-morphism iff f is an AlgLat_λ-morphism. ∎

As an immediate consequence, we obtain:

Theorem 4.5. $\mathsf{BDGM}_{\mathsf{SLat}}$ *is isomorphic to* AlgLat_λ.

As a corollary, we obtain:

Corollary 4.6.

(1) ([BD+24, Rem. 2.15]) *The Cornish isomorphism restricts to an isomorphism between* $\mathsf{JM}_{\mathsf{SLat}}$ *and* $\mathsf{BDGM}_{\mathsf{SLat}}$.
(2) ([BD+24, Thm. 2.11]) $\mathsf{BDGM}_{\mathsf{SLat}}$ *is dually equivalent to* SLat.

Proof. For (1) apply Theorems 2.14, 3.7 and 4.5, and for (2) apply Theorems 2.19(1) and 4.5. ∎

REMARK 4.7. The duality between $\mathsf{BDGM}_{\mathsf{SLat}}$ and SLat is obtained by sending X to $\mathsf{CLF}(X)$ and $f\colon X \to Y$ to $f^{-1}\colon \mathsf{CLF}(Y) \to \mathsf{CLF}(X)$. For the other direction, A is sent to $\mathsf{Filt}(A)$ and $\alpha\colon A \to B$ to $\alpha^{-1}\colon \mathsf{Filt}(B) \to \mathsf{Filt}(A)$.

We now consider a (nonfull) subcategory of $\mathsf{BDGM}_{\mathsf{SLat}}$ that is dually equivalent to Lat.

DEFINITION 4.8. Let BDGM be the category of BDGM-spaces satisfying

- finite joins of clopen filters are clopen,

and BDGM-morphisms $f\colon X \to Y$ satisfying

- $f(x) = 1$ implies $x = 1$;
- $y' \wedge z' \le f(x)$ implies that there are y, z with $y' \le f(y)$, $z' \le f(z)$, and $y \wedge z \le x$.

REMARK 4.9. The above spaces and morphisms were introduced in [BD⁺24] under the name of L-spaces and L-morphisms.

For the next lemma, observe that if f is a $\mathsf{BDGM}_{\mathsf{SLat}}$-morphism, it has a left adjoint by Proposition 4.4(2).

LEMMA 4.10. *Let $f\colon X \to Y$ be a $\mathsf{BDGM}_{\mathsf{SLat}}$-morphism and ℓ its left adjoint. The following are equivalent:*

(1) *f is a BDGM-morphism.*
(2) *f^{-1} preserves finite joins of principal filters.*
(3) *f^{-1} preserves finite joins of principal filters of compact elements.*
(4) *ℓ preserves finite meets.*
(5) *ℓ preserves finite meets of compact elements.*

Proof. By Lemma 2.16, conditions (2) to (5) are equivalent. Therefore, it is enough to show that (1) and (4) are equivalent.

(1)⇒(4) Since $1 \le f\ell(1)$, we have $f\ell(1) = 1$, which yields $\ell(1) = 1$. Therefore, ℓ preserves the empty meet. Let $y', z' \in Y$. Then $y' \wedge z' \le f\ell(y' \wedge z')$. Since f is a BDGM-morphism, there are $y, z \in X$ with $y' \le f(y)$, $z' \le f(z)$, and $y \wedge z \le \ell(y' \wedge z')$. Thus, $\ell(y') \le y$ and $\ell(z') \le z$, which imply that $\ell(y') \wedge \ell(z') \le \ell(y' \wedge z')$. Because ℓ is order-preserving, we conclude that it preserves finite meets.

(4)⇒(1) If $f(x) = 1$, then $1 = \ell(1) \le \ell f(x) \le x$. Therefore, $x = 1$. Next, suppose that $x \in X$ and $y', z' \in Y$ with $y' \wedge z' \le f(x)$. Then $\ell(y') \wedge \ell(z') = \ell(y' \wedge z') \le x$. We also find that $y' \le f\ell(y')$ and $z' \le f\ell(z')$. Letting $y = \ell(y')$ and $z = \ell(z')$ yields $y' \le f(y)$, $z' \le f(z)$, and $y \wedge z \le x$. Thus, f is a BDGM-morphism. ∎

PROPOSITION 4.11. *Let $X, Y \in \mathsf{BDGM}_{\mathsf{SLat}}$ and $f\colon X \to Y$ be a $\mathsf{BDGM}_{\mathsf{SLat}}$-morphism.*

(1) *$X \in \mathsf{BDGM}$ iff $X \in \mathsf{CohLat}_\lambda$.*
(2) *f is a BDGM-morphism iff f is a CohLat_λ-morphism.*

Proof. (1) By Proposition 4.4(1), $X \in \mathsf{AlgLat}_\lambda$. Now, $X \in \mathsf{BDGM}$ iff finite joins of clopen filters are clopen. By Lemma 4.3(2), this occurs iff $K(X)$ is closed under finite meets, which is equivalent to $X \in \mathsf{CohLat}_\lambda$.

(2) By Proposition 4.4(2), f is an AlgLat_λ-morphism. Thus, it is sufficient to apply Lemmas 2.16 and 4.10. ∎

As an immediate consequence, we obtain:

THEOREM 4.12.

(1) BDGM *is isomorphic to* CohLat_λ*.*
(2) ([BD$^+$24, Rem. 2.15]) *The Cornish isomorphism restricts to an isomorphism between* JM *and* BDGM*.*

Proof. For (1) apply Theorem 4.5 and Proposition 4.11; and for (2) apply (1), Theorems 2.18 and 3.14, and Corollary 4.6(1). ∎

Putting Theorem 4.12(1) together with Theorem 2.19(2) yields:

COROLLARY 4.13 ([BD$^+$24, Thm. 2.14]). BDGM *is dually equivalent to* Lat*.*

REMARK 4.14. The duality above is a restriction of the duality given in Remark 4.7.

The following diagrams show the relationships between the various categories of Chapters 2 to 4. Double arrows with no label represent isomorphisms of categories and dual equivalences when labeled with d.

$$\begin{tikzcd}
 & \mathsf{AlgLat}_\sigma \arrow[r, leftrightarrow] \arrow[dd, leftrightarrow] & \mathsf{JM}_{\mathsf{SLat}} \arrow[dd, leftrightarrow] \\
\mathsf{SLat} \arrow[ur, leftrightarrow, "d"] \arrow[dr, leftrightarrow, "d"'] & & \\
 & \mathsf{AlgLat}_\lambda \arrow[r, leftrightarrow] & \mathsf{BDGM}_{\mathsf{SLat}}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathsf{CohLat}_\sigma \arrow[r, leftrightarrow] \arrow[dd, leftrightarrow] & \mathsf{JM} \arrow[dd, leftrightarrow] \\
\mathsf{Lat} \arrow[ur, leftrightarrow, "d"] \arrow[dr, leftrightarrow, "d"'] & & \\
 & \mathsf{CohLat}_\lambda \arrow[r, leftrightarrow] & \mathsf{BDGM}
\end{tikzcd}$$

5. The Hartonas approach

In this chapter we consider the spaces introduced by Hartonas [Har97], giving rise to two categories that we show are isomorphic to $\mathsf{BDGM}_{\mathsf{SLat}}$ and BDGM, respectively.

As usual, for a poset $(X, \le)$ and $A \subseteq X$, we let A^u be the set of upper bounds and A^l the set of lower bounds of A. We then have the closure operator Γ on X given by $\Gamma(A) = A^{lu}$. So $\Gamma X = \{\Gamma(A) : A \subseteq X\}$ is a complete lattice, where meet is intersection, join is given by $\bigvee A_i = \Gamma(\bigcup_i A_i)$, top is $\Gamma(X) = X$, and bottom is $\Gamma(\varnothing) = X^u$, which is $\{1\}$ if X has a top, and $\varnothing$ otherwise. We note that $\Gamma(\{x\}) = {\uparrow}x$ for each $x \in X$.

DEFINITION 5.1. If X is a partially ordered Stone space, we set

$$\Lambda X = \{U \in \Gamma X : U \text{ is clopen}\}.$$

We recall that a subset S of a complete lattice X is *meet-dense* in X if each element of X is a meet from S.

Definition 5.2.

(1) Let X be a partially ordered Stone space. We call X an *Hartonas space* if
   (a) X is a complete lattice;
   (b) the topology on X is generated by $\Lambda X \cup \{-U : U \in \Lambda X\}$;
   (c) ΛX is meet-dense in ΓX.
(2) Let $\mathsf{Hs}_{\mathsf{SLat}}$ be the category of Hartonas spaces and continuous maps $f\colon X \to Y$ such that $U \in \Lambda Y$ implies $f^{-1}(U) \in \Lambda X$.

Remark 5.3.

(1) Our definition generalizes that of Hartonas [Har97, Def. 4.3] to obtain a category equivalent to $\mathsf{BDGM}_{\mathsf{SLat}}$ and dually equivalent to SLat. The axiom [Har97, Def. 4.3(2)] which is dropped from Definition 5.2 will be added in Definition 5.10 to obtain a category equivalent to BDGM and dually equivalent to Lat.
(2) In Definition 5.2, we start with a partial order $\leq$ on a Stone space X and define Γ from it. On the other hand, Hartonas starts with Γ on X which arises from a binary relation $\prec$, and defines $\leq$ by $x \leq y$ if $\Gamma(y) \subseteq \Gamma(x)$. The partial order $\leq$ and the closure operator Γ are definable from one another, but we prefer to start with $\leq$ since it is easier to connect the resulting duality with BDGM-duality.
(3) The condition that X is a complete lattice is equivalent to Hartonas's condition that $\Gamma X = \{{\uparrow}x : x \in X\}$. Also, the compactness condition in [Har97, Def. 4.2(1)] is superfluous since it follows from compactness of X.
(4) Let $x \in X$. Then ${\uparrow}x$ is closed since ${\uparrow}x \in \Gamma X$, so is an intersection from ΛX by Definition 5.2(1c), and hence is an intersection of clopen sets. (Note that ΛX is closed under finite intersections.)
(5) By Definition 5.2(1b, 2), each $\mathsf{Hs}_{\mathsf{SLat}}$-morphism is continuous.

Lemma 5.4. *Let X be an algebraic lattice. Viewing X as a topological space with the Lawson topology,* $\Lambda X = \{{\uparrow}k : k \in K(X)\} = \mathsf{CLF}(X)$.

Proof. By definition, $\Lambda X \subseteq \mathsf{CLF}(X)$. By Proposition 4.4(1), X is a BDGM-space. Therefore, $\mathsf{CLF}(X) = \{{\uparrow}k : k \in K(X)\}$ by Lemma 4.3(2). Finally, ${\uparrow}k \in \Lambda X$ for each $k \in K(X)$ since it is a clopen principal filter, hence an element of ΛX since $\Gamma X = \{{\uparrow}x : x \in X\}$ (see Remark 5.3(3)). ■

Remark 5.5. Putting Lemma 5.4 together with Lemma 3.5 yields $\mathsf{KOF}(X) = \mathsf{CLF}(X)$ for each algebraic lattice X.

Proposition 5.6.

(1) *A partially ordered Stone space is an Hartonas space iff it is a BDGM-space.*
(2) *A map $f\colon X \to Y$ is a $\mathsf{BDGM}_{\mathsf{SLat}}$-morphism iff f is an $\mathsf{Hs}_{\mathsf{SLat}}$-morphism.*

Proof. (1) Let X be an Hartonas space. Then X is compact. If $x \not\leq y$, then ${\uparrow}y \not\subseteq {\uparrow}x$. Since ΛX is meet-dense in ΓX, there is $F \in \Lambda X$ with ${\uparrow}y \not\subseteq F$ and ${\uparrow}x \subseteq F$. Therefore, F is a clopen filter containing x but not y, and hence X is a BDGM-space.

Conversely, let X be a BDGM-space. By Lemma 4.3(1), X is a complete lattice, so Definition 5.2(1a) follows. By Lemmas 4.3(3) and 5.4, $\Lambda X = \mathsf{CLF}(X)$. By Remark 4.2,

$\{F - G : F, G \in \mathsf{CLF}(X)\}$ is a subbasis for the topology of X, so Definition 5.2(1b) follows. Finally, for Definition 5.2(1c), if $A \in \Gamma X$, then $A = {\uparrow}x$ for some $x \in X$. Because ${\uparrow}x$ is the intersection of clopen filters, we see that ΛX is meet-dense in ΓX. Thus, X is an Hartonas space.

(2) Let f be a $\mathsf{BDGM}_{\mathsf{SLat}}$-morphism. Then $f^{-1}(F) \in \mathsf{CLF}(X)$ for each $F \in \mathsf{CLF}(Y)$. Now apply Lemma 5.4 to conclude that f is an $\mathsf{Hs}_{\mathsf{SLat}}$-morphism.

Conversely, let f be an $\mathsf{Hs}_{\mathsf{SLat}}$-morphism, so $f^{-1}(\Lambda Y) \subseteq \Lambda X$. Since ΛY is meet-dense in ΓY, we find that $f^{-1}(\Gamma Y) \subseteq \Gamma X$. Therefore, f^{-1} pulls principal filters to principal filters. Thus, there is a map $\ell : Y \to X$ satisfying $f^{-1}({\uparrow}y) = {\uparrow}\ell(y)$. This says exactly that ℓ is left adjoint to f. Therefore, f preserves all meets, and so is a SLat-morphism. Thus, f is a $\mathsf{BDGM}_{\mathsf{SLat}}$-morphism. ∎

Proposition 5.6 yields:

Theorem 5.7. $\mathsf{Hs}_{\mathsf{SLat}}$ *is isomorphic to* $\mathsf{BDGM}_{\mathsf{SLat}}$.

Corollary 5.8.

(1) $\mathsf{Hs}_{\mathsf{SLat}}$ *is isomorphic to* $\mathsf{AlgLat}_{\lambda}$ *and* $\mathsf{JM}_{\mathsf{SLat}}$.
(2) $\mathsf{Hs}_{\mathsf{SLat}}$ *is dually equivalent to* SLat.

Proof. (1) That $\mathsf{Hs}_{\mathsf{SLat}}$ is isomorphic to $\mathsf{AlgLat}_{\lambda}$ follows from Theorems 4.5 and 5.7, and that $\mathsf{Hs}_{\mathsf{SLat}}$ is isomorphic to $\mathsf{JM}_{\mathsf{SLat}}$ then follows from Theorems 2.14 and 3.7.

(2) Apply (1) and Theorem 2.19(1). ∎

Remark 5.9. The functors that yield the duality of Corollary 5.8(2) are those of BDGM-duality (see Remarks 4.7 and 4.14).

We now restrict the objects and morphisms of $\mathsf{Hs}_{\mathsf{SLat}}$ to yield a category dually equivalent to Lat.

Definition 5.10. Let Hs be the category of Hartonas spaces such that ΛX is a bounded sublattice of ΓX, and $\mathsf{Hs}_{\mathsf{SLat}}$-morphisms $f\colon X \to Y$ such that $f^{-1}\colon \Lambda Y \to \Lambda X$ preserves finite joins.

Remark 5.11. The spaces defined in Definition 5.10 are the same as those of [Har97, Def. 4.3] (see Remark 5.3(1)). Since Hartonas only works with semilattice morphisms between lattices, the extra condition on morphisms does not appear in [Har97]. This extra condition is necessary to establish duality between Hs and Lat.

Proposition 5.12. *Let* $X, Y \in \mathsf{Hs}_{\mathsf{SLat}}$.

(1) $X \in \mathsf{Hs}$ *iff* $X \in \mathsf{BDGM}$.
(2) *Let* $f\colon X \to Y$ *be an* $\mathsf{Hs}_{\mathsf{SLat}}$*-morphism. Then* f *is an* Hs*-morphism iff* f *is a* BDGM*-morphism.*

Proof. (1) Let $X \in \mathsf{Hs}_{\mathsf{SLat}}$. By Proposition 5.6(1), $X \in \mathsf{BDGM}_{\mathsf{SLat}}$. Moreover, $\mathsf{CLF}(X) = \Lambda X$ by Lemma 5.4. Therefore, $\mathsf{CLF}(X)$ is closed under finite joins iff ΛX is closed under finite joins. Thus, $X \in \mathsf{Hs}$ iff $X \in \mathsf{BDGM}$.

(2) Let $f\colon X \to Y$ be an $\mathsf{Hs}_{\mathsf{SLat}}$-morphism. By Proposition 5.6(2), f is a $\mathsf{BDGM}_{\mathsf{SLat}}$-morphism. Thus, by Lemmas 4.10 and 5.4, f is a BDGM-morphism iff f is an Hs-morphism. ■

As an immediate consequence of Theorem 5.7 and Proposition 5.12, we obtain:

Theorem 5.13. Hs *is isomorphic to* BDGM.

Corollary 5.14.

(1) Hs *is isomorphic to* JM.
(2) Hs *is dually equivalent to* Lat.

Proof. (1) Apply Theorems 5.13 and 4.12(2).
(2) Apply Theorem 5.13 and Corollary 4.13. ■

Remark 5.15.

(1) The functors yielding the duality of Corollary 5.14(2) are the restrictions of those given in Remark 5.9.
(2) Hartonas worked with meet-semilattice homomorphisms between lattices. Because of this, the lattices he considered only required having top but not necessarily bottom. His duality theorem [Har97, Thm. 4.4] can be obtained by replacing coherent lattices with arithmetic lattices and Hs-morphisms with $\mathsf{Hs}_{\mathsf{SLat}}$-morphisms.

6. The Celani–González approach

In this chapter we describe the Celani–González approach [CG20] (see also [Har25]). Like Jipsen and Moshier, they first describe the category $\mathsf{CG}_{\mathsf{SLat}}$ dual to SLat and show that $\mathsf{CG}_{\mathsf{SLat}}$ is equivalent to $\mathsf{JM}_{\mathsf{SLat}}$. Next, they restrict their attention to the category CG and prove that it is dual to Lat. It follows that CG is equivalent to JM. In this chapter we show that the functors yielding the equivalence between $\mathsf{CG}_{\mathsf{SLat}}$ and $\mathsf{JM}_{\mathsf{SLat}}$ restrict to give an equivalence between CG and JM. As a consequence, we conclude our series of equivalences of the upper part of Figure 1 (see the introduction).

Celani and González work with pairs $\mathcal{C} = (X, \mathcal{K})$ where X is a topological space and $\mathcal{K}$ is a fixed subbasis of X. We set

$$L\mathcal{C} = \{-U : U \in \mathcal{K}\}$$

and define the corresponding closure operator $\Delta_{\mathcal{C}}$ by

$$\Delta_{\mathcal{C}}(Y) = \bigcap\{A \in L\mathcal{C} : Y \subseteq A\}.$$

We call the fixpoints of this closure operator $\Delta_{\mathcal{C}}$*-closed sets*.

Let $Y \subseteq X$ be $\Delta_{\mathcal{C}}$-closed. A family $\mathcal{Z} \subseteq L\mathcal{C}$ is a Y*-family* if for all $A, B \in \mathcal{Z}$, there are $H \in L\mathcal{C}$ and $C \in \mathcal{Z}$ such that $Y \subseteq H$ and $A \cap H, B \cap H \subseteq C$. For a binary relation $S \subseteq X_1 \times X_2$, define $\Box_S\colon \wp(X_2) \to \wp(X_1)$ between the powersets by $\Box_S B = -S^{-1}[-B]$. The notation $\Box_S$ is motivated by its use in modal logic (see, e.g., [CZ97, p. 64]).

Definition 6.1 ([CG20, Defs. 3.5, 3.14]).

(1) Let X be a topological space with a fixed subbasis $\mathcal{K}$. The pair $\mathcal{C} = (X, \mathcal{K})$ is a *Celani–González space*, or *CG-space* for short, provided

   (a) X is a T_0-space and $X = \bigcup \mathcal{K}$;
   (b) each $U \in \mathcal{K}$ is compact open and $\mathcal{K}$ is closed under finite unions (so $\varnothing \in \mathcal{K}$);
   (c) for each $U, V \in \mathcal{K}$, if $x \in U \cap V$, then there are $W, D \in \mathcal{K}$ with $x \in D - W$ and $D \subseteq (U \cap V) \cup W$;
   (d) if Y is $\Delta_{\mathcal{C}}$-closed and $\mathcal{Z} \subseteq L\mathcal{C}$ is a Y-family such that $Y - A \neq \varnothing$ for all $A \in \mathcal{Z}$, then $Y \cap \bigcap \{-A : A \in \mathcal{Z}\} \neq \varnothing$.

(2) Let $\mathcal{C}_1 = (X_1, \mathcal{K}_1)$ and $\mathcal{C}_2 = (X_2, \mathcal{K}_2)$ be CG-spaces. A relation $S \subseteq X_1 \times X_2$ is a *CG-morphism* provided

   (1) if $B \in L\mathcal{C}_2$, then $\Box_S B \in L\mathcal{C}_1$;
   (2) $S[x]$ is $\Delta_{\mathcal{C}_2}$-closed for each $x \in X_1$.

(3) Let $\mathsf{CG}_{\mathsf{SLat}}$ be the category of CG-spaces and CG-morphisms.

It is proved in [CG20, Sec. 3] that $\mathsf{CG}_{\mathsf{SLat}}$ is indeed a category, where the composition $S_2 \star S_1$ of two CG-morphisms S_1, S_2 is defined by

$$x(\, S_2 \star S_1 \,)z \iff x \in \Box_{S_1}\Box_{S_2} B \text{ implies } z \in B \text{ for each } B \in L\mathcal{C}_3$$

and the identity morphism of $(X, \mathcal{K})$ is the dual $\leq$ of the specialization order $\sqsubseteq$ of X.

Remark 6.2. Let $\mathcal{C} = (X, \mathcal{K})$ be a CG-space.

(1) $\mathcal{K}$ is a join-semilattice, where join is union and bottom is $\varnothing$. Consequently, $L\mathcal{C}$ is a meet-semilattice, where meet is intersection and top is X. Since open sets are $\sqsubseteq$-upsets, elements of $L\mathcal{C}$ are $\leq$-upsets.
(2) If $x \in X$, then ${\uparrow}x = \bigcap \{B \in L\mathcal{C} : x \in B\}$, where $\uparrow$ is calculated with respect to $\leq$. To see this, the inclusion $\subseteq$ is clear since each $B \in L\mathcal{C}$ is a $\leq$-upset by (1). For the reverse inclusion, if $x \not\leq y$, then $y \not\sqsubseteq x$, and since $\mathcal{K}$ is a subbasis, there is $U \in \mathcal{K}$ with $y \in U$ and $x \notin U$. Letting $B = -U$, we have $B \in L\mathcal{C}$, $x \in B$, and $y \notin B$, completing the proof.
(3) The first condition of a CG-morphism is equivalent to: If $V \in \mathcal{K}_2$, then $S^{-1}[V] \in \mathcal{K}_1$.

Definition 6.3 ([CG20, Defs. 3.25, 3.29]). Let CG be the category of those $(X, \mathcal{K}) \in \mathsf{CG}_{\mathsf{SLat}}$ that satisfy

(1) $X \in \mathcal{K}$;
(2) $\bigcup \{W \in \mathcal{K} : W \subseteq U \cap V\} \in \mathcal{K}$ for all $U, V \in \mathcal{K}$,

and of those CG-morphisms $S \subseteq X_1 \times X_2$ that satisfy

(3) $S[x] \neq \varnothing$ for each $x \in X_1$ (that is, S is serial);
(4) $\Box_S \Delta_{\mathcal{C}_2}(B_1 \cup B_2) \subseteq \Delta_{\mathcal{C}_1}(\Box_S B_1 \cup \Box_S B_2)$ for all $B_1, B_2 \in L\mathcal{C}_2$.

Remark 6.4. Let $\mathcal{C} = (X, \mathcal{K})$ be a CG-space.

(1) Definition 6.3(2) says that if $U, V \in \mathcal{K}$, then the meet $U \wedge V$ exists in $\mathcal{K}$. Consequently, both $\mathcal{K}$ and $L\mathcal{C}$ are bounded lattices. The bottom of $L\mathcal{C}$ is $\varnothing$ by Definition 6.3(1).

(2) The reverse inclusion in Definition 6.3(4) holds automatically, so the condition says that $\Box_S$ preserves binary joins in $L\mathcal{C}$. This is equivalent to S^{-1} preserving binary meets in $\mathcal{K}$.

Clearly CG is a (nonfull) subcategory of $\mathsf{CG}_{\mathsf{SLat}}$. By [CG20, Thm. 3.24], $\mathsf{CG}_{\mathsf{SLat}}$ is dually equivalent to SLat, and CG is dually equivalent to Lat by [CG20, Thm. 3.32]. Consequently, CG is equivalent to JM. To obtain the equivalence of the upper part of Figure 1, we give a direct proof of CG being equivalent to JM, from which the duality between CG and Lat follows by Corollary 3.15.

We start by recalling the functor $\mathbb{M}\colon \mathsf{JM}_{\mathsf{SLat}} \to \mathsf{CG}_{\mathsf{SLat}}$ of [CG20]. For $X \in \mathsf{JM}_{\mathsf{SLat}}$ let X_m be the set of meet-irreducible elements of X, where we recall that $x \in X - \{1\}$ is *meet-irreducible* if $x = y \wedge z$ implies that $x = y$ or $x = z$. Set

$$\mathcal{K}_X = \{X_m - U : U \in \mathsf{KOF}(X)\},$$

so $\mathcal{K}_X = \{X_m - {\uparrow}k : k \in K(X)\}$ by Lemma 3.5. By [CG20, Thm. 5.20], $\mathcal{C}_X := (X_m, \mathcal{K}_X)$ is a CG-space.

Remark 6.5. The topology on X_m is the subspace topology of the lower topology on the algebraic lattice X. To see this, the lower topology on X has $\{{\uparrow}x : x \in X\}$ as a subbasis of closed sets. Since ${\uparrow}x = \bigcap\{{\uparrow}k : k \in K(X), k \leq x\}$, $-{\uparrow}x = \bigcup\{-{\uparrow}k : k \in K(X), k \leq x\}$. Thus, $\{-{\uparrow}k : k \in K(X)\}$ is a subbasis of open sets for X, and so $\{X_m - {\uparrow}k : k \in K(X)\}$ is a subbasis of open sets for X_m. Therefore, the topology on X_m is the subspace topology of the lower topology on X.

If $f\colon X \to Y$ is a $\mathsf{JM}_{\mathsf{SLat}}$-morphism, define $S_f \subseteq X_m \times Y_m$ by

$$x \, S_f \, y \iff f(x) \leq y.$$

Then S_f is a $\mathsf{CG}_{\mathsf{SLat}}$-morphism, and the assignment $X \mapsto \mathcal{C}_X$ and $f \mapsto S_f$ defines the functor $\mathbb{M}\colon \mathsf{JM}_{\mathsf{SLat}} \to \mathsf{CG}_{\mathsf{SLat}}$ [CG20, Thm. 5.30]. Proposition 6.8 shows that $\mathbb{M}$ restricts to a functor $\mathsf{JM} \to \mathsf{CG}$, the proof of which requires the following two lemmas.

Lemma 6.6. *Let $X \in \mathsf{JM}_{\mathsf{SLat}}$, $x \in X$, and $k, l \in K(X)$.*

(1) ([CG20, Lem. 5.11]) $x = \bigwedge({\uparrow}x \cap X_m)$.
(2) ([CG20, Prop. 5.13]) $X_m - {\uparrow}k \subseteq X_m - {\uparrow}l \iff k \leq l$.

Lemma 6.7. *Let $f\colon X \to Y$ be a JM-morphism and ℓ its left adjoint.*

(1) *If $k \in K(Y)$, then* $\Box_{S_f}({\uparrow}k \cap Y_m) = {\uparrow}\ell(k) \cap X_m$.
(2) *If $k, l \in K(Y)$, then* $\Delta_{\mathcal{C}_Y}(({\uparrow}k \cap Y_m) \cup ({\uparrow}l \cap Y_m)) = {\uparrow}(k \wedge l) \cap Y_m$.

Proof. (1) For $x \in X_m$, we have

$$\begin{aligned} x \in \Box_{S_f}({\uparrow}k \cap X_m) &\iff S_f[x] \subseteq {\uparrow}k \cap X_m \\ &\iff {\uparrow}f(x) \cap X_m \subseteq {\uparrow}k \cap X_m \\ &\iff k \leq f(x) \\ &\iff x \in f^{-1}({\uparrow}k) \cap X_m \\ &\iff x \in {\uparrow}\ell(k) \cap X_m, \end{aligned}$$

where the third equivalence follows from Lemma 6.6(1).

(2) By Lemma 6.6(2), for each $m \in K(Y)$ we have

$$\begin{aligned}(\uparrow k \cap Y_m) \cup (\uparrow l \cap Y_m) \subseteq \uparrow m \cap Y_m &\iff m \le k, l \\ &\iff m \le k \wedge l \\ &\iff \uparrow(k \wedge l) \subseteq \uparrow m.\end{aligned}$$

Because $k \wedge l \in K(Y)$ and $\Delta_{\mathcal{C}_Y}(B) = \bigcap\{\uparrow m \cap Y_m : B \subseteq \uparrow m\}$ for each $B \subseteq Y_m$ (see Lemma 3.5), the result follows. ∎

Proposition 6.8.

(1) *If $X \in \mathsf{JM}$, then $\mathcal{C}_X \in \mathsf{CG}$.*
(2) *If $f\colon X \to Y$ is a JM-morphism, then $S_f \subseteq X_m \times Y_m$ is a CG-morphism.*

Proof. (1) First, $\mathcal{C}_X \in \mathsf{CG}_{\mathsf{SLat}}$ by [CG20, Thm. 5.20]. Since $X \in \mathsf{JM}$, $\uparrow 1 \in \mathsf{KOF}(X)$, so $X_m = X_m - \uparrow 1 \in \mathcal{K}_X$. Next, let $U, V \in \mathcal{K}_X$. Then there are $k, l \in K(X)$ with $U = X_m - \uparrow k$ and $V = X_m - \uparrow l$. Let $W \in \mathcal{K}_X$ with $W \subseteq U \cap V$, and write $W = X_m - \uparrow m$ for some $m \in K(X)$. Then $m \le k, l$ by Lemma 6.6(2), so $m \le k \wedge l$. Because X is coherent, $k \wedge l \in K(X)$, so $\uparrow(k \wedge l) \in \mathsf{KOF}(X)$. Thus, $W \subseteq X_m - \uparrow(k \wedge l)$. This shows that $\bigcup\{W \in \mathcal{K}_X : W \subseteq U \cap V\} \subseteq X_m - \uparrow(k \wedge l)$, and the reverse inclusion holds since $X_m - \uparrow(k \wedge l) \in \mathcal{K}_X$. Consequently, $\mathcal{C}_X \in \mathsf{CG}$.

(2) By [CG20, Prop. 5.24], S_f is a $\mathsf{CG}_{\mathsf{SLat}}$-morphism. Let $x \in X_m$. If ℓ is the left adjoint of f, then $f(x) = 1$ implies $x = 1$ (because $\ell(1) = 1$). Thus, since $1 \notin X_m$, we have $f(x) \ne 1$. Therefore, Lemma 6.6(1) yields $y \in X_m$ with $f(x) \le y$. Consequently, $y \in S_f[x]$, so $S_f[x] \ne \varnothing$. Next, let $B_1, B_2 \in L\,\mathcal{C}_Y$. Then there are $m_1, m_2 \in K(Y)$ with $B_i = \uparrow m_i \cap Y_m$. By Lemma 6.7(2),

$$\Delta_{\mathcal{C}_Y}(B_1 \cup B_2) = \Delta_{\mathcal{C}_Y}((\uparrow m_1 \cap Y_m) \cup (\uparrow m_2 \cap Y_m)) = \uparrow(m_1 \wedge m_2) \cap Y_m,$$

so

$$\Box_{S_f} \Delta_{\mathcal{C}_Y}(B_1 \cup B_2) = \uparrow \ell(m_1 \wedge m_2) \cap X_m$$

by Lemma 6.7(1). Also, using Lemma 6.7 again,

$$\begin{aligned}\Delta_{\mathcal{C}_X}(\Box_{S_f} B_1 \cup \Box_{S_f} B_2) &= \Delta_{\mathcal{C}_X}(\Box_{S_f}(\uparrow m_1 \cap Y_m) \cup \Box_{S_f}(\uparrow m_2 \cap Y_m)) \\ &= \Delta_{\mathcal{C}_X}((\uparrow \ell(m_1) \cap X_m) \cup (\uparrow \ell(m_2) \cap X_m)) \\ &= \uparrow(\ell(m_1) \wedge \ell(m_2)) \cap X_m.\end{aligned}$$

Because $\ell(m_1 \wedge m_2) = \ell(m_1) \wedge \ell(m_2)$, we conclude

$$\begin{aligned}\Box_{S_f} \Delta_{\mathcal{C}_Y}(B_1 \cup B_2) &= \uparrow \ell(m_1 \wedge m_2) \cap X_m \\ &= (\uparrow \ell(m_1) \wedge \ell(m_2)) \cap X_m \\ &= \Delta_{\mathcal{C}_X}(\Box_{S_f} B_1 \cup \Box_{S_f} B_2).\end{aligned}$$

Consequently, S_f is a CG-morphism. ∎

We next recall the functor $\mathbb{H}\colon \mathsf{CG}_{\mathsf{SLat}} \to \mathsf{JM}_{\mathsf{SLat}}$ (defined immediately before [CG20, Thm. 5.30]). Let $\mathcal{C} = (X, \mathcal{K})$ be a CG-space. Set

$$\mathcal{H}(\mathcal{C}) = \left\{\bigcup \mathcal{S} \in \wp(X) : \mathcal{S} \subseteq \mathcal{K}\right\},$$

and for $U \in \mathcal{K}$ set

$$\Psi_U = \{H \in \mathcal{H}(\mathcal{C}) : U \subseteq H\}.$$

By [CG20, Thm. 5.9], if we topologize $\mathcal{H}(\mathcal{C})$ with the subbasis $\{\Psi_U : U \in \mathcal{K}\}$, we obtain $\mathcal{H}(\mathcal{C}) \in \mathsf{JM}_{\mathsf{SLat}}$. In particular, it is a complete lattice, where join is union, and meet is given by $\bigwedge \mathcal{S} = \bigcup \{U \in \mathcal{K} : U \subseteq \bigcap \mathcal{S}\}$.

Let $\mathcal{C}_1 = (X_1, \mathcal{K}_1)$ and $\mathcal{C}_2 = (X_2, \mathcal{K}_2)$ be CG-spaces. For a $\mathsf{CG}_{\mathsf{SLat}}$-morphism $S \subseteq X_1 \times X_2$, define $f_S \colon \mathcal{H}(\mathcal{C}_1) \to \mathcal{H}(\mathcal{C}_2)$ by

$$f_S(H) = \bigcup \{V \in \mathcal{K}_2 : S^{-1}[V] \subseteq H\}.$$

Then f_S is a $\mathsf{JM}_{\mathsf{SLat}}$-morphism, and the assignment $\mathcal{C} \mapsto \mathcal{H}(\mathcal{C})$ and $S \mapsto f_S$ defines the functor $\mathbb{H} \colon \mathsf{CG}_{\mathsf{SLat}} \to \mathsf{JM}_{\mathsf{SLat}}$ [CG20, Thm. 5.30]. Proposition 6.10 shows that $\mathbb{H}$ restricts to a functor $\mathsf{CG} \to \mathsf{JM}$, the proof of which requires the following:

Lemma 6.9. *Let $\mathcal{C} = (X, \mathcal{K})$ be a CG-space.*

(1) $\mathsf{KOF}(\mathcal{H}(\mathcal{C})) = \{\Psi_U : U \in \mathcal{K}\}$.
(2) $K(\mathcal{H}(\mathcal{C})) = \mathcal{K}$.

Proof. (1) This is proved in [CG20, Prop. 5.7].

(2) Since $\Psi_U = {\uparrow} U$ for each $U \in \mathcal{K}$, it is enough to observe that $\mathcal{H}(\mathcal{C}) \in \mathsf{JM}_{\mathsf{SLat}}$ and apply (1) and Lemma 3.5. ∎

Proposition 6.10.

(1) *If $\mathcal{C} \in \mathsf{CG}$, then $\mathcal{H}(\mathcal{C}) \in \mathsf{JM}$.*
(2) *If $\mathcal{C}_1, \mathcal{C}_2 \in \mathsf{CG}$ and $S \colon \mathcal{C}_1 \to \mathcal{C}_2$ is a CG-morphism, then $f_S \colon \mathcal{H}(\mathcal{C}_1) \to \mathcal{H}(\mathcal{C}_2)$ is a JM-morphism.*

Proof. (1) Let $\mathcal{C} = (X, \mathcal{K})$. By [CG20, Thm. 5.9], $\mathcal{H}(\mathcal{C}) \in \mathsf{JM}_{\mathsf{SLat}}$. To see that $\mathcal{H}(\mathcal{C}) \in \mathsf{JM}$, we need to see that $\mathsf{KOF}(\mathcal{H}(\mathcal{C}))$ is a bounded sublattice of $\mathsf{FSat}(\mathcal{H}(\mathcal{C}))$. By Lemma 6.9(1), we need to show that if $U, V \in \mathcal{K}$, then $\Psi_U \vee \Psi_V \in \mathsf{KOF}(\mathcal{H}(\mathcal{C}))$, where the join is taken in $\mathsf{FSat}(\mathcal{H}(\mathcal{C}))$. Because $U \wedge V \in \mathcal{K}$, the join is contained in $\Psi_{U \wedge V}$. If $\mathcal{F}$ is an open filter of $\mathcal{H}(\mathcal{C})$ containing $\Psi_U \cup \Psi_V$, then $U, V \in \mathcal{F}$, so $U \wedge V \in \mathcal{F}$, and hence $\Psi_{U \wedge V} \subseteq \mathcal{F}$. Thus, $\Psi_U \vee \Psi_V = \Psi_{U \wedge V} \in \mathsf{KOF}(\mathcal{H}(\mathcal{C}))$. Since $\{X\} = \Psi_X$, the result follows.

(2) Let $\mathcal{C}_1 = (X_1, \mathcal{K}_1)$ and $\mathcal{C}_2 = (X_2, \mathcal{K}_2)$. By [CG20, Prop. 5.27], f_S is a $\mathsf{JM}_{\mathsf{SLat}}$-morphism, and hence an AlgLat_σ-morphism. By Lemma 3.13, to see that f_S is a JM-morphism, it suffices to show that the left adjoint of f_S preserves finite meets of compact elements. But the left adjoint of f_S is given by $S^{-1} \colon \mathcal{H}(\mathcal{C}_2) \to \mathcal{H}(\mathcal{C}_1)$ (which is well defined by Remark 6.2(3)). By Remark 6.4(2), S^{-1} preserves binary meets from $\mathcal{K}_2$. The result follows since $K(\mathcal{H}(\mathcal{C}_2)) = \mathcal{K}_2$ by Lemma 6.9(2). ∎

We are ready to prove that CG is equivalent to JM.

Theorem 6.11. *The functors $\mathbb{M}$ and $\mathbb{H}$ restrict to yield an equivalence between CG and JM.*

Proof. By [CG20, Thm. 20], $\mathbb{M}$ and $\mathbb{H}$ yield an equivalence between $\mathsf{JM}_{\mathsf{SLat}}$ and $\mathsf{CG}_{\mathsf{SLat}}$. Therefore, in view of Propositions 6.8 and 6.10, it is enough to observe that each $\mathsf{JM}_{\mathsf{SLat}}$-isomorphism is a JM-isomorphism and each $\mathsf{CG}_{\mathsf{SLat}}$-isomorphism is a CG-isomorphism.

The former is obvious since these are order-isomorphisms. To see the latter, let S be a $\mathsf{CG}_{\mathsf{SLat}}$-isomorphism from $\mathcal{C}_1$ to $\mathcal{C}_2$ with inverse S'. It is sufficient to show that S is a CG-morphism. By [CG20, Thm. 3.24], $\Box_S\colon L\,\mathcal{C}_2 \to L\,\mathcal{C}_1$ is an SLat-isomorphism between lattices. Therefore, $\Box_S$ is a Lat-isomorphism, and so S satisfies Definition 6.3(4). To see that it also satisfies Definition 6.3(3), let $S[x] = \varnothing$. Then $x \in \Box_S \Box_{S'} B$ for each $B \in L\,\mathcal{C}_1$. Thus,

$$(\forall x' \in X_1, \forall B \in L\,\mathcal{C}_1)\,(x(\,S' \star S\,)x' \Rightarrow x' \in B).$$

Since $\leq\, = S' \star S$, we have

$$(\forall x' \in X_1, \forall B \in L\,\mathcal{C}_1)\,(x \leq x' \Rightarrow x' \in B),$$

and hence ${\uparrow}x = \bigcap L\,\mathcal{C}_1$. On the other hand, since $\bigcup \mathcal{K} = X$, we see that $\bigcap L\,\mathcal{C}_1 = \varnothing$. The resulting contradiction shows that $S[x] \neq \varnothing$. Thus, each $\mathsf{CG}_{\mathsf{SLat}}$-isomorphism is a CG-isomorphism, concluding the proof. ■

As an immediate consequence of the above theorem and Corollary 3.15, we obtain:

Corollary 6.12 ([CG20, Thm. 3.32]). CG *is dually equivalent to* Lat.

Remark 6.13. The functor from CG to Lat sends $\mathcal{C} = (X, \mathcal{K})$ to $L\,\mathcal{C}$ and $S\colon \mathcal{C}_1 \to \mathcal{C}_2$ to $\Box_S\colon L\,\mathcal{C}_2 \to L\,\mathcal{C}_1$. To describe the functor in the other direction, for a lattice A, let X_A be the set of meet-irreducible filters of A, let $\mathfrak{s}\colon A \to \wp(X_A)$ be the Stone map

$$\mathfrak{s}(a) = \{x \in X_A : a \in x\},$$

and let $\mathcal{K}_A = \{-\mathfrak{s}(a) : a \in A\}$. Then the functor from Lat to CG sends A to the pair $\mathcal{C}(A) = (X_A, \mathcal{K}_A)$ and $\alpha\colon A \to B$ to $S_\alpha\colon X_B \to X_A$ given by $y\, S_\alpha\, x$ if $\alpha^{-1}(y) \subseteq x$.

