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HOPF ALGEBRA STRUCTURES ON FREE UNITARY EXTENDED ROTA–BAXTER ALGEBRAS

BY

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Abstract. As a common generalization of operator forms of the classical Yang–Baxter equation and the modified classical Yang–Baxter equation, an extended Rota–Baxter operator has been found in various algebraic structures, such as Laurent series, owing to their importance in the Hopf algebraic approach of Connes and Kreimer to renormalization in quantum field theory. We present an explicit construction of free objects in the category of extended Rota–Baxter algebras via bracketed words. We then equip this free object with a bialgebra structure via its universal property, and further show that the free object possesses a Hopf algebra structure.

1. Introduction. As a remarkable coincidence, Rota–Baxter algebras and Hopf algebras simultaneously appear in the algebraic approach of Connes and Kreimer to renormalization in perturbative quantum field theory [12, 13]. Let λ be a scalar. A *Rota–Baxter algebra* of weight λ is an associative algebra A together with a linear operator P satisfying the *Rota–Baxter equation*

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \qquad x, y \in A. \tag{1.1}$$

In this case, P is called a *Rota–Baxter operator* of weight λ. The notion of a Rota–Baxter algebra was introduced in the work of G. Baxter in the fluctuation theory in probability [9, 39]. Thanks to the pioneering work of Atkinson, Cartier, Joni and Rota, Rota–Baxter algebras have found applications in a broad range of areas, such as bialgebras [3, 4], Hopf algebras [1], operads [2], combinatorics [11, 37, 40], quantum field theory [13], number theory [22], Yang–Baxter equations [5, 6, 7, 10] and group theory [8, 26]. See [24, 38] for more details and further references.

As an associative analogue of the modified classical Yang–Baxter equation, the *modified Rota–Baxter equation* is defined to be

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \kappa xy, \qquad x, y \in A. \tag{1.2}$$

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Here κ is a scalar. Then P is called a *modified Rota–Baxter operator* of weight κ, and (A, P) is called a *modified Rota–Baxter algebra* of weight κ. Under a suitable transformation, a modified Rota–Baxter operator is just a Rota–Baxter operator of a certain weight.

In order to put Rota–Baxter operators and modified Rota–Baxter operators in the same framework, the notion of an extended Rota–Baxter operator was introduced in [48], which can also be traced back to extended $\mathcal{O}$-operators [7]. More precisely, an *extended Rota–Baxter operator P* of weight (λ, κ) satisfies the *extended Rota–Baxter equation*

$$P(x)P(y) = P\big(xP(y)\big) + P\big(P(x)y\big) + \lambda P(xy) + \kappa xy, \qquad x, y \in R. \tag{1.3}$$

Then (A, P) is called an *extended Rota–Baxter algebra* of weight (λ, κ). When $\kappa = 0$, we recover Rota–Baxter algebras of weight λ. When $\lambda = 0$, we also recover modified Rota–Baxter algebras of weight κ.

Many classical examples of Hopf algebras are established on free objects in various categories. For example, the Connes–Kreimer Hopf algebra of rooted trees is built on free commutative associative algebras [12]. Similarly, the Loday–Ronco Hopf algebra of planar binary rooted trees, which is isomorphic to the noncommutative version of the Connes–Kreimer Hopf algebra (also known as the Foissy–Holtkamp Hopf algebra) [17, 18, 28, 30, 34], is based on free associative algebras [32]. Some other examples are obtained from free (tri)dendriform algebras [33]. More generally, in the operadic context, Holtkamp [29] constructed Hopf algebra structures on free objects over operads $\mathcal{P}$ equipped with coherent unit actions (see also [27, 31]), and proved that the free $\mathcal{P}$-algebra on a set X is a connected graded $\mathcal{P}$-Hopf algebra.

In particular, explicit constructions of free Rota–Baxter algebras on a set have been realized via different carriers, such as bracketed words, Motzkin paths and decorated rooted trees [16, 23]. More significantly, in the last couple of years, Hopf algebra structures on free noncommutative Rota–Baxter algebras have been obtained by means of bracketed words [20], leaf decorated forests [41] and angularly decorated rooted trees [44].