Putting together the results of Chapters 2 to 6, we arrive at the isomorphisms and equivalences of the upper part of Figure 1:

$$\begin{tikzcd}
\mathsf{CohLat}_\sigma \arrow[r, leftrightarrow, "3.14"] \arrow[d, leftrightarrow, "2.18"] & \mathsf{JM} \arrow[r, "6.11"] \arrow[d, leftrightarrow, "4.12(2)"] & \mathsf{CG} \\
\mathsf{CohLat}_\lambda \arrow[r, leftrightarrow, "4.12(1)"] & \mathsf{BDGM} \arrow[r, leftrightarrow, "5.13"] & \mathsf{Hs}
\end{tikzcd}$$

7. The Dunn–Hartonas approach

Prior to Hartonas [Har97], Dunn and Hartonas [HD97] developed another duality for Lat. For this they introduced the notion of an F-space (filter space), a canonical L-frame (lattice frame), and proved that the category of canonical L-frames is dually equivalent to Lat. An explicit axiomatization of the category of canonical L-frames was not given in [HD97]. We fill in this gap and show that it is equivalent to CohLat_λ, thus recovering Dunn–Hartonas duality from Hartonas duality.

Definition 7.1 ([HD97, Def. 2.1]). A topological space X is an *F-space* if X is a partially ordered Stone space such that

(1) X is a complete lattice;
(2) there is a family X^* of principal clopen upsets such that $X^* \cup \{-V : V \in X^*\}$ is a subbasis of open sets;
(3) the set $\{x \in X : {\uparrow}x \in X^*\}$ is join-dense in X;
(4) X^* is closed under finite intersections.

We show that F-spaces are precisely algebraic lattices with the Lawson topology. For this we require the following lemma.

Lemma 7.2. *Let X be an F-space.*

(1) *The set $\{x \in X : {\uparrow}x \in X^*\}$ is closed under finite joins.*
(2) *${\uparrow}x$ is closed for each $x \in X$.*
(3) *If ${\uparrow}x$ clopen, then $x \in K(X)$.*
(4) *$X^* = \{{\uparrow}x : x \in K(X)\}$.*

Proof. (1) If S is a finite subset of $\{x \in X : {\uparrow}x \in X^*\}$, then ${\uparrow}\bigvee S = \bigcap\{{\uparrow}x : x \in S\}$. Therefore, $\bigvee S \in \{x \in X : {\uparrow}x \in X^*\}$ by Definition 7.1(4).

(2) Let $x \in X$. By Definition 7.1(3), $x = \bigvee S$ for some $S \subseteq X$ with ${\uparrow}k \in X^*$ for each $k \in S$. Therefore, $\bigcap\{{\uparrow}k : k \in S\} = {\uparrow}x$. Because each ${\uparrow}k$ is clopen, ${\uparrow}x$ is closed.

(3) Let ${\uparrow}x$ be clopen and $x \leq \bigvee S$ for some $S \subseteq X$. Then $\bigcap\{{\uparrow}y : y \in S\} \subseteq {\uparrow}x$. Since each ${\uparrow}y$ is closed by (2) and ${\uparrow}x$ is open, compactness shows that there are $y_1, \dots, y_n \in S$ with ${\uparrow}y_1 \cap \cdots \cap {\uparrow}y_n \subseteq {\uparrow}x$. Therefore, $x \leq y_1 \vee \cdots \vee y_n$, and so $x \in K(X)$.

(4) If ${\uparrow}x \in X^*$, then $x \in K(X)$ by (3). For the reverse inclusion, $x \in K(X)$ and Definition 7.1(3) imply that x is a finite join $x = k_1 \vee \cdots \vee k_n$ for some $k_i \in X$ with ${\uparrow}k_i \in X^*$. Thus, ${\uparrow}x = {\uparrow}k_1 \cap \cdots \cap {\uparrow}k_n \in X^*$ by Definition 7.1(4). ∎

Proposition 7.3. *Let X be a partially ordered Stone space. Then X is an F-space iff X is an algebraic lattice with the Lawson topology, and $X^* = \mathsf{CLF}(X)$.*

Proof. Let X be an F-space. Then X is a complete lattice by Definition 7.1(1). By Definition 7.1(3) and Lemma 7.2(4), X is algebraic. Because

$$\{{\uparrow}k : k \in K(X)\} \cup \{-{\uparrow}l : l \in K(X)\}$$

is a subbasis for the Lawson topology, the topology of X is the Lawson topology by Definition 7.1(2). Moreover, Lemmas 5.4 and 7.2(4) show that $X^* = \mathsf{CLF}(X)$.

Conversely, suppose that X is an algebraic lattice. Then X is a complete lattice, so Definition 7.1(1) holds. We set $X^* = \{{\uparrow}k : k \in K(X)\}$. Then Definition 7.1(2) holds since $\{{\uparrow}k : k \in K(X)\} \cup \{-{\uparrow}l : l \in K(X)\}$ is a subbasis for the Lawson topology. Because X is an algebraic lattice, $K(X)$ is join-dense in X, so Definition 7.1(3) holds. Finally, X^* is closed under finite intersections since $K(X)$ is closed under finite joins. Thus, X is an F-space. ∎

By [HD97, p. 399], the canonical L-frame associated to a bounded lattice A is the triple $(\mathsf{Filt}(A), R, \mathsf{Idl}(A))$ such that $F\,R\,I$ iff $F \cap I \neq \varnothing$. In order to axiomatize such triples, it is more convenient to work with the complement $-R$ of the relation R since it turns out to be an interior relation. For Stone spaces X, Y we recall (see, e.g., [Esa78, Def. 6], [BG+19, Def. 5.1]) that a relation $R \subseteq X \times Y$ is *interior* provided that

(1) $R[x]$ is closed for each $x \in X$ and $R[U]$ is clopen for each clopen $U \subseteq X$;
(2) $R^{-1}[y]$ is closed for each $y \in Y$ and $R^{-1}[V]$ is clopen for each clopen $V \subseteq Y$.

Remark 7.4. We recall [Hal62, p. 53] that a relation R between Stone spaces X and Y is continuous if $R[x]$ is closed for each $x \in X$ and $R^{-1}[V]$ is clopen for each clopen $V \subseteq Y$. Interior relations are strengthening of continuous relations in that R is interior iff R and R^{-1} are continuous.

The next two definitions are reminiscent to what happens in positive modal logic; see [Gol89, CJ99] or more recent [Gol20, MB23].

Definition 7.5. Let $X, Y \in \mathsf{AlgLat}_\lambda$ and $R \subseteq X \times Y$ be a relation. We call R a *DH-relation* provided

(1) R is an interior relation;
(2) for each $x \in X$ and $y \in Y$, both $-R[x]$ and $-R^{-1}[y]$ are filters;
(3) $R[x] \subseteq R[x'] \Rightarrow x' \leq x$;
(4) $R^{-1}[y] \subseteq R^{-1}[y'] \Rightarrow y' \leq y$.

Remark 7.6. By Definition 7.5(2), the reverse implications in Definition 7.5(3, 4) also hold. To see the reverse implication of Definition 7.5(3), suppose $x' \leq x$. If $R[x] \not\subseteq R[x']$, then there is $y \in R[x]$ such that $y \notin R[x']$. Therefore, $x' \in -R^{-1}[y]$. Since $-R^{-1}[y]$ is a filter and $x' \leq x$, we have $x \in -R^{-1}[y]$, so $y \notin R[x]$, a contradiction. The argument for the reverse implication of Definition 7.5(4) is similar.

Definition 7.7.

(1) We call a triple (X, R, Y) a *Dunn–Hartonas space*, or *DH-space* for short, provided $X, Y \in \mathsf{CohLat}_\lambda$ and $R \subseteq X \times Y$ is a DH-relation.
(2) A *DH-morphism* between DH-spaces (X_1, R_1, Y_1) and (X_2, R_2, Y_2) is a pair of CohLat_λ-morphisms $f \colon X_1 \to X_2$ and $g \colon Y_1 \to Y_2$ which satisfies
   (a) if $x \, R_1 \, y$, then $f(x) \, R_2 \, g(y)$;
   (b) if $x' \, R_2 \, g(y)$, then there is $x \in X_1$ with $x \, R_1 \, y$ and $x' \leq f(x)$;
   (c) if $f(x) \, R_2 \, y'$, then there is $y \in Y_1$ with $x \, R_1 \, y$ and $y' \leq g(y)$.
(3) Let DH be the category of DH-spaces and DH-morphisms, where composition is given by $(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1)$ and $(1_X, 1_Y)$ is the identity morphism on (X, R, Y).

Remark 7.8. Let $(X_1, R_1, Y_1), (X_2, R_2, Y_2) \in \mathsf{DH}$ and $f \colon X_1 \to X_2$ and $g \colon Y_1 \to Y_2$ be a pair of functions. It is straightforward to verify that (f, g) is a DH-isomorphism iff f, g are order-isomorphisms such that $x \, R_1 \, y$ iff $f(x) \, R_2 \, g(y)$. When this occurs, (f^{-1}, g^{-1}) is the inverse of (f, g) in DH. This will be used throughout.

We now establish an equivalence between DH and CohLat_λ. It follows from Definition 7.7 that there is a forgetful functor $\mathsf{DH} \to \mathsf{CohLat}_\lambda$:

Proposition 7.9. *There is a functor* $\mathbb{F} \colon \mathsf{DH} \to \mathsf{CohLat}_\lambda$ *defined by* $\mathbb{F}(X, R, Y) = X$ *and* $\mathbb{F}(f, g) = f$ *for each* $(X, R, Y) \in \mathsf{DH}$ *and each* DH*-morphism* (f, g).

To define a functor in the opposite direction, we require some preparation.

Notation 7.10. Let $R \subseteq X \times Y$ be a DH-relation. Using the notation of [Bir79, p. 124], we denote by $\phi\colon \wp X \to \wp Y$ and $\psi\colon \wp Y \to \wp X$ the induced (antitone) Galois connection maps associated to the complement $-R$ of the relation R. Therefore,

$$\phi A = -R[A] \quad \text{and} \quad \psi B = -R^{-1}[B]$$

for each $A \subseteq X$ and $B \subseteq Y$. If $x \in X$ and $y \in Y$, we write $\phi(x)$ for $\phi(\{x\})$ and $\psi(y)$ for $\psi(\{y\})$.

Lemma 7.11. *Let $R \subseteq X \times Y$ be a DH-relation with $X, Y \in \mathsf{AlgLat}_\lambda$.*

(1) *If $y \in Y$, then $\psi(y)$ is an open filter of X.*
(2) *If $x \in X$, then $\phi(x)$ is an open filter of Y.*
(3) *$\psi\phi U = U$ for $U \in \mathsf{CLF}(X)$ and $\phi\psi V = V$ for $V \in \mathsf{CLF}(Y)$.*
(4) *ϕ and ψ restrict to yield dual isomorphisms between $\mathsf{CLF}(X)$ and $\mathsf{CLF}(Y)$.*

Proof. (1), (2) By Definition 7.5(2), $\psi(y)$ and $\phi(x)$ are filters. They are open by Definition 7.5(1).

(3) Let $U \in \mathsf{CLF}(X)$. Then $U = {\uparrow}k$ for some $k \in K(X)$ by Lemma 5.4. We have

$$\begin{aligned}\psi\phi U &= -R^{-1}[-R[U]] = \{x \in X : R[x] \subseteq R[{\uparrow}k]\} = \{x \in X : R[x] \subseteq R[k]\}\\ &= \{x \in X : k \leq x\} = U,\end{aligned}$$

where the third equality holds since $R^{-1}[y]$ is a downset for each $y \in Y$, and the fourth equality by Remark 7.6. Similarly, if $V \in \mathsf{CLF}(Y)$, then $V = \phi\psi V$.

(4) Let $U \in \mathsf{CLF}(X)$. Then ϕU is clopen since R is interior, and

$$\phi U = -R[U] = -\bigcup\{R[x] : x \in U\} = \bigcap\{\phi(x) : x \in U\}$$

is a filter by (2). Thus, $\phi U \in \mathsf{CLF}(Y)$. A similar argument, using (1) rather than (2), shows that $\psi V \in \mathsf{CLF}(X)$ for each $V \in \mathsf{CLF}(Y)$. Since ϕ and ψ are order-reversing maps, it suffices to apply (3). ■

Remark 7.12. Let $R \subseteq X \times Y$ be a DH-relation with X, Y compact algebraic lattices. By Lemma 7.11(4), $\mathsf{CLF}(X)$ and $\mathsf{CLF}(Y)$ are dually isomorphic meet-semilattices. Therefore, each is a bounded lattice. It then follows from Lemma 5.4 that $K(X)$ and $K(Y)$ are bounded lattices, and hence X and Y are coherent lattices. In particular, the bounds of $\mathsf{CLF}(X)$ are ${\uparrow}1$ and $X = {\uparrow}0$, and join and meet are given by

$${\uparrow}k \vee {\uparrow}l = {\uparrow}(k \wedge l) \quad \text{and} \quad {\uparrow}k \wedge {\uparrow}l = {\uparrow}k \cap {\uparrow}l = {\uparrow}(k \vee l)$$

for $k, l \in K(X)$.

Lemma 7.13. *Let $X \in \mathsf{CohLat}_\lambda$. Then $\mathsf{OF}(X)$ is isomorphic to $\mathsf{Idl}(\mathsf{CLF}(X))$. Consequently, $\mathsf{OF}(X) \in \mathsf{CohLat}_\lambda$ and $K(\mathsf{OF}(X)) = \mathsf{CLF}(X)$.*

Proof. Every open filter is a Lawson-open upset, so a Scott-open set by Theorem 2.10(2). Therefore, it is a union from $\mathsf{CLF}(X)$ by Theorem 2.8(3) and Lemma 5.4. This union is a directed union by the description of join in $\mathsf{CLF}(X)$ given in Remark 7.12. It is then straightforward to see that the desired isomorphism is the map $f\colon \mathsf{OF}(X) \to \mathsf{Idl}(\mathsf{CLF}(X))$ defined by $f(F) = \{U \in \mathsf{CLF}(X) : U \subseteq F\}$, whose inverse $g\colon \mathsf{Idl}(\mathsf{CLF}(X)) \to \mathsf{OF}(X)$ is given by $g(I) = \bigcup\{U \in \mathsf{CLF}(X) : U \in I\}$. Thus, $\mathsf{OF}(X) \in \mathsf{CohLat}_\lambda$ by Theorem 2.5(3), and $K(\mathsf{OF}(X)) = \mathsf{CLF}(X)$ because the compact elements of $\mathsf{Idl}(\mathsf{CLF}(X))$ are the principal downsets, which are sent to $\mathsf{CLF}(X)$ via g. ■

Recalling that we are axiomatizing the complement of the relation of a canonical L-frame, we define:

Definition 7.14. If $X \in \mathsf{CohLat}_\lambda$, define $R \subseteq X \times \mathsf{OF}(X)$ by $x\,R\,U$ if $x \notin U$.

Proposition 7.15. *If $X \in \mathsf{CohLat}_\lambda$, then $(X, R, \mathsf{OF}(X)) \in \mathsf{DH}$.*

Proof. Since $X \in \mathsf{CohLat}_\lambda$, we have $\mathsf{OF}(X) \in \mathsf{CohLat}_\lambda$ by Lemma 7.13. We show that R is a DH-relation. For Definition 7.5(2), if $x \in X$, then $-R[x] = \{U \in \mathsf{OF}(X) : x \in U\}$ is clearly a filter. If $U \in \mathsf{OF}(X)$, then $-R^{-1}[U] = \{x \in X : x \in U\} = U$, so is a filter. For Definition 7.5(3), suppose that $R[x] \subseteq R[x']$. If $x' \not\leq x$, then there is $k \in K(X)$ with $k \leq x'$ and $k \not\leq x$. If $V = {\uparrow}k$, then $V \in \mathsf{OF}(X)$ with $V \in R[x]$ and $V \notin R[x']$. This contradiction shows that $x' \leq x$. To see Definition 7.5(4), let $U, U' \in \mathsf{OF}(X)$ with $R^{-1}[U] \subseteq R^{-1}[U']$. If $U' \not\subseteq U$, then there is $x \in X$ with $x \in U' - U$. Therefore, $x \in R^{-1}[U]$ and $x \notin R^{-1}[U']$, a contradiction. Thus, $U' \subseteq U$.

It is left to verify Definition 7.5(1). Let $x \in X$. To see that $R[x]$ is closed, let $V \in -R[x]$, so $x \in V$. Because ${\uparrow}x = \bigcap\{{\uparrow}k : k \in K(X),\ k \leq x\}$, compactness of the Lawson topology shows that there are $k_1, \dots, k_n \in K(X)$ with each $k_i \leq x$ and ${\uparrow}k_1 \cap \dots \cap {\uparrow}k_n \subseteq V$. Set $k = k_1 \vee \dots \vee k_n$. Then $k \in K(X)$ with $k \leq x$ and ${\uparrow}k = {\uparrow}k_1 \cap \dots \cap {\uparrow}k_n \subseteq V$. Let $U = {\uparrow}k$, so $U \in \mathsf{CLF}(X)$ by Lemma 5.4. Then $\mathcal{W}_U := \{F \in \mathsf{OF}(X) : U \subseteq F\}$ is Scott-open by Lemma 7.13, and hence Lawson-open. Therefore, $\mathcal{W}_U$ is an open neighborhood of V. Furthermore, if $F \in \mathcal{W}_U$, then $k \in F$, so $x \in F$, and hence $x \not\!R\, F$. Thus, $\mathcal{W}_U \subseteq -R[x]$, and so $R[x]$ is closed.

Next, let C be clopen in $\mathsf{OF}(X)$. By Theorem 2.10(1), we may assume that $C = \mathcal{W}_U - \mathcal{W}_V$ with $U, V \in \mathsf{CLF}(X)$. If $C = \varnothing$, then $R^{-1}[C] = \varnothing$ is clopen. Otherwise, $V \not\subseteq U$. We show that $R^{-1}[C] = -U$. If $x \in R^{-1}[C]$, then there is $W \in C$ with $x\,R\,W$. This means that $U \subseteq W$, $V \not\subseteq W$, and $x \notin W$. Therefore, $x \notin U$, and so $x \in -U$. Conversely, if $x \in -U$, then $U \in C$ and $x\,R\,U$, so $x \in R^{-1}[C]$. Thus, $R^{-1}[C] = -U$, and hence is clopen. This implies that R is a continuous relation, and so $R^{-1}[F]$ is closed for each $F \in \mathsf{OF}(X)$ (see, e.g., [Esa19, Thm. 3.1.2]).

Finally, let U be clopen in X. To see that $R[U]$ is clopen in $\mathsf{OF}(X)$, by Theorem 2.10(1) it suffices to assume that $U = {\uparrow}k - {\uparrow}l$ for some $k, l \in K(X)$. If $U = \varnothing$ then $R[U] = \varnothing$ is clopen, so we assume that U is nonempty, which means that $l \not\leq k$. We claim that $R[U] = -\mathcal{W}_{{\uparrow}k}$. To see this, if $V \in \mathsf{OF}(X)$ with $V \in R[U]$, then there is $x \notin V$ with $k \leq x$ and $l \not\leq x$. Therefore, $k \notin V$, so ${\uparrow}k \not\subseteq V$, and hence $V \in -\mathcal{W}_{{\uparrow}k}$. For the reverse inclusion, if $V \in -\mathcal{W}_{{\uparrow}k}$, then $k \notin V$. Therefore, $k \in U$ and $k \notin V$, so $V \in R[U]$. Thus, $R[U]$ is clopen, and hence R is an interior relation.

Consequently, R is a DH-relation, yielding $(X, R, \mathsf{OF}(X)) \in \mathsf{DH}$. ■

Lemma 7.16. *If $f\colon X \to Y$ is a CohLat_λ-morphism, then $f^{-1}\colon \mathsf{OF}(Y) \to \mathsf{OF}(X)$ has a right adjoint $r\colon \mathsf{OF}(X) \to \mathsf{OF}(Y)$ which is a CohLat_λ-morphism.*

Proof. Since f is a CohLat_λ-morphism, it is a JM-morphism by Theorem 2.19(2) and Lemma 3.13, and hence $f^{-1}\colon \mathsf{OF}(Y) \to \mathsf{OF}(X)$ preserves arbitrary joins by [MJ14, Lem. 5.3]. It therefore has a right adjoint $r\colon \mathsf{OF}(X) \to \mathsf{OF}(Y)$ which preserves all meets. Recall that if $U \in \mathsf{OF}(X)$, then $r(U) = \bigvee\{W \in \mathsf{OF}(Y) : f^{-1}(W) \subseteq U\}$. Because this family is directed and each $W \in \mathsf{OF}(Y)$ is a union from $\mathsf{CLF}(Y)$, we may write

$$r(U) = \bigcup\{V \in \mathsf{CLF}(Y) : f^{-1}(V) \subseteq U\}. \tag{7.1}$$

To show that r is a CohLat_λ-morphism, it suffices to show that r preserves directed joins (see Theorems 2.8(5) and 2.10(3)). Let $\mathcal{S} \subseteq \mathsf{OF}(X)$ be directed and set $U = \bigcup \mathcal{S}$. Since f^{-1} preserves joins, $\{V \in \mathsf{CLF}(Y) : f^{-1}(V) \subseteq U\}$ is directed. Therefore,

$$\begin{aligned} r(U) &= \bigcup\{V \in \mathsf{CLF}(Y) : f^{-1}(V) \subseteq U\} \\ &= \bigcup\{V \in \mathsf{CLF}(Y) : f^{-1}(V) \subseteq W \text{ for some } W \in \mathcal{S}\} \\ &= \bigcup\{r(W) : W \in \mathcal{S}\} = \bigvee\{r(W) : W \in \mathcal{S}\}, \end{aligned}$$

where the second equality holds since $f^{-1}(V)$ is compact for each $V \in \mathsf{CLF}(Y)$ and $\mathcal{S}$ is directed, and the final equality holds since the set $\{r(W) : W \in \mathcal{S}\}$ is directed. ∎

Proposition 7.17. *There is a functor* $\mathbb{E}\colon \mathsf{CohLat}_\lambda \to \mathsf{DH}$ *which sends each* $X \in \mathsf{CohLat}_\lambda$ *to* $\mathbb{E}(X) = (X, R_X, \mathsf{OF}(X))$ *and each* CohLat_λ*-morphism* $f\colon X_1 \to X_2$ *to* $\mathbb{E}(f) = (f, r)$, *where* R_X *is defined in Definition* 7.14 *and* r *is right adjoint to* $f^{-1}\colon \mathsf{OF}(X_2) \to \mathsf{OF}(X_1)$.

Proof. By Proposition 7.15, $\mathbb{E}(X) \in \mathsf{DH}$. Let $f\colon X_1 \to X_2$ be a CohLat_λ-morphism. We show that (f, r) is a DH-morphism. The map r is a CohLat_λ-morphism by Lemma 7.16. Recall from (7.1) that $r(U) = \bigcup\{V \in \mathsf{CLF}(X_2) : f^{-1}(V) \subseteq U\}$ for each $U \in \mathsf{OF}(X_1)$. If $x \in X_1$ and $U \in \mathsf{OF}(X_1)$ with $x \, R_{X_1} \, U$, then $x \notin U$. If $V \in \mathsf{OF}(X_2)$ with $f^{-1}(V) \subseteq U$, then $x \notin f^{-1}(V)$, so $f(x) \notin V$. Therefore, $f(x) \notin r(U)$, and hence $f(x) \, R_{X_2} \, r(U)$, so Definition 7.7(2a) holds.

Next, suppose that $x' \, R_{X_2} \, r(U)$ for some $x' \in X_2$ and $U \in \mathsf{OF}(X_1)$. Then $x' \notin V$ for each $V \in \mathsf{CLF}(X_2)$ with $f^{-1}(V) \subseteq U$. We have ${\uparrow}x' = \bigcap\{V \in \mathsf{CLF}(X_2) : x' \in V\}$, so $f^{-1}({\uparrow}x') = \bigcap\{f^{-1}(V) : x' \in V\}$. If $f^{-1}({\uparrow}x') \subseteq U$, then compactness of X_1 implies that there are $V_1, \dots, V_n \in \mathsf{CLF}(X_2)$ with $f^{-1}(V_1) \cap \dots \cap f^{-1}(V_n) \subseteq U$. If $V = V_1 \cap \dots \cap V_n$, then $V \in \mathsf{CLF}(X_2)$, $x' \in V$, and $f^{-1}(V) \subseteq U$, a contradiction to $x' \notin r(U)$. Therefore, $f^{-1}({\uparrow}x') \not\subseteq U$, so there is $x \in f^{-1}({\uparrow}x') - U$. Thus, $x \, R_{X_1} \, U$ and $x' \leq f(x)$, and hence Definition 7.7(2b) holds.

Finally, suppose that $f(x) \, R_{X_2} \, V$ for some $x \in X_1$ and $V \in \mathsf{OF}(X_2)$. Then $f(x) \notin V$, so $x \notin f^{-1}(V)$. If $U = f^{-1}(V)$, then $U \in \mathsf{OF}(X_1)$, so $V \subseteq r(U)$ and $x \, R_{X_1} \, U$. Consequently, Definition 7.7(2c) holds, and thus (f, r) is a DH-morphism. ∎

In what follows we will require the following result from modal logic ([Esa74, Lem. 3], [BBH15, Lem. 2.17]). We recall that a relation $R \subseteq X \times Y$ between topological spaces is *point-closed* if $R[x]$ is closed for each $x \in X$.

Lemma 7.18 (Esakia's Lemma). *Let* $R \subseteq X \times Y$ *be a point-closed relation between compact Hausdorff spaces. If* $\{C_i : i \in I\}$ *is a (nonempty) down-directed family of closed subsets of* Y, *then*

$$R^{-1}\left[\bigcap C_i\right] = \bigcap R^{-1}[C_i].$$

Lemma 7.19. *Let* $(X, R, Y) \in \mathsf{DH}$.

(1) *If* $W \in \mathsf{OF}(X)$, *then there is* $y \in Y$ *with* $W = \psi(y)$.
(2) *The map* $\nu\colon Y \to \mathsf{OF}(X)$, *defined by* $\nu(y) = \psi(y)$, *is a* CohLat_λ*-isomorphism.*

Proof. (1) Let W be an open filter of X. Because W is a union from $\mathsf{CLF}(X)$ and R commutes with unions, $\phi W = \bigcap\{\phi U : U \in \mathsf{CLF}(X), U \subseteq W\}$. Since X is coherent, $\{U \in \mathsf{CLF}(X) : U \subseteq W\}$ is up-directed. Therefore, $\{\phi U : U \in \mathsf{CLF}(X), U \subseteq W\}$ is a down-directed family of closed sets, as each ϕU is clopen (since R is interior). Consequently, by Esakia's lemma,

$$R^{-1}[\phi W] = \bigcap\{R^{-1}[\phi U] : U \in \mathsf{CLF}(X),\, U \subseteq W\},$$

and so

$$\begin{aligned}\psi\phi W = -R^{-1}[\phi W] &= \bigcup\{-R^{-1}[\phi U] : U \in \mathsf{CLF}(X),\, U \subseteq W\}\\ &= \bigcup\{\psi\phi U : U \in \mathsf{CLF}(X),\, U \subseteq W\} = W\end{aligned}$$

since $\psi\phi U = U$ for each $U \in \mathsf{CLF}(X)$ by Lemma 7.11(3). Also, Lemma 7.11(4) and Lemma 5.4 show that $\phi U = {\uparrow} l_U$ for some $l_U \in Y$. Therefore, $\phi W = \bigcap\{{\uparrow} l_U : U \subseteq W\} = {\uparrow} y$ for $y = \bigvee\{l_U : U \subseteq W\}$. Thus, $W = \psi\phi W = \psi({\uparrow} y) = -R^{-1}[{\uparrow} y] = -R^{-1}[y] = \psi(y)$.

(2) By Lemma 7.11(1), $\nu\colon Y \to \mathsf{OF}(X)$ is well defined, and is onto by (1). If $y \le y'$, then $R^{-1}[y'] \subseteq R^{-1}[y]$ since $R^{-1}[y]$ is a downset. Therefore, $\nu(y) \subseteq \nu(y')$. Conversely, if $y \not\le y'$, then $R^{-1}[y'] \not\subseteq R^{-1}[y]$ by Definition 7.5(4). Consequently, $\nu(y) \not\subseteq \nu(y')$. Thus, ν is an order-isomorphism, and hence a CohLat_λ-isomorphism. ∎

Proposition 7.20. *Let* $\mathcal{D} = (X, R, Y)$ *be a DH-space. Define*

$$\kappa_{\mathcal{D}}\colon (X, R, Y) \to (X, R_X, \mathsf{OF}(X))$$

by $\kappa_{\mathcal{D}} = (1_X, \nu_{\mathcal{D}})$, *where* $\nu_{\mathcal{D}}$ *is defined in Lemma* 7.19. *Then both* $\kappa_{\mathcal{D}}$ *and its inverse* $(1_X, \nu_{\mathcal{D}}^{-1})$ *are* DH*-morphisms, and* $\kappa\colon 1_{\mathsf{DH}} \to \mathbb{EF}$ *is a natural isomorphism.*

Proof. Let $(x, y) \in X \times Y$. Then

$$x \, R \, y \iff x \in R^{-1}[y] \iff x \notin -R^{-1}[y] = \nu_{\mathcal{D}}(y) \iff x \, R_X \, \nu_{\mathcal{D}}(y).$$

Therefore, Lemma 7.19 and Remark 7.8 show that both $\kappa_{\mathcal{D}}$ and $\kappa_{\mathcal{D}}^{-1}$ are DH-isomorphisms.

To show that κ is natural, we need to show that the following diagram commutes for each pair of DH-objects $\mathcal{D} = (X, R, Y), \mathcal{D}' = (X', R', Y')$ and each DH-morphism $(f, g)\colon \mathcal{D} \to \mathcal{D}'$:

$$\begin{tikzcd}
(X, R, Y) \arrow[r, "{(f,g)}"] \arrow[d, "\kappa_{\mathcal{D}}"'] & (X', R', Y') \arrow[d, "\kappa_{\mathcal{D}'}"] \\
(X, R_X, \mathsf{OF}(X)) \arrow[r, "{(f,r)}"] & (X', R_{X'}, \mathsf{OF}(X'))
\end{tikzcd}$$

This amounts to showing that $r \circ \nu_{\mathcal{D}} = \nu_{\mathcal{D}'} \circ g$. By (7.1) we have

$$r(\nu_{\mathcal{D}}(y)) = \bigcup\{V' \in \mathsf{CLF}(X') : f^{-1}(V') \subseteq \nu_{\mathcal{D}}(y)\}.$$

Let $x' \in r(\nu_{\mathcal{D}}(y))$. If $x' \notin \nu_{\mathcal{D}'}(g(y))$, then $x' \, R' \, g(y)$. Therefore, by Definition 7.7(2b), there is $x \in X$ with $x \, R \, y$ and $x' \le f(x)$. Because $x' \in r(\nu_{\mathcal{D}}(y))$, there is $V' \in \mathsf{CLF}(X')$ with $x' \in V'$ and $f^{-1}(V') \subseteq \nu_{\mathcal{D}}(y)$. Thus, $f(x) \in V'$ since V' is an upset, so $x \in f^{-1}(V')$, and hence $x \in \nu_{\mathcal{D}}(y) = -R^{-1}[y]$. This means that $x \not R\, y$, a contradiction. Consequently, $x' \in \nu_{\mathcal{D}'}(g(y))$.

For the reverse inclusion, let $x' \in \nu_{\mathcal{D}'}(g(y))$. Then there is $V' \in \mathsf{CLF}(X')$ such that $x' \in V' \subseteq \nu_{\mathcal{D}'}(g(y))$. If $x \in f^{-1}(V')$, then $f(x) \in \nu_{\mathcal{D}'}(g(y))$, so $f(x) \not R'\, g(y)$. Therefore, $x \not R\, y$ by Definition 7.7(2a). This means that $x \in \nu_{\mathcal{D}}(y)$, and so $f^{-1}(V') \subseteq \nu_D(y)$. Thus, $x' \in r(\nu_{\mathcal{D}}(y))$. ■

Proposition 7.20 together with $\mathbb{F}\mathbb{E} = 1_{\mathsf{CohLat}_\lambda}$ yields:

THEOREM 7.21. $\mathbb{E}$ *and* $\mathbb{F}$ *establish an equivalence of categories between* CohLat_λ *and* DH.

COROLLARY 7.22.

(1) DH *is equivalent to* JM, BDGM, Hs, *and* CG.
(2) [HD97, Thm. 2.15] DH *is dually equivalent to* Lat.

Proof. The first item is an immediate consequence of Theorems 2.18, 3.14, 4.12, 5.13, 6.11 and 7.21, and the second item follows from Theorems 2.19(2) and 7.21. ■

REMARK 7.23. The duality in Corollary 7.22(2) can be described directly as follows. A triple $(X, R, Y) \in \mathsf{DH}$ is sent to $\mathsf{CLF}(X)$ and a DH-morphism $(f, g)\colon (X, R, Y) \to (X', R', Y')$ to $f^{-1}\colon \mathsf{CLF}(X') \to \mathsf{CLF}(X)$. Going the other direction, $A \in \mathsf{Lat}$ is sent to $(\mathsf{Filt}(A), R, \mathsf{Idl}(A)) \in \mathsf{DH}$, where R is given by $F\,R\,I$ if $F \cap I = \varnothing$, and a Lat-morphism $\alpha\colon A \to B$ to $\big(\alpha^{-1}|_{\mathsf{Filt}(B)}, \alpha^{-1}|_{\mathsf{Idl}(B)}\big)$.

8. From Dunn–Hartonas to Gehrke–van Gool

In this chapter we connect the Dunn–Hartonas approach to the Gehrke–van Gool approach. The resulting spaces were introduced and studied in [GvG14]. The notion of morphism considered by Gehrke and van Gool dually characterizes special lattice homomorphisms [GvG14, Def. 3.19]. We generalize their notion of morphism by working with relations instead of functions, thus giving rise to the category GvG. We show that there is a functor from DH to GvG. In Chapter 13 we will show that this functor is an equivalence and in Chapter 14 that GvG is dually equivalent to Lat. Therefore, our generalized notion of morphism captures all bounded lattice homomorphisms.

Let $R \subseteq X \times Y$. Motivated by a standard technique in modal logic (see, e.g., [CZ97, p. 64]), we define $\Diamond_R, \Box_R\colon \wp(Y) \to \wp(X)$ by

$$\Diamond_R B = R^{-1}[B] \quad \text{and} \quad \Box_R B = -R^{-1}[-B]$$

for each $B \subseteq Y$. Similarly, define $\blacklozenge_R, \blacksquare_R\colon \wp(X) \to \wp(Y)$ by

$$\blacklozenge_R A = R[A] \quad \text{and} \quad \blacksquare_R A = -R[-A]$$

for each $A \subseteq X$. We will mainly work with $\blacklozenge_R$ and $\Box_R$, and when there is no danger of confusion, we simply write $\blacklozenge$ and $\Box$.

The following well-known lemma (see, e.g., [Pri84, Prop. 2.6], [Esa19, Thm. 3.2.1]) will be used throughout.

Lemma 8.1. *Let X be a Priestley space and let C be a closed subset of X. For each $x \in C$ there is $m \in \max C$ with $x \leq m$.*

Definition 8.2. Let X, Y be Priestley spaces and $R \subseteq X \times Y$ a relation. Define

$$X_0 = \bigcup \{\max R^{-1}[y] : y \in Y\} \quad \text{and} \quad Y_0 = \bigcup \{\max R[x] : x \in X\}.$$

Definition 8.3 ([GvG14, Def. 4.18]). We call a triple (X, R, Y) a *Gehrke–van Gool space*, or *GvG-space* for short, if X, Y are Priestley spaces and $R \subseteq X \times Y$ is a relation satisfying

(1) $x \leq x'$ iff $R[x'] \subseteq R[x]$;
(2) $y \leq y'$ iff $R^{-1}[y'] \subseteq R^{-1}[y]$;
(3) if U is a clopen upset of X, then $\blacklozenge U$ is a clopen downset of Y;
(4) if V is a clopen downset of Y, then $\Box V$ is a clopen upset of X;
(5) if $x \not R y$, then there is a clopen upset $U \subseteq X$ such that $U = \Box\blacklozenge U$, $x \in U$, and $y \notin \blacklozenge U$;
(6) if U is a clopen upset of X with $X_0 \cap \Box\blacklozenge U \subseteq U$, then $U = \Box\blacklozenge U$;
(7) if V is a clopen downset of Y with $Y_0 \cap V \subseteq \blacklozenge\Box V$, then $V = \blacklozenge\Box V$.

Remark 8.4.

(1) The order on X in the definition above is dual to the order used in [GvG14] and is more convenient to compare GvG-spaces to DH-spaces.
(2) In [GvG14] Condition (5) is phrased using clopen sets $U \subseteq X$ and $V \subseteq Y$. The latter can be recovered as $V = \blacklozenge U$.
(3) R is a closed subset of $X \times Y$, and so both R and R^{-1} are point-closed.
(4) For each $U \subseteq X$ and $V \subseteq Y$, we have $\Box\blacklozenge U = \{x \in X : R[x] \subseteq R[U]\}$ and $\blacklozenge\Box V = \{y \in Y : \exists x \text{ with } x \, R \, y \text{ and } R[x] \subseteq V\}$.
(5) Since $\blacklozenge\Box V = -\blacksquare\Diamond - V$, Definition 8.3(7) is equivalent to $Y_0 \cap \blacksquare\Diamond - V \subseteq -V$ implies $V = \blacklozenge\Box V$, which is how it is phrased in [GvG14, Def. 4.18].

Definition 8.5. Suppose that $\mathcal{G} = (X, R, Y)$ is a GvG-space. We let $\mathsf{ClopUp}(X)$ be the clopen upsets of X, and set

$$L\,\mathcal{G} = \{U \in \mathsf{ClopUp}(X) : U = \Box\blacklozenge U\}.$$

Remark 8.6. If $\mathcal{G}$ is a GvG-space, then $L\,\mathcal{G}$ is a bounded lattice, where meet and join are given by

$$U_1 \wedge U_2 = U_1 \cap U_2 \quad \text{and} \quad U_1 \vee U_2 = \Box\blacklozenge(U_1 \cup U_2)$$

for each $U_1, U_2 \in L\,\mathcal{G}$. Moreover, X is the top and $\varnothing$ is the bottom of $L\,\mathcal{G}$. To see the latter, let $x \in X_0$. Then $x \, R \, y$ for some $y \in Y$, so $R[x] \neq \varnothing$. Therefore, $x \notin \Box\varnothing = \Box\blacklozenge\varnothing$, and so $X_0 \cap \Box\blacklozenge\varnothing = \varnothing$. By Definition 8.3(6), $\varnothing = \Box\blacklozenge\varnothing$, and hence $\varnothing \in L\,\mathcal{G}$.

Lemma 8.7. *Let $\mathcal{G} = (X, R, Y)$ be a GvG-space.*

(1) ${\uparrow}x = \bigcap \{U \in L\,\mathcal{G} : x \in U\}$ *for each* $x \in X$.
(2) $L\,\mathcal{G}$ *is a basis for the open upset topology of* X.
(3) ${\downarrow}y = \bigcap \{\blacklozenge U : U \in L\,\mathcal{G}, y \in \blacklozenge U\}$ *for each* $y \in Y$.
(4) $\{\blacklozenge U : U \in L\,\mathcal{G}\}$ *is a basis for the open downset topology of* Y.
(5) *If* $U \in L\,\mathcal{G}$ *and* $x \notin U$, *then there is* $x' \in X_0$ *with* $x \leq x'$ *and* $x' \notin U$.

Proof. (1) The inclusion $\subseteq$ is clear. For the inclusion $\supseteq$, let $x \not\leq x'$. Then $R[x'] \not\subseteq R[x]$ by Definition 8.3(1), so there is $y \in Y$ with $x' \, R \, y$ and $x \not R \, y$. Therefore, by Definition 8.3(5), there is $U \in L\,\mathcal{G}$ with $x \in U$ and $y \notin \blacklozenge U$. Since $x' \, R \, y$, we have $x' \notin U$.

(2) Because each open upset is a union of clopen upsets, it is enough to see that each clopen upset V is a union from $L\,\mathcal{G}$. Let $x \in V$. If $x' \notin V$, then $x \not\leq x'$, so (1) implies that there is $U_{x'} \in L\,\mathcal{G}$ with $x \in U_{x'}$ and $x' \notin U_{x'}$. The set $\{-U_{x'} : x' \notin V\}$ is an open cover of $-V$, so compactness implies that there are $x'_1, \dots, x'_n$ with $-V \subseteq -U_{x'_1} \cup \cdots \cup -U_{x'_n}$. Let $U = U_{x'_1} \cap \cdots \cap U_{x'_n}$. Then $x \in U$, $U \in L\,\mathcal{G}$ (since $L\,\mathcal{G}$ is closed under finite intersections), and $U \subseteq V$.

The proofs of (3) and (4) are similar.

(5) Let $x \notin U$. Since $U = \Box \blacklozenge U$, there is $y \in Y$ with $x \, R \, y$ and $y \notin R[U]$. Clearly ${\uparrow} x \cap R^{-1}[y]$ is nonempty, and it is closed by Remark 8.4(3). Therefore, by Lemma 8.1, there is $x' \in \max({\uparrow} x \cap R^{-1}[y])$. Thus, $x \leq x'$ and $x' \in \max R^{-1}[y] \subseteq X_0$. Since $x' \, R \, y$ and $y \notin R[U]$, we have $x' \notin U$, concluding the proof. ∎

Morphisms considered by Gehrke and van Gool are pairs of functions between GvG-spaces that dually characterize special bounded lattice homomorphisms (called admissible homomorphisms in [GvG14, Def. 3.19]). To characterize all bounded lattice homomorphisms in the language of GvG-spaces, we need to work with pairs of relations.

Definition 8.8. Let $\mathcal{G} = (X, R, Y)$ and $\mathcal{G}' = (X', R', Y')$ be GvG-spaces. A pair (S, T) of relations $S \subseteq X \times X'$ and $T \subseteq Y \times Y'$ is a GvG-morphism provided

(1) if $U' \in L\,\mathcal{G}'$, then $\Box_S U' \in L\,\mathcal{G}$ and $\blacklozenge \Box_S U' = \Diamond_T \blacklozenge' U'$;
(2) if $x \not S \, x'$, then there is $U' \in L\,\mathcal{G}'$ with $x \in \Box_S U'$ and $x' \notin U'$;
(3) if $y \not T \, y'$, then there is $U' \in L\,\mathcal{G}'$ with $y \notin \Diamond_T \blacklozenge' U'$ and $y' \in \blacklozenge' U'$;
(4) if $x \, R \, y$, then there are $x' \in X'$ and $y' \in Y'$ with $x \, S \, x'$, $y \, T \, y'$, and $x' \, R' \, y'$.

Remark 8.9. Let $(S, T)\colon \mathcal{G} \to \mathcal{G}'$ be a GvG-morphism.

(1) Both S and T are closed relations. To see that S is closed, let $(x, x') \notin S$. Then $x \not S \, x'$, so by Definition 8.8(2), there is $U' \in L\,\mathcal{G}'$ with $x \in \Box_S U'$ and $x' \notin U'$. By Definition 8.8(1), $\Box_S U' \times -U'$ is an open neighborhood of (x, x') disjoint from S. Thus, S is closed. A similar argument using Definition 8.8(3) shows that T is closed.
(2) Because the converse of Definition 8.8(2) is obvious, we have $x \, S \, x'$ iff for each $U' \in L\,\mathcal{G}'$, if $x \in \Box_S U'$, then $x' \in U'$. The same applies to Definition 8.8(3).
(3) If $x \in X$ and $y \in Y$, then $S[x]$ and $T[y]$ are upsets. To see this, suppose that $x \, S \, x'_1$ and $x'_1 \leq x'_2$. If $x \not S \, x'_2$, by Definition 8.8(2) there is $U' \in L\,\mathcal{G}'$ with $x \in \Box_S U'$ and $x'_2 \notin U'$. The former implies that $S[x] \subseteq U'$, so $x'_1 \in U'$. Because U' is an upset, $x'_2 \in U'$. This contradicts the latter, yielding $x \, S \, x'_2$. A similar argument shows that $T[y]$ is an upset.
(4) If $x' \in X'$ and $y' \in Y'$, then $S^{-1}[x']$ and $T^{-1}[y']$ are downsets. For this, suppose $x_1 \leq x_2$ and $x_2 \in S^{-1}[x']$. Then $x' \in S[x_2]$. If $x_1 \not S \, x'$, by Definition 8.8(2) there is $U' \in L\,\mathcal{G}'$ with $x_1 \in \Box_S U'$ and $x' \notin U'$. Because $\Box_S U' \in L\,\mathcal{G}$, it is an upset, so $x_2 \in \Box_S U'$. This implies that $S[x_2] \subseteq U'$, a contradiction since $x' \notin U'$ and $x' \in S[x_2]$. Thus, $x_1 \in S^{-1}[x']$. A similar argument shows that $T^{-1}[y']$ is a downset.

Composition of two GvG-morphisms is not relation composition, which is reminiscent of what happens in the category of generalized Priestley spaces [BJ11, p. 106].

Definition 8.10. Let $(S_1, T_1)\colon \mathcal{G}_1 \to \mathcal{G}_2$ and $(S_2, T_2)\colon \mathcal{G}_2 \to \mathcal{G}_3$ be GvG-morphisms.

(1) Define $S_2 \star S_1$ by $x\,(S_2 \star S_1)\,z$ if $x \in \Box_{S_1}\Box_{S_2} U$ implies $z \in U$ for each $U \in L\,\mathcal{G}_3$.
(2) Define $T_2 \star T_1$ by $y\,(T_2 \star T_1)\,w$ if $w \in \blacklozenge U$ implies $y \in \lozenge_{T_1}\lozenge_{T_2}\blacklozenge U$ for each $U \in L\,\mathcal{G}_3$.
(3) Set $(S_2, T_2) \star (S_1, T_1) = (S_2 \star S_1, T_2 \star T_1)$.

Remark 8.11. Let $(S_1, T_1)\colon \mathcal{G}_1 \to \mathcal{G}_2$ and $(S_2, T_2)\colon \mathcal{G}_2 \to \mathcal{G}_3$ be GvG-morphisms.

(1) $S_2 \circ S_1 \subseteq S_2 \star S_1$ and $T_2 \circ T_1 \subseteq T_2 \star T_1$, but the equalities may not hold in general.
(2) $(S_2 \star S_1)[x] = \bigcap\{U \in L\,\mathcal{G}_3 : x \in \Box_{S_1}\Box_{S_2} U\} = \bigcap\{U \in L\,\mathcal{G}_3 : S_2 S_1[x] \subseteq U\}$.
(3) $(T_2 \star T_1)^{-1}[w] = \bigcap\{(T_2 \circ T_1)^{-1}\blacklozenge U : w \in \blacklozenge U,\ U \in L\,\mathcal{G}_3\}$.

Lemma 8.12. *Suppose that* $(S_1, T_1)\colon \mathcal{G}_1 \to \mathcal{G}_2$ *and* $(S_2, T_2)\colon \mathcal{G}_2 \to \mathcal{G}_3$ *are GvG-morphisms. If* $U \in L\,\mathcal{G}_3$, *then*

$$\Box_{S_2\star S_1} U = \Box_{S_1}\Box_{S_2} U \quad \text{and} \quad \lozenge_{T_2\star T_1}\blacklozenge U = \lozenge_{T_1}\lozenge_{T_2}\blacklozenge U.$$

Proof. For each $x \in X_1$, by Remark 8.11(2),

$$(S_2 \star S_1)[x] \subseteq U \iff \bigcap\{W \in L\,\mathcal{G}_3 : S_2 S_1[x] \subseteq W\} \subseteq U \iff S_2 S_1[x] \subseteq U.$$

Thus, $\Box_{S_2\star S_1} U = \Box_{S_1}\Box_{S_2} U$.

For each $y \in Y_1$ we show that $y \in \lozenge_{T_2\star T_1}\blacklozenge U$ iff $y \in \lozenge_{T_1}\lozenge_{T_2}\blacklozenge U$. It is enough to show that $(T_2 \star T_1)[y] \cap \blacklozenge U \neq \varnothing$ iff $(T_2 \circ T_1)[y] \cap \blacklozenge U \neq \varnothing$. By Remark 8.11(1), $(T_2 \circ T_1)[y] \subseteq (T_2 \star T_1)[y]$, and hence $(T_2 \circ T_1)[y] \cap \blacklozenge U \neq \varnothing$ implies that $(T_2 \star T_1)[y] \cap \blacklozenge U \neq \varnothing$. For the converse, suppose that $(T_2 \star T_1)[y] \cap \blacklozenge U \neq \varnothing$. Then there is $w \in \blacklozenge U$ with $y \in (T_2 \star T_1)^{-1}[w]$. Therefore, by Remark 8.11(3),

$$y \in \bigcap\{(T_2 \circ T_1)^{-1}\blacklozenge W : w \in \blacklozenge W,\ W \in L\,\mathcal{G}_3\} \subseteq (T_2 \circ T_1)^{-1}\blacklozenge U,$$

which gives $(T_2 \circ T_1)[y] \cap \blacklozenge U \neq \varnothing$. ∎

Proposition 8.13. *Suppose that* $(S_1, T_1)\colon \mathcal{G}_1 \to \mathcal{G}_2$ *and* $(S_2, T_2)\colon \mathcal{G}_2 \to \mathcal{G}_3$ *are GvG-morphisms.*

(1) $(S_2, T_2) \star (S_1, T_1)$ *is a GvG-morphism.*
(2) *If* $\mathcal{G} = (X, R, Y)$ *is a GvG-space, then* $(\leq_X, \leq_Y)\colon \mathcal{G} \to \mathcal{G}$ *is a GvG-morphism and is the identity morphism for* $\mathcal{G}$.
(3) *The operation* $\star$ *is associative.*

Proof. (1) To simplify notation, we write $S = S_2 \star S_1$ and $T = T_2 \star T_1$. Let $U'' \in L\,\mathcal{G}_3$ and set $U' = \Box_{S_2} U''$. Then $U' \in L\,\mathcal{G}_2$ since (S_2, T_2) is a GvG-morphism. By Lemma 8.12,

$$\Box_S U'' = \Box_{S_1}\Box_{S_2} U'' = \Box_{S_1} U' \in L\,\mathcal{G}_1$$

because (S_1, T_1) is a GvG-morphism.

In addition, since (S_1, T_1) and (S_2, T_2) are GvG-morphisms, by Lemma 8.12

$$\blacklozenge\Box_S U = \blacklozenge\Box_{S_1}(\Box_{S_2} U) = \lozenge_{T_1}\blacklozenge\Box_{S_2} U = \lozenge_{T_1}\lozenge_{T_2}\blacklozenge U = \lozenge_T\blacklozenge U.$$

This shows that Definition 8.8(1) holds.

Suppose $x \not\mathrel{S} z$. By Remark 8.11(2), there is $U \in L\,\mathcal{G}_3$ with $x \in \Box_{S_1}\Box_{S_2} U$ and $z \notin U$. By Lemma 8.12, $\Box_{S_1}\Box_{S_2} U = \Box_S U$, so $x \in \Box_S U$ and $z \notin U$. Hence, Definition 8.8(2) holds.

Suppose $y \not\!T\, w$. By Remark 8.11(3), there is $U \in L\,\mathcal{G}_3$ with $w \in \blacklozenge U$ and $y \notin \lozenge_{T_1}\lozenge_{T_2}\blacklozenge U$. By Lemma 8.12, $y \notin \lozenge_T \blacklozenge U$, Thus, Definition 8.8(3) holds.

Suppose $x\,R_1\,y$. Then there are $x' \in X_2$ and $y' \in Y_2$ with $x'\,R_2\,y'$, $x\,S_1\,x'$, and $y\,T_1\,y'$. From this there are $x'' \in X_3$ and $y'' \in Y_3$ with $x''\,R_3\,y''$, $x'\,S_2\,x''$, and $y'\,T_2\,y''$. Since $S_2 \circ S_1 \subseteq S$ and $T_2 \circ T_1 \subseteq T$ by Remark 8.11(1), we have $x\,S\,x''$ and $y\,T\,y''$. This verifies Definition 8.8(4). Consequently, $(S_2 \star S_1, T_2 \star T_1)$ is a GvG-morphism.

(2) Let $\mathcal{G} = (X, R, Y)$ be a GvG-space. We have $\Box_{\leq_X} U = U$ for any upset $U \subseteq X$ and $\lozenge_{\leq_Y} V = V$ for any downset $V \subseteq Y$. Therefore, Definition 8.8 is trivially satisfied. Thus, $(\leq_X, \leq_Y)$ is a GvG-morphism.

Let $(S,T)\colon \mathcal{G} \to \mathcal{G}'$ and $(S',T')\colon \mathcal{G}' \to \mathcal{G}$ be GvG-morphisms. Since $\Box_{\leq_X} W = W$ for any upset $W \subseteq X$, we have $x\,(S \star \leq_X)\,y$ iff $x \in \Box_S U$ implies $y \in U$ for each $U \in L\,\mathcal{G}'$. On the other hand, $S \circ {\leq_X} = S$ by Remark 8.9(4), which by Remark 8.9(2) is equivalent to $x \in \Box_S U$ implies $y \in U$ for each $U \in L\,\mathcal{G}'$. Therefore, $S \star {\leq_X} = S \circ {\leq_X} = S$, and similarly, $T \star {\leq_Y} = T \circ {\leq_Y} = T$. Thus, $(S,T) \star (\leq_X, \leq_Y) = (S,T)$, and an analogous argument gives $(\leq_X, \leq_Y) \star (S', T') = (S', T')$. Consequently, $(\leq_X, \leq_Y)$ is the identity morphism for $\mathcal{G}$.

(3) Let $(S_1, T_1)\colon \mathcal{G}_1 \to \mathcal{G}_2$, $(S_2, T_2)\colon \mathcal{G}_2 \to \mathcal{G}_3$, and $(S_3, T_3)\colon \mathcal{G}_3 \to \mathcal{G}_4$ be GvG-morphisms. By Lemma 8.12, for each $U \in L\,\mathcal{G}_4$, we have

$$\Box_{(S_3 \star S_2) \star S_1} U = \Box_{S_1} \Box_{S_3 \star S_2} U = \Box_{S_1} \Box_{S_2} \Box_{S_3} U.$$

Therefore, $x\,((S_3 \star S_2) \star S_1)\,w$ iff $x \in \Box_{S_1}\Box_{S_2}\Box_{S_3} U$ implies $w \in U$ for each $U \in L\,\mathcal{G}_4$. A similar calculation shows that $x\,(S_3 \star (S_2 \star S_1))\,w$ iff $x \in \Box_{S_1}\Box_{S_2}\Box_{S_3} U$ implies $w \in U$. Thus, $(S_3 \star S_2) \star S_1 = S_3 \star (S_2 \star S_1)$. A similar argument shows that $(T_3 \star T_2) \star T_1 = T_3 \star (T_2 \star T_1)$. Consequently, $\star$ is associative. ∎

The previous lemma shows that we have a category, motivating the following definition.

Definition 8.14. Let GvG denote the category of GvG-objects and GvG-morphisms.

The rest of the chapter is dedicated to defining a functor $\mathbb{G}\colon \mathsf{DH} \to \mathsf{GvG}$. To define $\mathbb{G}$ on objects, we recall the notion of distributive joins and meets, which goes back to MacNeille [Mac37, Def. 3.10] (see also [BL70, HK71]). Distributive joins and meets are also known as exact [Bal84] (see also [BPP14, BPW16]). As in [GvG14], we will be working with finite distributive joins and meets. Let L be a lattice. For $M \subseteq L$ finite, we say that $\bigwedge M$ is a *distributive meet* if

$$a \vee \bigwedge M = \bigwedge \{a \vee m : m \in M\}$$

for each $a \in L$. Distributive joins are defined similarly. The next definition weakens the standard definition of a (meet-)prime element [GH+03, Def. I.3.11].

Definition 8.15. Let Z be a coherent lattice.

(1) An element $x \in Z$ is *d-prime* if $x \neq 1$ and whenever $\bigwedge M$ is a finite distributive meet in $K(Z)$ and $\bigwedge M \leq x$, then $m \leq x$ for some $m \in M$.
(2) Let Z_p be the set of d-prime elements of Z.

Lemma 8.16. *Let $Z \in \mathsf{CohLat}_\lambda$. Then Z_p is Lawson-closed in Z. Therefore, Z_p is a Priestley space with the subspace topology and induced order.*

Proof. Let $x \notin Z_p$. Then x is not d-prime, so there is a distributive meet $k = k_1 \wedge \cdots \wedge k_n$ in $K(Z)$ with $k \le x$ but all $k_i \not\le x$. Let $U = {\uparrow}k - ({\uparrow}k_1 \cup \cdots \cup {\uparrow}k_n)$. Then U is Lawson-open and $x \in U$. Moreover, if $x' \in U$, then $k \le x'$ and $k_i \not\le x'$ for all i, so $x' \notin Z_p$, and hence $U \cap Z_p = \varnothing$. Thus, Z_p is closed in Z, implying that Z_p is a Priestley space. ∎

To associate with each DH-space a GvG-space, we need the following lemma.

Lemma 8.17. *Let X, Y be Priestley spaces and $R \subseteq X \times Y$ a closed relation.*

(1) *If $x \,R\, y$, then there is $x' \in \max R^{-1}[y]$ with $x \le x'$.*
(2) *If $x \,R\, y$ then there is $y' \in \max R[x]$ with $y \le y'$.*

If in addition (X, R, Y) is a DH-space, then

(3) *$x = \bigwedge({\uparrow}x \cap X_0)$ for each $x \in X$;*
(4) *$y = \bigwedge({\uparrow}y \cap Y_0)$ for each $y \in Y$.*

Proof. (1) Suppose that $x \,R\, y$. Because ${\uparrow}x \cap R^{-1}[y]$ is a nonempty closed subset of X by Remark 8.4(3), there is $x' \in \max({\uparrow}x \cap R^{-1}[y])$ by Lemma 8.1. Therefore, $x \le x'$ and it is clear that $x' \in \max R^{-1}[y]$.

(2) The proof is similar to that of (1).

(3) The inequality $x \le \bigwedge({\uparrow}x \cap X_0)$ is clear. For the reverse inequality, suppose that $z \le \bigwedge({\uparrow}x \cap X_0)$. If $z \not\le x$, then $R[x] \not\subseteq R[z]$ by Definition 7.5(3). Therefore, there is $y \in Y$ with $x \,R\, y$ and $z \not{R}\, y$. By (1), there is $x' \in \max R^{-1}[y] \subseteq X_0$ with $x \le x'$. Thus, $z \not\le x'$ since $R^{-1}[y]$ is a downset. This contradiction shows that $z \le x$, so $x = \bigwedge({\uparrow}x \cap X_0)$.

(4) The proof is similar to that of (3). ∎

Definition 8.18. For each $(X, R, Y) \in \mathsf{DH}$ we set $R_p = R \cap (X_p \times Y_p)$.

If $(X, R, Y) \in \mathsf{DH}$, then we can apply Definition 8.2 starting from (X, R, Y) or (X_p, R_p, Y_p). The following lemma shows that we get the same result.

Lemma 8.19. *Let $(X, R, Y) \in \mathsf{DH}$. Then $X_0 = (X_p)_0$ and $Y_0 = (Y_p)_0$.*

Proof. To start, we prove that $X_0 \subseteq X_p$ and $Y_0 \subseteq Y_p$. Let $x \in X_0$. Then $x \in \max R^{-1}[y]$ for some $y \in Y$. Suppose that $k = k_1 \wedge \cdots \wedge k_n$ is a distributive meet in $K(X)$ with $k \le x$. Suppose that $k_i \not\le x$. Then $x < x \vee k_i$, so $(x \vee k_i) \not{R}\, y$. If this happens for each i, then $[(x \vee k_1) \wedge \cdots \wedge (x \vee k_n)] \not{R}\, y$ by Definition 7.5(2). We show that $x \vee k = (x \vee k_1) \wedge \cdots \wedge (x \vee k_n)$, which will show that $(x \vee k) \not{R}\, y$, a contradiction since $k \le x$ and $x \,R\, y$. The inequality $x \vee k \le (x \vee k_1) \wedge \cdots \wedge (x \vee k_n)$ is clear. For the reverse inequality, it suffices to show that $m \le x \vee k$ for each $m \in K(X)$ with $m \le (x \vee k_1) \wedge \cdots \wedge (x \vee k_n)$. Given such m, we have $m \le x \vee k_i$ for each i. Because $m \in K(X)$ and X is algebraic, there is $m_i \in K(X)$ with $m_i \le x$ and $m \le m_i \vee k_i$. Set $s = m_1 \vee \cdots \vee m_n$. Then $s \le x$ and $m \le s \vee k_i$ for each i. Therefore, since $k = k_1 \wedge \cdots \wedge k_n$ is a distributive meet, $s \vee k = (s \vee k_1) \wedge \cdots \wedge (s \vee k_n)$. This implies that $m \le s \vee k \le x \vee k$. Thus, $x \in X_p$. A similar argument shows that $Y_0 \subseteq Y_p$.

We next show that $(X_p)_0 \subseteq X_0$. Let $x \in (X_p)_0$. Then $x \in \max R_p^{-1}[y]$ for some $y \in Y_p$. Suppose there is $x' \in X$ with $x \le x'$ and $x' \,R\, y$. By Lemma 8.17(1), there is

$x'' \in \max R^{-1}[y]$ with $x' \leq x''$. Therefore, $x'' \in X_0 \subseteq X_p$, which forces $x'' = x$. Thus, $x \in \max R^{-1}[y]$, and hence $x \in X_0$.

For the reverse inclusion, let $x \in X_0$. Then $x \in \max R^{-1}[y]$ for some $y \in Y$. Therefore, $x \in X_0 \subseteq X_p$. By Lemma 8.17(2), there is $y' \in Y_0$ with $x\,R\,y'$ and $y \leq y'$. Thus, $y' \in Y_0 \subseteq Y_p$, so $x\,R_p\,y'$. By Lemma 8.16, X_p and Y_p are Priestley spaces. Moreover, $R_p \subseteq X_p \times Y_p$ is a closed relation by definition. Hence, Lemma 8.17(1) applies, by which there is $x' \in \max R_p^{-1}[y']$ with $x \leq x'$. Because $x'\,R\,y'$ and $y \leq y'$, we have $x'\,R\,y$. From $x \in \max R^{-1}[y]$ it follows that $x' = x$. Therefore, $x \in \max R_p^{-1}[y'] \subseteq (X_p)_0$. This yields the reverse inclusion. A similar argument shows that $(Y_p)_0 = Y_0$. ■

Given $(X, R, Y) \in \mathsf{DH}$, we recall from the beginning of this chapter that we write $\Box, \blacklozenge$ for $\Box_R, \blacklozenge_R$. We also write $\Box_p, \blacklozenge_p$ for $\Box_{R_p}, \blacklozenge_{R_p}$.

Lemma 8.20. *Let $(X, R, Y) \in \mathsf{DH}$.*

(1) *If $U = U' \cap X_p$ with U' an upset of X, then $\blacklozenge_p U = \blacklozenge U' \cap Y_p$.*
(2) *If $V = V' \cap Y_p$ with V' a downset of Y, then $\Box_p V = \Box V' \cap X_p$.*
(3) *Let $U' = {\uparrow}k_1 \cup \cdots \cup {\uparrow}k_n$ with $k_i \in K(X)$ and set $U = U' \cap X_p$. Then $\blacklozenge_p U = Y_p - {\uparrow}l$ for some $l \in K(Y)$. Therefore, if U is a clopen upset of X_p, then $\blacklozenge_p U$ is a clopen downset of Y_p.*
(4) *Let $V' = -({\uparrow}l_1 \cup \cdots \cup {\uparrow}l_n)$ with $l_i \in K(Y)$ and set $V = V' \cap Y_p$. Then $\Box_p V = {\uparrow}k \cap X_p$ for some $k \in K(X)$. Therefore, if V' is a clopen downset of Y_p, then $\Box_p V$ is a clopen upset of X_p.*
(5) *If $U = {\uparrow}k \cap X_p$ with $k \in K(X)$, then $\Box_p \blacklozenge_p U = U$.*
(6) *If $V = Y_p - {\uparrow}l$ with $l \in K(Y)$, then $\blacklozenge_p \Box_p V = V$.*

Proof. (1) Since $\blacklozenge_p U = R_p[U]$, the inclusion $\blacklozenge_p U \subseteq \blacklozenge U' \cap Y_p$ is clear. For the reverse inclusion, suppose that $y \in Y_p$ with $x\,R\,y$ for some $x \in U'$. By Lemma 8.17(1), there is $x' \geq x$ with $x' \in X_0$ and $x'\,R\,y$. Therefore, $x'\,R_p\,y$ because $X_0 \subseteq X_p$ by Lemma 8.19. Since U' is an upset, $x' \in U'$, so $x' \in U$. Thus, $y \in \blacklozenge_p U$.

(2) We have $\Box_p V = \{x \in X_p : R_p[x] \subseteq V\}$ and $\Box V' = \{x \in X : R[x] \subseteq V'\}$. If $x \in \Box V' \cap X_p$, then $x \in X_p$ and $R[x] \subseteq V'$. Therefore, $R_p[x] = R[x] \cap Y_p \subseteq V' \cap Y_p = V$. Thus, $x \in \Box_p V$. For the reverse inclusion, let $x \in \Box_p V$ and $y \in Y$ with $x\,R\,y$. By Lemma 8.17(2), there is $y' \geq y$ with $y' \in Y_0$ and $x\,R\,y'$. By Lemma 8.19, $Y_0 \subseteq Y_p$. Therefore, $y' \in R_p[x]$, so $y' \in V \subseteq V'$. Because V' is a downset, $y \in V'$. Thus, $R[x] \subseteq V'$, so $x \in \Box V' \cap X_p$.

(3) By (1) and Lemma 7.11(4),

$$\begin{aligned}\blacklozenge_p U = \blacklozenge U' \cap Y_p &= (\blacklozenge{\uparrow}k_1 \cup \cdots \cup \blacklozenge{\uparrow}k_n) \cap Y_p \\ &= [(-{\uparrow}l_1) \cup \cdots \cup (-{\uparrow}l_n)] \cap Y_p \\ &= Y_p - \bigcap {\uparrow}l_i = Y_p - {\uparrow}l\end{aligned}$$

for some $l_i \in K(Y)$, where $l = l_1 \vee \cdots \vee l_n$, so $l \in K(Y)$.

If U is a clopen upset of X_p, then $U = U' \cap X_p$ for some clopen upset U' of X. Therefore, $U = ({\uparrow}k_1 \cup \cdots \cup {\uparrow}k_n) \cap X_p$ for some $k_i \in K(X)$. The previous paragraph shows that $\blacklozenge_p U$ is a clopen downset of Y_p.

(4) By Lemma 7.11(4),

$$\begin{aligned}\Box V' &= -R^{-1}[\uparrow l_1 \cup \cdots \cup \uparrow l_n] = -(R^{-1}[\uparrow l_1] \cup \cdots \cup R^{-1}[\uparrow l_n]) \\ &= -R^{-1}[\uparrow l_1] \cap \cdots \cap -R^{-1}[\uparrow l_n] = \uparrow k_1 \cap \cdots \cap \uparrow k_n\end{aligned}$$

for some $k_i \in K(X)$. By (2), $\Box_p V = \uparrow k \cap X_p$, where $k = k_1 \vee \cdots \vee k_n$, so $k \in K(X)$.

If V is a clopen downset of Y_p, then $V = V' \cap Y_p$ for some clopen downset V' of Y. Therefore, $V = -(\uparrow l_1 \cup \cdots \cup \uparrow l_n) \cap Y_p$ for some $l_i \in K(Y)$. The previous paragraph shows that $\Box_p V$ is a clopen upset of X_p.

(5) Set $U = \uparrow k \cap X_p$ with $k \in K(X)$. By (1), (2), and Lemma 7.11(3),

$$\begin{aligned}\Box_p \blacklozenge_p U &= \Box_p \blacklozenge_p (\uparrow k \cap X_p) = \Box_p (\blacklozenge \uparrow k \cap Y_p) = \Box \blacklozenge (\uparrow k) \cap X_p \\ &= \psi\phi(\uparrow k) \cap X_p = \uparrow k \cap X_p = U.\end{aligned}$$

(6) Let $V = Y_p - \uparrow l$ with $l \in K(Y)$. By (1), (2), and Lemma 7.11(3),

$$\begin{aligned}\blacklozenge_p \Box_p V &= \blacklozenge_p \Box_p (-\uparrow l \cap Y_p) = \blacklozenge_p (\Box(-\uparrow l) \cap X_p) = \blacklozenge \Box (-\uparrow l) \cap Y_p \\ &= -\blacksquare \Diamond \uparrow l \cap Y_p = -\phi\psi(\uparrow l) \cap Y_p = -\uparrow l \cap Y_p = V. \ \blacksquare\end{aligned}$$

Lemma 8.21. *Let $(X, R, Y) \in \mathsf{DH}$. If $x \not R_p\, y$, then there is a clopen upset $U \subseteq X_p$ with $x \in U$, $y \notin \blacklozenge_p U$, and $\Box_p \blacklozenge_p U = U$.*

Proof. Suppose that $x \not R_p\, y$. Then $x \notin R^{-1}[y]$, and since $R^{-1}[y]$ is a closed downset of X, there is $k \in K(X)$ with $k \le x$ and $\uparrow k \cap R^{-1}[y] = \varnothing$. Let $U = \uparrow k \cap X_p$. Then U is a clopen upset of X_p and $x \in U$. By Lemma 8.20(5), $U = \Box_p \blacklozenge_p U$. If $y \in \blacklozenge_p U$, then $x' R_p\, y$ for some $x' \in U$. Therefore, $k \le x'$, so $k \,R\, y$ by Definition 7.5(2). This contradicts the choice of k. Thus, $y \notin \blacklozenge_p U$. $\blacksquare$

Let $Z \in \mathsf{CohLat}_\lambda$ and U be a clopen upset of Z_p. Since Z_p is a closed subset of Z by Lemma 8.16, $U = U' \cap Z_p$ for some clopen upset U' of Z by Priestley duality. Therefore, $U' = \uparrow k_1 \cup \cdots \cup \uparrow k_n$ for some $k_i \in K(Z)$ by Theorem 2.8(3) and Theorem 2.10(2).

Lemma 8.22. *Let $(X, R, Y) \in \mathsf{DH}$.*

(1) *Let U be a clopen upset of X_p. Write $U = U' \cap X_p$ with $U' = \uparrow k_1 \cup \cdots \cup \uparrow k_n$ for some $k_i \in K(X)$. If $\Box_p \blacklozenge_p U \cap X_0 \subseteq U$ then $k := k_1 \wedge \cdots \wedge k_n$ is a distributive meet and $U = \uparrow k \cap X_p$. Therefore, $\Box_p \blacklozenge_p U = U$.*
(2) *Let V be a clopen downset of Y_p. Write $V = V' \cap X_p$ with $V' = -(\uparrow l_1 \cup \cdots \cup \uparrow l_n)$ for some $l_i \in K(Y)$. If $Y_0 \cap V \subseteq \blacklozenge_p \Box_p V$ then $l := l_1 \wedge \cdots \wedge l_n$ is a distributive meet and $V = Y_p - \uparrow l$. Therefore, $\blacklozenge_p \Box_p V = V$.*

Proof. (1) Suppose that $\Box_p \blacklozenge_p U \cap X_0 \subseteq U$. If k is not a distributive meet, then there is $l \in K(X)$ with $l \vee k < (l \vee k_1) \wedge \cdots \wedge (l \vee k_n)$. By Lemma 8.17(3), there is $x \in X_0$ with $l \vee k \le x$ and $(l \vee k_1) \wedge \cdots \wedge (l \vee k_n) \not\le x$. We have $k \le l \vee k \le x$. If $k_i \le x$ for some i, then $l \vee k_i \le x$, a contradiction, so $k \le x$ and each $k_i \not\le x$. This shows that $x \notin U$. If $y \in R_p[x]$, then $x \,R\, y$, so $k \,R\, y$ by Definition 7.5(2). If $k_i \not R\, y$ for each i, then $k \not R\, y$, again by Definition 7.5(2), which is false. Therefore, $k_i \,R\, y$ for some i. This yields $y \in R_p[U]$, and so $R_p[x] \subseteq R_p[U]$. Thus, $X_0 \cap \Box_p \blacklozenge_p U \not\subseteq U$, a contradiction. Consequently, $k = k_1 \wedge \cdots \wedge k_n$ is a distributive meet. We finish by showing that this implies that $U = \uparrow k \cap X_p$. The inclusion $U \subseteq \uparrow k \cap X_p$ is clear. For the other inclusion, if $x \in \uparrow k \cap X_p$, then $k \le x$ and x

is d-prime. Therefore, some $k_i \leq x$, so $x \in U$. Thus, $U = {\uparrow}k \cap X_p$, and hence $\Box_p\blacklozenge_p U = U$ by Lemma 8.20(5).

(2) is proved similarly. ∎

PROPOSITION 8.23. *If (X,R,Y) is a DH-space, then (X_p,R_p,Y_p) is a GvG-space.*

Proof. Since the orders on X_p and Y_p are the restrictions of the orders on X and Y, respectively, the first two properties of Definition 8.3 hold. The remaining properties hold by Lemmas 8.16 and 8.20 to 8.22. ∎

The above proposition associates a GvG-space with each DH-space. We extend this correspondence to a functor from DH to GvG. For this we require the following two facts.

PROPOSITION 8.24. *Let $\mathcal{D} = (X,R,Y) \in \mathsf{DH}$ and $\mathcal{G} = (X_p,R_p,Y_p)$. Define the map $\gamma_D\colon \mathsf{CLF}(X) \to L\,\mathcal{G}$ by $\gamma_D(U) = U \cap X_p$ for each $U \in \mathsf{CLF}(X)$. Then γ_D is a Lat-isomorphism.*

Proof. That γ_D is well defined follows from Lemmas 5.4 and 8.20(5), and it is clearly order-preserving. To see that it is order-reflecting, let $U,V \in \mathsf{CLF}(X)$ with $U \not\subseteq V$. By Lemma 5.4, $U = {\uparrow}k$ and $V = {\uparrow}l$ for some $k,l \in K(X)$. Therefore, $l \not\leq k$, so Lemma 8.17(3) implies that there is $x \in X_0$ with $k \leq x$ and $l \not\leq x$. By Lemma 8.19, $X_0 \subseteq X_p$, so $x \in X_p$, and hence $U \cap X_p \not\subseteq V \cap X_p$. Thus, $\gamma_D(U) \not\subseteq \gamma_D(V)$. Finally, to see that γ_D is onto, let $U \in L\,\mathcal{G}$. Then $\Box_p\blacklozenge_p U = U$, so $X_0 \cap \Box_p\blacklozenge_p U \subseteq U$. By Lemma 8.22(1), $U = {\uparrow}k \cap X_p$ for some $k \in K(X)$. Therefore, ${\uparrow}k \in \mathsf{CLF}(X)$ and $\gamma_D({\uparrow}k) = U$. Thus, γ_D is an order-isomorphism, and hence a Lat-isomorphism. ∎

PROPOSITION 8.25. *Let $(f,g)\colon (X,R,Y) \to (X',R',Y')$ be a DH-morphism. Define relations $S_f \subseteq X_p \times X'_p$ and $T_g \subseteq Y_p \times Y'_p$ by*

$$x\,S_f\,x' \iff f(x) \leq x' \quad \text{and} \quad y\,T_g\,y' \iff g(y) \leq y'.$$

Then (S_f,T_g) is a GvG-morphism.

Proof. Set $\mathcal{G} = (X_p,R_p,Y_p)$ and $\mathcal{G}' = (X'_p,R'_p,Y'_p)$, where R_p and R'_p are the appropriate restrictions of R and R', respectively. We first verify Definition 8.8(1). Let $U' \in L\,\mathcal{G}'$. By Lemma 5.4 and Proposition 8.24, there is $l \in K(X')$ with $U' = {\uparrow}l \cap X'_p$. Since $S_f[x] = {\uparrow}f(x) \cap X'_p$,

$$\Box_{S_f} U' = \{x \in X_p : S_f[x] \subseteq U'\} = \{x \in X_p : l \leq f(x)\} = f^{-1}({\uparrow}l) \cap X_p.$$

By Lemma 5.4, ${\uparrow}l \in \mathsf{CLF}(X')$. Thus, since f is a CohLat_λ-morphism, $f^{-1}({\uparrow}l) \in \mathsf{CLF}(X)$. Therefore, $f^{-1}({\uparrow}l) = {\uparrow}k$ for some $k \in K(X)$, and hence $\Box_{S_f} U' = {\uparrow}k \cap X_p$, which implies that $\Box_{S_f} U' \in L\,\mathcal{G}$.

Next, let $y \in \Diamond_{T_g}\blacklozenge'_p U'$. Then there is $y' \in Y'_p$ with $y\,T_g\,y'$ and $x' \in U'$ with $x'\,R'_p\,y'$. We have $g(y) \leq y'$, so $x'\,R'\,g(y)$ by Definition 7.5(2). Therefore, there is $x \in X$ with $x\,R\,y$ and $x' \leq f(x)$ by Definition 7.7(2b). Furthermore, we may assume that $x \in X_p$ by Lemma 8.17(1) and Lemma 8.19. Because $x' \in U'$, we have $S_f[x] = {\uparrow}f(x) \cap X'_p \subseteq U'$, so $y \in \blacklozenge_p\Box_{S_f} U'$.

Conversely, if $y \in \blacklozenge_p\Box_{S_f} U'$, then there is $x \in X_p$ with $x\,R_p\,y$ and $S_f[x] \subseteq U'$. Since $x\,R_p\,y$, we have $x\,R\,y$, so $f(x)\,R'\,g(y)$ by Definition 7.7(2a). By Lemmas 8.17(1,2) and 8.19, there are $x' \in X'_p$ and $y' \in Y'_p$ such that $f(x) \leq x'$, $g(y) \leq y'$, and $x'\,R'_p\,y'$. Therefore,

$x\, S_f\, x'$ and $y\, T_g\, y'$. Since $S_f[x] \subseteq U'$, we have $x' \in U'$. This shows that $y' \in T_g[y] \cap \blacklozenge'_p U'$. Consequently, $y \in \Diamond_{T_g} \blacklozenge'_p U'$ and hence $\blacklozenge_p \Box_{S_f} U' = \Diamond_{T_g} \blacklozenge'_p U'$.

We next verify Definition 8.8(2). If $x \not\!S_f\, x'$, then $f(x) \not\le x'$. Therefore, there is $l \in K(X')$ with $l \le f(x)$ and $l \not\le x'$. Set $U' = {\uparrow} l \cap X'_p$. By Lemma 8.22(1), $U' = \Box'_p \blacklozenge'_p U'$, ${\uparrow} f(x) \cap X'_p \subseteq U'$, and $x' \notin U'$. Thus, $S_f[x] = {\uparrow} f(x) \cap X'_p \subseteq U'$, so $x \in \Box_{S_f} U'$.

We now verify Definition 8.8(3). If $y \not\!T_g\, y'$, then $g(y) \not\le y'$, so there is $l \in K(Y')$ with $l \le g(y)$ and $l \not\le y'$. Set $U' = \Box'_p(Y'_p - {\uparrow} l)$. Then $\blacklozenge'_p U' = Y'_p - {\uparrow} l$ by Lemma 8.22(2), and hence $y' \in \blacklozenge'_p U'$. On the other hand, $T_g[y] = {\uparrow} g(y) \cap Y'_p \subseteq {\uparrow} l \cap Y'_p$, so $T_g[y] \cap \blacklozenge'_p U' = \varnothing$, and hence $y \notin \Diamond_{T_g} \blacklozenge'_p U'$.

We finally verify Definition 8.8(4). If $x\, R\, y$, then $f(x)\, R'\, g(y)$ by Definition 7.7(2a). There are $x' \in X'_0$ and $y' \in Y'_0$ with $f(x) \le x'$, $g(y) \le y'$, and $x'\, R'\, y'$ by Definition 7.7(2b, 2c). Thus, by Lemma 8.19, $x' \in X'_p$, $y' \in Y'_p$, $x\, S_f\, x'$, and $y\, T_g\, y'$. ■

THEOREM 8.26. *There is a functor* $\mathbb{G}\colon \mathsf{DH} \to \mathsf{GvG}$ *sending* (X, R, Y) *to* (X_p, R_p, Y_p) *and* (f, g) *to* (S_f, T_g).

Proof. Proposition 8.23 shows that $\mathbb{G}$ is well defined on objects, and Proposition 8.25 proves that it is well defined on morphisms. Let $(X, R, Y) \in \mathsf{DH}$. The identity morphism for (X, R, Y) is $(1_X, 1_Y)$. Since $\mathbb{G}(1_X, 1_Y) = (\le_X, \le_Y)$, we see that $\mathbb{G}$ sends identity morphisms to identity morphisms by Proposition 8.13(2).

It is left to show that composition is preserved. Let $(f, g)\colon (X, R, Y) \to (X', R', Y')$ and $(f', g')\colon (X', R', Y') \to (X'', R'', Y'')$ be DH-morphisms. Set $S = S_{f'} \star S_f$ and $T = T_{g'} \star T_g$. We wish to show that $(S, T) = (S_{f'f}, T_{g'g})$. We prove that $S = S_{f'f}$; the proof for T is similar. First suppose that $x\, S_{f'f}\, x''$, so $f'f(x) \le x''$. Let $U \in L\,\mathcal{G}''$, where $\mathcal{G}'' = (X''_p, R''_p, Y''_p)$. By Lemma 5.4 and Proposition 8.24, $U = {\uparrow} k \cap X''_p$ for some $k \in K(X'')$. If $x \in \Box_S U$, then $x \in \Box_{S_f} \Box_{S_{f'}} U$ by Lemma 8.12, so $S_{f'} S_f[x] \subseteq {\uparrow} k$, and hence $k \le f'f(x)$. Therefore, $k \le x''$, so $x'' \in {\uparrow} k \cap X''_p = U$. Thus, $x\, S\, x''$. Conversely, suppose that $x\, S\, x''$. If $f'f(x) \not\le x''$, then there is $k \in K(X'')$ with $k \le f'f(x)$ and $k \not\le x''$. Set $U = {\uparrow} k \cap X''_p$. Then $U \in L\,\mathcal{G}''$ by Lemma 5.4 and Proposition 8.24. If $z \in S_{f'} S_f[x]$, then there is $x' \in X'_p$ with $f(x) \le x'$ and $f'(x') \le z$. Therefore, $f'f(x) \le z$. Thus, $S_{f'} S_f[x] \subseteq {\uparrow} k$, and so $x \in \Box_S U$ by Lemma 8.12. Since $x\, S\, x''$, we then have $x'' \in U \subseteq {\uparrow} k$, which is a contradiction. Consequently, $f'f(x) \le x''$, and so $x\, S_{f'f}\, x''$. Hence, $S_{f'f} = S_{f'} \star S_f$, completing the proof that $\mathbb{G}$ is a functor. ■

9. From Gehrke–van Gool to Hartung

In this chapter we connect the Gehrke–van Gool approach to that of Hartung. The resulting spaces were introduced and studied in [Har92]. The morphisms that Hartung considered dually correspond to onto bounded lattice homomorphisms [Har92, Sec. 3.1]. We introduce a more general notion of morphism between Hartung spaces, which dually correspond to all bounded lattice homomorphisms. This gives rise to the category Hg. We show that there is a functor from GvG to Hg. In Chapter 13 we will show that this functor is an equivalence and in Chapter 14 that Hg is dually equivalent to Lat.

Let $R \subseteq X \times Y$ be a relation. We write ϕ, ψ for the Galois connection associated to $-R$ (see Notation 7.10). We then have $\phi A = -\blacklozenge A$ for each $A \subseteq X$ and $\psi B = -\Diamond B$ for each $B \subseteq Y$. As in Chapter 8, we write $\Box$ for $\Box_R$ and $\blacklozenge$ for $\blacklozenge_R$.

Definition 9.1. Let X, Y be topological spaces and $R \subseteq X \times Y$ a relation. Setting $\mathcal{H} = (X, R, Y)$, let

$$L\mathcal{H} = \{A \subseteq X : A = \Box\blacklozenge A,\ A \text{ and } -\blacklozenge A \text{ are closed}\}.$$

Definition 9.2. Let $R \subseteq X \times Y$ be a relation.

(1) Define $\leq_X$ on X by $x \leq_X x'$ if $R[x'] \subseteq R[x]$.
(2) Define $\leq_Y$ on Y by $y \leq_Y y'$ if $R^{-1}[y'] \subseteq R^{-1}[y]$.
(3) If $x \in X$ and $y \in Y$ with $x\,R\,y$, then x is *y-maximal* if whenever $x \leq_X x'$ and $x'\,R\,y$, then $x' = x$.
(4) If $x \in X$ and $y \in Y$ with $x\,R\,y$, then y is *x-maximal* if whenever $y \leq_Y y'$ and $x\,R\,y'$, then $y' = y$.

Definition 9.3. Let X, Y be topological spaces and $R \subseteq X \times Y$ a relation. The triple $\mathcal{H} = (X, R, Y)$ is a *Hartung space* provided

(1) $\leq_X$ and $\leq_Y$ are partial orders;
(2) R is a compact subset of $X \times Y$ (with respect to the product topology);
(3) $L\mathcal{H}$ is a subbasis of closed sets for X, and $\{-\blacklozenge A : A \in L\mathcal{H}\}$ is a subbasis of closed sets for Y;
(4) $A \subseteq X$ closed implies $\Box\blacklozenge A$ closed, and $B \subseteq Y$ closed implies $\blacksquare\Diamond B$ closed;
(5) if $x \in X$, then there is $y \in Y$ such that x is y-maximal, and if $y \in Y$, there is $x \in X$ such that y is x-maximal;
(6) if $x \not R\, y$, then there is $A \in L\mathcal{H}$ with $x \in A$ and $y \notin \blacklozenge A$.

Remark 9.4.

(1) The definition above is a reinterpretation of [Har92, Def. 2.2.3] to connect more directly with the notion of GvG-space.
(2) If $\mathcal{H}$ is a Hartung space, then $L\mathcal{H}$ is a bounded lattice whose top is X, bottom is $\varnothing$, join is given by $A_1 \vee A_2 = \Box\blacklozenge(A_1 \cup A_2)$, and meet by $A_1 \wedge A_2 = A_1 \cap A_2$.
(3) Hartung associates to $\mathcal{H} = (X, R, Y)$ the lattice

$$\{(A, B) \in \wp(X) \times \wp(Y) : B = \phi A,\ A = \psi B, \text{ and } A, B \text{ are closed}\}.$$

Because B is determined from A, our definition is isomorphic as a lattice to Hartung's, and is more convenient for us.
(4) We work with the complement of Hartung's relation, which is more convenient to compare with the other categories we consider.

Lemma 9.5. *Let $\mathcal{H} = (X, R, Y)$ be a Hartung space, $x \in X$, and $y \in Y$.*

(1) *If $A \in L\mathcal{H}$, then A is an upset and $\blacklozenge A$ is a downset.*
(2) $\uparrow x = \bigcap\{A \in L\mathcal{H} : x \in A\}$.
(3) $\uparrow y = \bigcap\{-\blacklozenge A : A \in L\mathcal{H},\ y \in -\blacklozenge A\}$.
(4) *$R[x] \cap \uparrow y$ and $\uparrow x \cap R^{-1}[y]$ are compact.*

(5) *If $x\,R\,y$, then there are $x' \in X$ and $y' \in Y$ such that $x \leq_X x'$, x' is y-maximal, $y \leq_Y y'$, and y' is x-maximal.*

Proof. (1) Since $A \in L\mathcal{H}$, $A = \Box\blacklozenge A$, so

$$A = -R^{-1}[-\blacklozenge A] = -\bigcup\{R^{-1}[y] : y \in -\blacklozenge A\} = \bigcap\{-R^{-1}[y] : y \in -\blacklozenge A\}.$$

Definition 9.2(1) implies that $R^{-1}[y]$ is a downset for each y. Therefore, A is an upset. A similar argument shows that $\blacklozenge A$ is a downset.

(2) The inclusion $\subseteq$ is clear since each $A \in L\mathcal{H}$ is an upset by (1). For the reverse inclusion, suppose that $x \not\leq_X x'$. Then $R[x'] \not\subseteq R[x]$, so there is y with $x'\,R\,y$ and $x \not\!R\, y$. Therefore, there is $A \in L\mathcal{H}$ with $x \in A$ and $y \notin \blacklozenge A$. Since $x'\,R\,y$, we have $x' \notin A$, completing the proof.

(3) This is proved similarly to (2).

(4) By (2) and (3), ${\uparrow}x \times {\uparrow}y$ is closed in $X \times Y$, so $R \cap ({\uparrow}x \times {\uparrow}y)$ is closed in R, and hence compact because R is compact by Definition 9.3(2). Let $\pi\colon X \times Y \to Y$ be the projection map. Since π is continuous and $R \cap ({\uparrow}x \times {\uparrow}y)$ is compact, $\pi[R \cap ({\uparrow}x \times {\uparrow}y)]$ is compact. But $\pi[R \cap ({\uparrow}x \times {\uparrow}y)] = R[x] \cap {\uparrow}y$, so $R[x] \cap {\uparrow}y$ is compact. A similar argument shows that ${\uparrow}x \cap R^{-1}[y]$ is compact.

(5) Let $x\,R\,y$. We show that there is $y' \in Y$ such that $y \leq_Y y'$ and y' is x-maximal. Let $\mathcal{T}$ be the set of all totally ordered subsets of $R[x] \cap {\uparrow}y$, ordered by inclusion. Then $\mathcal{T}$ is nonempty since $\{y\} \in \mathcal{T}$. Because the union of a chain of totally ordered subsets of $R[x] \cap {\uparrow}y$ is again totally ordered, Zorn's lemma shows that there is a maximal element $C \in \mathcal{T}$. The collection $\{R[x] \cap {\uparrow}z : z \in C\}$ is a family of closed subsets of $R[x] \cap {\uparrow}y$ by (3), and it clearly has the finite intersection property. Therefore, by (4), $\bigcap\{R[x] \cap {\uparrow}z : z \in C\} \neq \varnothing$. Take $y' \in \bigcap\{R[x] \cap {\uparrow}z : z \in C\}$. Then $y \leq_Y y'$ and $x\,R\,y'$. To see that y' is x-maximal, let $y' \leq_Y y''$ with $x\,R\,y''$. Then $y'' \geq_Y y' \geq_Y z$ for each $z \in C$, so $C \cup \{y''\}$ is a totally ordered subset of $R[x] \cap {\uparrow}y$ containing C. Maximality of C shows that $y'' \in C$, forcing $y'' \leq_Y y'$, and hence $y'' = y'$. Thus, y' is x-maximal. That there is $x' \in X$ such that $x \leq_X x'$ and x' is y-maximal is proved similarly but uses that ${\uparrow}x \cap R^{-1}[y]$ is compact. ∎

We next define the notion of a Hartung morphism which is based on that of a GvG-morphism (see Definition 8.8).

Definition 9.6. Let $\mathcal{H} = (X, R, Y)$ and $\mathcal{H}' = (X', R', Y')$ be Hartung spaces. We call a pair (S, T) of relations with $S \subseteq X \times X'$ and $T \subseteq Y \times Y'$ a *Hartung morphism* provided

(1) if $A' \in L\mathcal{H}'$, then $\Box_S A' \in L\mathcal{H}$ and $\blacklozenge\Box_S A' = \Diamond_T \blacklozenge' A'$;
(2) if $x \not\!S\, x'$, then there is $A' \in L\mathcal{H}'$ with $x \in \Box_S A'$ and $x' \notin A'$;
(3) if $y \not\!T\, y'$, then there is $A' \in L\mathcal{H}'$ with $y \notin \Diamond_T \blacklozenge' A'$ and $y' \in \blacklozenge' A'$;
(4) if $x\,R\,y$, then there are $x' \in X'$ and $y' \in Y'$ with $x\,S\,x'$, $y\,T\,y'$, and $x'\,R'\,y'$.

As in Remark 8.9, we have:

Remark 9.7. Let $(S, T)\colon \mathcal{H} \to \mathcal{H}'$ be a Hartung morphism between Hartung spaces $\mathcal{H} = (X, R, Y)$ and $\mathcal{H}' = (X', R', Y')$.

(1) Definition 9.6(2) is equivalent to
$$x\,S\,x' \iff (\forall A' \in L\,\mathcal{H}')\,(x \in \Box_S A' \Rightarrow x' \in A'),$$
and an analogous statement holds for Definition 9.6(3).
(2) $S[x]$ is an upset of X' for each $x \in X$ and $T[y]$ is an upset of Y' for each $y \in Y$.
(3) $S^{-1}[x']$ is a downset of X for each $x' \in X'$ and $T^{-1}[y']$ is a downset of Y for each $y' \in Y'$.

Composition of Hartung morphisms is defined as that of GvG-morphisms:

Definition 9.8. Let $(S_1, T_1)\colon \mathcal{H}_1 \to \mathcal{H}_2$ and $(S_2, T_2)\colon \mathcal{H}_2 \to \mathcal{H}_3$ be Hartung morphisms.

(1) Define $S_2 \star S_1$ by $x\,(S_2 \star S_1)\,z$ provided that $x \in \Box_{S_1}\Box_{S_2} A$ implies $z \in A$ for each $A \in L\,\mathcal{H}_3$.
(2) Define $T_2 \star T_1$ by $y\,(T_2 \star T_1)\,w$ provided that $w \in \blacklozenge A$ implies $y \in \Diamond_{T_1}\Diamond_{T_2}\blacklozenge A$ for each $A \in L\,\mathcal{H}_3$.
(3) Set $(S_2, T_2) \star (S_1, T_1) = (S_2 \star S_1, T_2 \star T_1)$.

An argument similar to that of Lemma 8.12 yields:

Lemma 9.9. *Suppose that $(S_1, T_1)\colon \mathcal{H}_1 \to \mathcal{H}_2$ and $(S_2, T_2)\colon \mathcal{H}_2 \to \mathcal{H}_3$ are Hartung morphisms. If $A \in L\,\mathcal{H}_3$, then $\Box_{S_2 \star S_1} A = \Box_{S_1}\Box_{S_2} A$ and $\Diamond_{T_2 \star T_1}\blacklozenge A = \Diamond_{T_1}\Diamond_{T_2}\blacklozenge A$.*

An analogous proof to Proposition 8.13 yields:

Proposition 9.10. *Suppose that $(S_1, T_1)\colon \mathcal{H}_1 \to \mathcal{H}_2$ and $(S_2, T_2)\colon \mathcal{H}_2 \to \mathcal{H}_3$ are Hartung morphisms.*

(1) *$(S_2, T_2) \star (S_1, T_1)$ is a Hartung morphism.*
(2) *If $\mathcal{H} = (X, R, Y)$ is a Hartung space, then $(\leq_X, \leq_Y)\colon \mathcal{H} \to \mathcal{H}$ is a Hartung morphism and is the identity morphism for $\mathcal{H}$.*
(3) *The operation $\star$ is associative.*

The previous proposition shows that we have a category, motivating the following definition.

Definition 9.11. Let Hg denote the category of Hartung spaces and Hartung morphisms.

In the rest of the chapter we define a functor $\mathbb{H}\colon \mathsf{GvG} \to \mathsf{Hg}$. Let $\mathcal{G} := (X, R, Y) \in \mathsf{GvG}$ and, as in Definition 8.2, let
$$X_0 = \bigcup\{\max R^{-1}[y] : y \in Y\} \quad \text{and} \quad Y_0 = \bigcup\{\max R[x] : x \in X\}.$$
We topologize X_0 with the subspace topology of the open downset topology on X, and topologize Y_0 similarly. Set $R_0 = R \cap (X_0 \times Y_0)$. Our aim is to show that $\mathcal{H} := (X_0, R_0, Y_0)$ is a Hartung space, which we do in a series of lemmas.

Lemma 9.12. *Define $\leq_{X_0}$ on X_0 and $\leq_{Y_0}$ on Y_0 by Definition 9.2(1, 2) applied to R_0. Then $\leq_{X_0}$ and $\leq_{Y_0}$ are the restrictions of the partial orders of X and Y, respectively, and thus are partial orders.*

Proof. Let $x, x' \in X_0$. By Definition 8.3(1), if $x \leq x'$, then $R[x'] \subseteq R[x]$. Because $R_0[x'] = R[x'] \cap Y_0$ and $R_0[x] = R[x] \cap Y_0$, we have $R_0[x'] \subseteq R_0[x]$, so $x \leq_{X_0} x'$.

Conversely, suppose $x \leq_{X_0} x'$, so $R_0[x'] \subseteq R_0[x]$. Let $y \in R[x']$. By Lemma 8.17(2), there is $y' \in Y_0$ with $y \leq y'$ and $x' \mathrel{R} y'$. This yields $x' \mathrel{R_0} y'$, so

$$y' \in R_0[x'] \subseteq R_0[x] \subseteq R[x].$$

Because $y \leq y'$, Definition 8.3(2) gives $y \in R[x]$. Thus, $R[x'] \subseteq R[x]$, and hence $x \leq x'$ by Definition 8.3(1).

A similar argument shows that $\leq_{Y_0}$ is the restriction of the partial order of Y. ∎

Lemma 9.13. *R_0 is a compact subset of $X_0 \times Y_0$ with respect to the product topology on $X_0 \times Y_0$.*

Proof. By Remark 8.4(3), R is a closed subset of $X \times Y$, hence is compact. Therefore, R is a compact subspace of the weaker topology of the product of the open downset topologies on X and Y. We show that $R = {\downarrow} R_0$, where $\downarrow$ is calculated with respect to the product order on $X \times Y$ of the Priestley orders of X and Y. Let $(x, y) \in R$, so $x \mathrel{R} y$. By Lemma 8.17(1, 2), there are $x' \in X_0$ and $y' \in Y_0$ with $x \leq x', y \leq y'$, and $x' \mathrel{R} y'$, so $x' \mathrel{R_0} y'$. Hence, $(x', y') \in R_0$ and $(x, y) \leq (x', y')$. Thus, $R \subseteq {\downarrow} R_0$. For the reverse inclusion, suppose that $(x, y) \leq (x', y') \in R_0$. Then $x' \mathrel{R_0} y'$, so $x' \mathrel{R} y'$. This yields $x \mathrel{R} y'$ by Definition 9.2(1), and hence $x \mathrel{R} y$ by Definition 9.2(2), concluding the proof that $R = {\downarrow} R_0$. By Lemma 9.12, $\leq_{X_0}$ is the restriction to X_0 of $\leq$ on X, and similarly for $\leq_{Y_0}$. Since $\leq$ is the specialization order of the open upset topology, which is dual to the specialization order of the open downset topology, we find that the specialization order of $X_0 \times Y_0$ is the dual of the restriction of the product order on $X \times Y$. Therefore, from $R = {\downarrow} R_0$ it follows that R is the saturation of R_0. Thus, since R is compact, we conclude that R_0 is compact (see, e.g., [GH$^+$03, p. 43]). ∎

The previous two lemmas show that $\mathcal{H}$ satisfies Definition 9.3(1, 2).

Lemma 9.14. *Suppose that $A' \subseteq X$ is an upset and $B' \subseteq Y$ is a downset. Set $A = A' \cap X_0$ and $B = B' \cap Y_0$.*

(1) $\blacklozenge_0 A = \blacklozenge A' \cap Y_0$.
(2) $\Box_0 B = \Box B' \cap X_0$.
(3) $\Box_0 \blacklozenge_0 A = \Box \blacklozenge A' \cap X_0$.
(4) $\blacksquare_0 \Diamond_0 B = \blacksquare \Diamond B' \cap Y_0$.
(5) *If A is closed, then $\Box_0 \blacklozenge_0 A$ is closed.*
(6) *If B is closed, then $\blacksquare_0 \Diamond_0 B$ is closed.*

Proof. (1) The inclusion $\blacklozenge_0 A \subseteq \blacklozenge A' \cap Y_0$ is clear. For the reverse inclusion, let $y \in \blacklozenge A' \cap Y_0$. Then $x \mathrel{R} y$ for some $x \in A'$. By Lemma 8.17(1), there is $x' \in X_0$ with $x \leq x'$ and $x' \mathrel{R} y$, so $x' \mathrel{R_0} y$. Since A' is an upset, $x' \in A' \cap X_0 = A$. Therefore, $y \in R_0[A]$, yielding the equality.

(2) The proof is similar to that of (1).

(3) By (1) and (2) applied to $B' = \blacklozenge A'$, we have

$$\Box_0 \blacklozenge_0 A = \Box_0(\blacklozenge A' \cap Y_0) = \Box \blacklozenge A' \cap X_0.$$

(4) The proof is similar to that of (3).

(5) If A is closed, we may assume that A' is closed (since the topology on X_0 is the subspace topology of the open downset topology on X). Therefore, $A' = \bigcap U_i$ for some family $\{U_i\}$ of clopen upsets of X (see, e.g., [DP02, Lem. 11.21]), and we may assume the family is down-directed. Since $R^{-1}[y]$ is closed for each $y \in Y$ (see Remark 8.4(3)), Lemma 7.18 shows that $\blacklozenge A' = \bigcap \blacklozenge U_i$, and so $\Box\blacklozenge A' = \bigcap \Box\blacklozenge U_i$ since $\Box$ commutes with arbitrary intersections. Each $\Box\blacklozenge U_i$ is a clopen upset, so $\Box\blacklozenge A'$ is a closed upset. Therefore, $\Box_0\blacklozenge_0 A = \Box\blacklozenge A' \cap X_0$ is closed.

(6) The proof is similar to that of (5). ■

LEMMA 9.15.

(1) $L\,\mathcal{H}$ *is a subbasis of closed sets for* X_0.
(2) $\{Y_0 - \blacklozenge U : U \in L\,\mathcal{G}\}$ *is a subbasis of closed sets for* Y_0.

Proof. (1) Let $C \subseteq X_0$ be a closed set. Then $C = D \cap X_0$ for some closed upset D of X, and $D = \bigcap U_i$ for some family $\{U_i\}$ of clopen upsets of X. By Lemma 8.7(2), each U_i is a finite union from $L\,\mathcal{G}$, and the result follows.

(2) The proof is similar, but uses Lemma 8.7(4) instead of 8.7(2). ■

Lemma 9.15 shows that $\mathcal{H}$ satisfies Definition 9.3(3), and 9.14(5, 6) that $\mathcal{H}$ satisfies Definition 9.3(4).

LEMMA 9.16. $\mathcal{H}$ *satisfies Definition* 9.3(5, 6).

Proof. For Definition 9.3(5), let $x \in X_0$. Then x is y-maximal for some $y \in Y$. By Lemma 8.17(2), there is $y' \geq y$ such that y' is x-maximal, so $y' \in Y_0$. Suppose that $x \leq x'$ with $x' \mathrel{R} y'$. Then $x' \mathrel{R} y$ (see Definition 8.3(2)). Thus, $x' = x$ since x is y-maximal. Consequently, x is y'-maximal for some $y \in Y_0$. The other half of Definition 9.3(5) is proved similarly.

For Definition 9.3(6), if $x \not\mathrel{R}_0 y$, then $x \not\mathrel{R} y$, so by Definition 8.3(5) there is $U = \Box\blacklozenge U$ with $x \in U$ and $y \notin \blacklozenge U$. Set $A = U \cap X_0$. Then $x \in A$ and $\blacklozenge_0 A = \blacklozenge U \cap Y_0$ by Lemma 9.14(1), so $y \notin \blacklozenge_0 A$. ■

The above lemmas verify that $\mathcal{H}$ satisfies the conditions of Definition 9.3. We thus have:

PROPOSITION 9.17. *If* $\mathcal{G} = (X, R, Y)$ *is a GvG-space, then* $\mathcal{H} = (X_0, R_0, Y_0)$ *is a Hartung space.*

It follows from [GvG14, Ex. 4.1] that in general X_0 is properly contained in X. The example of Gehrke–van Gool is infinite. The next proposition shows that this is unavoidable as $X_0 = X$ in the finite case (see [GvG14, Ex. 5.4]):

PROPOSITION 9.18. *If* $\mathcal{G} = (X, R, Y)$ *is a finite GvG-space, then* $X_0 = X$.

Proof. Let $x \in X$ and set $U = {\uparrow}({\uparrow}x \cap X_0)$. Then U is an upset, and hence a clopen upset of X because X is finite, hence discrete. Clearly $U \subseteq {\uparrow}x$. For the other inclusion, observe that ${\uparrow}x \in L\,\mathcal{G}$ since the intersection in Lemma 8.7(1) is finite and $L\,\mathcal{G}$ is closed under finite intersections. Therefore, $U \subseteq {\uparrow}x$ implies $\Box\blacklozenge U \subseteq {\uparrow}x$. Thus, $X_0 \cap \Box\blacklozenge U \subseteq X_0 \cap {\uparrow}x \subseteq U$, so $\Box\blacklozenge U = U$ by Definition 8.3(6), and hence $U \in L\,\mathcal{G}$. If $x \notin U$, then by Lemma 8.7(5) there is $x' \in X_0$ with $x \le x'$ and $x' \notin U$, contradicting the definition of U. Consequently, $U = {\uparrow}x$, and so $x \in X_0$ since $U = {\uparrow}({\uparrow}x \cap X_0)$. ∎

We define a functor $\mathbb{H}\colon \mathsf{GvG} \to \mathsf{Hg}$ by setting $\mathbb{H}(X, R, Y) = (X_0, R_0, Y_0)$ for each $(X, R, Y) \in \mathsf{GvG}$ and $\mathbb{H}(S, T) = (S_0, T_0)$ for each GvG-morphism $(S, T)\colon (X, R, Y) \to (X', R', Y')$, where $S_0 = S \cap (X_0 \times X_0')$ and $T_0 = T \cap (Y_0 \times Y_0')$. That $\mathbb{H}$ is well defined on objects follows from Proposition 9.17. In order to see that $\mathbb{H}$ is well defined on morphisms, we require the following two facts.

Proposition 9.19. *Let $\mathcal{G} \in \mathsf{GvG}$. Then the map $\delta_{\mathcal{G}}\colon L\,\mathcal{G} \to L\,\mathcal{H}$ defined by $\delta_{\mathcal{G}}(U) = U \cap X_0$ is a Lat-isomorphism.*

Proof. Lemma 9.14(3) shows that $\delta_{\mathcal{G}}$ is well defined, and clearly it is order-preserving. To show it is order-reflecting, let $U, V \in L\,\mathcal{G}$ with $U \not\subseteq V$. Then there is $x \in U - V$. By Lemma 8.7(5), there is $x' \in X_0$ with $x \le x'$ and $x' \notin V$. We have $x' \in U$ since U is an upset. Consequently, $\delta_{\mathcal{G}}(U) \not\subseteq \delta_{\mathcal{G}}(V)$.

To see that $\delta_{\mathcal{G}}$ is onto, let $A \in L\,\mathcal{H}$. Since A is closed in X_0, there is a closed upset A' of X with $A = A' \cap X_0$. Therefore, there is a down-directed family $\{U_i\}$ of clopen upsets of X with $A' = \bigcap U_i$. Because $A \times -\blacklozenge_0 A \subseteq -R_0$, we have

$$R_0 \cap \bigcap \{(U_i \cap X_0) \times -\blacklozenge_0 A\} = \varnothing.$$

Compactness of R_0 implies that a finite intersection is empty. Since $\{U_i\}$ is a down-directed family, there is a clopen upset $U \supseteq A'$ with $R_0 \cap ((U \cap X_0) \times -\blacklozenge_0 A) = \varnothing$. This forces $U \cap X_0 \subseteq \Box_0 \blacklozenge_0 A = A$, and so $A = U \cap X_0$. We then have $A = \Box_0 \blacklozenge_0 A = \Box\blacklozenge U \cap X_0$ by Lemma 9.14(3). Because $\Box\blacklozenge U \in L\,\mathcal{G}$, we conclude that $A = \delta_{\mathcal{G}}(U)$. Thus, $\delta_{\mathcal{G}}$ is an order-isomorphism, and hence a Lat-isomorphism. ∎

Lemma 9.20. *Let $\mathcal{G} = (X, R, Y)$ and $\mathcal{G}' = (X', R', Y')$ be GvG-spaces. Let $(S, T)\colon \mathcal{G} \to \mathcal{G}'$ be a GvG-morphism. If $U' \in L\,\mathcal{G}'$ and $A' = U' \cap X_0'$, then*

$$\Box_{S_0} A' = \Box_S U' \cap X_0 \quad \textit{and} \quad \Box_{T_0} \blacklozenge_0' A' = \Box_T \blacklozenge' U' \cap Y_0.$$

Proof. Let $x \in \Box_S U' \cap X_0$. Then $S_0[x] = S[x] \cap X_0' \subseteq U' \cap X_0' = A'$, so $x \in \Box_{S_0} A'$. Conversely, suppose that $x \in \Box_{S_0} A'$. If $x \notin \Box_S U'$, then $S[x] \not\subseteq U'$, so there is $x' \in X'$ with $x \mathrel{S} x'$ and $x' \notin U'$. Since $U' \in L\mathcal{G}'$ and $x' \notin U'$, $R[x'] \not\subseteq R[U']$ (see Remark 8.4(4)). Therefore, there is $y \in Y$ with $x' \mathrel{R} y$ and $y \notin R[U']$. By Lemma 8.17(1), there is $z' \in X_0'$ with $x' \le z'$ and $z' \mathrel{R} y$. Thus, $z' \notin U'$ since $y \notin R[U']$. By Remark 8.9(3), $S[x]$ is an upset, so $z' \in S[x] \cap X_0' = S_0[x]$. This forces $z' \in A' \subseteq U'$, a contradiction. Consequently, $x \in \Box_S U' \cap X_0$.

For the second statement, the inclusion $\Box_T \blacklozenge' U' \cap Y_0 \subseteq \Box_{T_0} \blacklozenge_0' A'$ is straightforward. For the reverse inclusion, let $y \in \Box_{T_0} \blacklozenge_0' A'$. If $y \notin \Box_T \blacklozenge' U'$, then $T[y] \not\subseteq \blacklozenge U'$, so there is y' with $y \mathrel{T} y'$ and $y' \notin \blacklozenge U'$. By Remark 8.9(1) and Lemma 8.17(2), there is $z' \in Y_0'$ with $y' \le z'$ and $y \mathrel{T} z'$. Therefore, $y \mathrel{T_0} z'$, so $z' \in T_0[y]$. Because $\blacklozenge U'$ is a downset and $y' \notin \blacklozenge U'$, we have $z' \notin \blacklozenge U'$. This is a contradiction to $y \in \Box_{T_0} \blacklozenge_0' A'$. Thus, $y \in \Box_T \blacklozenge' U'$. ∎

Proposition 9.21. *Let $\mathcal{G}, \mathcal{G}' \in \mathsf{GvG}$, $\mathcal{H} = \mathbb{H}(\mathcal{G})$, and $\mathcal{H}' = \mathbb{H}(\mathcal{G}')$. If $(S, T)\colon \mathcal{G} \to \mathcal{G}'$ is a GvG-morphism, then $(S_0, T_0)\colon \mathcal{H} \to \mathcal{H}'$ is an Hg-morphism.*

Proof. We first verify Definition 9.6(1). Let $A' \in L\mathcal{H}'$. By Proposition 9.19, we may write $A' = U' \cap X_0$ with $U' \in L\mathcal{G}'$. Then $\Box_{S_0} A' = \Box_S U' \cap X_0$ and $\Box_{T_0} \blacklozenge'_0 A' = \Box_T \blacklozenge' U' \cap Y_0$ by Lemma 9.20, and $\Box_{S_0} A' \in L\mathcal{H}$ by Proposition 9.19 and Definition 8.8(1). Thus, by Lemma 9.14(1) and Definition 8.8(1),

$$\blacklozenge_0 \Box_{S_0} A' = \blacklozenge_0(\Box_S U' \cap X_0) = \blacklozenge \Box_S U' \cap Y_0 = \Box_T \blacklozenge' U' \cap Y_0 = \Diamond_{T_0} \blacklozenge'_0 A'.$$

We next verify Definition 9.6(2). Suppose that $x \not\mathrel{S}_0 x'$. Then $x \not\mathrel{S} x'$, and since (S, T) is a GvG-morphism, there is $U \in L\mathcal{G}'$ with $x \in \Box_S U'$ and $x' \notin U'$. If $A' = U' \cap X_0$, then $A' \in L\mathcal{H}$ by Proposition 9.19. We have $\Box_{S_0} A' = \Box_S U' \cap X_0$ by the previous paragraph. Therefore, $x \in \Box_{S_0} A'$ and $x' \notin A'$. A similar argument verifies Definition 9.6(3).

Finally, we verify Definition 9.6(4). Suppose that $x \mathrel{R_0} y$. Then $x \mathrel{R} y$, and since (S, T) is a GvG-morphism, there are x', y' with $x \mathrel{S} x'$, $y \mathrel{T} y'$, and $x' \mathrel{R'} y$. By Lemma 8.17, there are $x'' \in X'_0$ and $y'' \in Y'_0$ such that $x' \le x''$, $y' \le y''$, and $x'' \mathrel{R'} y''$, so $x'' \mathrel{R'_0} y''$. Also, $x \mathrel{S} x''$ and $y \mathrel{T} y''$, hence $x \mathrel{S_0} x''$ and $y \mathrel{T_0} y''$. This completes the proof that (S_0, T_0) is an Hg-morphism. ∎

THEOREM 9.22. *There is a functor* $\mathbb{H}\colon \mathsf{GvG} \to \mathsf{Hg}$ *sending* (X, R, Y) *to* (X_0, R_0, Y_0) *and* $(S, T)\colon \mathcal{G} \to \mathcal{G}'$ *to* (S_0, T_0).

Proof. By Propositions 9.17 and 9.21, $\mathbb{H}$ is well defined on objects and morphisms. Let $\mathcal{G} = (X, R, Y) \in \mathsf{GvG}$. Then $1_{\mathcal{G}} = (\le_X, \le_Y)$. Therefore, by Proposition 9.10(2),

$$\mathbb{H}(\le_X, \le_Y) = (\le_{X_0}, \le_{Y_0}) = 1_{\mathbb{H}(\mathcal{G})},$$

so $\mathbb{H}$ preserves identity morphisms.

To show that composition is preserved, let $(S, T)\colon \mathcal{G} \to \mathcal{G}'$ and $(S', T')\colon \mathcal{G}' \to \mathcal{G}''$ be GvG-morphisms. Set $S'' = S' \star S$ and $T'' = T' \star T$. We show that $S''_0 = S'_0 \star S_0$; the proof that $T''_0 = T'_0 \star T_0$ is similar. Let $x \in X_0$ and $x'' \in X''_0$. Then $x\,(S'_0 \star S_0)\,x''$ iff $x \in \Box_{S_0}\Box_{S'_0} A''$ implies $x'' \in A''$ for each $A'' \in L\mathcal{H}''$. By Proposition 9.19, we may write $A'' = U'' \cap X''_0$ for some $U'' \in L\mathcal{G}''$. Then $x\,(S' \star S)\,x''$ iff $x \in \Box_S \Box_{S'} U''$ implies $x'' \in U''$ for each $U'' \in L\mathcal{G}'$. We show that $x\,(S'_0 \star S_0)\,x''$ iff $x\,(S' \star S)\,x''$, which will show that $S''_0 = S'_0 \star S_0$. First, let $x\,(S'_0 \star S_0)\,x''$. Let $U'' \in L\mathcal{G}''$ and set $A'' = U'' \cap X''_0$. Suppose that $x \in \Box_S \Box_{S'} U''$. By Lemma 9.20,

$$\Box_{S_0}\Box_{S'_0} A'' = \Box_{S_0}(\Box_{S'} U'' \cap X'_0) = \Box_S \Box_{S'} U'' \cap X_0. \tag{9.1}$$

Therefore, $x \in \Box_S \Box_{S'} U'' \cap X_0 = \Box_{S_0}\Box_{S'_0} A''$, so $x'' \in A''$, and hence $x'' \in U''$. This shows that $x\,(S' \star S)\,x''$.

Conversely, let $x\,(S' \star S)\,x''$ and $A'' \in L\mathcal{H}''$. Suppose $x \in \Box_{S_0}\Box_{S'_0} A''$. By Proposition 9.19, there is $U'' \in L\mathcal{G}''$ with $A'' = U'' \cap X''_0$. By (9.1), $\Box_{S_0}\Box_{S'_0} A'' = \Box_S \Box_{S'} U'' \cap X_0$. Thus, $x \in \Box_S \Box_{S'} U''$, so $x'' \in U''$, and hence $x'' \in U'' \cap X''_0 = A''$. This shows that $x\,(S'_0 \star S_0)\,x''$. Consequently, $S''_0 = S'_0 \star S_0$, as desired. Therefore, $\mathbb{H}$ preserves composition, and hence is a functor. ∎

10. From Hartung to Urquhart

In this chapter we connect the Hartung approach to that of Urquhart. The resulting spaces were introduced and studied in [Urq78]. The morphisms that Urquhart considered dually correspond to onto bounded lattice homomorphisms [Urq78, Sec. 5]. We introduce a more general notion of morphism between Urquhart spaces, which dually correspond to all bounded lattice homomorphisms. We show that there is a functor from Hg to the resulting category Urq of Urquhart spaces. In Chapter 13 we will show that this functor is an equivalence and in Chapter 14 that Urq is dually equivalent to Lat.

Let Z be a topological space with two quasi-orders $\le_1$ and $\le_2$. We write $\mathsf{Up}(Z, \le_i)$ for the upsets of $(Z, \le_i)$ for $i = 1, 2$. By [Urq78, Lem. 1], we have the Galois connection $\phi\colon \mathsf{Up}(Z, \le_1) \to \mathsf{Up}(Z, \le_2)$ and $\psi\colon \mathsf{Up}(Z, \le_2) \to \mathsf{Up}(Z, \le_1)$ given by

$$\phi A = -{\downarrow_2} A \quad \text{and} \quad \psi A = -{\downarrow_1} A,$$

where ${\downarrow_i} A$ is the $\le_i$-downset of A.

Definition 10.1. For a topological space $\mathcal{U} = (Z, \le_1, \le_2)$ with two quasi-orders, let

$$L\mathcal{U} = \{C \in \mathsf{Up}(Z, \le_1) : C = \psi\phi C,\ C \text{ and } \phi C \text{ are closed}\}.$$

Following [Urq78, p. 46], we call the triple $(Z, \le_1, \le_2)$ a *doubly ordered space* if

$$x \le_1 y \text{ and } x \le_2 y \text{ imply } x = y.$$

Definition 10.2. Let $\mathcal{U} = (Z, \le_1, \le_2)$ be a topological space with two quasi-orders. Then $\mathcal{U}$ is an *Urquhart space* provided

(1) Z is compact and doubly ordered;
(2) the family $L\mathcal{U} \cup \{\phi C : C \in L\mathcal{U}\}$ is a subbasis of closed sets for the topology on Z;
(3) if $C_1, C_2 \in L\mathcal{U}$, then $\phi(C_1 \cap C_2)$ and $\psi(\phi C_1 \cap \phi C_2)$ are closed;
(4) if $x \not\le_1 y$, then there is $C \in L\mathcal{U}$ with $x \in C$ and $y \notin C$;
(5) if $x \not\le_2 y$, then there is $C \in L\mathcal{U}$ with $x \in \phi C$ and $y \notin \phi C$.

Remark 10.3. Urquhart calls a doubly ordered space doubly disconnected when it satisfies 10.2(4, 5). Observe that every Urquhart space is T_1. Indeed, if $x, y \in Z$ with $x \ne y$, then $x \not\le_1 y$ or $x \not\le_2 y$. If $x \not\le_1 y$, there is $C \in L\mathcal{U}$ with $x \in C$ and $y \notin C$. Therefore, $-C$ is an open set containing y but not x. If $x \not\le_2 y$, there is $C \in L\mathcal{U}$ with $x \in \phi C$ and $y \notin C$. Thus, $-\phi C$ is an open set containing y but not x. It follows that Z is T_1.

Also observe that if $\mathcal{U}$ is an Urquhart space, then $L\mathcal{U}$ is a bounded lattice, where top is X, bottom is $\varnothing$, $C_1 \wedge C_2 = C_1 \cap C_2$, and $C_1 \vee C_2 = \psi(\phi C_1 \cap \phi C_2)$ for each $C_1, C_2 \in L\mathcal{U}$.

We next define Urquhart morphisms.

Definition 10.4. Let $\mathcal{U} = (Z, \le_1, \le_2)$ and $\mathcal{U}' = (Z', \le_1', \le_2')$ be Urquhart spaces. Let also ϕ, ψ be the Galois connection maps for $\mathcal{U}$ and ϕ', ψ' for $\mathcal{U}'$. A pair (P, Q) of relations $P, Q \subseteq Z \times Z'$ is called an *Urquhart morphism* provided

(1) if $C' \in L\mathcal{U}'$, then $\Box_P C' \in L\mathcal{U}$ and $\phi \Box_P C' = \Box_Q \phi' C'$;
(2) if $z \not\mathrel{P} z'$, then there is $C' \in L\mathcal{U}'$ with $z \in \Box_P C'$ and $z' \notin C'$;
(3) if $z \not\mathrel{Q} z'$, then there is $C' \in L\mathcal{U}'$ with $z \in \Box_Q \phi C'$ and $z' \notin \phi C'$;
(4) if $z \in Z$, then $P[z] \cap Q[z] \ne \varnothing$.

Note that in Definition 10.4(3),

$$z \not\mathrel{Q} z' \iff \text{there is } C' \in L\mathcal{U}' \text{ with } z \notin \Diamond_Q - \phi C' \text{ and } z' \in -\phi C',$$

which is more in line with Definition 9.6(3).

Composition of Urquhart morphisms is defined similarly to that of GvG-morphisms and Hartung morphisms:

Definition 10.5. Let $(P_1, Q_1)\colon \mathcal{U}_1 \to \mathcal{U}_2$ and $(P_2, Q_2)\colon \mathcal{U}_2 \to \mathcal{U}_3$ be Urquhart morphisms.

(1) Define $P_2 \star P_1$ by $x\,(P_2 \star P_1)\,z$ if $x \in \Box_{P_1}\Box_{P_2} C$ implies $z \in C$ for each $C \in L\mathcal{U}_3$.
(2) Define $Q_2 \star Q_1$ by $y\,(Q_2 \star Q_1)\,w$ if $y \in \Box_{Q_1}\Box_{Q_2}\phi C$ implies $w \in \phi C$ for each $C \in L\mathcal{U}_3$.
(3) Set $(P_2, Q_2) \star (P_1, Q_1) = (P_2 \star P_1, Q_2 \star Q_1)$.

Observe that in Definition 10.5(2),

$$y\,(Q_2 \star Q_1)\,w \iff (\forall C \in L\mathcal{U}_3)\,(w \in -\phi C \Longrightarrow y \in \Diamond_{Q_1}\Diamond_{Q_2} - \phi C),$$

which is more in line with Definition 9.8(2). We have the following analogue of Remark 8.9 for Urquhart morphisms:

Remark 10.6. Let $(P, Q)\colon \mathcal{U} \to \mathcal{U}'$ be an Urquhart morphism.

(1) $z\,P\,z' \Leftrightarrow (\forall C \in L\mathcal{U}')\,(z \in \Box_P C \Rightarrow z' \in C)$.
(2) $z\,Q\,z' \Leftrightarrow (\forall C \in L\mathcal{U}')\,(z \in \Box_Q \phi C \Rightarrow z' \in \phi C)$.
(3) $P[z]$ is a $\leq_1$-upset and $Q[z]$ is a $\leq_2$-upset of Z' for each $z \in Z$.
(4) $P^{-1}[z']$ is a $\leq_1$-downset and $Q^{-1}[z']$ is a $\leq_2$-downset of Z for each $z' \in Z'$.

An argument similar to that of Lemma 8.12 yields:

Lemma 10.7. *Let $(P_1, Q_1)\colon \mathcal{U}_1 \to \mathcal{U}_2$ and $(P_2, Q_2)\colon \mathcal{U}_2 \to \mathcal{U}_3$ be Urquhart morphisms. If $C \in L\mathcal{U}_3$, then $\Box_{P_2 \star P_1} C = \Box_{P_1}\Box_{P_2} C$ and $\Box_{Q_2 \star Q_1}\phi C = \Box_{Q_1}\Box_{Q_2}\phi C$.*

An analogous proof to Proposition 8.13 yields:

Proposition 10.8. *Let $(P_1, Q_1)\colon \mathcal{U}_1 \to \mathcal{U}_2$ and $(P_2, Q_2)\colon \mathcal{U}_2 \to \mathcal{U}_3$ be Urquhart morphisms.*

(1) *$(P_2, Q_2) \star (P_1, Q_1)$ is an Urquhart morphism.*
(2) *If $\mathcal{U} = (Z, \leq_1, \leq_2)$ is an Urquhart space, then $(\leq_1, \leq_2)$ is the identity morphism for $\mathcal{U}$.*
(3) *The operation $\star$ is associative.*

We thus arrive at the following definition.

Definition 10.9. Let Urq denote the category of Urquhart spaces and Urquhart morphisms.

In the rest of the chapter we define a functor $\mathbb{U}\colon \mathsf{Hg} \to \mathsf{Urq}$.

Definition 10.10. Let $\mathcal{H} = (X, R, Y)$ be a Hartung space.

(1) Set $Z_{\mathcal{H}} = \{(x, y) \in X \times Y : x \text{ is } y\text{-maximal and } y \text{ is } x\text{-maximal}\}$.
(2) Topologize $Z_{\mathcal{H}}$ with the subspace topology of the product topology on $X \times Y$.
(3) Define $\leq_1$ and $\leq_2$ on $Z_{\mathcal{H}}$ by setting

$$(x, y) \leq_1 (x', y') \text{ if } x \leq_X x' \quad \text{and} \quad (x, y) \leq_2 (x', y') \text{ if } y \leq_Y y'.$$

Remark 10.11. Let $\mathcal{H} = (X, R, Y) \in \mathsf{Hg}$.

(1) $Z_{\mathcal{H}}$ is a subset of R and $\leq_1$, $\leq_2$ are quasi-orders on $Z_{\mathcal{H}}$.
(2) Lemma 9.5(5) shows that if $x\,R\,y$, then there is y' with $y \leq_Y y'$ such that y' is x-maximal. Applying it again yields x' with $x \leq_X x'$ such that x' is y'-maximal. It then follows that $(x', y') \in Z_{\mathcal{H}}$.
(3) For each $x \in X$, there is $y' \in Y$ with $(x, y') \in Z_{\mathcal{H}}$. To see this, by Definition 9.3(5), there is $y \in Y$ such that x is y-maximal. Since $x\,R\,y$, (2) yields $y' \geq_Y y$ such that y' is x-maximal. To see that x is y'-maximal, suppose $x \leq_X x'$ with $x'\,R\,y'$. Then $x'\,R\,y$ by Definition 9.2(2), so $x' = x$ as x is y-maximal. Thus, x is y'-maximal, and hence $(x, y') \in Z_{\mathcal{H}}$. A similar argument shows that for each $y \in Y$, there is $x' \in X$ with $(x', y) \in Z_{\mathcal{H}}$.

Lemma 10.12. *For $\mathcal{H} = (X, R, Y) \in \mathsf{Hg}$ set $\mathcal{U} = (Z_{\mathcal{H}}, \leq_1, \leq_2)$. Suppose that $A \subseteq X$ and $B \subseteq Y$ are upsets.*

(1) $\phi((A \times Y) \cap Z_{\mathcal{H}}) = (X \times -\blacklozenge A) \cap Z_{\mathcal{H}}$.
(2) $\psi((X \times B) \cap Z_{\mathcal{H}}) = (-\Diamond B \times Y) \cap Z_{\mathcal{H}}$.
(3) $\psi\phi((A \times Y) \cap Z_{\mathcal{H}}) = (\Box\blacklozenge A \times Y) \cap Z_{\mathcal{H}}$.

Proof. (1) Set $C = (A \times Y) \cap Z_{\mathcal{H}}$ and let $(x, y) \in \phi C$. To show that $y \in -\blacklozenge A$, let $w \in A$. If $w\,R\,y$, then by Remark 10.11(3) there is $y_1 \geq_Y y$ with $(w, y_1) \in Z_{\mathcal{H}}$. Because ϕC is a $\leq_2$-upset and $(x, y) \leq_2 (w, y_1)$, we have $(w, y_1) \in \phi C$. From $\phi C \cap C = \varnothing$, it follows that $(w, y_1) \notin C$, so $w \notin A$. This contradiction shows that $w\,\not R\,y$. Therefore, $y \in -\blacklozenge A$, so $(x, y) \in (X \times -\blacklozenge A) \cap Z_{\mathcal{H}}$, and hence $\phi C \subseteq (X \times -\blacklozenge A) \cap Z_{\mathcal{H}}$.

For the reverse inclusion, suppose that $(x, y) \notin \phi C$. Because $-\phi C = {\downarrow_2} C$, there is $(x', y') \in C$ with $(x, y) \leq_2 (x', y')$. We then have $y \leq_Y y'$ and $x' \in A$. Since $x'\,R\,y'$ and $y \leq_Y y'$, it follows that $x'\,R\,y$ (see Definition 9.2(2)), so $y \in \blacklozenge A$. Consequently, $(x, y) \notin (X \times -\blacklozenge A) \cap Z_{\mathcal{H}}$.

(2) The proof is similar to (1).

(3) By (1) and (2) applied to $B = -\blacklozenge A$, we have

$$\begin{aligned}\psi\phi((A \times Y) \cap Z_{\mathcal{H}}) &= \psi((X \times -\blacklozenge A) \cap Z_{\mathcal{H}}) = (-\Diamond - \blacklozenge A \times Y) \cap Z_{\mathcal{H}} \\ &= (\Box\blacklozenge A \times Y) \cap Z_{\mathcal{H}}. \ \blacksquare\end{aligned}$$

Let $(X, R, Y) \in \mathsf{Hg}$ and $\mathcal{U} = (Z_{\mathcal{H}}, \leq_1, \leq_2)$. To show that $\mathcal{U}$ is an Urquhart space, we require the following two results.

Lemma 10.13 ([Urq78, Lem. 5]). *Let $\mathcal{U} = (Z, \leq_1, \leq_2)$ be a compact doubly ordered space. Suppose that $\mathcal{S}$ is a bounded sublattice of $L\mathcal{U}$ such that whenever $x \not\leq_1 y$, there is $C \in \mathcal{S}$ with $x \in C$ and $y \notin C$. Then $\mathcal{S} = L\mathcal{U}$.*

Proposition 10.14. *Let $\mathcal{H} = (X, R, Y) \in \mathsf{Hg}$ and $\mathcal{U} = (Z_{\mathcal{H}}, \leq_1, \leq_2)$. Then the map $\mu_{\mathcal{H}} \colon L\,\mathcal{H} \to L\mathcal{U}$, defined by $\mu_{\mathcal{H}}(A) = (A \times Y) \cap Z_{\mathcal{H}}$, is a Lat-isomorphism.*

Proof. The map $\mu_{\mathcal{H}}$ is well defined by Lemma 10.12(3), and it is clearly order-preserving. To see that it is order-reflecting, let $A_1, A_2 \in L\,\mathcal{H}$ with $A_1 \not\subseteq A_2$. Then there is $x \in A_1 - A_2$. By Remark 10.11(3), there is $y \in Y$ with $(x, y) \in Z_{\mathcal{H}}$. Consequently, $(x, y) \in (A_1 \times Y) \cap Z_{\mathcal{H}}$ and $(x, y) \notin (A_2 \times Y) \cap Z_{\mathcal{H}}$. Thus, $\mu_{\mathcal{H}}(A_1) \not\subseteq \mu_{\mathcal{H}}(A_2)$.

To see that $\mu_{\mathcal{H}}$ is onto, let $\mathcal{S} = \{(A \times Y) \cap Z_{\mathcal{H}} : A \in L\mathcal{H}\}$, so $\mathcal{S} \subseteq L\mathcal{U}$ by Lemma 10.12(3). By Lemma 10.13, to see that $L\mathcal{U} = \mathcal{S}$, it suffices to show that $\mathcal{S}$ is a bounded sublattice of $L\mathcal{U}$ such that if $(x_1, y_1), (x_2, y_2) \in Z_{\mathcal{H}}$ with $(x_1, y_1) \not\leq_1 (x_2, y_2)$, then there is $C \in \mathcal{S}$ with $(x_1, y_1) \in C$ and $(x_2, y_2) \notin C$. First, since $\varnothing, X \in L\mathcal{H}$, we have $\varnothing = (\varnothing \times Y) \cap Z_{\mathcal{H}}$ and $Z_{\mathcal{H}} = (X \times Y) \cap Z_{\mathcal{H}}$ are both in $\mathcal{S}$. Next, let $C_1, C_2 \in \mathcal{S}$, so $C_1 = (A_1 \times Y) \cap Z_{\mathcal{H}}$ and $C_2 = (A_2 \times Y) \cap Z_{\mathcal{H}}$ for some $A_1, A_2 \in L\mathcal{H}$. Then

$$C_1 \wedge C_2 = C_1 \cap C_2 = ((A_1 \cap A_2) \times Y) \cap Z_{\mathcal{H}} \in \mathcal{S}$$

since $A_1 \cap A_2 \in L\mathcal{H}$. Also, by Remark 10.3 and Lemma 10.12,

$$\begin{aligned} C_1 \vee C_2 &= \psi(\phi C_1 \cap \phi C_2) = \psi((X \times -\blacklozenge A_1) \cap Z_{\mathcal{H}} \cap (X \times -\blacklozenge A_2) \cap Z_{\mathcal{H}}) \\ &= \psi((X \times (-\blacklozenge A_1 \cap -\blacklozenge A_2)) \cap Z_{\mathcal{H}}) = \psi((X \times -\blacklozenge(A_1 \cup A_2)) \cap Z_{\mathcal{H}}) \\ &= (-\Diamond - \blacklozenge(A_1 \cup A_2) \times Y) \cap Z_{\mathcal{H}} = (\Box\blacklozenge(A_1 \cup A_2) \times Y) \cap Z_{\mathcal{H}} \in \mathcal{S} \end{aligned} \tag{10.1}$$

since $\Box\blacklozenge(A_1 \cup A_2) \in L\mathcal{H}$. Finally, suppose that $(x_1, y_1) \not\leq_1 (x_2, y_2)$. Then $x_1 \not\leq_X x_2$. Therefore, by Definition 9.3(6), there is $A \in L\mathcal{H}$ with $x_1 \in A$ and $x_2 \notin A$. If $C = (A \times Y) \cap Z_{\mathcal{H}}$, then $C \in \mathcal{S}$ and $(x_1, y_1) \in C$ but $(x_2, y_2) \notin C$. Thus, $L\mathcal{H} = \mathcal{S}$, and hence $\mu_{\mathcal{H}}$ is onto. Consequently, $\mu_{\mathcal{H}}$ is an order-isomorphism, and so a Lat-isomorphism. ■

Proposition 10.15. *If* $\mathcal{H} = (X, R, Y) \in \mathsf{Hg}$, *then* $\mathcal{U} = (Z_{\mathcal{H}}, \leq_1, \leq_2) \in \mathsf{Urq}$.

Proof. To verify Definition 10.2(1), R is a compact subset of $X \times Y$ by Definition 9.3(2). To see that $Z_{\mathcal{H}}$ is compact, by [GH+03, p. 43] it suffices to show that $R = {\uparrow}Z_{\mathcal{H}}$, where $\uparrow$ is taken with respect to the product order on $X \times Y$. If $(x, y) \in R$, by Remark 10.11(2) there is $(x', y') \in Z_{\mathcal{H}}$ with $x \leq_X x'$ and $y \leq_Y y'$. Therefore, $(x, y) \leq (x', y')$, and so $R = {\uparrow}Z_{\mathcal{H}}$.

To see that $\mathcal{U}$ is doubly ordered, suppose that $(x, y) \leq_1 (x', y')$ and $(x, y) \leq_2 (x', y')$. Then $x \leq_X x'$ and $y \leq_Y y'$. We have $x' \, R \, y'$, so $x' \, R \, y$ by Definition 9.2(2). Because x is y-maximal, $x' = x$. Similarly, $y' = y$.

Next, to verify Definition 10.2(2), the topology on $Z_{\mathcal{H}}$ is the subspace topology of the product topology of $X \times Y$. Definition 9.3(3) shows that

$$\{A \times Y : A \in L\mathcal{H}\} \cup \{X \times -\blacklozenge A : A \in L\mathcal{H}\}$$

is a subbasis of closed sets for $X \times Y$. We have $L\mathcal{U} = \{(A \times Y) \cap Z_{\mathcal{H}} : A \in L\mathcal{H}\}$ by Proposition 10.14, and

$$\{\phi C : C \in L\mathcal{U}\} = \{(X \times -\blacklozenge A) \cap Z_{\mathcal{H}} : A \in L\mathcal{H}\}$$

by Lemmas 10.12(1) and 10.14. Thus, $L\mathcal{U} \cup \{\phi C : C \in L\mathcal{U}\}$ is a subbasis of closed sets for $Z_{\mathcal{H}}$.

To verify Definition 10.2(3), let $C_1, C_2 \in L\mathcal{U}$, and write $C_i = (A_i \times Y) \cap Z_{\mathcal{H}}$ for $i = 1, 2$. By Lemma 10.12(1),

$$\phi(C_1 \cap C_2) = \phi(((A_1 \cap A_2) \times Y) \cap Z_{\mathcal{H}}) = (X \times -\blacklozenge(A_1 \cap A_2)) \cap Z_{\mathcal{H}},$$

so is closed since $A_1 \cap A_2 \in L\mathcal{H}$, and hence $-\blacklozenge(A_1 \cap A_2)$ is closed in Y by Definition 9.3(4). Furthermore, by (10.1), $\psi(\phi C_1 \cap \phi C_2) = (\Box\blacklozenge(A_1 \cup A_2) \times Y) \cap Z_{\mathcal{H}}$, so is closed because $\Box\blacklozenge(A_1 \cup A_2) \in L\mathcal{H}$.

We next verify Definition 10.2(4). Let $(x_1, y_1), (x_2, y_2) \in Z_{\mathcal{H}}$ with $(x_1, y_1) \not\leq_1 (x_2, y_2)$. Then $x_1 \not\leq_X x_2$. By Lemma 9.5(2), there is $A \in L\mathcal{H}$ with $x_1 \in A$ and $x_2 \notin A$. Letting $C = (A \times Y) \cap Z_{\mathcal{H}}$, we have $C \in L\mathcal{U}$ (see Proposition 10.14), $(x_1, y_1) \in C$, and $(x_2, y_2) \notin C$.

Finally, we verify Definition 10.2(5). Suppose that $(x_1, y_1) \not\leq_2 (x_2, y_2)$. Then $y_1 \not\leq_Y y_2$. By Lemma 9.5(3), there is $A \in L\mathcal{H}$ with $y_1 \notin \blacklozenge A$ and $y_2 \in \blacklozenge A$. Letting $C = (A \times Y) \cap Z_{\mathcal{H}}$, we find that $C \in L\mathcal{U}$, $(x_1, y_1) \in \phi C$, and $(x_2, y_2) \notin \phi C$. This completes the proof that $\mathcal{U} \in \mathsf{Urq}$. ∎

THEOREM 10.16. *There is a functor* $\mathbb{U}\colon \mathsf{Hg} \to \mathsf{Urq}$ *which sends* $\mathcal{H} = (X, R, Y)$ *to* $\mathcal{U} = (Z_{\mathcal{H}}, \leq_1, \leq_2)$ *and* $(S, T)\colon \mathcal{H} \to \mathcal{H}'$ *to* $(P, Q)\colon \mathcal{U} \to \mathcal{U}'$, *where*

$$(x, y)\, P\, (x', y') \iff x\, S\, x' \quad \text{and} \quad (x, y)\, Q\, (x', y') \iff y\, T\, y'.$$

Proof. If $\mathcal{H} \in \mathsf{Hg}$, then $\mathcal{U} \in \mathsf{Urq}$ by Proposition 10.15. Let $(S, T)\colon \mathcal{H} \to \mathcal{H}'$ be an Hg-morphism. We show that $(P, Q)\colon \mathcal{U} \to \mathcal{U}'$ is an Urq-morphism. To verify Definition 10.4(1), let $C' \in L\mathcal{U}'$. By Proposition 10.14, $C' = (A' \times Y') \cap Z_{\mathcal{H}'}$ for some $A' \in L\mathcal{H}'$. We have

$$(x, y) \in \Box_P C' \iff P[(x, y)] \subseteq C' \iff S[x] \subseteq A' \iff x \in \Box_S A'.$$

Therefore,

$$\Box_P C' = (\Box_S A' \times Y) \cap Z_{\mathcal{H}} \in L\mathcal{U} \tag{10.2}$$

by Definition 9.6(1) and Proposition 10.14.

For the second statement of Definition 10.4(1), we have $\phi' C' = (X' \times -\blacklozenge' A') \cap Z_{\mathcal{H}'}$ by Lemma 10.12(1). Moreover,

$$\begin{aligned}(x, y) \in \Box_Q \phi' C' &\iff Q[(x, y)] \subseteq \phi' C' \iff T[y] \subseteq -\blacklozenge' A' \\ &\iff y \in \Box_T - \blacklozenge' A' \iff y \in -\Diamond_T \blacklozenge' A' \\ &\iff (x, y) \in (X \times -\Diamond_T \blacklozenge' A') \cap Z_{\mathcal{H}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Box_Q \phi' C' &= (X \times -\Diamond_T \blacklozenge' A') \cap Z_{\mathcal{H}} = (X \times -\blacklozenge \Box_S A') \cap Z_{\mathcal{H}} \\ &= \phi((\Box_S A' \times Y) \cap Z_{\mathcal{H}}) = \phi \Box_P C',\end{aligned} \tag{10.3}$$

where the second equality holds by Definition 9.6(1), the third by Lemma 10.12(1), and the final one by (10.2).

For Definition 10.4(2), suppose $(x, y) \not P (x', y')$. Then $x \not S x'$. By Definition 9.6(2), there is $A' \in L\mathcal{H}'$ with $x \in \Box_S A'$ and $x' \notin A'$. Set $C' = (A' \times Y') \cap Z_{\mathcal{H}'}$. Then $C' \in L\mathcal{U}'$ by Proposition 10.14, and $(x', y') \notin C'$. Moreover,

$$\begin{aligned}P[(x, y)] &= \{(x_1, y_1) \in Z_{\mathcal{H}'} : (x, y)\, P\, (x_1, y_1)\} = \{(x_1, y_1) \in Z_{\mathcal{H}'} : x\, S\, x_1\} \\ &= \{(x_1, y_1) \in Z_{\mathcal{H}'} : x_1 \in S[x]\} \subseteq (A' \times Y') \cap Z_{\mathcal{H}'} = C',\end{aligned}$$

so $(x, y) \in \Box_P C'$.

To verify Definition 10.4(3), suppose that $(x, y) \not Q (x', y')$. Then $y \not T y'$, so by Definition 9.6(3), there is $A' \in L\mathcal{H}'$ with $y \notin \Diamond_T \blacklozenge' A'$ and $y' \in \blacklozenge' A'$. Set $C' = (A' \times Y') \cap Z_{\mathcal{H}'}$. Then $\phi' C' = (X' \times -\blacklozenge' A') \cap Z_{\mathcal{H}'}$ by Lemma 10.12(1). Therefore, $(x', y') \notin \phi' C'$, and $(x, y) \in \Box_Q \phi' C'$ by (10.3).

Finally, to verify Definition 10.4(4), let $(x,y) \in Z_{\mathcal{H}}$. Then $x\,R\,y$ and by Definition 9.6(4), there are $x' \in X'$, $y' \in Y'$ with $x\,S\,x'$, $y\,T\,y'$, and $x'\,R'\,y'$. By Remark 10.11(2), there is $(x'',y'') \in Z_{\mathcal{H}'}$ with $x' \leq_X x''$ and $y' \leq_Y y''$. By Remark 9.7(2), $x\,S\,x''$ and $y\,T\,y''$. Therefore, $(x'',y'') \in P[(x,y)] \cap Q[(x,y)]$. Thus, (P,Q) is an Urquhart morphism.

We next show that $\mathbb{U}$ sends identity morphisms to identity morphisms. Let $\mathcal{H} \in \mathsf{Hg}$. By Proposition 9.10(2), the identity morphism for $\mathcal{H}$ is $(\leq_X, \leq_Y)$. If $(P,Q) = \mathbb{U}(\leq_X, \leq_Y)$, then

$$(x,y)\,P\,(x',y') \iff x \leq_X x' \quad \text{and} \quad (x,y)\,Q\,(x',y') \iff y \leq_Y y'.$$

Therefore, $\mathbb{U}(\leq_X, \leq_Y) = (\leq_1, \leq_2)$, which by Proposition 10.8(2) is the identity morphism for $\mathbb{U}(\mathcal{H})$.

To finish the proof, we show that $\mathbb{U}$ preserves composition. Consider Hg-morphisms $(S_1,T_1)\colon \mathcal{H}_1 \to \mathcal{H}_2$ and $(S_2,T_2)\colon \mathcal{H}_2 \to \mathcal{H}_3$. Set $(S,T) = (S_2,T_2)\star(S_1,T_1)$, so $S = S_2 \star S_1$ and $T = T_2 \star T_1$. If $(x,y) \in Z_1$ and $(x',y') \in Z_3$, then

$$(x,y)\,P\,(x',y') \iff x\,S\,x' \iff x\,(S_2 \star S_1)\,x'.$$

Also,

$$(x,y)\,(P_2 \star P_1)\,(x',y') \iff (\forall C \in L\,\mathcal{U}_3)\ ((x,y) \in \Box_{P_1}\Box_{P_2}C \Longrightarrow (x',y') \in C)\,.$$

Given such C there is $A \in L\,\mathcal{H}_3$ with $C = (A \times Y_3) \cap Z_3$. By (10.2),

$$\Box_{P_1}\Box_{P_2}C = \Box_{P_1}((\Box_{S_2}A \times Y_2) \cap Z_2) = (\Box_{S_1}\Box_{S_2}A \times Y_1) \cap Z_1.$$

Therefore,

$$\begin{aligned}(x,y)\,(P_2 \star P_1)\,(x',y') &\iff (\forall A \in L\,\mathcal{H}_3)\ (x \in \Box_{S_1}\Box_{S_2}A \Longrightarrow x' \in A)\\ &\iff x\,(S_2 \star S_1)\,x'.\end{aligned}$$

A similar calculation shows that

$$(x,y)\,(Q_2 \star Q_1)\,(x',y') \iff y\,(T_2 \star T_1)\,y'.$$

Thus, $(P,Q) = (P_2,Q_2) \star (P_1,Q_1)$, which shows that $\mathbb{U}$ preserves composition. ∎

11. Ploščica spaces: an alternative approach to Urquhart spaces

In [Plo95], Ploščica showed that the two quasi-orders of an Urquhart space can be replaced by a reflexive relation. In this chapter we introduce the category Plo of Ploščica spaces and Ploščica morphisms and show that it is isomorphic to Urq. Following Chapter 8, for a relation R, we write $\Box, \blacklozenge$ instead of $\Box_R, \blacklozenge_R$ when the context is clear.

Definition 11.1. For the pair $\mathcal{P} = (Z,R)$, where Z is a topological space and R is a relation on Z, we set

$$L\,\mathcal{P} = \{C \subseteq Z : C = \Box\blacklozenge C \text{ and } C, -\blacklozenge C \text{ are closed}\}.$$

Definition 11.2. Let $\mathcal{P} = (Z,R)$ be a topological space with a reflexive relation R. We call $\mathcal{P}$ a *Ploščica space* provided

(1) Z is compact and T_1;
(2) the family $L\,\mathcal{P} \cup \{-\blacklozenge C : C \in L\,\mathcal{P}\}$ is a subbasis of closed sets for the topology on Z;

(3) if $C_1, C_2 \in L\mathcal{P}$, then $-\blacklozenge(C_1 \cap C_2)$ and $\Box\blacklozenge(C_1 \cup C_2)$ are closed;
(4) if $R[y] \not\subseteq R[x]$, then there is $C \in L\mathcal{P}$ with $x \in C$ and $y \notin C$;
(5) if $R^{-1}[y] \not\subseteq R^{-1}[x]$, then there is $C \in L\mathcal{P}$ with $x \notin \blacklozenge C$ and $y \in \blacklozenge C$;
(6) if $x\,R\,y$, then there is $z \in Z$ with $R[z] \subseteq R[x]$ and $R^{-1}[z] \subseteq R^{-1}[y]$.

Remark 11.3. Definition 11.2(3) implies that $L\mathcal{P}$ is a bounded lattice whose top is X, bottom is $\varnothing$, $C_1 \wedge C_2 = C_1 \cap C_2$, and $C_1 \vee C_2 = \Box\blacklozenge(C_1 \cup C_2)$ for each $C_1, C_2 \in L\mathcal{P}$.

Definition 11.4. Let $\mathcal{P} = (Z, R)$ and $\mathcal{P}' = (Z', R')$ be Ploščica spaces. We call a pair (P, Q) of relations $P, Q \subseteq Z \times Z'$ a *Ploščica morphism* provided

(1) if $C' \in L\mathcal{P}'$, then $\Box_P C' \in L\mathcal{P}$ and $\blacklozenge\Box_P C' = \Diamond_Q \blacklozenge' C'$;
(2) if $z \not{P} z'$, then there is $C' \in L\mathcal{P}'$ with $z \in \Box_P C'$ and $z' \notin C'$;
(3) if $z \not{Q} z'$, then there is $C' \in L\mathcal{P}'$ with $z \notin \Diamond_Q \blacklozenge' C'$ and $z' \in \blacklozenge' C'$;
(4) if $z \in Z$, then $P[z] \cap Q[z] \neq \varnothing$.

We define composition of Ploščica morphisms exactly as for Urq-morphisms.

Definition 11.5. Let $(P_1, Q_1)\colon \mathcal{P}_1 \to \mathcal{P}_2$ and $(P_2, Q_2)\colon \mathcal{P}_2 \to \mathcal{P}_3$ be Ploščica morphisms.

(1) Define $P_2 \star P_1$ by $x\,(P_2 \star P_1)\,z$ provided that $x \in \Box_{P_1}\Box_{P_2} C$ implies $z \in C$ for each $C \in L\mathcal{P}_3$.
(2) Define $Q_2 \star Q_1$ by $y\,(Q_2 \star Q_1)\,w$ provided that $y \in \Box_{Q_1}\Box_{Q_2} - \blacklozenge C$ implies $w \in -\blacklozenge C$ for each $C \in L\mathcal{P}_3$.
(3) Set $(P_2, Q_2) \star (P_1, Q_1) = (P_2 \star P_1, Q_2 \star Q_1)$.

Definition 11.6. For a Ploščica space $\mathcal{P} = (Z, R)$, define $\leq_1$ and $\leq_2$ on Z by

$$x \leq_1 z \iff R[z] \subseteq R[x] \quad \text{and} \quad y \leq_2 z \iff R^{-1}[z] \subseteq R^{-1}[y].$$

Remark 11.7.

(1) By the above definition, we can phrase Definition 11.2(6) as follows:

$$\text{If } x\,R\,y\text{, then there is } z \text{ with } x \leq_1 z \text{ and } y \leq_2 z.$$

(2) If $C \in L\mathcal{P}$, then C is a $\leq_1$-upset and $\blacklozenge C$ is a $\leq_2$-downset.

The same argument as in Proposition 10.8 (which is analogous to that in Proposition 8.13) yields:

Proposition 11.8. *The Ploščica spaces and Ploščica morphisms form a category* Plo *where composition is given by* $\star$ *and the identity morphism for* $\mathcal{P}$ *is* $(\leq_1, \leq_2)$.

We next define a functor $\mathbb{P}\colon \mathsf{Urq} \to \mathsf{Plo}$.

Definition 11.9. Let $\mathcal{U} = (Z, \leq_1, \leq_2)$ be a topological space with two quasi-orders. Define $R_{\mathcal{U}} \subseteq Z \times Z$ by

$$x\,R_{\mathcal{U}}\,y \iff \exists z \in Z : x \leq_1 z \text{ and } y \leq_2 z.$$

Remark 11.10. It is immediate from the above definition that $R_{\mathcal{U}} = {\geq_2} \circ {\leq_1}$, and hence $R_{\mathcal{U}}[C] = {\downarrow_2}{\uparrow_1} C$ and $R_{\mathcal{U}}^{-1}[C] = {\downarrow_1}{\uparrow_2} C$ for each $C \subseteq X$.

As follows from [Plo95, Thm. 2.1], we can recover $\leq_1$ and $\leq_2$ from $R_{\mathcal{U}}$:

Lemma 11.11. *Let* $\mathcal{U} = (Z, \leq_1, \leq_2)$ *be an Urquhart space and* $x, y \in Z$.

(1) $x \leq_1 y$ *iff* $R_{\mathcal{U}}[y] \subseteq R_{\mathcal{U}}[x]$.
(2) $x \leq_2 y$ *iff* $R_{\mathcal{U}}^{-1}[y] \subseteq R_{\mathcal{U}}^{-1}[x]$.

Proof. (1) Suppose that $x \leq_1 y$. Then $\uparrow_1 y \subseteq \uparrow_1 x$, so $R_{\mathcal{U}}[y] = \downarrow_2\uparrow_1 y \subseteq \downarrow_2\uparrow_1 x = R_{\mathcal{U}}[x]$. Conversely, suppose that $x \not\leq_1 y$. By Definition 10.2(4), there is $C \in L\mathcal{U}$ with $x \in C$ and $y \notin C$. Because $C = \psi\phi C = -\downarrow_1\phi C$, there is $z \in \phi C$ with $y \leq_1 z$. On the one hand, $z \in \uparrow_1 y \subseteq \downarrow_2\uparrow_1 y = R_{\mathcal{U}}[y]$. On the other hand, since $z \in \phi C = -\downarrow_2 C$ and $\uparrow_1 x \subseteq C$, we have $z \notin \downarrow_2\uparrow_1 x = R_{\mathcal{U}}[x]$. Thus, $R_{\mathcal{U}}[y] \not\subseteq R_{\mathcal{U}}[x]$.

(2) The proof is similar to that of (1). ∎

If $\mathcal{U} \in \mathsf{Urq}$, it is clear that $R_{\mathcal{U}}$ is a reflexive relation, and that $\leq_1, \leq_2$ are definable in terms of $R_{\mathcal{U}}$ by Lemma 11.11.

Proposition 11.12. *Let Z be a set, R a relation on Z, and $\leq_1, \leq_2$ quasi-orders on Z such that for all* $x, y \in Z$,

$$x\,R\,y \iff \exists z \in Z : x \leq_1 z \text{ and } y \leq_2 z.$$

Let also ϕ, ψ be the Galois connection maps defined in Chapter 10, *C a $\leq_1$-upset, and D a $\leq_2$-upset.*

(1) $\phi C = -\blacklozenge C$.
(2) $\psi D = \Box - D$.
(3) $\psi\phi C = \Box\blacklozenge C$.

Proof. (1) By Remark 11.10, $\blacklozenge C = R[C] = \downarrow_2\uparrow_1 C = \downarrow_2 C$. Thus, $-\blacklozenge C = -\downarrow_2 C = \phi C$.

(2) The proof is similar to that of (1).

(3) Let C be a $\leq_1$-upset. Then $-\blacklozenge C$ is a $\leq_2$-upset by (1). Consequently, by (1) and (2) applied to $D = -\blacklozenge C$, we have $\psi\phi C = \psi(-\blacklozenge C) = \Box\blacklozenge C$. ∎

Lemma 11.13.

(1) *Let* $\mathcal{U} = (Z, \leq_1, \leq_2) \in \mathsf{Urq}$ *and* $\mathcal{P} = (Z, R_{\mathcal{U}})$. *Then* $L\mathcal{P} = L\mathcal{U}$.
(2) *Let* $\mathcal{P} = (Z, R) \in \mathsf{Plo}$, *and define* $\leq_1, \leq_2$ *as in Definition* 11.6. *If* $\mathcal{U} = (Z, \leq_1, \leq_2)$, *then* $L\mathcal{U} = L\mathcal{P}$.

Proof. (1) Let $C \in L\mathcal{U}$. By definition, $C, \phi C$ are closed and $\psi\phi C = C$. Therefore, $C \in L\mathcal{P}$ by Proposition 11.12. Conversely, if $C \in L\mathcal{P}$, then $C, -\blacklozenge C$ are closed and $\Box\blacklozenge C = C$. Remark 11.10 shows that C is a $\leq_1$-upset, so Proposition 11.12 implies that ϕC is closed and $\psi\phi C = C$. Thus, $C \in L\mathcal{U}$, and hence $L\mathcal{P} = L\mathcal{U}$.

(2) The proof is similar to that of (1). ∎

Proposition 11.14. *If* $\mathcal{U} = (Z, \leq_1, \leq_2) \in \mathsf{Urq}$, *then* $\mathcal{P} := (Z, R_{\mathcal{U}}) \in \mathsf{Plo}$.

Proof. To verify Definition 11.2(1), the space Z is compact by definition, and it is T_1 by Remark 10.3. Lemma 11.13(1) shows that $L\mathcal{P} = L\mathcal{U}$, and so Definition 11.2(2-5) follows from Proposition 11.12 because $\mathcal{U} \in \mathsf{Urq}$ (see Definition 10.2). To verify Definition 11.2(6), suppose that $x\,R_{\mathcal{U}}\,y$. Then there is $z \in Z$ with $x \leq_1 z$ and $y \leq_2 z$. By Lemma 11.11, $R_{\mathcal{U}}[z] \subseteq R_{\mathcal{U}}[x]$ and $R_{\mathcal{U}}^{-1}[z] \subseteq R_{\mathcal{U}}^{-1}[y]$. Thus, $\mathcal{P} \in \mathsf{Plo}$. ∎

Proposition 11.15. *If $(P, Q)\colon \mathcal{U} \to \mathcal{U}'$ is an* Urq*-morphism, then $(P, Q)\colon \mathcal{P} \to \mathcal{P}'$ is a* Plo*-morphism, where $\mathcal{P} = (Z, R_{\mathcal{U}})$ and $\mathcal{P}' = (Z', R_{\mathcal{U}'})$.*

Proof. By Lemma 11.13(1), $L\mathcal{P} = L\mathcal{U}$. Thus, by Proposition 11.12, Definition 10.4(i) implies Definition 11.4(i) for each $1 \le i \le 4$. Consequently, (P, Q) is a Plo-morphism. ∎

Theorem 11.16. *There is a functor $\mathbb{P}\colon \mathsf{Urq} \to \mathsf{Plo}$ given by $\mathbb{P}(\mathcal{U}) = (Z, R_{\mathcal{U}})$ for each $\mathcal{U} \in \mathsf{Urq}$, and $\mathbb{P}(P, Q) = (P, Q)$ for each* Urq*-morphism (P, Q).*

Proof. Proposition 11.14 shows that $\mathbb{P}$ is well defined on objects, and Proposition 11.8 that it is well defined on morphisms. It is clear that $\mathbb{P}$ preserves composition and identity morphisms. Thus, $\mathbb{P}$ is a functor. ∎

We now show that $\mathbb{P}$ is an isomorphism of categories. For this we require the following lemma.

Lemma 11.17. *Let $\mathcal{P} = (Z, R) \in \mathsf{Plo}$. If $x, y \in Z$, then $x \, R \, y$ iff there is $z \in Z$ with $x \le_1 z$ and $y \le_2 z$.*

Proof. One direction follows from Remark 11.7(1). For the other direction, suppose that there is $z \in Z$ with $x \le_1 z$ and $y \le_2 z$. From $x \le_1 z$ we get $R[z] \subseteq R[x]$. Because R is reflexive, $z \in R[z]$, so $z \in R[x]$. Therefore, $x \, R \, z$, and hence $x \in R^{-1}[z]$. Because $y \le_2 z$, we have $R^{-1}[z] \subseteq R^{-1}[y]$. Consequently, $x \in R^{-1}[y]$, yielding $x \, R \, y$. ∎

Theorem 11.18. *$\mathbb{P}\colon \mathsf{Urq} \to \mathsf{Plo}$ is an isomorphism of categories.*

Proof. By [ML71, p. 14], to see that $\mathbb{P}$ is an isomorphism of categories, it suffices to show that $\mathbb{P}$ is a bijection on objects and on morphisms. If $\mathcal{U} = (Z, \le_1, \le_2)$, then $\le_1, \le_2$ are determined by $R_{\mathcal{U}}$ by Definition 11.6 and Lemma 11.11. This shows that $\mathbb{P}$ is 1-1 on objects. To see that $\mathbb{P}$ is onto on objects, let $(Z, R) \in \mathsf{Plo}$, and set $\mathcal{U} = (Z, \le_1, \le_2)$, where $\le_1, \le_2$ are given in Definition 11.6. We show that $\mathcal{U} \in \mathsf{Urq}$. To see that $\mathcal{U}$ is a doubly ordered space, suppose that $x, y \in Z$ with $x \le_1 y$ and $x \le_2 y$. If $x \ne y$, since Z is T_1, there is a closed set C containing x but not y. We may assume that C is a subbasic closed set. Therefore, Definition 10.2(2) yields that either $C \in L\mathcal{P}$ or $C = -\blacklozenge D$ for some $D \in L\mathcal{P}$. If $C \in L\mathcal{P}$, then C is a $\le_1$-upset by Remark 11.7(2); and C contains x but not y, a contradiction to $x \le_1 y$. If $C = -\blacklozenge D$, then C is a $\le_2$-upset, again yielding a contradiction since $x \le_2 y$. Therefore, $x = y$. This shows that $\mathcal{U}$ is a doubly ordered space. By Lemma 11.13(2), $L\mathcal{U} = L\mathcal{P}$. Thus, Definition 11.2 and Proposition 11.12 show that $\mathcal{U} \in \mathsf{Urq}$. If $x, y \in Z$, then by Lemma 11.17, $x \, R \, y$ iff there is $z \in Z$ with $x \le_1 z$ and $y \le_2 z$, which happens iff $x \, R_{\mathcal{U}} \, y$. Consequently, $R = R_{\mathcal{U}}$, so $\mathcal{P} = \mathbb{P}(\mathcal{U})$, and hence $\mathbb{P}$ is onto on objects.

By Proposition 11.15, if $(P, Q)\colon \mathcal{U} \to \mathcal{U}'$ is an Urq-morphism, then $(P, Q)\colon \mathcal{P} \to \mathcal{P}'$ is a Plo-morphism. Conversely, if $(P, Q)\colon \mathcal{P} \to \mathcal{P}'$ is a Plo-morphism, then Lemma 11.13(2) and Proposition 11.12 show that $(P, Q)\colon \mathcal{U} \to \mathcal{U}'$ is an Urquhart morphism, so $\mathbb{P}$ is one-to-one and onto on morphisms, completing the proof that $\mathbb{P}$ is an isomorphism of categories. ∎

12. From Urquhart back to Dunn–Hartonas

In Chapters 8 to 10 we defined the functors

$$\mathsf{DH} \xrightarrow{\mathbb{G}} \mathsf{GvG} \xrightarrow{\mathbb{H}} \mathsf{Hg} \xrightarrow{\mathbb{U}} \mathsf{Urq}$$

and in Chapter 11 we saw that $\mathbb{P}\colon \mathsf{Urq} \to \mathsf{Plo}$ is an isomorphism. In this chapter we complete the circle by defining a functor $\mathbb{D}\colon \mathsf{Urq} \to \mathsf{DH}$.

Definition 12.1. Let $\mathcal{U} = (Z, \leq_1, \leq_2) \in \mathsf{Urq}$. Set

$$\mathsf{LC}\mathcal{U} = \{D \in \mathsf{Up}(Z, \leq_1) : D \text{ is closed, } \psi\phi D = D\},$$
$$\mathsf{RC}\mathcal{U} = \{E \in \mathsf{Up}(Z, \leq_2) : E \text{ is closed, } \phi\psi E = E\}.$$

We order $\mathsf{LC}\mathcal{U}$ and $\mathsf{RC}\mathcal{U}$ by reverse inclusion.

Remark 12.2. In [Urq78, Thm. 3], elements of $\mathsf{LC}\mathcal{U}$ are called left-closed stable sets and elements of $\mathsf{RC}\mathcal{U}$ right-closed stable sets, hence the notation.

We show that $\mathsf{LC}\mathcal{U}$ and $\mathsf{RC}\mathcal{U}$ are coherent lattices. For this we require the following lemma.

Lemma 12.3. *Let* $\mathcal{U} = (Z, \leq_1, \leq_2) \in \mathsf{Urq}$.

(1) *If* $D \in \mathsf{LC}\mathcal{U}$ *and* $C \in L\mathcal{U}$, *then* $D \subseteq C$ *iff* $D \cap \phi C = \varnothing$.
(2) *If* $E \in \mathsf{RC}\mathcal{U}$ *and* $C \in L\mathcal{U}$, *then* $E \subseteq \phi C$ *iff* $C \cap E = \varnothing$.
(3) *If* $\mathcal{S} \subseteq \mathsf{LC}\mathcal{U}$ *and* $C \in L\mathcal{U}$ *with* $\bigcap \mathcal{S} \subseteq C$, *then there are* $D_1, \dots, D_n \in \mathcal{S}$ *with* $D_1 \cap \dots \cap D_n \subseteq C$.
(4) *If* $\mathcal{S} \subseteq \mathsf{RC}\mathcal{U}$ *and* $C \in L\mathcal{U}$ *with* $\bigcap \mathcal{S} \subseteq \phi C$, *then there are* $E_1, \dots, E_n \in \mathcal{S}$ *with* $E_1 \cap \dots \cap E_n \subseteq \phi C$.
(5) *If* $D \in \mathsf{LC}\mathcal{U}$, *then* $D = \bigcap \{C \in L\mathcal{U} : D \subseteq C\}$.
(6) *If* $E \in \mathsf{RC}\mathcal{U}$, *then* $E = \bigcap \{\phi C : C \in L\mathcal{U}, E \subseteq \phi C\}$.

Proof. (1) This follows from [Urq78, Thm. 2b]. For the reader's convenience, we give a short proof. If $D \subseteq C$, then $D \cap \phi C \subseteq C \cap \phi C = \varnothing$. Conversely, suppose that $D \cap \phi C = \varnothing$. Then $D \subseteq -\phi C = {\downarrow_2} C$. Therefore, $\phi({\downarrow_2} C) \subseteq \phi D$. But $\phi({\downarrow_2} C) = -{\downarrow_2}{\downarrow_2} C = -{\downarrow_2} C = \phi C$. Thus, $\phi C \subseteq \phi D$, so $\psi\phi D \subseteq \psi\phi C$, and hence $D \subseteq C$ since $C \in L\mathcal{U}$ and $D \in \mathsf{LC}\mathcal{U}$.

(2) The proof is similar to that of (1).

(3) If $\bigcap \mathcal{S} \subseteq C$, then $\bigcap \mathcal{S} \subseteq -\phi C$ because $C \cap \phi C = \varnothing$. Since $-\phi C$ is open, compactness implies that there are $D_1, \dots, D_n \in \mathcal{S}$ with $D_1 \cap \dots \cap D_n \subseteq -\phi C$. Since $\mathsf{LC}\mathcal{U}$ is closed under finite intersections, (1) shows that $D_1 \cap \dots \cap D_n \subseteq C$.

(4) The proof is similar to that of (3).

(5) The inclusion $\subseteq$ is obvious. For the reverse inclusion, suppose that $x \notin D$. Since $D = \psi\phi D$, we have $x \in {\downarrow_1}\phi D = {\downarrow_1}(-{\downarrow_2} D)$. Therefore, $x \leq_1 y$ for some $y \in -{\downarrow_2} D$. Consequently, $y \not\leq_2 z$ for each $z \in D$. By Definition 10.2(5), there is $C_z \in L\mathcal{U}$ with $y \in \phi C_z$ and $z \notin \phi C_z$. Because $D \subseteq \bigcup\{-\phi C_z : z \in D\}$, compactness implies that there are $z_1, \dots, z_n \in D$ with $D \subseteq -\phi C_{z_1} \cup \dots \cup -\phi C_{z_n}$. Let $C = \psi(\phi C_{z_1} \cap \dots \cap \phi C_{z_n})$. Then $C \in L\mathcal{U}$ by Remark 10.3. Since $\phi C_{z_1} \cap \dots \cap \phi C_{z_n} = \phi(C_{z_1} \cup \dots \cup C_{z_n})$ and $\phi\psi\phi = \phi$, we

have

$$\begin{aligned}\phi C &= \phi\psi(\phi C_{z_1} \cap \cdots \cap \phi C_{z_n}) = \phi\psi\phi(C_{z_1} \cup \cdots \cup C_{z_n}) \\ &= \phi(C_{z_1} \cup \cdots \cup C_{z_n}) = \phi C_{z_1} \cap \cdots \cap \phi C_{z_n},\end{aligned}$$

so $D \cap \phi C = \varnothing$. This implies that $D \subseteq C$ by (1). Moreover, $y \in \phi C$, yielding $y \notin C$. Finally, since C is an upset and $x \leq_1 y$, it follows that $x \notin C$.

(6) The proof is similar to that of (5). ∎

Proposition 12.4. *Let $\mathcal{U} \in \mathsf{Urq}$.*

(1) $\mathsf{LC}\mathcal{U}, \mathsf{RC}\mathcal{U} \in \mathsf{CohLat}_\lambda$.
(2) $K(\mathsf{LC}\mathcal{U}) = L\mathcal{U}$ *and* $K(\mathsf{RC}\mathcal{U}) = \{\phi C : C \in L\mathcal{U}\}$.
(3) *The map* $\xi_U \colon L\mathcal{U} \to \mathsf{CLF}(\mathsf{LC}\mathcal{U})$, *defined by* $\xi_U(C) = {\uparrow}C$, *is a* Lat*-isomorphism.*

Proof. (1) Define $u \colon \mathsf{Filt}(L\mathcal{U}) \to \mathsf{LC}\mathcal{U}$ and $v \colon \mathsf{Idl}(L\mathcal{U}) \to \mathsf{RC}\mathcal{U}$ by

$$u(\mathcal{F}) = \bigcap \mathcal{F} \quad \text{and} \quad v(\mathcal{I}) = \bigcap \{\phi C : C \in \mathcal{I}\}$$

for each $\mathcal{F} \in \mathsf{Filt}(L\mathcal{U})$ and $\mathcal{I} \in \mathsf{Idl}(L\mathcal{U})$. It is straightforward to see that u and v are well defined and, with respect to reverse inclusion on $\mathsf{LC}\mathcal{U}$ and $\mathsf{RC}\mathcal{U}$, both are order-preserving. To see that u is onto, let $D \in \mathsf{LC}\mathcal{U}$. If $\mathcal{F} = \{C \in L\mathcal{U} : D \subseteq C\}$, then it is straightforward to see that $\mathcal{F}$ is a filter and $D = \bigcap \mathcal{F}$ follows from Lemma 12.3(5). A similar argument using Lemma 12.3(6) shows that v is onto.

To see that u is order-reflecting, we show that $\mathcal{F} = \{C \in L\mathcal{U} : u(\mathcal{F}) \subseteq C\}$ for each $\mathcal{F} \in \mathsf{Filt}(L\mathcal{U})$. The inclusion $\subseteq$ is clear by the definition of u. For the reverse inclusion, suppose that $u(\mathcal{F}) \subseteq C$ for some $C \in L\mathcal{U}$. Because $u(\mathcal{F}) = \bigcap \mathcal{F}$, Lemma 12.3(3) implies that there are $C_1, \dots, C_n \in \mathcal{F}$ with $C_1 \cap \cdots \cap C_n \subseteq C$. Therefore, $C \in \mathcal{F}$. Thus, for $\mathcal{F}, \mathcal{F}' \in \mathsf{Filt}(L\mathcal{U})$,

$$\mathcal{F} \subseteq \mathcal{F}' \iff u(\mathcal{F}') \subseteq u(\mathcal{F}) \iff u(\mathcal{F}) \leq u(\mathcal{F}').$$

A similar argument, using Lemma 12.3(4) instead of Lemma 12.3(3), shows that v is order-reflecting. Thus, u and v are order-isomorphisms, and hence $\mathsf{LC}\mathcal{U}, \mathsf{RC}\mathcal{U} \in \mathsf{CohLat}_\lambda$.

(2) Since $\mathsf{LC}\mathcal{U}$ is ordered by $\supseteq$, it follows from Lemma 12.3(3) that $L\mathcal{U} \subseteq K(\mathsf{LC}\mathcal{U})$. For the reverse inclusion, by Lemma 12.3(5), each $D \in \mathsf{LC}\mathcal{U}$ is a join in $\mathsf{LC}\mathcal{U}$ from $L\mathcal{U}$. Therefore, if $D \in K(\mathsf{LC}\mathcal{U})$, then it is a finite join from $L\mathcal{U}$. Since a finite join in $\mathsf{LC}\mathcal{U}$ is a finite intersection and $L\mathcal{U}$ is closed under finite intersections, $D \in L\mathcal{U}$. Thus, $K(\mathsf{LC}\mathcal{U}) = L\mathcal{U}$. The argument for $\mathsf{RC}\mathcal{U}$ is similar but uses Lemma 12.3(4, 6) instead of Lemma 12.3(3, 5).

(3) Apply (1), (2), and Lemma 5.4. ∎

For $\mathcal{U} \in \mathsf{Urq}$, define $R_\mathcal{U} \subseteq \mathsf{LC}\mathcal{U} \times \mathsf{RC}\mathcal{U}$ by

$$D \, R_\mathcal{U} \, E \iff D \cap E \neq \varnothing,$$

and set $\mathcal{D}(\mathcal{U}) = (\mathsf{LC}\mathcal{U}, R_\mathcal{U}, \mathsf{RC}\mathcal{U})$.

Proposition 12.5. *If $\mathcal{U} \in \mathsf{Urq}$, then $\mathcal{D}(\mathcal{U}) \in \mathsf{DH}$.*

Proof. By Proposition 12.4(1), $\mathsf{LC}\mathcal{U}, \mathsf{RC}\mathcal{U} \in \mathsf{CohLat}_\lambda$. It is left to show that $R_\mathcal{U}$ is a DH-relation. For Definition 7.5(1) let $D \in \mathsf{LC}\mathcal{U}$. Then

$$\begin{aligned} R_\mathcal{U}[D] &= \{E \in \mathsf{RC}\mathcal{U} : D \cap E \neq \varnothing\} \\ &= \{E \in \mathsf{RC}\mathcal{U} : C \cap E \neq \varnothing \ \forall C \in L\mathcal{U} \text{ with } D \subseteq C\}, \end{aligned}$$

where the second equality holds by Lemma 12.3(5) and compactness. For $E \in \mathsf{RC}\mathcal{U}$ and $C \in L\mathcal{U}$, Lemma 12.3(2) shows that $C \cap E \neq \varnothing$ iff $E \not\subseteq \phi C$ iff $\phi C \not\leq E$. Consequently,

$$\begin{aligned} R_\mathcal{U}[D] &= \{E \in \mathsf{RC}\mathcal{U} : \phi C \not\leq E \ \forall C \in L\mathcal{U} \text{ with } D \subseteq C\} \\ &= \{E \in \mathsf{RC}\mathcal{U} : E \in -{\uparrow}\phi C \ \forall C \in L\mathcal{U} \text{ with } D \subseteq C\} \\ &= \bigcap \{-{\uparrow}\phi C : C \in L\mathcal{U} \text{ with } D \subseteq C\}. \end{aligned}$$

Since each $-{\uparrow}\phi C$ is clopen by Lemma 5.4 and Proposition 12.4(2), we conclude that $R_\mathcal{U}[D]$ is closed. A similar argument shows that $R_\mathcal{U}^{-1}[E]$ is closed for each $E \in \mathsf{RC}\mathcal{U}$.

Next, we show that $R_\mathcal{U}[\mathcal{V}]$ is clopen for each clopen $\mathcal{V} \subseteq \mathsf{LC}\mathcal{U}$. If $\mathcal{V} = \varnothing$, then $R_\mathcal{U}[\mathcal{V}] = \varnothing$, so is clopen. Suppose $\mathcal{V} \neq \varnothing$. By Proposition 12.4(2) and Theorems 2.8(3), 2.10(1),

$$\{{\uparrow}C \cap -{\uparrow}C_1 \cap \cdots \cap -{\uparrow}C_n : C, C_1, \dots, C_n \in L\mathcal{U}\}$$

is a basis for the Lawson topology of $\mathsf{LC}\mathcal{U}$. Since $R_\mathcal{U}$ preserves unions, it suffices to assume that $\mathcal{V} = {\uparrow}C \cap -{\uparrow}C_1 \cap \cdots \cap -{\uparrow}C_n$ for some $C, C_1, \dots, C_n \in L\mathcal{U}$ with $C \not\subseteq C_i$ for each i. This implies that $C \in \mathcal{V}$ (recall that $\mathsf{LC}\mathcal{U}$ is ordered by reverse inclusion). We show that $R_\mathcal{U}[\mathcal{V}] = -{\uparrow}\phi C$. For one inclusion, by Lemma 12.3(2),

$$E \in -{\uparrow}\phi C \iff E \not\subseteq \phi C \iff C \cap E \neq \varnothing,$$

and so $C \, R_\mathcal{U} \, E$. Therefore, $-{\uparrow}\phi C \subseteq R_\mathcal{U}[\mathcal{V}]$ since $C \in \mathcal{V}$. For the reverse inclusion, if $E \in R_\mathcal{U}[\mathcal{V}]$, then $D \cap E \neq \varnothing$ for some $D \in \mathcal{V}$. Because $D \in \mathcal{V}$, we have $D \subseteq C$, and so $C \cap E \neq \varnothing$. Thus, $E \not\subseteq \phi C$, and hence $E \in -{\uparrow}\phi C$. Consequently, $R_\mathcal{U}[\mathcal{V}] = -{\uparrow}\phi C$, and so $R_\mathcal{U}[\mathcal{V}]$ is clopen. A similar argument shows that $R_\mathcal{U}^{-1}[\mathcal{W}]$ is clopen for each clopen $\mathcal{W} \subseteq \mathsf{RC}\mathcal{U}$. This completes the proof that $R_\mathcal{U}$ is an interior relation.

We next verify Definition 7.5(2). Let $D \in \mathsf{LC}\mathcal{U}$ and $E \in \mathsf{RC}\mathcal{U}$. Since the top of $\mathsf{RC}\mathcal{U}$ is $\varnothing$, we have $D \not\!R_\mathcal{U} \varnothing$ for each $D \in \mathsf{LC}\mathcal{U}$. Similarly, $\varnothing \not\!R_\mathcal{U} E$ for each $E \in \mathsf{RC}\mathcal{U}$. Thus, $\mathsf{LC}\mathcal{U} - R_\mathcal{U}[D]$ and $\mathsf{RC}\mathcal{U} - R_\mathcal{U}^{-1}[E]$ are nonempty. Because $D \, R_\mathcal{U} \, E$ iff $D \cap E \neq \varnothing$ and the orders on $\mathsf{LC}\mathcal{U}$ and $\mathsf{RC}\mathcal{U}$ are each reverse inclusion, $R_\mathcal{U}[D]$ is a downset of $\mathsf{RC}\mathcal{U}$ and $R_\mathcal{U}^{-1}[E]$ is a downset of $\mathsf{LC}\mathcal{U}$. Therefore, $\mathsf{LC}\mathcal{U} - R_\mathcal{U}[D]$ and $\mathsf{RC}\mathcal{U} - R_\mathcal{U}^{-1}[E]$ are upsets. Finally, to see that $\mathsf{LC}\mathcal{U} - R_\mathcal{U}[D]$ is closed under binary meets, let $D_1, D_2 \in \mathsf{LC}\mathcal{U}$ and $E \in \mathsf{RC}\mathcal{U}$. Suppose that $D_1 \not\!R_\mathcal{U} E$ and $D_2 \not\!R_\mathcal{U} E$. Then $D_1 \cap E = \varnothing = D_2 \cap E$. By Lemma 12.3(5) and compactness, there are $C_i \in L\mathcal{U}$ with $D_i \subseteq C_i$ and $C_i \cap E = \varnothing$ $(i = 1, 2)$. Therefore, $E \subseteq \phi C_i$ by Lemma 12.3(2), so $E \subseteq \phi C_1 \cap \phi C_2$. Then $C := \psi(\phi C_1 \cap \phi C_2) \subseteq \psi E$. By Remark 10.3, $C \in L\mathcal{U}$. From $C \subseteq \psi E$ we get $C \cap E = \varnothing$. Because $D_1, D_2 \subseteq C$, we have $C \leq D_1, D_2$ in $\mathsf{LC}\mathcal{U}$, and hence $C \leq D_1 \wedge D_2$, which means that $D_1 \wedge D_2 \subseteq C$. Consequently, $(D_1 \wedge D_2) \cap E = \varnothing$. Thus, $(D_1 \wedge D_2) \not\!R_\mathcal{U} E$. This proves that $\mathsf{LC}\mathcal{U} - R_\mathcal{U}[D]$ is closed under binary meets. The proof that $\mathsf{RC}\mathcal{U} - R_\mathcal{U}^{-1}[E]$ is closed under binary meets for each $E \in \mathsf{RC}\mathcal{U}$ is similar.

For Definition 7.5(3), let $D, D' \in \mathsf{LC}\mathcal{U}$. Suppose that $R_{\mathcal{U}}[D] \subseteq R_{\mathcal{U}}[D']$. Let $C \in L\mathcal{U}$ with $D' \subseteq C$. By Lemma 12.3(1), $D' \not R_{\mathcal{U}}\, \phi C$, so $D \not R_{\mathcal{U}}\, \phi C$, and hence $D \subseteq C$. Therefore, $D \subseteq D'$ by Lemma 12.3(5), so $D' \leq D$. The proof of Definition 7.5(4) is similar but uses Lemma 12.3(2, 6) instead of Lemma 12.3(1, 5). This completes the proof that $R_{\mathcal{U}}$ is a DH-relation, and thus $(\mathsf{LC}\mathcal{U}, R_{\mathcal{U}}, \mathsf{RC}\mathcal{U}) \in \mathsf{DH}$. ∎

We next associate a DH-morphism to each Urq-morphism.

Definition 12.6. Let $(P, Q)\colon \mathcal{U} \to \mathcal{U}'$ be an Urq-morphism. For each $D \in \mathsf{LC}\mathcal{U}$ and $E \in \mathsf{RC}\mathcal{U}$, define

$$f_{PQ}(D) = \bigcap \{C' \in L\mathcal{U}' : D \subseteq \Box_P C'\},$$
$$g_{PQ}(E) = \bigcap \{\phi' C' : C' \in L\mathcal{U}',\, E \subseteq \Box_Q \phi' C'\}.$$

Because $f_{PQ}(D)$ is an intersection from $L\mathcal{U}'$, we have $f_{PQ}(D) \in \mathsf{LC}\mathcal{U}'$, and similarly $g_{PQ}(E) \in \mathsf{RC}\mathcal{U}'$. Therefore, $f_{PQ}\colon \mathsf{LC}\mathcal{U} \to \mathsf{LC}\mathcal{U}'$ and $g_{PQ}\colon \mathsf{RC}\mathcal{U} \to \mathsf{RC}\mathcal{U}'$ are well-defined functions. In order to prove that $(f_{PQ}, g_{PQ})\colon \mathcal{D}(\mathcal{U}) \to \mathcal{D}(\mathcal{U}')$ is a DH-morphism, we require the following lemma.

Lemma 12.7. *Let $(P, Q)\colon \mathcal{U} \to \mathcal{U}'$ be an Urq-morphism.*

(1) *If $D \in \mathsf{LC}\mathcal{U}$ and $C' \in L\mathcal{U}'$, then $D \subseteq \Box_P C'$ iff $f_{PQ}(D) \subseteq C'$.*
(2) *If $E \in \mathsf{RC}\mathcal{U}$ and $C' \in L\mathcal{U}'$, then $E \subseteq \Box_Q \phi' C'$ iff $g_{PQ}(E) \subseteq \phi' C'$.*

Proof. (1) If $D \subseteq \Box_P C'$, then $f_{PQ}(D) \subseteq C'$ by definition of f_{PQ}. Conversely, suppose $f_{PQ}(D) \subseteq C'$. By definition of $f_{PQ}(D)$ and Lemma 12.3(3), there are $C'_1, \dots, C'_n \in L\mathcal{U}'$ with $D \subseteq \Box_P C'_i$ for each i and $C'_1 \cap \cdots \cap C'_n \subseteq C'$. Therefore,

$$D \subseteq \Box_P C'_1 \cap \cdots \cap \Box_P C'_n = \Box_P (C'_1 \cap \cdots \cap C'_n) \subseteq \Box_P C'.$$

(2) The proof is similar to that of (1). ∎

Proposition 12.8. *If $(P, Q)\colon \mathcal{U} \to \mathcal{U}'$ is an Urq-morphism, then $(f_{PQ}, g_{PQ})\colon \mathcal{D}(\mathcal{U}) \to \mathcal{D}(\mathcal{U}')$ is a DH-morphism.*

Proof. We first point out that f_{PQ} is order-preserving. If $D, D' \in \mathsf{LC}\mathcal{U}$ with $D \subseteq D'$, then $f_{PQ}(D) \subseteq f_{PQ}(D')$ follows from the definition of f_{PQ}. Because the orders on $\mathsf{LC}\mathcal{U}$ and $\mathsf{LC}\mathcal{U}'$ are reverse inclusions, this shows that f_{PQ} is order-preserving.

We prove that f_{PQ} is a CohLat_λ-morphism. Let $D = \bigvee D_i$ be a directed join. Then $D = \bigcap D_i$ is a directed intersection. Since f_{PQ} is order-preserving, $\bigvee f_{PQ}(D_i) \leq f_{PQ}(D)$, which says that $f_{PQ}(D) \subseteq \bigcap f_{PQ}(D_i)$. For the reverse inclusion, by the definition of $f_{PQ}(D)$, it suffices to show that if $C' \in L\mathcal{U}'$ with $D \subseteq \Box_P C'$, then there is i with $f_{PQ}(D_i) \subseteq C'$. Because $D = \bigcap D_i$ is a directed intersection, Lemma 12.3(3) shows that $D_i \subseteq \Box_P C'$ for some i, and so $f_{PQ}(D_i) \subseteq C'$. Thus, f_{PQ} preserves directed joins.

To see that f_{PQ} preserves meets, suppose that $D = \bigwedge D_i$. Then $f_{PQ}(D) \leq \bigwedge f_{PQ}(D_i)$ because f_{PQ} is order-preserving. Since $\leq$ on $\mathsf{LC}\mathcal{U}$ is reverse inclusion, Lemma 12.3(5) implies that a meet $\bigwedge \mathcal{S}$ in $\mathsf{LC}\mathcal{U}$ is the intersection of all $C \in L\mathcal{U}$ containing $\bigcup \mathcal{S}$. Therefore, for the reverse inequality, it suffices to show that if $C' \in L\mathcal{U}'$ such that $f_{PQ}(D_i) \subseteq C'$ for each i, then $f_{PQ}(D) \subseteq C'$. If $f_{PQ}(D_i) \subseteq C'$, then $D_i \subseteq \Box_P C'$ by

Lemma 12.7(1). If this happens for each i, then $D \subseteq \Box_P C'$ because $D = \bigwedge D_i$, and hence $f_{PQ}(D) \subseteq C'$. Thus, f_{PQ} preserves arbitrary meets.

Observe that the left adjoint ℓ of f_{PQ} is given by $\ell(D') = \Box_P D'$ for each $D' \in \mathsf{LC}\mathcal{U}'$. To see this, since $\Box_P$ preserves arbitrary intersections, by Lemmas 12.3(5) and 12.7(1), for each $D \in \mathsf{LC}\mathcal{U}$ we have

$$\begin{aligned} D \subseteq \Box_P D' &\iff D \subseteq \Box_P C' \ \forall C' \in L\mathcal{U}' \text{ with } D' \subseteq C' \\ &\iff f_{PQ}(D) \subseteq C' \ \forall C' \in L\mathcal{U}' \text{ with } D' \subseteq C' \iff f_{PQ}(D) \subseteq D'. \end{aligned}$$

Thus, $\Box_P D' \leq D$ iff $D' \leq f_{PQ}(D)$, showing that $\ell(D') = \Box_P D'$.

We now show that ℓ preserves finite meets of compact elements. We have $K(\mathsf{LC}\mathcal{U}') = L\mathcal{U}'$ by Proposition 12.4(2). If $\mathcal{S} \subseteq L\mathcal{U}'$ is finite, then $\bigwedge \mathcal{S}$ in $\mathsf{LC}\mathcal{U}'$ is equal to $\bigvee \mathcal{S}$ in $L\mathcal{U}'$ since the order on $\mathsf{LC}\mathcal{U}'$ is reverse inclusion while the order on $L\mathcal{U}'$ is inclusion. Thus, it is enough to prove the following:

Claim 12.9. *ℓ preserves finite joins from $L\mathcal{U}$.*

Proof. Let $\mathcal{S}$ be a finite subset of $L\mathcal{U}$. If $\mathcal{S} = \varnothing$, then $\Box_P (\bigvee \varnothing) = \Box_P \varnothing = \varnothing = \bigvee \varnothing$. Thus, it suffices to prove that ℓ preserves binary joins from $L\mathcal{U}'$. Let $C_1, C_2 \in L\mathcal{U}'$. The inclusion $\Box_P C_1 \vee \Box_P C_2 \subseteq \Box_P (C_1 \vee C_2)$ is clear since $\Box_P$ is order-preserving. For the other inclusion, let $x \in \Box_P(C_1 \vee C_2)$. Since $\Box_P(C_1 \vee C_2) = \Box_P \psi'(\phi' C_1 \cap \phi' C_2)$,

$$P[x] \subseteq \psi'(\phi' C_1 \cap \phi' C_2) = -{\downarrow_1}(\phi' C_1 \cap \phi' C_2),$$

so $P[x] \cap \phi' C_1 \cap \phi' C_2 = \varnothing$ because $P[x]$ is a $\leq_1$-upset.

Suppose that $x \notin \Box_P C_1 \vee \Box_P C_2$. Since $\Box_P C_1 \vee \Box_P C_2 = \psi(\phi \Box_P C_1 \cap \phi \Box_P C_2)$, we have $x \notin \psi(\phi \Box_P C_1 \cap \phi \Box_P C_2)$, so $x \in {\downarrow_1}(\phi \Box_P C_1 \cap \phi \Box_P C_2)$. Therefore, there is y with $x \leq_1 y$ and $y \in \phi \Box_P C_1 \cap \phi \Box_P C_2$. By Definition 10.4(1),

$$\phi \Box_P C_1 \cap \phi \Box_P C_2 = \Box_Q \phi' C_1 \cap \Box_Q \phi' C_2 = \Box_Q (\phi' C_1 \cap \phi' C_2).$$

Thus, $Q[y] \subseteq \phi' C_1 \cap \phi' C_2$. Since $x \leq_1 y$, we have $P[y] \subseteq P[x]$ by Remark 10.6(4). Because

$$P[x] \cap \phi' C_1 \cap \phi' C_2 = \varnothing \quad \text{and} \quad Q[y] \subseteq \phi' C_1 \cap \phi' C_2,$$

we conclude that $P[y] \cap Q[y] \subseteq P[x] \cap Q[y] = \varnothing$, contradicting Definition 10.4(4). Consequently, $\Box_P(C_1 \vee C_2) \subseteq \Box_P C_1 \vee \Box_P C_2$, and hence $\Box_P$ preserves binary joins. ∎

Applying Lemma 2.16 completes the proof that f_{PQ} is a CohLat_λ-morphism. Similar arguments show that g_{PQ} is a CohLat_λ-morphism.

Finally, we show that (f_{PQ}, g_{PQ}) is a DH-morphism. Because f_{PQ}, g_{PQ} are CohLat_λ-morphisms, it remains to verify Definition 7.7(2). For Definition 7.7(2a), let $D \in \mathsf{LC}\mathcal{U}$ and $E \in \mathsf{RC}\mathcal{U}$ with $D \, R_{\mathcal{U}} \, E$. Then $D \cap E \neq \varnothing$. If $f_{PQ}(D) \cap g_{PQ}(E) = \varnothing$, then

$$\bigcap \{C_1' \in L\mathcal{U}' : D \subseteq \Box_P C_1'\} \cap \bigcap \{\phi' C_2' : C_2' \in L\mathcal{U}', \ E \subseteq \Box_Q \phi' C_2'\} = \varnothing.$$

These intersections are directed, so compactness implies that there are $C_1', C_2' \in L\mathcal{U}'$ with $D \subseteq \Box_P C_1'$, $E \subseteq \Box_Q \phi C_2'$, and $C_1' \cap \phi' C_2' = \varnothing$. Since there is $z \in D \cap E$, we have $z \in \Box_P C_1' \cap \Box_Q \phi' C_2'$. Therefore, $P[z] \cap Q[z] \subseteq C_1' \cap \phi' C_2' = \varnothing$. This is a contradiction since $P[z] \cap Q[z] \neq \varnothing$ by Definition 10.4(4). Thus, $f_{PQ}(D) \cap f_{PQ}(E) \neq \varnothing$, so $f_{PQ}(D) \, R_{\mathcal{U}'} \, f_{PQ}(E)$.

To verify Definition 7.7(2b), suppose $D' \, R_{\mathcal{U}'} \, g_{PQ}(E)$ and set $D = \ell(D') \in \mathsf{LC}\mathcal{U}$. Then $D' \leq f_{PQ}(D)$. If $D \cap E = \varnothing$, because ℓ preserves arbitrary joins in $\mathsf{LC}\mathcal{U}'$ (which are intersections), compactness implies that there is $C' \in L\mathcal{U}'$ such that $D' \subseteq C'$ and $\ell(C') \cap E = \varnothing$. Since $\ell(C') = \Box_P C'$, we have $\Box_P C' \cap E = \varnothing$. Therefore, $E \subseteq \phi \Box_P C'$ by Lemma 12.3(2), and so $E \subseteq \Box_Q \phi' C'$ by Definition 10.4(1). Thus, $g_{PQ}(E) \subseteq \phi' C'$ by Lemma 12.7(2), and so $D' \cap g_{PQ}(E) \subseteq C' \cap \phi' C' = \varnothing$. This is a contradiction since $D' \, R_{\mathcal{U}'} \, g_{PQ}(E)$. Consequently, $D \cap E \neq \varnothing$, and hence $D \, R_{\mathcal{U}} \, E$.

A similar argument verifies Definition 7.7(2c), completing the proof that (f_{PQ}, g_{PQ}) is a DH-morphism. ∎

Theorem 12.10. *There is a functor* $\mathbb{D}\colon \mathsf{Urq} \to \mathsf{DH}$ *given by* $\mathbb{D}(\mathcal{U}) = \mathcal{D}(\mathcal{U})$ *for each* $\mathcal{U} \in \mathsf{Urq}$ *and* $\mathbb{D}(P,Q) = (f_{PQ}, g_{PQ})$ *for each* Urq*-morphism* (P,Q)*, where* f_{PQ}, g_{PQ} *are given in Definition* 12.6.

Proof. By Proposition 12.5, $\mathbb{D}$ is well defined on objects, and it is well defined on morphisms by Proposition 12.8. We show that $\mathbb{D}$ sends identity morphisms to identity morphisms. Let $\mathcal{U} = (Z, \leq_1, \leq_2) \in \mathsf{Urq}$. By Proposition 10.8(2), the identity morphism of $\mathcal{U}$ is $(P,Q) := (\leq_1, \leq_2)$. If $z, z' \in Z$, then $z \, P \, z'$ iff $z \leq_1 z'$. Consequently, $\Box_P C = C$ for each $C \in \mathsf{LC}\mathcal{U}$ because C is a $\leq_1$-upset. Therefore, if $D \in \mathsf{LC}\mathcal{U}$, by Lemma 12.3(5),

$$f_{PQ}(D) = \bigcap \{C \in L\mathcal{U} : D \subseteq \Box_P C\} = \bigcap \{C \in L\mathcal{U} : D \subseteq C\} = D.$$

This shows that f_{PQ} is the identity on $\mathsf{LC}\mathcal{U}$. A similar argument shows that g_{PQ} is the identity on $\mathsf{RC}\mathcal{U}$. Therefore, (f_{PQ}, g_{PQ}) is the identity on $\mathbb{D}(\mathcal{U})$.

Next let $(P_1, Q_1)\colon \mathcal{U}_1 \to \mathcal{U}_2$ and $(P_2, Q_2)\colon \mathcal{U}_2 \to \mathcal{U}_3$ be Urq-morphisms. Set $P = P_2 \star P_1$ and $Q = Q_2 \star Q_1$. To show that $\mathbb{D}$ preserves composition, by definition of composition in DH it suffices to show that $f_{PQ} = f_{P_2Q_2} \circ f_{P_1Q_1}$ and $g_{PQ} = g_{P_2Q_2} \circ g_{P_1Q_1}$. If $D \in \mathsf{LC}\mathcal{U}_1$, then

$$f_{PQ}(D) = \bigcap \{C'' \in L\mathcal{U}_3 : D \subseteq \Box_P C''\} = \bigcap \{C'' \in L\mathcal{U}_3 : D \subseteq \Box_{P_1} \Box_{P_2} C''\}$$

by Lemma 10.7. Thus,

$$\begin{aligned}(f_{P_2Q_2} \circ f_{P_1Q_1})(D) = f_{P_2Q_2}(f_{P_1Q_1}(D)) &= \bigcap \{C'' \in L\mathcal{U}_3 : f_{P_1Q_1}(D) \subseteq \Box_{P_2} C''\} \\ &= \bigcap \{C'' \in L\mathcal{U}_3 : D \subseteq \Box_{P_1} \Box_{P_2} C''\} = f_{PQ}(D),\end{aligned}$$

where the third equality holds by Lemma 12.7(1). Thus, $f_{PQ} = f_{P_2Q_2} \circ f_{P_1Q_1}$. A similar argument gives $g_{PQ} = g_{P_2Q_2} \circ g_{P_1Q_1}$, completing the proof that $\mathbb{D}$ is a functor. ∎

13. The resulting category equivalences

In Chapters 8 to 10 we defined the functors $\mathbb{G}\colon \mathsf{DH} \to \mathsf{GvG}$, $\mathbb{H}\colon \mathsf{GvG} \to \mathsf{Hg}$, and $\mathbb{U}\colon \mathsf{Hg} \to \mathsf{Urq}$; in Chapter 11 we showed that the functor $\mathbb{P}\colon \mathsf{Urq} \to \mathsf{Plo}$ is an isomorphism; and in Chapter 12 we defined the functor $\mathbb{D}\colon \mathsf{Urq} \to \mathsf{DH}$. In this chapter we show that the functors

$$\begin{tikzcd} \mathsf{DH} \arrow[r, "\mathbb{G}"] & \mathsf{GvG} \arrow[r, "\mathbb{H}"] & \mathsf{Hg} \arrow[r, "\mathbb{U}"] & \mathsf{Urq} \arrow[lll, bend left, "\mathbb{D}"] \end{tikzcd}$$

yield equivalences of $\mathsf{DH}, \mathsf{GvG}, \mathsf{Hg}$, and Urq, and hence each of these categories is also equivalent to Plo. This concludes the circle of our equivalences.

We show that the above four functors are equivalences by producing natural isomorphisms between the identity functor of each of the above four categories and the corresponding composition of the four functors. Since various of these natural isomorphisms have similar proofs, we skip some details towards the end of the chapter.

Let $\mathcal{U} \in \mathsf{Urq}$. By Proposition 12.5, $(\mathsf{LC}\mathcal{U}, R_{\mathcal{U}}, \mathsf{RC}\mathcal{U}) \in \mathsf{DH}$. Therefore, by Proposition 8.23, $((\mathsf{LC}\mathcal{U})_p, (R_{\mathcal{U}})_p, (\mathsf{RC}\mathcal{U})_p) \in \mathsf{GvG}$. Thus, by Lemma 8.19 and Proposition 9.17, $((\mathsf{LC}\mathcal{U})_0, (R_{\mathcal{U}})_0, (\mathsf{RC}\mathcal{U})_0) \in \mathsf{Hg}$. Throughout the chapter, we assume that an Urquhart space is the triple $\mathcal{U} = (Z, \leq_1, \leq_2)$.

Lemma 13.1.

(1) *Let $\mathcal{D} = (X, R, Y) \in \mathsf{DH}$. Set $\mathcal{U} = \mathbb{UHG}(\mathcal{D})$ and $\overline{\mathcal{D}} = \mathbb{DUHG}(\mathcal{D})$. Then*

$$L\mathcal{U} = \{(U \times Y) \cap Z : U \in \mathsf{CLF}(X)\},$$
$$\mathsf{CLF}(\mathsf{LC}\mathcal{U}) = \{\uparrow[(U \times Y) \cap Z] : U \in \mathsf{CLF}(X)\}.$$

(2) *Let $\mathcal{G} = (X, R, Y) \in \mathsf{GvG}$. Set $\mathcal{U} = \mathbb{UH}(\mathcal{G})$ and $\overline{\mathcal{G}} = \mathbb{GDUH}(\mathcal{G})$. Then*

$$L\mathcal{U} = \{(U \times Y) \cap Z : U \in L\mathcal{G}\},$$
$$L\overline{\mathcal{G}} = \{\uparrow[(U \times Y) \cap Z] \cap (\mathsf{LC}\mathcal{U})_p : U \in L\mathcal{G}\}.$$

(3) *Let $\mathcal{H} = (X, R, Y) \in \mathsf{Hg}$. Set $\mathcal{U} = \mathbb{U}(\mathcal{H})$, and $\overline{\mathcal{H}} = \mathbb{HGDU}(\mathcal{H})$. Then*

$$L\mathcal{U} = \{(A \times Y) \cap Z : A \in L\mathcal{H}\},$$
$$L\overline{\mathcal{H}} = \{\uparrow[(A \times Y) \cap Z] \cap (\mathsf{LC}\mathcal{U})_0 : A \in L\mathcal{H}\}.$$

Proof. (1) Let $\mathcal{G} = \mathbb{G}(\mathcal{D})$ and $\mathcal{H} = \mathbb{H}(\mathcal{G})$. Then $L\mathcal{G} = \{U \cap X_p : U \in \mathsf{CLF}(X)\}$ by Proposition 8.24 and $L\mathcal{H} = \{U \cap X_0 : U \in \mathsf{CLF}(X)\}$ by Lemma 8.19 and Proposition 9.19. Therefore, Proposition 10.14 yields

$$L\mathcal{U} = \{((U \cap X_0) \times Y) \cap Z : U \in \mathsf{CLF}(X)\}.$$

We show that $((U \cap X_0) \times Y) \cap Z = (U \times Y) \cap Z$. One inclusion is obvious. For the reverse inclusion, let $(x, y) \in Z$ with $x \in U$. Because $Z \subseteq X_0 \times Y_0$, we have $x \in U \cap X_0$. Therefore,

$$L\mathcal{U} = \{(U \times Y) \cap Z : U \in \mathsf{CLF}(X)\}.$$

By Proposition 12.4(2), $K(\mathsf{LC}\mathcal{U}) = L\mathcal{U}$. Thus, Lemma 5.4 implies that

$$\mathsf{CLF}(\mathsf{LC}\mathcal{U}) = \{\uparrow[(U \times Y) \cap Z] : U \in \mathsf{CLF}(X)\}.$$

The proofs of (2) and (3) are similar to that of (1). ■

Lemma 13.2.

(1) *Let $\mathcal{D} = (X, R, Y) \in \mathsf{DH}$ and set $\mathbb{UHG}(\mathcal{D}) = (Z, \leq_1, \leq_2)$. If $x \in X$ and $U \in \mathsf{CLF}(X)$, then*

$$x \in U \iff (\uparrow x \times Y) \cap Z \subseteq (U \times Y) \cap Z.$$

(2) *Let $\mathcal{G} = (X, R, Y) \in \mathsf{GvG}$ and set $\mathbb{UH}(\mathcal{G}) = (Z, \leq_1, \leq_2)$. If $x \in X$ and $U \in L\mathcal{G}$, then*

$$x \in U \iff (\uparrow x \times Y) \cap Z \subseteq (U \times Y) \cap Z.$$

(3) *Let* $\mathcal{H} = (X, R, Y) \in \mathsf{Hg}$ *and set* $\mathbb{U}(\mathcal{H}) = (Z, \leq_1, \leq_2)$. *If* $x \in X$ *and* $A \in L\,\mathcal{H}$, *then*

$$x \in A \iff (\uparrow x \times Y) \cap Z \subseteq (A \times Y) \cap Z.$$

Proof. (1) One direction is clear since U is an upset. For the other, suppose that $x \notin U$. By Lemma 7.11(3), $U = \psi\phi U$. Therefore, there is $y \in \phi U$ with $x\, R\, y$. By Remark 10.11(2), there are $x' \geq x$ and $y' \geq y$ with $(x', y') \in Z$. Thus, $(x', y') \in (\uparrow x \times Y) \cap Z$. On the other hand, $x'\, R\, y'$ and $y \leq y'$ imply that $x'\, R\, y$. Hence, $y \in \phi U$ yields $x' \notin U$. Consequently, $(x', y') \notin (U \times Y) \cap Z$, and so $(\uparrow x \times Y) \cap Z \not\subseteq (U \times Y) \cap Z$.

The proofs of (2) and (3) are similar to that of (1). ■

13.1. The natural isomorphism $1_{\mathsf{DH}} \to \mathbb{DUHG}$

LEMMA 13.3. *For each* $\mathcal{D} \in \mathsf{DH}$, *there is a* DH-*isomorphism* $\epsilon_{\mathcal{D}} \colon \mathcal{D} \to \mathbb{DUHG}(\mathcal{D})$.

Proof. Let $\mathcal{D} = (X, R, Y)$. Set $\mathcal{U} = \mathbb{UHG}(\mathcal{D}) := (Z, \leq_1, \leq_2)$ and $\overline{\mathcal{D}} = \mathbb{DUHG}(\mathcal{D})$, so $\overline{\mathcal{D}} = (\mathsf{LC}\mathcal{U}, R_{\mathcal{U}}, \mathsf{RC}\mathcal{U})$. For $U \in \mathsf{CLF}(X)$ let $\overline{U} = (U \times Y) \cap Z$. By Lemma 13.1(1), we have $L\mathcal{U} = \{\overline{U} : U \in \mathsf{CLF}(X)\}$ and $\mathsf{CLF}(\mathsf{LC}\mathcal{U}) = \{\uparrow\overline{U} : U \in \mathsf{CLF}(X)\}$.

Define $\alpha \colon X \to \mathsf{LC}\mathcal{U}$ and $\beta \colon Y \to \mathsf{RC}\mathcal{U}$ by

$$\alpha(x) = (\uparrow x \times Y) \cap Z \quad \text{and} \quad \beta(y) = (X \times \uparrow y) \cap Z.$$

Since $\mathsf{CLF}(X) = \{\uparrow k : k \in K(X)\}$ (see Lemma 5.4), $\uparrow x = \bigcap\{U \in \mathsf{CLF}(X) : x \in U\}$. Therefore, $\alpha(x) = \bigcap\{\overline{U} : U \in \mathsf{CLF}(X), x \in U\}$, an intersection from $L\mathcal{U}$, and hence $\alpha(x) \in \mathsf{LC}\mathcal{U}$ since $\mathsf{LC}\mathcal{U}$ is closed under intersections. It is easy to see that α is order-preserving since the order on $\mathsf{LC}\mathcal{U}$ is reverse inclusion. To see that α is order-reflecting, suppose that $\alpha(x_1) \leq \alpha(x_2)$ in $\mathsf{LC}\mathcal{U}$. Then $\alpha(x_2) \subseteq \alpha(x_1)$. By Lemma 8.17(3), for each $x \in X$, we have $x = \bigwedge(\uparrow x \cap X_0)$; and by Remark 10.11(3), for each $z \in X_0$, there is $y \in Y_0$ with $(z, y) \in Z$. Therefore,

$$\begin{aligned} x_1 &= \bigwedge\{z \in X_0 : \exists y \in Y_0,\ (z, y) \in \alpha(x_1)\} \\ &\leq \bigwedge\{z \in X_0 : \exists y \in Y_0,\ (z, y) \in \alpha(x_2)\} = x_2. \end{aligned}$$

Thus, α is order-reflecting. To show that α is onto, let $D \in \mathsf{LC}\mathcal{U}$. By Lemmas 12.3(5) and 13.1(1), there is a family $\{U_i\} \subseteq \mathsf{CLF}(X)$ with $D = \bigcap \overline{U_i}$. By Lemma 5.4, for each i there is $k_i \in K(X)$ with $U_i = \uparrow k_i$. If $x = \bigvee k_i$, then $\uparrow x = \bigcap U_i$. Consequently, $\alpha(x) = D$. This shows that α is an order-isomorphism, hence a CohLat_λ-isomorphism. Similar arguments show that β is a CohLat_λ-isomorphism.

To see that (α, β) is a DH-isomorphism, by Remark 7.8, it is enough to show that $x\, R\, y$ iff $\alpha(x)\, R_{\mathcal{U}}\, \beta(y)$. If $\alpha(x)\, R_{\mathcal{U}}\, \beta(y)$, then $\alpha(x) \cap \beta(y) \neq \varnothing$, so there is $(x', y') \in Z$ with $x \leq x'$ and $y \leq y'$. Since $(x', y') \in Z$, we have $x'\, R\, y'$, so $x\, R\, y$ by Definition 7.5(2). Conversely, if $x\, R\, y$, by Remark 10.11(2), there are $x' \geq x$ and $y' \geq y$ with $(x', y') \in Z$, so $\alpha(x) \cap \beta(y) \neq \varnothing$, and hence $\alpha(x)\, R_{\mathcal{U}}\, \beta(y)$. Setting $\epsilon_{\mathcal{D}} = (\alpha, \beta)$ yields the desired DH-isomorphism. ■

PROPOSITION 13.4. *There is a natural isomorphism* $\epsilon \colon 1_{\mathsf{DH}} \to \mathbb{DUHG}$.

Proof. Let $\mathcal{D} = (X, R, Y) \in \mathsf{DH}$. Then $\epsilon_{\mathcal{D}}$ is a DH-isomorphism by Lemma 13.3. To see that ϵ is natural, let $(f, g) \colon \mathcal{D} \to \mathcal{D}'$ be a DH-morphism. Set $\mathcal{U} = \mathbb{UHG}(\mathcal{D})$,

$\mathcal{U}' = \mathbb{UHG}(\mathcal{D}')$, and $\mathbb{UHG}(f,g) = (P,Q)$. Then $\mathbb{DUHG}(f,g) = (f_{PQ}, g_{PQ})$. We need to show that the following diagram commutes:

$$\begin{tikzcd} \mathcal{D} \arrow[r, "{(f,g)}"] \arrow[d, "\epsilon_{\mathcal{D}}"'] & \mathcal{D}' \arrow[d, "\epsilon_{\mathcal{D}'}"] \\ \overline{\mathcal{D}} \arrow[r, "{(f_{PQ},g_{PQ})}"] & \overline{\mathcal{D}'} \end{tikzcd}$$

Write $\epsilon_{\mathcal{D}} = (\alpha, \beta)$ and $\epsilon_{\mathcal{D}'} = (\alpha', \beta')$. We then need to show that $\alpha' \circ f = f_{PQ} \circ \alpha$ and $\beta' \circ g = g_{PQ} \circ \beta$. Let $x \in X$. Then $\alpha'(f(x)), f_{PQ}(\alpha(x)) \in \mathsf{LC}\mathcal{U}'$. To see that they are equal, by Lemma 12.3(5) it suffices to show that

$$\alpha'(f(x)) \subseteq C' \iff f_{PQ}(\alpha(x)) \subseteq C'$$

for each $C' \in L\mathcal{U}'$. Recall that $\mathcal{U}' = (Z', \leq_1', \leq_2')$. Using the notation of Lemma 13.3, it follows from Lemma 13.1(1) that $C' = (U' \times Y') \cap Z' = \overline{U'}$ for some $U' \in \mathsf{CLF}(X')$. Since $\alpha'(f(x)) = (\uparrow f(x) \times Y') \cap Z'$, we have $\alpha'(f(x)) \subseteq \overline{U'}$ iff $f(x) \in U'$ by Lemma 13.2(1). By Lemma 12.7(1), $f_{PQ}(\alpha(x)) \subseteq \overline{U'}$ iff $\alpha(x) \subseteq \Box_P \overline{U'}$. Let $(x_1, y_1) \in Z$ and $(x_1', y_1') \in Z'$. Recalling how $\mathbb{U}$, $\mathbb{H}$, and $\mathbb{G}$ act on morphisms, we have

$$(x_1, y_1)\, P\, (x_1', y_1') \iff x_1\, S_f\, x_1' \iff f(x_1) \leq x_1',$$

where S_f is defined in Proposition 8.25. Therefore,

$$\begin{aligned} \Box_P \overline{U'} &= \{(x_1, y_1) \in Z : (x_1', y_1') \in Z' \text{ and } f(x_1) \leq x_1' \Rightarrow x_1' \in U'\} \\ &= \{(x_1, y_1) \in Z : (\uparrow f(x_1) \times Y') \cap Z' \subseteq (U' \times Y') \cap Z'\} \\ &= \{(x_1, y_1) \in Z : f(x_1) \in U'\} = (f^{-1}(U') \times Y) \cap Z, \end{aligned}$$

where the second-to-last equality follows from Lemma 13.2(1). Thus,

$$\begin{aligned} f_{PQ}(\alpha(x)) \subseteq \overline{U'} &\iff \alpha(x) \subseteq \Box_P \overline{U'} \\ &\iff (\uparrow x \times Y) \cap Z \subseteq (f^{-1}(U') \times Y) \cap Z \\ &\iff x \in f^{-1}(U') \iff f(x) \in U' \\ &\iff \alpha'(f(x)) \subseteq \overline{U'}, \end{aligned}$$

again by Lemma 13.2(1). Consequently, $f_{PQ}(\alpha(x)) = \alpha'(f(x))$, and hence $f_{PQ} \circ \alpha = \alpha' \circ f$. Similar calculations show that $g_{PQ} \circ \beta = \beta' \circ g$. Thus, ϵ is natural. ∎

13.2. The natural isomorphism $1_{\mathsf{GvG}} \to \mathbb{GDUH}$

Lemma 13.5. *Let $\mathcal{G} = (X, R, Y)$ and $\mathcal{G}' = (X', R', Y')$ be GvG-spaces. Suppose that $\alpha\colon X \to X'$ and $\beta\colon Y \to Y'$ are* Pries*-isomorphisms satisfying $x\, R\, y$ iff $\alpha(x)\, R'\, \beta(y)$. Define $S_\alpha \subseteq X \times X'$ and $T_\beta \subseteq Y \times Y'$ by*

$$x\, S_\alpha\, x' \iff \alpha(x) \leq x' \quad \text{and} \quad y\, T_\beta\, y' \iff \beta(y) \leq y'.$$

Then $(S_\alpha, T_\beta)\colon \mathcal{G} \to \mathcal{G}'$ is a GvG*-isomorphism.*

Proof. Let $U' \in L\,\mathcal{G}'$. Since U' is an upset,

$$\begin{aligned} \Box_{S_\alpha} U' &= \{x \in X : S_\alpha[x] \subseteq U'\} = \{x \in X : \uparrow\alpha(x) \subseteq U'\} \\ &= \{x \in X : \alpha(x) \in U'\} = \alpha^{-1}(U'). \end{aligned}$$

Therefore, because U' is a clopen upset and α is a Pries-morphism, $\Box_{S_\alpha} U'$ is a clopen upset. Similarly, $\Diamond_{T_\beta} \blacklozenge' U' = \beta^{-1}(\blacklozenge' U')$, and hence $\Diamond_{T_\beta} \blacklozenge' U'$ is a clopen downset. We show that (S_α, T_β) is a GvG-morphism. Since α, β are bijections such that $x \, R \, y$ iff $\alpha(x) \, R' \, \beta(y)$,

$$\alpha^{-1} \Box' V' = \Box \beta^{-1} V' \quad \text{and} \quad \blacklozenge \alpha^{-1} U' = \beta^{-1} \blacklozenge' U'$$

for each $U' \subseteq X'$ and $V' \subseteq Y'$. Therefore, since $\Box' \blacklozenge' U' = U'$,

$$\begin{aligned} \Box \blacklozenge (\Box_{S_\alpha} U') &= \Box \blacklozenge \alpha^{-1}(U') = \Box \beta^{-1}(\blacklozenge' U') = \alpha^{-1}(\Box' \blacklozenge' U') \\ &= \alpha^{-1}(U') = \Box_{S_\alpha} U'. \end{aligned}$$

Thus, $\Box_{S_\alpha} U' \in L\,\mathcal{G}$. Moreover,

$$\blacklozenge \Box_{S_\alpha} U' = \blacklozenge \alpha^{-1}(U') = \beta^{-1}(\blacklozenge' U') = \Diamond_{T_\beta} \blacklozenge' U'.$$

Consequently, Definition 8.8(1) is satisfied.

For Definition 8.8(2), suppose that $x \not S_\alpha x'$, so $\alpha(x) \not\leq x'$. By Lemma 8.7(1), there is $U' \in L\,\mathcal{G}'$ with $\alpha(x) \in U'$ and $x' \notin U'$. Since U' is an upset, $S_\alpha[x] = {\uparrow}\alpha(x) \subseteq U'$, so $x \in \Box_{S_\alpha} U'$. A similar proof holds for T_β, so Definition 8.8(3) is satisfied. Suppose that $x \, R \, y$. Then $\alpha(x) \, R' \, \beta(y)$. Because $x \, S_\alpha \, \alpha(x)$ and $y \, T_\beta \, \beta(y)$, Definition 8.8(4) holds. Thus, (S_α, T_β) is a GvG-morphism.

Finally, to show that (S_α, T_β) is a GvG-isomorphism, $(S_{\alpha^{-1}}, T_{\beta^{-1}}) \colon \mathcal{G}' \to \mathcal{G}$ is a GvG-morphism by the above argument. We prove that $(S_{\alpha^{-1}}, T_{\beta^{-1}}) \star (S_\alpha, T_\beta) = (\leq_X, \leq_Y)$, the identity morphism for $\mathcal{G}$ by Proposition 8.13(2). Since $S_{\alpha^{-1}} S_\alpha[x] = {\uparrow}\alpha^{-1}\alpha(x) = {\uparrow}x$,

$$\begin{aligned} x \, (S_{\alpha^{-1}} \star S_\alpha) \, z &\iff (\forall U \in L\,\mathcal{G}) \, (S_{\alpha^{-1}} S_\alpha[x] \subseteq U \Longrightarrow z \in U) \\ &\iff (\forall U \in L\,\mathcal{G}) \, ({\uparrow}x \subseteq U \Longrightarrow z \in U) \iff x \leq z, \end{aligned}$$

where the final equivalence follows from Lemma 8.7(1). Thus, $S_{\alpha^{-1}} \star S_\alpha = {\leq_X}$. Similarly, $T_{\beta^{-1}} \star T_\beta = {\leq_Y}$, so $(S_{\alpha^{-1}}, T_{\beta^{-1}}) \star (S_\alpha, T_\beta) = (\leq_X, \leq_Y)$. A parallel argument shows that $(S_\alpha, T_\beta) \star (S_{\alpha^{-1}}, T_{\beta^{-1}}) = (\leq_{X'}, \leq_{Y'})$. Therefore, (S_α, T_β) is a GvG-isomorphism. ■

LEMMA 13.6. *Let $\mathcal{G} = (X, R, Y)$ be a GvG-space. If $U = U_1 \vee \cdots \vee U_n$ is a join in $L\,\mathcal{G}$, then U is a distributive join iff $U = U_1 \cup \cdots \cup U_n$.*

Proof. One implication is clear since if a finite union of elements of $L\,\mathcal{G}$ is in $L\,\mathcal{G}$, then it is the join in $L\,\mathcal{G}$ and it is clearly distributive. For the other, set $V = U_1 \cup \cdots \cup U_n$, so $U = \Box \blacklozenge V$ by Remark 8.6. We show that $U = V$. If not, then there is $x \in U - V$. By Lemma 8.7(5), we may assume that $x \in X_0$. By definition of X_0, there is $y \in Y$ with $x \in \max R^{-1}[y]$. Therefore, $R^{-1}[y] \cap {\uparrow}x = \{x\}$, so $R^{-1}[y] \cap {\uparrow}x \cap V = \varnothing$. Since V is clopen and ${\uparrow}x = \bigcap \{W \in L\,\mathcal{G} : x \in W\}$ by Lemma 8.7(1), compactness of X implies that there is $W \in L\,\mathcal{G}$ with $x \in W$ and $R^{-1}[y] \cap W \cap V = \varnothing$. Thus, $y \in R[x]$ and $y \notin R[W \cap V]$, so $x \notin \Box \blacklozenge (W \cap V)$. On the other hand, since U is a distributive join,

$$\begin{aligned} W \cap U &= (W \cap U_1) \vee \cdots \vee (W \cap U_n) = \Box \blacklozenge ((W \cap U_1) \cup \cdots \cup (W \cap U_n)) \\ &= \Box \blacklozenge (W \cap V). \end{aligned}$$

This is a contradiction since $x \in W \cap U$ and $x \notin \Box \blacklozenge (W \cap V)$. Therefore, $U = U_1 \cup \cdots \cup U_n$. ■

Lemma 13.7. *Let $\mathcal{G} = (X, R, Y)$ be a GvG-space and set $\mathcal{U} = \mathbb{UH}(\mathcal{G})$.*

(1) *There are* Pries*-isomorphisms $\alpha\colon X \to (\mathsf{LC}\mathcal{U})_p$ and $\beta\colon Y \to (\mathsf{RC}\mathcal{U})_p$ satisfying $x\,R\,y$ iff $\alpha(x)\,R_{\mathcal{U}}\,\beta(y)$.*
(2) *There is a* GvG*-isomorphism $\zeta_{\mathcal{G}}\colon \mathcal{G} \to \mathbb{GDUH}(\mathcal{G})$ given by $\zeta_{\mathcal{G}} = (S_\alpha, T_\beta)$ in the notation of Lemma* 13.5*.*

Proof. (1) Since $\mathcal{U} := (Z, \le_1, \le_2)$ is an Urquhart space, $\mathsf{LC}\mathcal{U}$ and $\mathsf{RC}\mathcal{U}$ are coherent lattices by Proposition 12.4(1), and hence $(\mathsf{LC}\mathcal{U})_p$ and $(\mathsf{RC}\mathcal{U})_p$ are Priestley spaces by Lemma 8.16. For $x \in X$ and $y \in Y$ define

$$\alpha(x) = ({\uparrow}x \times Y) \cap Z \quad \text{and} \quad \beta(y) = (X \times {\uparrow}y) \cap Z.$$

By Lemma 8.7(1), ${\uparrow}x = \bigcap\{U \in L\,\mathcal{G} : x \in U\}$, which shows that

$$\alpha(x) = \bigcap\{(U \times Y) \cap Z : U \in L\,\mathcal{G}, x \in U\},$$

so is an intersection from $L\mathcal{U}$ by Lemma 13.1(2). Thus, $\alpha(x) \in \mathsf{LC}\mathcal{U}$. We show that $\alpha(x) \in (\mathsf{LC}\mathcal{U})_p$. Let $C_1, \dots, C_n \in L\mathcal{U}$ such that $C := C_1 \wedge \cdots \wedge C_n$ is a distributive meet in $\mathsf{LC}\mathcal{U}$ with $C \le \alpha(x)$. By Lemma 13.1(2), there are $U, U_1, \dots, U_n \in L\,\mathcal{G}$ with $C_i = (U_i \times Y) \cap Z$ for each i and $C = (U \times Y) \cap Z$. Since C is a distributive meet in $\mathsf{LC}\mathcal{U}$, it is a distributive join in $L\mathcal{U}$. Because the map $L\,\mathcal{G} \to L\mathcal{U}$ sending U to $(U \times Y) \cap Z$ is an isomorphism by Propositions 9.19 and 10.14, $U = U_1 \vee \cdots \vee U_n$ is a distributive join in $L\,\mathcal{G}$. Therefore, $U = U_1 \cup \cdots \cup U_n$ by Lemma 13.6. From $C \le \alpha(x)$ we get $\alpha(x) \subseteq C$, so $\alpha(x) \subseteq (U \times Y) \cap Z$, and hence $x \in U$ by Lemma 13.2(2). Therefore, $x \in U_i$ for some i, so

$$\alpha(x) = ({\uparrow}x \times Y) \cap Z \subseteq (U_i \times Y) \cap Z = C_i,$$

which yields $C_i \le \alpha(x)$. Thus, $\alpha(x) \in (\mathsf{LC}\mathcal{U})_p$.

It is clear that α is order-preserving. To see that α is order-reflecting, suppose that $x \not\le x'$. By Lemma 8.7(1), there is $U \in L\,\mathcal{G}$ with $x \in U$ and $x' \notin U$. By Lemma 13.2(2), $\alpha(x) \subseteq (U \times Y) \cap Z$ and $\alpha(x') \not\subseteq (U \times Y) \cap Z$. Therefore, $\alpha(x') \not\subseteq \alpha(x)$, so $\alpha(x) \not\le \alpha(x')$, and hence α is order-reflecting.

We next show that α is onto. Let $D \in (\mathsf{LC}\mathcal{U})_p$. Then D is an intersection from $L\mathcal{U}$ by Lemma 12.3(5), and so $D = (C \times Y) \cap Z$ with C an intersection from $L\,\mathcal{G}$ by Lemma 13.1(2). Set

$$\mathcal{F} = \{U \in \mathsf{ClopUp}(X) : C \subseteq U\}.$$

Then $\mathcal{F}$ is a filter. We show that $\mathcal{F}$ is prime. By Lemma 8.7(2), each clopen upset of X is a finite union from $L\,\mathcal{G}$. Therefore, it suffices to show that if $U_1, \dots, U_n \in L\,\mathcal{G}$ with $U_1 \cup \cdots \cup U_n \in \mathcal{F}$, then some U_i is in $\mathcal{F}$. When this occurs, $C \subseteq U_1 \cup \cdots \cup U_n$. Since C is an intersection from $L\,\mathcal{G}$, compactness of X implies that there is $V \in L\,\mathcal{G}$ with $C \subseteq V$ and $V \subseteq U_1 \cup \cdots \cup U_n$. We have $V = (V \cap U_1) \cup \cdots \cup (V \cap U_n)$, which is a distributive join in $L\,\mathcal{G}$ by Lemma 13.6. Thus,

$$(V \times Y) \cap Z = ([(V \cap U_1) \times Y] \cap Z) \cup \cdots \cup ([(V \cap U_n) \times Y] \cap Z)$$

is a distributive join in $L\mathcal{U}$. Since $D = (C \times Y) \cap Z \subseteq (V \times Y) \cap Z$, in $\mathsf{LC}\mathcal{U}$ we have

$$([(V \cap U_1) \times Y] \cap Z) \wedge \cdots \wedge ([(V \cap U_n) \times Y] \cap Z) \le D.$$

Because $D \in (\mathsf{LC}\mathcal{U})_p$, $[(V \cap U_i) \times Y] \cap Z \leq D$ for some i. Therefore,

$$(C \times Y) \cap Z \subseteq ((V \cap U_i) \times Y) \cap Z \subseteq (U_i \times Y) \cap Z,$$

so $C \subseteq U_i$ by Lemma 13.2(2), and hence $U_i \in \mathcal{F}$. Thus, $\mathcal{F}$ is a prime filter. By Priestley duality, there is $x \in X$ with $\mathcal{F} = \{U \in \mathsf{ClopUp}(X) : x \in U\}$. Therefore, $C = \bigcap \mathcal{F} = {\uparrow}x$, so $D = (C \times Y) \cap Z = ({\uparrow}x \times Y) \cap Z = \alpha(x)$, and hence α is onto.

To see that α is continuous, by Priestley duality it suffices to show that $\alpha^{-1}(V) \in \mathsf{ClopUp}(X)$ for each $V \in \mathsf{ClopUp}((\mathsf{LC}\mathcal{U})_p)$. By Lemma 8.7(2), it is enough to assume that $V \in L\overline{\mathcal{G}}$, where we recall that $\overline{\mathcal{G}} = \mathbb{GDUH}(\mathcal{G})$. By Lemma 13.1(2), we may assume that $V = {\uparrow}C \cap (\mathsf{LC}\mathcal{U})_p$, where $C = (U \times Y) \cap Z$ for some $U \in L\mathcal{G}$. Therefore,

$$\begin{aligned}\alpha^{-1}(V) &= \alpha^{-1}({\uparrow}C \cap (\mathsf{LC}\mathcal{U})_p) = \{x \in X : C \leq \alpha(x)\} \\ &= \{x \in X : \alpha(x) \subseteq C\} = \{x \in X : {\uparrow}(x \times Y) \cap Z \subseteq (U \times Y) \cap Z\} \\ &= \{x \in X : x \in U\} = U,\end{aligned}$$

where the fifth equality holds by Lemma 13.2(2). Thus, α is continuous. It is then a homeomorphism since it is a continuous bijection between compact Hausdorff spaces. Consequently, α is a Pries-isomorphism. Similar arguments show that β is a Pries-isomorphism.

Finally, we show that $x\,R\,y$ iff $\alpha(x)\,R_{\mathcal{U}}\,\beta(y)$ for each $x \in X, y \in Y$. First suppose $x\,R\,y$. By Remark 10.11(2), there are $x' \geq x$ and $y' \geq y$ with $(x', y') \in Z$, so $\alpha(x) \cap \beta(y) \neq \varnothing$, and hence $\alpha(x)\,R_{\mathcal{U}}\,\beta(y)$. Conversely, if $\alpha(x)\,R_{\mathcal{U}}\,\beta(y)$, then $\alpha(x) \cap \beta(y) \neq \varnothing$. Therefore, by Remark 10.11(2), there are $x' \geq x$ and $y' \geq y$ with $(x', y') \in Z$. Since $(x', y') \in Z$ implies that $x'\,R\,y'$, we then have $x\,R\,y$ by Definition 8.3(1,2).

(2) By (1) and Lemma 13.5, $\zeta_{\mathcal{G}}$ is a GvG-isomorphism. ■

Proposition 13.8. *There is a natural isomorphism* $\zeta\colon 1_{\mathsf{GvG}} \to \mathbb{GDUH}$.

Proof. Let $\mathcal{G} = (X, R, Y) \in \mathsf{GvG}$. Set $\mathcal{U} = \mathbb{UH}(\mathcal{G}) := (Z, \leq_1, \leq_2)$. Also, set $\overline{\mathcal{G}} = \mathbb{GDUH}(\mathcal{G})$, so $\overline{\mathcal{G}} = ((\mathsf{LC}\mathcal{U})_p, (R_{\mathcal{U}})_p, (\mathsf{RC}\mathcal{U})_p)$. By Lemma 13.7, there is a GvG-isomorphism $\zeta_{\mathcal{G}} = (S_\alpha, T_\beta)$, where $\alpha\colon X \to (\mathsf{LC}\mathcal{U})_p$ and $\beta\colon Y \to (\mathsf{RC}\mathcal{U})_p$ are Pries-isomorphisms given by

$$\alpha(x) = ({\uparrow}x \times Y) \cap Z \quad \text{and} \quad \beta(y) = (X \times {\uparrow}y) \cap Z$$

for all $x \in X$ and $y \in Y$. The relation $S_\alpha \subseteq X \times (\mathsf{LC}\mathcal{U})_p$ is defined by $x\,S_\alpha\,D$ if $\alpha(x) \leq D$. Since α is a Pries-isomorphism, we may write $D = \alpha(z)$ for some $z \in X$, and so

$$x\,S_\alpha\,\alpha(z) \iff \alpha(x) \leq \alpha(z) \iff x \leq z.$$

The relation T_β is given similarly. To show that ζ is natural, let $(S, T)\colon \mathcal{G} \to \mathcal{G}'$ be a GvG-morphism and set $(\overline{S}, \overline{T}) = \mathbb{GDUH}(S, T)$. If $(P, Q) = \mathbb{UH}(S, T) \subseteq Z \times Z'$, then

$$(x, y)\,P\,(x', y') \iff x\,S\,x' \quad \text{and} \quad (x, y)\,Q\,(x', y') \iff y\,T\,y'.$$

Set $(f_{PQ}, g_{PQ}) = \mathbb{D}(P, Q)$. Then $(\overline{S}, \overline{T}) = \mathbb{G}(f_{PQ}, g_{PQ})$, so $\overline{S} = S_{f_{PQ}}$ and $\overline{T} = T_{g_{PQ}}$ (see Proposition 8.25), where $f_{PQ}\colon \mathsf{LC}\mathcal{U} \to \mathsf{LC}\mathcal{U}'$ and $g_{PQ}\colon \mathsf{RC}\mathcal{U} \to \mathsf{RC}\mathcal{U}'$ are given by

$$f_{PQ}(D) = \bigcap \{C' \in L\mathcal{U}' : D \subseteq \Box_P C'\}, \qquad g_{PQ}(E) = \bigcap \{\phi C' : C' \in L\mathcal{U}',\, E \subseteq \Box_Q \phi' C'\}$$

for each $D \in \mathsf{LC}\mathcal{U}$ and $E \in \mathsf{RC}\mathcal{U}$ (see Definition 12.6). For $U \in L\mathcal{G}$, set $\overline{U} = (U \times Y) \cap Z$. By Lemma 13.1(2), $L\mathcal{U} = \{\overline{U} : U \in L\mathcal{G}\}$ and $L\overline{\mathcal{G}} = \{{\uparrow}\overline{U} \cap (\mathsf{LC}\mathcal{U})_p : U \in L\mathcal{G}\}$. Let

$x \in X$. Then, by Lemma 13.2(2),

$$\alpha(x) \in {\uparrow}\overline{U} \iff \overline{U} \le ({\uparrow}x \times Y) \cap Z \iff ({\uparrow}x \times Y) \cap Z \subseteq \overline{U} \iff x \in U.$$

Therefore, ${\uparrow}\overline{U} \cap (\mathsf{LC}\,\mathcal{U})_p = \alpha[U]$, and so

$$L\,\overline{\mathcal{G}} = \{\alpha[U] : U \in L\,\mathcal{G}\}. \tag{13.1}$$

We show that $\overline{S} \star S_\alpha = S_{\alpha'} \star S$ and $\overline{T} \star T_\beta = T_{\beta'} \star T$. This will show that the diagram below commutes, and hence that ζ is natural.

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "{(S,T)}"] \arrow[d, "\zeta_{\mathcal{G}}"'] & \mathcal{G}' \arrow[d, "\zeta_{\mathcal{G}'}"] \\
\overline{\mathcal{G}} \arrow[r, "{(\overline{S},\overline{T})}"] & \overline{\mathcal{G}'}
\end{tikzcd}$$

We verify the first equality by using the following claim. The proof of the second equality is similar.

Claim 13.9. *Let $x \in X$, $x' \in X'$, $U \in L\,\mathcal{G}$, and $U' \in L\,\mathcal{G}'$.*

(1) *$\alpha(x)\,\overline{S}\,\alpha'(x')$ iff $x\,S\,x'$.*
(2) *$\Box_{\overline{S}}\alpha'[U'] = \alpha[\Box_S U']$.*
(3) *$\Box_{S_\alpha}\alpha[U] = U$.*

Proof of Claim. (1) We first show that $\Box_P\overline{U'} = (\Box_S U' \times Y) \cap Z$. To see this, $(x,y) \in \Box_P\overline{U'}$ iff whenever $(x',y') \in Z'$ and $x\,S\,x'$, then $(x',y') \in \overline{U'}$. Therefore, $(x,y) \in \Box_P\overline{U'}$ iff $x\,S\,x'$ implies $x' \in U'$. Thus, $\Box_P\overline{U'} = (\Box_S U' \times Y) \cap Z$.

Now let $x \in X$ and $x' \in X'$. Then

$$\alpha(x)\,\overline{S}\,\alpha'(x') \iff f_{PQ}\alpha(x) \le \alpha'(x') \iff \alpha'(x') \subseteq f_{PQ}\alpha(x).$$

Since $\Box_P\overline{U'} = (\Box_S U' \times Y) \cap Z$, we have

$$\begin{aligned}
f_{PQ}\alpha(x) &= \bigcap\{\overline{U'} \in L\,\mathcal{U}' : \alpha(x) \subseteq \Box_P\overline{U'}\} \\
&= \bigcap\{\overline{U'} : U' \in L\,\mathcal{G}',\ ({\uparrow}x \times Y) \cap Z \subseteq (\Box_S U' \times Y) \cap Z\} \\
&= \bigcap\{\overline{U'} : U' \in L\,\mathcal{G}',\ x \in \Box_S U'\},
\end{aligned}$$

where the last equality holds by Lemma 13.2(2). Because $\alpha'(x') \subseteq \overline{U'}$ iff $x' \in U'$,

$$\begin{aligned}
\alpha'(x') \subseteq f_{PQ}\alpha(x) &\iff \alpha'(x') \subseteq \overline{U'} \quad \forall U' \in L\,\mathcal{G}' \text{ with } x \in \Box_S U' \\
&\iff x' \in U' \quad \forall U' \in L\,\mathcal{G}' \text{ with } x \in \Box_S U' \\
&\iff x\,S\,x',
\end{aligned}$$

where the last equivalence holds by Remark 8.9(2). Thus, $\alpha(x)\,\overline{S}\,\alpha'(x')$ iff $x\,S\,x'$.

(2) Since α' is a Pries-isomorphism, by (1) we have

$$\begin{aligned}
\Box_{\overline{S}}\alpha'[U'] &= \{\alpha(x) : \alpha(x)\,\overline{S}\,\alpha'(x) \Longrightarrow \alpha'(x') \in \alpha'[U']\} \\
&= \{\alpha(x) : x\,S\,x' \Longrightarrow x' \in U'\} = \alpha[\Box_S U'].
\end{aligned}$$

(3) We have

$$\begin{aligned}\Box_{S_\alpha}\alpha[U] &= \{x \in X : S_\alpha[x] \subseteq \alpha[U]\} = \{x : {\uparrow}\alpha(x) \subseteq \alpha[U]\} \\ &= \{x : \alpha(x) \in \alpha[U]\} = U,\end{aligned}$$

where the third equality holds since $\alpha[U]$ is an upset. ■

By definition, $x\,(\overline{S} \star S_\alpha)\,\alpha'(x')$ iff $(\forall V \in L\,\overline{\mathcal{G}'})\,(x \in \Box_{S_\alpha}\Box_{\overline{S}}V \Longrightarrow \alpha'(x') \in V)$. By (13.1), $L\,\overline{\mathcal{G}'} = \{\alpha'[U'] : U' \in L\,\mathcal{G}'\}$. Therefore,

$$x\,(\overline{S} \star S_\alpha)\,\alpha'(x') \iff (\forall U' \in L\,\mathcal{G}')\,(x \in \Box_{S_\alpha}\Box_{\overline{S}}\alpha'[U'] \Longrightarrow \alpha'(x') \in \alpha'[U']).$$

By Claim 13.9(2, 3),

$$\Box_{S_\alpha}\Box_{\overline{S}}\alpha'[U'] = \Box_{S_\alpha}\alpha[\Box_S U'] = \Box_S U'.$$

This yields

$$x\,(\overline{S} \star S_\alpha)\,\alpha'(x') \iff (\forall U' \in L\,\mathcal{G}')\,(x \in \Box_S U' \Longrightarrow x' \in U') \iff x\,S\,x',$$

where the last equivalence follows from Remark 8.9(2). On the other hand,

$$\begin{aligned} x\,(S_{\alpha'} \star S)\,\alpha'(x') &\iff (\forall U' \in L\,\mathcal{G}')\,(x \in \Box_S\Box_{S_{\alpha'}}\alpha'[U'] \Longrightarrow \alpha'(x') \in \alpha'[U']) \\ &\iff (\forall U' \in L\,\mathcal{G}')\,(x \in \Box_S U' \Longrightarrow x' \in U') \\ &\iff x\,S\,x'. \end{aligned}$$

Therefore, $\overline{S} \star S_\alpha = S_{\alpha'} \star S$. A similar argument shows that $\overline{T} \star T_\beta = T_{\beta'} \star T$. Thus, ζ is natural, and hence is a natural isomorphism. ■

13.3. The natural isomorphisms $\eta\colon 1_{\mathsf{Hg}} \to \mathbb{HGDU}$ and $\theta\colon 1_{\mathsf{Urq}} \to \mathbb{UHGD}$. In this section we show that there are natural isomorphisms $\eta\colon 1_{\mathsf{Hg}} \to \mathbb{HGDU}$ and $\theta\colon 1_{\mathsf{Urq}} \to \mathbb{UHGD}$, which together with the natural isomorphisms of the previous two sections yield our desired equivalences (see Theorem 13.16). The proofs in this section are largely similar to those in Section 13.2, so we mostly skip them and only indicate the differences. We start with $\eta\colon 1_{\mathsf{Hg}} \to \mathbb{HGDU}$ for which we have the following analogues of Lemmas 13.5, 13.7, and Proposition 13.8.

Lemma 13.10. *Let $\mathcal{H} = (X, R, Y), \mathcal{H}' = (X', R', Y') \in \mathsf{Hg}$ and suppose that $\alpha\colon X \to X'$ and $\beta\colon Y \to Y'$ are order-homeomorphisms which satisfy $x\,R\,y$ iff $\alpha(x)\,R'\,\beta(y)$. If (S_α, T_β) is defined by $x\,S_\alpha\,x'$ if $\alpha(x) \le x'$ and $y\,T_\beta\,y'$ if $\beta(y) \le y'$, then $(S_\alpha, T_\beta)\colon \mathcal{H} \to \mathcal{H}'$ is an Hg-isomorphism.*

Lemma 13.11. *Let $\mathcal{H} = (X, R, Y) \in \mathsf{Hg}$ and set $\mathcal{U} = \mathbb{U}(\mathcal{H})$.*

(1) *There are order-homeomorphisms $\alpha\colon X \to (\mathsf{LC}\mathcal{U})_0$ and $\beta\colon Y \to (\mathsf{RC}\mathcal{U})_0$ satisfying $x\,R\,y$ iff $\alpha(x)\,R_{\mathcal{U}}\,\beta(y)$.*
(2) *There is an Hg-isomorphism $\eta_{\mathcal{H}}\colon \mathcal{H} \to \mathbb{HGDU}(\mathcal{H})$ given by $\eta_{\mathcal{H}} = (S_\alpha, T_\beta)$ in the notation of Lemma 13.10.*

Proposition 13.12. *There is a natural isomorphism $\eta\colon 1_{\mathsf{Hg}} \to \mathbb{HGDU}$.*

While the proofs of these three results are largely the same as the corresponding proofs in Section 13.2, an additional argument is needed in Lemma 13.11(1) to guarantee that α and β are well defined and onto. Recalling that $\mathcal{U} = (Z, \leq_1, \leq_2)$, for $x \in X$ and $y \in Y$ define

$$\alpha(x) = ({\uparrow}x \times Y) \cap Z \quad \text{and} \quad \beta(y) = (X \times {\uparrow}y) \cap Z.$$

We only show that α is well defined and onto since the argument for β is similar. By Lemma 9.5(2), ${\uparrow}x = \bigcap\{A \in L\mathcal{H} : x \in A\}$, which shows that

$$\alpha(x) = \bigcap\{(A \times Y) \cap Z : x \in A \in L\mathcal{H}\},$$

so is an intersection from $L\mathcal{U}$ by Proposition 10.14. Thus, $\alpha(x) \in \mathsf{LC}\mathcal{U}$. It is left to show that $\alpha(x) \in (\mathsf{LC}\mathcal{U})_0$. By Remark 10.11(3), there is $y \in Y$ with $(x, y) \in Z$. Therefore, $\alpha(x)\, R_{\mathcal{U}}\, \beta(y)$. Suppose there is $D \in \mathsf{LC}\mathcal{U}$ with $\alpha(x) \leq D$ and $D\, R_{\mathcal{U}}\, \beta(y)$, so $D \cap \beta(y) \neq \varnothing$. Because the order on $\mathsf{LC}\mathcal{U}$ is reverse inclusion, $D \subseteq \alpha(x)$. If $(x', y') \in D \cap \beta(y)$, then $y \leq y'$ and $x \leq x'$. Thus, $(x, y) \leq_1 (x', y')$ and $(x, y) \leq_2 (x', y')$, so $(x, y) = (x', y')$ because $\mathcal{U}$ is doubly ordered. Consequently, $(x, y) \in D$. By Proposition 10.14 and Lemma 12.3(5), $D = (B \times Y) \cap Z$ with B an intersection from $L\mathcal{H}$. Therefore, $x \in B$, so $\alpha(x) \subseteq D$. Thus, $D = \alpha(x)$, and hence $\alpha(x)$ is $\beta(y)$-maximal. Consequently, $\alpha(x) \in (\mathsf{LC}\mathcal{U})_0$.

To see that α is onto, let $D \in (\mathsf{LC}\mathcal{U})_0$. Then there is $E \in (\mathsf{RC}\mathcal{U})_0$ such that D is E-maximal. This implies that $D\, R_{\mathcal{U}}\, E$, so $D \cap E \neq \varnothing$. Let $(x, y) \in D \cap E$. As we saw in the previous paragraph, there is $B \subseteq X$ with $D = (B \times Y) \cap Z$ and B is an intersection from $L\mathcal{H}$. Since $x \in B$ and B is an upset, ${\uparrow}x \subseteq B$, so $\alpha(x) \subseteq D$. If this is a proper inclusion, then $D < \alpha(x)$, so $\alpha(x) \cap E = \varnothing$ because D is E-maximal. This is false since $(x, y) \in \alpha(x) \cap E$. Therefore, $D = \alpha(x)$. Consequently, $\alpha \colon X \to (\mathsf{LC}\mathcal{U})_0$ is onto.

We next turn our attention to $\theta \colon 1_{\mathsf{Urq}} \to \mathbb{UHGD}$. We have the following analogues of Lemmas 13.5, 13.7, and Proposition 13.8.

Lemma 13.13. *Let $\mathcal{U} = (Z, \leq_1, \leq_2), \mathcal{U}' = (Z', \leq_1', \leq_2') \in \mathsf{Urq}$ and $h \colon Z \to Z'$ be a homeomorphism and order-isomorphism with respect to both $\leq_1$ and $\leq_2$. Define $P_h, Q_h \subseteq Z \times Z'$ by $z\, P_h\, z'$ if $h(z) \leq_1 z'$ and $z\, Q_h\, z'$ if $h(z) \leq_2 z'$. Then (P_h, Q_h) is an Urq-isomorphism.*

Lemma 13.14. *Let $\mathcal{U} := (Z, \leq_1, \leq_2) \in \mathsf{Urq}$ and $\mathbb{UHGD}(\mathcal{U}) := (\overline{Z}, \leq_1, \leq_2)$.*

(1) *There is $h \colon Z \to \overline{Z}$ that is a homeomorphism and an order-isomorphism with respect to both $\leq_1$ and $\leq_2$.*
(2) *There is an Urq-isomorphism $\theta_{\mathcal{U}} \colon \mathcal{U} \to \mathbb{UHGD}(\mathcal{U})$ given by $\theta_{\mathcal{U}} = (P_h, Q_h)$ in the notation of Lemma* 13.13.

Proposition 13.15. *There is a natural isomorphism $\theta \colon 1_{\mathsf{Urq}} \to \mathbb{UHGD}$.*

Since in the case of Urquhart spaces we work with two quasi-orders instead of a binary relation, the corresponding proofs simplify. For the proof of Lemma 13.13, since the Galois connections ϕ, ψ are described by the two quasi-orders, a quick argument shows that for each $C' \in L\mathcal{U}'$ we have $\phi\Box_{P_h} C' = \Box_{Q_h} \phi' C'$ and $\psi\phi\Box_{P_h} C' = \Box_{P_h} C'$. Thus, $\Box_{P_h} C' \in L\mathcal{U}$, verifying that (P_h, Q_h) satisfies Definition 10.4(1). The proof that (P_h, Q_h) satisfies the remaining conditions of Definition 10.4 is similar to that of (S_α, T_β) being a GvG-morphism given in Lemma 13.5.

For the proof of Lemma 13.14(1), define $h \colon Z \to \overline{Z}$ by $h(z) = ({\uparrow}_1 z, {\uparrow}_2 z)$. The argument that h is well defined and onto is similar to that of Lemma 13.11(1), and the rest to that of Lemma 13.7(1). Lastly, the proof of Proposition 13.15 is similar to that of Proposition 13.8.

Finally, putting Propositions 13.4, 13.8, 13.12 and 13.15 together yields the main result of this chapter:

THEOREM 13.16. *The categories* DH, GvG, Hg, *and* Urq *are equivalent.*

14. Overview of the resulting dualities for lattices

We have provided explicit equivalences between various categories that were proposed in the literature as extensions of Priestley duality to arbitrary (not necessarily distributive) bounded lattices. This included developing the duals of bounded lattice homomorphisms where they were lacking. As a result, we arrive at the diagram in Figure 2, where the horizontal arrows represent equivalence of categories and the upward arrows dual equivalence. In this chapter we show that each triangle in the diagram commutes up to natural isomorphism.

$$\begin{tikzcd}
& & & & \mathsf{Lat} & & & & \\
\mathsf{CG} \arrow[r] \arrow[urrrr] & \mathsf{JM} \arrow[r] \arrow[urrr] & \mathsf{BDGM} \arrow[r] \arrow[urr] & \mathsf{Hs} \arrow[r] \arrow[ur] & \mathsf{DH} \arrow[r] \arrow[u] & \mathsf{GvG} \arrow[r] \arrow[ul] & \mathsf{Hg} \arrow[r] \arrow[ull] & \mathsf{Urq} \arrow[r] \arrow[ulll] & \mathsf{Plo} \arrow[ullll]
\end{tikzcd}$$

Fig. 2. Various dualities for Lat

By Theorem 6.11, there is an equivalence of categories between CG and JM. Remark 3.16 shows that there is a functor $\mathbb{L}_{\mathsf{JM}} \colon \mathsf{JM} \to \mathsf{Lat}$ which sends $X \in \mathsf{JM}$ to $\mathsf{KOF}(X)$ and a JM-morphism $f \colon X \to Y$ to $f^{-1} \colon \mathsf{KOF}(Y) \to \mathsf{KOF}(X)$. By Remark 6.13, there is a functor $\mathbb{L}_{\mathsf{CG}} \colon \mathsf{CG} \to \mathsf{Lat}$ which sends $\mathcal{C} \in \mathsf{CG}$ to $L\mathcal{C}$ and a CG-morphism $S \colon \mathcal{C}_1 \to \mathcal{C}_2$ to $\Box_S \colon L\mathcal{C}_2 \to L\mathcal{C}_1$.

THEOREM 14.1. *The following diagram commutes up to natural isomorphism:*

$$\begin{tikzcd}
& \mathsf{Lat} & \\
\mathsf{CG} \arrow[ur, "\mathbb{L}_{\mathsf{CG}}"] \arrow[rr, "\mathbb{H}"] & & \mathsf{JM} \arrow[ul, "\mathbb{L}_{\mathsf{JM}}"']
\end{tikzcd}$$

Proof. Let $\mathcal{C} = (X, \mathcal{K}) \in \mathsf{CG}$. Then

$$\mathbb{L}_{\mathsf{CG}}(\mathcal{C}) = L\mathcal{C} = \{-U : U \in \mathcal{K}\}$$

and

$$\mathbb{L}_{\mathsf{JM}}\mathcal{H}(\mathcal{C}) = \{\Psi_U : U \in \mathcal{K}\} = \{\{H \in \mathcal{H}(\mathcal{C}) : U \subseteq H\} : U \in \mathcal{K}\}.$$

Define $\alpha_{\mathcal{C}} \colon L\,\mathcal{C} \to \mathsf{KOF}(\mathcal{H}(\mathcal{C}))$ by $\alpha_{\mathcal{C}}(-U) = \Psi_U$ for each $U \in \mathcal{K}$. It is straightforward to see that

$$-U \subseteq -V \iff \alpha_{\mathcal{C}}(-U) \subseteq \alpha_{\mathcal{C}}(-V)$$

for all $U, V \in \mathcal{K}$. Moreover, $\alpha_{\mathcal{C}}$ is onto by Lemma 6.9(1). Consequently, $\alpha_{\mathcal{C}}$ is an order-isomorphism, hence a Lat-isomorphism. To see that $\alpha \colon \mathbb{L}_{\mathsf{CG}} \to \mathbb{L}_{\mathsf{JM}} \circ \mathbb{H}$ is natural, we show that the following diagram commutes:

$$\begin{tikzcd}
L\,\mathcal{C}' \arrow[r, "\Box_S"] \arrow[d, "\alpha_{\mathcal{C}'}"'] & L\,\mathcal{C} \arrow[d, "\alpha_{\mathcal{C}}"] \\
\mathsf{KOF}(\mathcal{H}(\mathcal{C}')) \arrow[r, "f_S^{-1}"] & \mathsf{KOF}(\mathcal{H}(\mathcal{C}))
\end{tikzcd}$$

Let $U \in \mathcal{K}'$. Recalling from Chapter 6 that $f_S(H) = \bigcup\{V \in \mathcal{K}' : S^{-1}[V] \subseteq H\}$ for each $H \in \mathcal{H}(\mathcal{C})$, we have

$$\begin{aligned}
\alpha_{\mathcal{C}}\Box_S(-U) &= \alpha_{\mathcal{C}}(-S^{-1}[U]) = \Psi_{S^{-1}[U]} = \{H \in \mathcal{H}(\mathcal{C}) : S^{-1}[U] \subseteq H\} \\
&= \{H \in \mathcal{H}(\mathcal{C}) : U \subseteq f_S(H)\} = \{H \in \mathcal{H}(\mathcal{C}) : f_S(H) \in \Psi_U\} \\
&= f_S^{-1}[\Psi_U].
\end{aligned}$$

Thus, α is a natural isomorphism, and hence the triangle above commutes up to natural isomorphism. ■

By Theorem 4.12(2), there is an isomorphism of categories between JM and BDGM, which we denote by $\mathbb{B}$. The functor $\mathbb{B}$ sends $X \in \mathsf{JM}$, which is an algebraic lattice with the Scott topology, to (X, λ), where λ is the Lawson topology. By Remark 4.7, there is a functor $\mathbb{L}_{\mathsf{BDGM}} \colon \mathsf{BDGM} \to \mathsf{Lat}$ which sends $X \in \mathsf{BDGM}$ to $\mathsf{CLF}(X)$ and a BDGM-morphism $f \colon X \to Y$ to $f^{-1} \colon \mathsf{CLF}(Y) \to \mathsf{CLF}(X)$. Since $\mathsf{KOF}(X) = \mathsf{CLF}(X)$ by Remark 5.5, we have $\mathbb{L}_{\mathsf{BDGM}} \circ \mathbb{B} = \mathbb{L}_{\mathsf{JM}}$, yielding the next result.

THEOREM 14.2. *The following diagram commutes:*

$$\begin{tikzcd}
 & \mathsf{Lat} & \\
\mathsf{JM} \arrow[ur, "\mathbb{L}_{\mathsf{JM}}"] \arrow[rr, "\mathbb{B}"] & & \mathsf{BDGM} \arrow[ul, "\mathbb{L}_{\mathsf{BDGM}}"']
\end{tikzcd}$$

By Theorem 5.13, there is an isomorphism of categories between BDGM and Hs, which we denote by $\mathbb{H}\mathbb{s}$. By Remark 5.15(1), there is a functor $\mathbb{L}_{\mathsf{Hs}} \colon \mathsf{Hs} \to \mathsf{Lat}$ which sends $X \in \mathsf{Hs}$ to $\mathsf{CLF}(X)$ and an Hs-morphism $f \colon X \to Y$ to $f^{-1} \colon \mathsf{CLF}(Y) \to \mathsf{CLF}(X)$. The next result is then immediate from these facts.

THEOREM 14.3. *The following diagram commutes:*

$$\begin{tikzcd}
 & \mathsf{Lat} & \\
\mathsf{BDGM} \arrow[ur, "\mathbb{L}_{\mathsf{BDGM}}"] \arrow[rr, "\mathbb{H}\mathbb{s}"] & & \mathsf{Hs} \arrow[ul, "\mathbb{L}_{\mathsf{Hs}}"']
\end{tikzcd}$$

By Theorems 5.13, 4.12(1), and 7.21, there are functors

$$\mathsf{Hs} \to \mathsf{BDGM} \to \mathsf{CohLat}_{\lambda} \to \mathsf{DH}.$$

Denote the composition $\mathsf{Hs} \to \mathsf{DH}$ by $\mathbb{DH}$. This functor sends $X \in \mathsf{Hs}$ to $(X, R_X, \mathsf{OF}(X))$ (see Definition 7.14) and an Hs-morphism $f\colon X \to Y$ to

$$(f, r)\colon (X, R_X, \mathsf{OF}(X)) \to (Y, R_Y, \mathsf{OF}(Y)),$$

where r is the right adjoint of $f^{-1}\colon \mathsf{OF}(Y) \to \mathsf{OF}(X)$ (see Lemma 7.16). By Corollary 7.22(1), this functor is an equivalence. Remark 7.23 shows that there is a functor $\mathbb{L}_{\mathsf{DH}}\colon \mathsf{DH} \to \mathsf{Lat}$ which sends $(X, R, Y) \in \mathsf{DH}$ to $\mathsf{CLF}(X)$ and a DH-morphism $(f, g)\colon (X, R, Y) \to (X', R', Y')$ to $f^{-1}\colon \mathsf{CLF}(X') \to \mathsf{CLF}(X)$. The next result is then immediate.

Theorem 14.4. *The following diagram commutes:*

$$\begin{tikzcd} & \mathsf{Lat} & \\ \mathsf{Hs} \arrow[ur, "\mathbb{L}_{\mathsf{Hs}}"] \arrow[rr, "\mathbb{DH}"] & & \mathsf{DH} \arrow[ul, "\mathbb{L}_{\mathsf{DH}}"'] \end{tikzcd}$$

In order to finish proving the commutativity of Figure 2, we need to define functors from each of GvG, Hg, Urq, and Plo to Lat. We start with the functor from GvG to Lat.

Proposition 14.5. *There is a functor* $\mathbb{L}_{\mathsf{GvG}}\colon \mathsf{GvG} \to \mathsf{Lat}$ *which sends* $\mathcal{G} \in \mathsf{GvG}$ *to* $L\,\mathcal{G}$ *and a* GvG*-morphism* (S, T) *to* $\Box_S$.

Proof. By Remark 8.6, $\mathbb{L}_{\mathsf{GvG}}$ is well defined on objects. If $(S, T)\colon \mathcal{G} \to \mathcal{G}'$ is a GvG-morphism, then $\Box_S\colon L\,\mathcal{G}' \to L\,\mathcal{G}$ is a well-defined function by Definition 8.8(1). We show that $\Box_S$ is a Lat-morphism. It is clear that $\Box_S X' = X$. Since binary meet in $L\,\mathcal{G}$ is intersection, we also find that $\Box_S(U \wedge V) = \Box_S U \wedge \Box_S V$ for each $U, V \in L\,\mathcal{G}$. To see that $\Box_S \varnothing = \varnothing$, if $x \in \Box_S \varnothing$, then $S[x] = \varnothing$. Consequently, $R[x] = \varnothing$ by Definition 8.8(4). Therefore, $x \in \Box \blacklozenge \varnothing$, which is false as $\Box \blacklozenge \varnothing = \varnothing$ by Remark 8.6. Thus, $\Box_S \varnothing = \varnothing$. It remains to show that $\Box_S$ preserves binary joins. The inclusion $\Box_S U \vee \Box_S V \subseteq \Box_S(U \vee V)$ is clear since $\Box_S$ is order-preserving. For the other inclusion, let $x \notin \Box_S U \vee \Box_S V$. Since $\Box_S U \vee \Box_S V = \Box \blacklozenge (\Box_S U \cup \Box_S V)$, there is $y \in Y$ with $x\,R\,y$ and $y \notin \blacklozenge(\Box_S U \cup \Box_S V)$. By Definition 8.8(1),

$$\begin{aligned} \blacklozenge(\Box_S U \cup \Box_S V) &= \blacklozenge \Box_S U \cup \blacklozenge \Box_S V = \Diamond_T \blacklozenge' U \cup \Diamond_T \blacklozenge' V \\ &= \Diamond_T(\blacklozenge' U \cup \blacklozenge' V) = \Diamond_T(\blacklozenge'(U \cup V)). \end{aligned}$$

Therefore, $y \notin \Diamond_T(\blacklozenge'(U \cup V))$. Because $x\,R\,y$, by Definition 8.8(4) there are $x' \in X'$ and $y' \in Y'$ with $x\,S\,x'$, $y\,T\,y'$, and $x'\,R'\,y'$. If $x \in \Box_S(U \vee V) = \Box_S(\Box' \blacklozenge'(U \cup V))$, then $x' \in \Box' \blacklozenge'(U \cup V)$, so $y' \in \blacklozenge'(U \cup V)$. This implies that $y \in \Diamond_T(\blacklozenge'(U \cup V))$, which is false. Thus, $x \notin \Box_S(U \vee V)$. Consequently, $\Box_S$ preserves binary joins, and hence is a Lat-morphism. This shows that $\mathbb{L}_{\mathsf{GvG}}$ is well defined on morphisms.

By Lemma 8.12, $\mathbb{L}_{\mathsf{GvG}}$ preserves composition. If $\mathcal{G} = (X, R, Y) \in \mathsf{GvG}$, then $(\leq_X, \leq_Y)$ is the identity morphism for $\mathcal{G}$ by Proposition 8.13(2). It is also immediate that $\Box_{\leq_X} U = U$ for each $U \in L\,\mathcal{G}$. Therefore, $\Box_{\leq_X}$ is the identity morphism of $L\,\mathcal{G}$, completing the proof that $\mathbb{L}_{\mathsf{GvG}}$ is a functor. ∎

THEOREM 14.6. *The following diagram commutes up to natural isomorphism:*

$$\begin{tikzcd} & \mathsf{Lat} & \\ \mathsf{DH} \arrow[ur, "\mathbb{L}_{\mathsf{DH}}"] \arrow[rr, "\mathbb{G}"] & & \mathsf{GvG} \arrow[ul, "\mathbb{L}_{\mathsf{GvG}}"'] \end{tikzcd}$$

Proof. Let $\mathcal{D} = (X, R, Y) \in \mathsf{DH}$ and $\mathcal{G} = \mathbb{G}(\mathcal{D})$, so $\mathcal{G} = (X_p, R_p, Y_p)$. By Proposition 8.24, there is a Lat-isomorphism $\gamma_D\colon \mathsf{CLF}(X) \to L\,\mathcal{G}$ given by $\gamma_D(U) = U \cap X_p$ for each $U \in \mathsf{CLF}(X)$. Let $(f, g)\colon \mathcal{D} \to \mathcal{D}'$ be a DH-morphism and $(S_f, T_g) = \mathbb{G}(f, g)$. By Proposition 8.25,

$$x\, S_f\, x' \iff f(x) \le x' \quad \text{and} \quad y\, T_g\, y' \iff g(y) \le y'.$$

To see that $\gamma\colon \mathbb{L}_{\mathsf{DH}} \to \mathbb{L}_{\mathsf{GvG}} \circ \mathbb{G}$ is natural, we show that the following diagram commutes:

$$\begin{tikzcd} \mathsf{CLF}(X') \arrow[r, "f^{-1}"] \arrow[d, "\gamma_{\mathcal{D}'}"'] & \mathsf{CLF}(X) \arrow[d, "\gamma_{\mathcal{D}}"] \\ L\,\mathcal{G}' \arrow[r, "\Box_{S_f}"] & L\,\mathcal{G} \end{tikzcd}$$

Let $U \in \mathsf{CLF}(X')$. Then $\Box_{S_f}\gamma_{\mathcal{D}'}(U) = \Box_{S_f}(U \cap X'_p)$ while $\gamma_{\mathcal{D}}(f^{-1}(U)) = f^{-1}(U) \cap X_p$. We show that these are equal. If $x \in f^{-1}(U) \cap X_p$, then $f(x) \in U$, so

$$S_f[x] = {\uparrow} f(x) \cap X'_p \subseteq U \cap X'_p,$$

and hence $x \in \Box_{S_f}(U \cap X'_p)$. For the reverse inclusion, suppose that $x \in \Box_{S_f}(U \cap X'_p)$, so $S_f[x] \subseteq U \cap X'_p$. By Lemma 5.4, U is a principal upset. Therefore, to show that $f(x) \in U$, by Lemma 8.17(3) it suffices to show that ${\uparrow} f(x) \cap X'_0 \subseteq U$. Since $X'_0 \subseteq X'_p$ (see Lemma 8.19),

$${\uparrow} f(x) \cap X'_0 \subseteq {\uparrow} f(x) \cap X'_p = S_f[x] \subseteq U \cap X'_p \subseteq U.$$

Thus, $f(x) \in U$, so $\gamma\colon \mathbb{L}_{\mathsf{DH}} \to \mathbb{L}_{\mathsf{GvG}} \circ \mathbb{G}$ is a natural isomorphism, and hence the triangle above commutes up to natural isomorphism. ∎

We next turn our attention to the functor from Hg to Lat. An argument similar to that of Proposition 14.5 yields:

PROPOSITION 14.7. *There is a functor* $\mathbb{L}_{\mathsf{Hg}}\colon \mathsf{Hg} \to \mathsf{Lat}$ *which sends* $\mathcal{H} \in \mathsf{Hg}$ *to* $L\,\mathcal{H}$ *and an* Hg*-morphism* (S, T) *to* $\Box_S$.

THEOREM 14.8. *The following diagram commutes up to natural isomorphism:*

$$\begin{tikzcd} & \mathsf{Lat} & \\ \mathsf{GvG} \arrow[ur, "\mathbb{L}_{\mathsf{GvG}}"] \arrow[rr, "\mathbb{H}"] & & \mathsf{Hg} \arrow[ul, "\mathbb{L}_{\mathsf{Hg}}"'] \end{tikzcd}$$

Proof. Let $\mathcal{G} = (X, R, Y) \in \mathsf{GvG}$ and $\mathcal{H} = \mathbb{H}(\mathcal{G}) = (X_0, R_0, Y_0)$. By Proposition 9.19, there is a Lat-isomorphism $\delta_{\mathcal{G}}\colon L\,\mathcal{G} \to L\,\mathcal{H}$ given by $\delta_{\mathcal{G}}(U) = U \cap X_0$ for each $U \in L\,\mathcal{G}$. To show that $\delta\colon \mathbb{L}_{\mathsf{GvG}} \to \mathbb{L}_{\mathsf{Hg}} \circ \mathbb{H}$ is natural, let $(S, T)\colon \mathcal{G} \to \mathcal{G}'$ be a GvG-morphism and

$(S_0, T_0) = \mathbb{H}(S, T)$. We show that the following diagram commutes:

$$\begin{tikzcd} L\,\mathcal{G}' \arrow[r, "\Box_S"] \arrow[d, "\delta_{\mathcal{G}'}"'] & L\,\mathcal{G} \arrow[d, "\delta_{\mathcal{G}}"] \\ L\,\mathcal{H}' \arrow[r, "\Box_{S_0}"] & L\,\mathcal{H} \end{tikzcd}$$

Let $U' \in L\,\mathcal{G}'$. Then $\Box_{S_0}\delta_{\mathcal{G}'}(U') = \Box_{S_0}(U' \cap X_0')$ and $\delta_{\mathcal{G}}(\Box_S U') = \Box_S U' \cap X_0$. To see that the two are equal, if $x \in \Box_S U' \cap X_0$, then $S[x] \subseteq U'$, so $S_0[x] = S[x] \cap X_0' \subseteq U' \cap X_0'$, and hence $x \in \Box_{S_0}(U' \cap X_0')$. For the reverse inclusion, suppose that $x \in \Box_{S_0}(U' \cap X_0')$. Then $S_0[x] \subseteq U'$. Let $x' \in S[x]$. If $x' \notin U'$, by Lemma 8.7(5) there is $x_0 \in X_0'$ with $x' \le x_0$ and $x_0 \notin U'$. On the other hand, by Remark 8.9(3), $S[x]$ is an upset. Therefore, $x_0 \in S[x] \cap X_0' = S_0[x]$, which implies that $x_0 \in U'$. The resulting contradiction shows that $x' \in U'$. Consequently, $\delta\colon \mathbb{L}_{\mathsf{GvG}} \to \mathbb{L}_{\mathsf{Hg}} \circ \mathbb{H}$ is a natural isomorphism, and hence the triangle above commutes up to natural isomorphism. ∎

We now concentrate on the functor from Urq to Lat.

PROPOSITION 14.9. *There is a functor* $\mathbb{L}_{\mathsf{Urq}}\colon \mathsf{Urq} \to \mathsf{Lat}$ *which sends* $\mathcal{U} \in \mathsf{Hg}$ *to* $L\mathcal{U}$ *and an* Urq*-morphism* (P, Q) *to* $\Box_P$*.*

Proof. The proof is essentially the same as that of Proposition 14.5, but uses Claim 12.9 to show that if (P, Q) is an Urq-morphism, then $\Box_P$ preserves binary joins. ∎

THEOREM 14.10. *The following diagram commutes up to natural isomorphism:*

$$\begin{tikzcd} & \mathsf{Lat} & \\ \mathsf{Hg} \arrow[ur, "\mathbb{L}_{\mathsf{Hg}}"] \arrow[rr, "\mathbb{U}"] & & \mathsf{Urq} \arrow[ul, "\mathbb{L}_{\mathsf{Urq}}"'] \end{tikzcd}$$

Proof. Let $\mathcal{H} = (X, R, Y) \in \mathsf{Hg}$ and $\mathcal{U} = \mathbb{U}(\mathcal{H}) = (Z, \le_1, \le_2)$. By Proposition 10.14, there is a Lat-isomorphism $\mu_{\mathcal{H}}\colon L\,\mathcal{H} \to L\mathcal{U}$, given by $\mu_{\mathcal{H}}(A) = (A \times Y) \cap Z$. Let $(S, T)\colon \mathcal{H} \to \mathcal{H}'$ be an Hg-morphism and $(P, Q) = \mathbb{U}(S, T)$. By Theorem 10.16,

$$(x, y)\, P\, (x', y') \iff x\, S\, x' \quad \text{and} \quad (x, y)\, Q\, (x', y') \iff y\, T\, y'.$$

We show that the following diagram commutes:

$$\begin{tikzcd} L\,\mathcal{H}' \arrow[r, "\Box_S"] \arrow[d, "\mu_{\mathcal{H}'}"'] & L\,\mathcal{H} \arrow[d, "\mu_{\mathcal{H}}"] \\ L\mathcal{U}' \arrow[r, "\Box_P"] & L\mathcal{U} \end{tikzcd}$$

Let $A' \in L\,\mathcal{H}'$. Then $\Box_P \mu_{\mathcal{H}'}(A') = \Box_P((A' \times Y') \cap Z')$ and $\mu_{\mathcal{H}}(\Box_S A') = (\Box_S A' \times Y) \cap Z$, which are equal by (10.2). Thus, $\mu\colon \mathbb{L}_{\mathsf{Hg}} \to \mathbb{L}_{\mathsf{Urq}} \circ \mathbb{U}$ is natural, and hence the triangle above commutes up to natural isomorphism. ∎

By composing $\mathbb{P}^{-1}\colon \mathsf{Plo} \to \mathsf{Urq}$ and $\mathbb{L}_{\mathsf{Urq}}\colon \mathsf{Urq} \to \mathsf{Lat}$, we get a functor $\mathbb{L}_{\mathsf{Plo}}\colon \mathsf{Plo} \to \mathsf{Lat}$ which sends $\mathcal{P}$ to $L\,\mathcal{P}$ (see Lemma 11.13(2)) and a Plo-morphism (P, Q) to $\Box_P$. The next result is then immediate.

THEOREM 14.11. *The following diagram commutes:*

$$\begin{tikzcd} & \mathsf{Lat} & \\ \mathsf{Urq} \arrow[ur, "\mathbb{L}_{\mathsf{Urq}}"] \arrow[rr, "\mathbb{P}"] & & \mathsf{Plo} \arrow[ul, "\mathbb{L}_{\mathsf{Plo}}"'] \end{tikzcd}$$

We complete the circle of equivalences with our final result of this chapter.

THEOREM 14.12. *The following diagram commutes up to natural isomorphism:*

$$\begin{tikzcd} & \mathsf{Lat} & \\ \mathsf{Urq} \arrow[ur, "\mathbb{L}_{\mathsf{Urq}}"] \arrow[rr, "\mathbb{D}"] & & \mathsf{DH} \arrow[ul, "\mathbb{L}_{\mathsf{DH}}"'] \end{tikzcd}$$

Proof. By Proposition 12.4(3) there is a Lat-isomorphism $\xi_U : L\mathcal{U} \to \mathsf{CLF}(\mathsf{LC}\mathcal{U})$, given by $\xi_U(C) = {\uparrow}C$. For naturality, let $(P,Q) : \mathcal{U} \to \mathcal{U}'$ be an Urq-morphism and define $(f,g) = \mathbb{D}(P,Q)$. By Definition 12.6,

$$f(D) = \bigcap\{C' \in \mathsf{LC}\mathcal{U}' : D \subseteq \Box_P C'\} \quad \text{and} \quad g(E) = \bigcap\{\phi' C' : C' \in L\mathcal{U}', E \subseteq \Box_Q \phi' C'\}.$$

We show that the following diagram commutes:

$$\begin{tikzcd} L\mathcal{U}' \arrow[r, "\Box_P"] \arrow[d, "\xi_{\mathcal{U}'}"'] & L\mathcal{U} \arrow[d, "\xi_{\mathcal{U}}"] \\ \mathsf{CLF}(\mathsf{LC}\mathcal{U}') \arrow[r, "f^{-1}"] & \mathsf{CLF}(\mathsf{LC}\mathcal{U}) \end{tikzcd}$$

Let $C' \in L\mathcal{U}'$. Then

$$\begin{aligned} f^{-1}(\xi_{\mathcal{U}'}(C')) &= f^{-1}({\uparrow}C') = \{D \in \mathsf{LC}\mathcal{U} : C' \le f(D)\} = \{D \in \mathsf{LC}\mathcal{U} : \Box_P C' \le D\} \\ &= {\uparrow}(\Box_P C') = \xi_{\mathcal{U}}(\Box_P C'), \end{aligned}$$

where the third equality follows from Lemma 12.7(1). Thus, $\xi : \mathbb{L}_{\mathsf{Urq}} \to \mathbb{L}_{\mathsf{DH}} \circ \mathbb{D}$ is natural, and hence the triangle above commutes up to natural isomorphism. ∎

15. Illustrating similarities and differences

In this final chapter, we discuss how each of the dualities for bounded lattices restricts to the distributive case. As we will see, in this case, Gehrke–van Gool, Urquhart, and Ploščica dualities collapse to Priestley duality, while Celani–González and Hartung dualities to Stone duality for distributive lattices. We then consider the action of each of the dual equivalences on a nondistributive lattice, thus showcasing the similarities and differences of the various dualities for bounded lattices considered in the literature.

15.1. The distributive case. Let A be a bounded distributive lattice and X its Priestley space. We recall (see, e.g., [Pri84] or [BB+10]) that there is an isomorphism between $\mathsf{Filt}(A)$ and the frame $\mathsf{ClosedUp}(X)$ of closed upsets of X (ordered by $\supseteq$). By identifying A with $\mathsf{ClopUp}(X)$, this isomorphism is given by

$$\mathcal{F} \mapsto \bigcap\{U \in \mathsf{ClopUp}(X) : U \in \mathcal{F}\} \quad \text{and} \quad C \mapsto \{U \in \mathsf{ClopUp}(X) : C \subseteq U\}.$$

Similarly, there is an isomorphism between $\mathsf{Idl}(A)$ and the frame $\mathsf{OpenUp}(X)$ of open upsets of X, given by

$$\mathcal{I} \mapsto \bigcup\{U \in \mathsf{ClopUp}(X) : U \in \mathcal{I}\} \quad \text{and} \quad O \mapsto \{U \in \mathsf{ClopUp}(X) : U \subseteq O\}.$$

Jipsen–Moshier approach. The lattice A is sent to $(\mathsf{Filt}(A), \sigma)$, where σ is the Scott topology on $\mathsf{Filt}(A)$. Therefore, the Jipsen–Moshier dual of A can be interpreted as $\mathsf{ClosedUp}(X)$ with the Scott topology, whose basis is given by $\{\uparrow U : U \in \mathsf{ClopUp}(X)\}$, where $\uparrow$ is calculated with respect to $\supseteq$.

BDGM-approach. The lattice A is sent to $(\mathsf{Filt}(A), \lambda)$, where λ is the Lawson topology on $\mathsf{Filt}(A)$. Thus, the BDGM-dual of A can be interpreted as $\mathsf{ClosedUp}(X)$ with the Lawson topology, which is generated by $\{\uparrow U : U \in \mathsf{ClopUp}(X)\} \cup \{-\uparrow U : U \in \mathsf{ClopUp}(X)\}$, where again $\uparrow$ is calculated with respect to $\supseteq$.

Hartonas approach. The Hartonas dual is the same as the BDGM-dual.

Celani–González approach. The lattice A is sent to $\mathcal{C}(A) = (X_A, \mathcal{K}_A)$ (see Remark 6.13). Since A is distributive, X_A is exactly the set of prime filters of A. Therefore, X_A is the Stone dual of A and $\mathcal{K}_A$ is the collection of complements of compact opens of the spectral topology on X_A. Thus, the topology on X_A is the co-compact topology of the spectral topology on X_A (which, when viewing X_A as a Priestley space, corresponds to the topology of open downsets).

Dunn–Hartonas approach. The lattice A is sent to the triple $(\mathsf{Filt}(A), R, \mathsf{Idl}(A))$, where recalling from Chapter 7 that we are working with the complement of their relation, R is given by $F\,R\,I$ iff $F \cap I = \varnothing$. Interpreting F as $\{U \in \mathsf{ClopUp}(X) : C \subseteq U\}$ for some $C \in \mathsf{ClosedUp}(X)$ and I as $\{V \in \mathsf{ClopUp}(X) : V \subseteq O\}$ for some $O \in \mathsf{OpenUp}(X)$,

$$\begin{aligned} C \subseteq O &\iff \bigcap\{U \in \mathsf{ClopUp}(X) : C \subseteq U\} \subseteq \bigcup\{V \in \mathsf{ClopUp}(X) : V \subseteq O\} \\ &\iff F \cap I \neq \varnothing, \end{aligned}$$

where the second equivalence follows by compactness. Thus, the Dunn–Hartonas dual of A can be interpreted as $(\mathsf{ClosedUp}(X), R, \mathsf{OpenUp}(X))$, where R is given by $C\,R\,O$ iff $C \nsubseteq O$.

Gehrke–van Gool approach. The lattice A is sent to $(\mathsf{Filt}(A)_p, R_p, \mathsf{Idl}(A)_p)$, where $F\,R_p\,I$ iff $F \cap I = \varnothing$. Since A is distributive, a filter is d-prime iff it is prime. Thus, $\mathsf{Filt}(A)_p = X$. Moreover, $\mathsf{Idl}(A)_p := Y$ is the set of prime ideals of A. Therefore, the Gehrke–van Gool dual of A is (X, R, Y), where $x\,R\,y$ iff $x \cap y = \varnothing$ for each $x \in X$ and $y \in Y$. Since there is an order-reversing bijection $Y \to X$ sending y to $-y$, we may identify (X, R, Y) with X, where R is identified with the order on X (see [GvG14, p. 452]), and hence interpret the Gehrke–van Gool dual as the Priestley dual of A.

Hartung approach. The lattice A is sent to $(\mathsf{Filt}(A)_0, R_0, \mathsf{Idl}(A)_0)$, where recalling from Remark 9.4(4) that we are working with the complement of the Hartung relation, $\mathsf{Filt}(A)_0$

is the set of I-maximal filters for some ideal I, $\mathsf{Idl}(A)_0$ is the set of F-maximal ideals for some filter F, and $F\,R_0\,I$ iff $F \cap I = \varnothing$ (see [Har92, Def. 2.1.6]). Because A is distributive, these are precisely the sets of prime filters and prime ideals of A (see [Har92, p. 278]). Therefore, $\mathsf{Filt}(A)_0 = \mathsf{Filt}(A)_p$, $\mathsf{Idl}(A)_0 = \mathsf{Idl}(A)_p$, and $R_0 = R_p$. Thus, as in the Gehrke–van Gool approach, we may identify the Hartung dual of A with $\mathsf{Filt}(A)_0$, but the topology of this space is the topology of open downsets. Consequently, the topology of the Hartung dual of A may be interpreted as the Stone dual of A with the co-compact topology of the spectral topology.

Urquhart approach. The lattice A is sent to $(Z, \leq_1, \leq_2)$, where Z is the set of maximal filter-ideal pairs of A, $(F, I) \leq_1 (F', I')$ iff $F \subseteq F'$, and $(F, I) \leq_2 (F', I')$ iff $I \subseteq I'$ (see [Urq78, p. 47]). Since A is distributive, $Z = \{(x, -x) : x \in X\}$. Thus, Z may be identified with X, the quasi-order $\leq_1$ on Z with the order $\leq$ on X, and $\leq_2$ with $\geq$ (see [Urq78, Thm. 5]). Consequently, we may interpret the Urquhart dual as the Priestley dual of A.

Ploščica approach. The Ploščica dual of A utilizes the same space as Urquhart (see [Plo95, p. 76]), but replaces the two quasi-orders with a relation R. Since A is distributive, it follows from the Urquhart approach that each maximal filter-ideal pair is of the form $(x, -x)$ for some $x \in X$. Thus, by Definition 11.9, $(x, -x)\,R\,(x', -x')$ iff there is $x'' \in X$ such that $x \subseteq x''$ and $-x' \subseteq -x''$, which happens iff $x \leq x'$. Consequently, in view of the Urquhart approach, we may interpret the Ploščica dual as the Priestley dual of A.

We now turn our attention to morphisms. Let $\alpha\colon A' \to A$ be a morphism of bounded distributive lattices. In each of Jipsen–Moshier, BDGM, and Hartonas dualities, α is sent to $\alpha^{-1}\colon \mathsf{Filt}(A) \to \mathsf{Filt}(A')$. The Dunn–Hartonas dual of α is the DH-morphism

$$(f, g)\colon (\mathsf{Filt}(A), R, \mathsf{Idl}(A)) \to (\mathsf{Filt}(A'), R', \mathsf{Idl}(A')),$$

where $f = \alpha^{-1}\colon \mathsf{Filt}(A) \to \mathsf{Filt}(A')$ and $g = \alpha^{-1}\colon \mathsf{Idl}(A) \to \mathsf{Idl}(A')$. Let $\varphi\colon X \to X'$ be the Priestley dual of $\alpha\colon A' \to A$. Interpreting F as $\{U \in \mathsf{ClopUp}(X) : C \subseteq U\}$ for some $C \in \mathsf{ClosedUp}(X)$, $\alpha^{-1}[F]$ is interpreted as the least closed upset of X' containing $\varphi[C]$. Also, interpreting I as $\{V \in \mathsf{ClopUp}(X) : V \subseteq O\}$ for some $O \in \mathsf{OpenUp}(X)$, $\alpha^{-1}[I]$ is interpreted as the greatest open upset of X' contained in $\varphi[O]$. Thus, we may view the Dunn–Hartonas dual of α as the DH-morphism

$$(f, g)\colon (\mathsf{ClosedUp}(X), R, \mathsf{OpenUp}(X)) \to (\mathsf{ClosedUp}(X'), R', \mathsf{OpenUp}(X')),$$

where f and g are given by $f(C) = {\uparrow}\varphi[C]$ for each $C \in \mathsf{ClosedUp}(X)$ and $g(O) = -{\downarrow}\varphi[-O]$ for each $O \in \mathsf{OpenUp}(X)$.

The GvG-morphism corresponding to α is

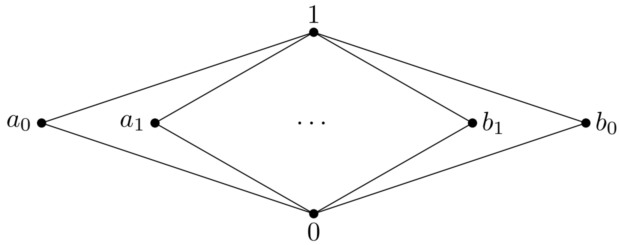
$$(S, T)\colon (\mathsf{Filt}(A)_p, R_p, \mathsf{Idl}(A)_p) \to (\mathsf{Filt}(A')_p, R'_p, \mathsf{Idl}(A')_p).$$

With the identifications above, we may view these as the relation $S \subseteq X \times X'$ given by

$$x\,S\,x' \iff \varphi(x) \subseteq x'.$$

This relation, in turn, may be identified with φ. Thus, the Gehrke–van Gool dual of α may be thought of as its Priestley dual φ. The same applies to the Hartung, Urquhart, and Ploščica duals of α.

15.2. Illustrating the functors. We conclude by illustrating the action of the functors described in this paper on the various duals of the lattice A consisting of a countable antichain A_0 together with $0, 1$ shown in the following picture:



This lattice is sometimes denoted by M_∞ (where M stands for modular and ∞ for the infinite antichain in the middle), and was considered by Gehrke and van Gool [GvG14, Ex. 4.1] in contrasting their duality with Hartung duality. Letting $X = \mathsf{Filt}(A)$, we see that $X = \{{\uparrow}c : c \in A\}$ is dually isomorphic to A, and $Y = \mathsf{Idl}(A)$ is isomorphic to A.

Jipsen–Moshier, BDGM, and Hartonas duals. The Jipsen–Moshier dual of A is X (see Figure 3) with the Scott topology. Because $K(X) = X$, the Scott topology is simply the topology of upsets of X. The BDGM-dual of A is X with the Lawson topology. Since the lower topology has subbasis $\{-{\uparrow}x : x \in X\}$, it follows that the lower topology consists of all cofinite subsets of X containing ${\uparrow}1$. Consequently, each point in $\{{\uparrow}c : c \in A_0\}$ is isolated, as is ${\uparrow}0$. Therefore, the Lawson topology is the one-point compactification of the discrete space $X - \{{\uparrow}1\}$. The Hartonas dual of A is the same as the BDGM-dual.

Celani–González dual. We know that $X_m = \{{\uparrow}c : c \in A_0\}$ is the set of meet-irreducible elements of X. The topology on X_m is the restriction of the lower topology on X, and so is the cofinite topology on X_m. Because each element of X is compact, the subbasis $\mathcal{K}$ of the topology on X_m is $\{X_m - {\uparrow}x : x \in X\}$.

Dunn–Hartonas dual. The Dunn–Hartonas dual of A is (X, R, Y). By identifying Y with A via $d \mapsto {\downarrow}d$, from the previous case we see that the Lawson topology on Y is the one-point compactification of the discrete space $Y - \{0\}$ and R is given by

$$ {\uparrow}c\, R\, d \iff {\uparrow}c \cap {\downarrow}d = \varnothing \iff c \not\leq d. $$

Gehrke–van Gool dual. It is straightforward to see that there are no finite distributive meets of incomparable elements in X. Consequently, $X_p = X - \{{\uparrow}0\}$. Similarly, $Y_p = Y - \{1\}$. Because the topologies on X_p and Y_p are the subspace topologies of the Lawson topologies, the topology on X_p is the one-point compactification of $X_p - \{{\uparrow}1\}$, and similarly for Y_p. Moreover, R_p is the restriction of R to $X_p \times Y_p$.

Hartung dual. It is easy to see that ${\uparrow}1$ is not d-maximal for any $d \in A$. Moreover, ${\uparrow}a_n$ is b_0-maximal and ${\uparrow}b_n$ is a_0-maximal for each n. Therefore, $X_0 = X_p - \{{\uparrow}1\} = X - \{{\uparrow}0, {\uparrow}1\}$. Similarly, $Y_0 = Y - \{0, 1\}$. The topologies on X_0 and Y_0 are the restrictions of the open downset topologies, which are exactly the cofinite topologies, and the relation R_0 is the restriction of R_p. Thus, R_0 is given by $\uparrow c\, R_0\, d$ iff $c \neq d$. The following picture describes X, X_p, and X_0:

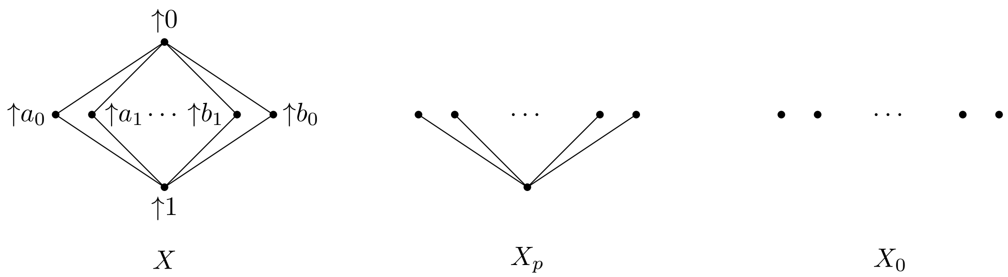


Fig. 3. X, X_p, and X_0

Observe that X_0 is exactly the Celani–González space X_m. More generally, if X is a coherent lattice, then $X_0 \subseteq X_m \subseteq X_p$. To see the second inclusion, let $x \in X_m$ and let $k_1 \wedge \cdots \wedge k_n$ be a distributive meet in $K(X)$ with $k_1 \wedge \cdots \wedge k_n \leq x$. Then

$$x = x \vee (k_1 \wedge \cdots \wedge k_n) = (x \vee k_1) \wedge \cdots \wedge (x \vee k_n),$$

where the second equality can be proved much as in the proof of Lemma 8.19. Since $x \in X_m$, we have $x = x \vee k_i$ for some i, and so $k_i \leq x$. Thus, $x \in X_p$.

To see the first inclusion, let $Y = \mathsf{OF}(X)$ and define

$$x\, R\, U \iff x \notin U.$$

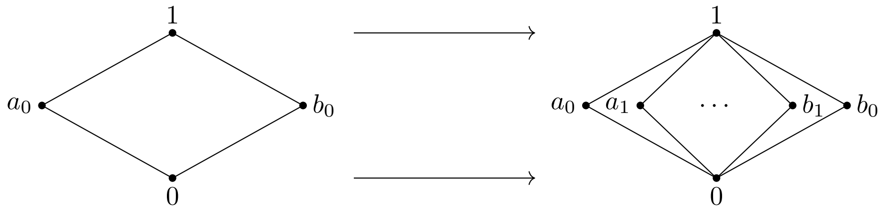
Then $(X, R, Y) \in \mathsf{DH}$ by Proposition 7.15. Now, let $x \in X_0$. Then $x \in \max R^{-1}[U]$ for some $U \in Y$. If $x \notin X_m$, then $x = x_1 \wedge x_2$ for some $x_1, x_2 \in X$ with $x < x_1, x_2$. Since $x \in \max R^{-1}[U]$, we see that $x_i \not\!R\, U$, so $x_1, x_2 \in U$, and hence $x = x_1 \wedge x_2 \in U$. Thus, $x \not\!R\, U$. The resulting contradiction proves that $x \in X_m$.

Our example shows that the inclusion of X_m into X_p may in general be proper. We currently do not have an example showing that the inclusion X_0 into X_m may also be proper.

Urquhart dual. If $c, d \in A_0$ with $c \neq d$, then $\uparrow c$ is d-maximal and vice versa. From this it follows that $Z = \{(\uparrow c, d) : c, d \in A_0,\ c \neq d\}$. The topology on Z is the subspace topology induced from the inclusion $Z \subseteq X_0 \times Y_0$, where $X_0 \times Y_0$ is given the product topology (of two cofinite topologies). The quasi-order $\leq_1$ is given by $(\uparrow c, d) \leq_1 (\uparrow c', d')$ iff $\uparrow c \subseteq \uparrow c'$, which happens iff $c = c'$ since $c, c' \in A_0$. Similarly, $(\uparrow c, d) \leq_2 (\uparrow c', d')$ iff $d = d'$.

Ploščica dual. The Ploščica relation R on Z is given by $(\uparrow c, d)\, R\, (\uparrow c', d')$ iff there is $(\uparrow c'', d'') \in Z$ with $(\uparrow c, d) \leq_1 (\uparrow c'', d'')$ and $(\uparrow c', d') \leq_2 (\uparrow c'', d'')$. From the description of $\leq_1, \leq_2$ above, we see that R is the equality relation on Z.

Finally, we illustrate how the various functors act on morphisms by considering the inclusion $\alpha \colon A' \to A$, where A' is the bounded sublattice $\{0, a_0, b_0, 1\}$ of A.



Jipsen–Moshier, BDGM, and Hartonas duals. Recalling that $X = \mathsf{Filt}(A)$ and $X' = \mathsf{Filt}(A')$, in all three of these cases $\alpha\colon A' \to A$ induces the morphism $f\colon X \to X'$ given by $f(F) = \alpha^{-1}(F)$ for each $F \in X$.

Dunn–Hartonas dual. Let $\mathcal{D} = (X, R, Y)$ and $\mathcal{D}' = (X', R', Y')$ be the Dunn–Hartonas duals of A and A', where using the above identifications, $Y = A$ and $Y' = A'$. Then the Dunn–Hartonas dual of α is the DH-morphism $(f, g)\colon \mathcal{D} \to \mathcal{D}'$, where $f(F) = \alpha^{-1}(F)$ for each $F \in X$ and g is given by

$$g(d) = \begin{cases} d & \text{if } d \in A', \\ 0 & \text{otherwise.} \end{cases}$$

Gehrke–van Gool dual. Let $\mathcal{G}$ and $\mathcal{G}'$ be the Gehrke–van Gool duals of A and A'. The description of $\mathcal{G}$ is given above, while $\mathcal{G}' = (X'_p, R'_p, Y'_p)$, where $X'_p = \{\uparrow a_0, \uparrow b_0\}$ and $Y'_p = \{a_0, b_0\}$. The GvG-morphism (S, T) corresponding to α is given by

$$\uparrow c\, S\, \uparrow c' \text{ if } c \in \{a_0, b_0\} \Longrightarrow c' \le c \quad \text{and} \quad d\, T\, d' \text{ if } d \in \{a_0, b_0\} \Longrightarrow d \le d'.$$

Celani–González dual. The Celani–González duals of A and A' are X_m and X'_m, where X_m is described above and $X'_m = X'_p$. The CG-morphism corresponding to α is then the restriction of S to $X_m \times X'_m$.

Hartung dual. Let $\mathcal{H}$ and $\mathcal{H}'$ be the Hartung duals of A and A'. The description of $\mathcal{H}$ is given above and $\mathcal{H}' = \mathcal{G}'$. The Hartung morphism (S_0, T_0) corresponding to α is given by $S_0 = S \cap (X_0 \times X'_0)$ and $T_0 = T \cap (Y_0 \times Y'_0)$.

Urquhart and Ploščica duals. Let $\mathcal{U}$ and $\mathcal{U}'$ be the Urquhart duals of A and A'. The description of $\mathcal{U}$ is given above, while $\mathcal{U}' = (Z', \le'_1, \le'_2)$, where $Z' = \{(\uparrow a_0, b_0), (\uparrow b_0, a_0)\}$. The two quasi-orders $\le'_1, \le'_2$ are both the equality relation. The Urquhart morphism (P, Q) corresponding to α is given by

$$\begin{aligned} (\uparrow c, d)\, P\, (\uparrow c', d') &\iff c \in \{a_0, b_0\} \text{ implies } c' \le c; \\ (\uparrow c, d)\, Q\, (\uparrow c', d') &\iff d \in \{a_0, b_0\} \text{ implies } d \le d'. \end{aligned}$$

Finally, let $\mathcal{P}$ and $\mathcal{P}'$ be the Ploščica duals of A and A'. Since the Ploščica dual of α is (P, Q), the Plo-morphism is the same as the Urquhart morphism described above.

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