For the commutative case, the free object in the category of commutative Rota–Baxter algebras was first constructed by Rota [38]. It has been proven that its construction is closely related to Spitzer's identity and Waring's formula for symmetric functions. Moreover, it is worth mentioning that free commutative Rota–Baxter algebras with arbitrary weight are constructed by mixable shuffles [25], and have also been equipped with a Hopf algebra structure [1, 14].

In recent years, Zhang, Gao and Guo [42, 43] gave explicit constructions of free commutative modified Rota–Baxter algebras by a variation of the shuffle product, as well as in the noncommutative case by bracketed words. Meanwhile, a Hopf algebra structure on the free (noncommutative)

commutative modified Rota–Baxter algebra was established by applying a Hochschild 1-cocycle condition.

What is more, through different variations of Hochschild 1-cocycle conditions, several Hopf-like algebra structures are also provided on free (noncommutative) commutative Nijenhuis algebras [21, 47] and free Rota–Baxter systems [35, 36], as well as a cocycle Hopf algebra on free matching Rota–Baxter algebras [45].

In [48], a free commutative extended Rota–Baxter algebra on a commutative algebra is constructed by a generalization of the quasi-shuffle product. On the other hand, in order to explore the derived structures from extended Rota–Baxter algebras, an explicit construction of free nonunitary extended Rota–Baxter algebras on a nonunitary algebra was obtained by bracketed words [46].

In view of the great importance of free (modified) Rota–Baxter algebras, this paper first aims to construct free unitary extended Rota–Baxter algebras by combinatorical methods such as bracketed words. On the basis of the fact that Rota–Baxter algebras have played a crucial role in Connes–Kreimer's Hopf algebraic approach to renormalization, it is desirable to investigate a Hopf algebra structure on free unitary extended Rota–Baxter algebras. This is the second purpose of this paper.

Here is the layout of the paper. Section 2 gives an explicit construction of free unitary extended Rota–Baxter algebras on a unitary algebra by Rota–Baxter bracketed words (Theorem 2.5). In Section 3, the tensor product of free unitary extended Rota–Baxter algebras is first equipped with an extended Rota–Baxter operator of weight (λ, κ) (Lemma 3.2). Then we provide a comultiplication on free extended Rota–Baxter algebras via its universal property. This comultiplication also satisfies a weighted 1-cocycle condition, and then makes the free extended Rota–Baxter algebra into a bialgebra. We finally prove that this bialgebra is a connected filtered bialgebra (Theorem 3.5), and hence a Hopf algebra (Theorem 3.9).

Convention. Throughout this paper, unless otherwise specified, $\mathbf{k}$ is a unitary commutative ring, which will be the base ring of all modules, algebras, coalgebras, bialgebras, as well as linear maps. All algebras are assumed to be unitary $\mathbf{k}$-algebras.

2. Free unitary extended Rota–Baxter algebras. In this section, we first recall the notion of a Rota–Baxter bracketed word, and then give an explicit construction of free unitary extended Rota–Baxter algebras on a unitary algebra by Rota–Baxter bracketed words.

2.1. Rota–Baxter bracketed words. Let A be a unitary algebra. We always assume that A has a basis $X \cup \{\mathbf{1}\}$, where $\mathbf{1}$ is the identity of A. Set

$X' := X \cup \{\lfloor, \rfloor\}$, where $\lfloor$ and $\rfloor$ are symbols and are not in X. For any set Y, let $S(Y)$ be the free semigroup on Y.

DEFINITION 2.1. Let Y, Z be two subsets of $S(X')$. Define the *alternating product* of Y and Z to be

$$\begin{aligned}\Lambda(Y, Z) = \Big(\bigcup_{r\geq 1}(Y\lfloor Z\rfloor)^r\Big) \cup \Big(\bigcup_{r\geq 0}(Y\lfloor Z\rfloor)^r Y\Big)\\ \cup \Big(\bigcup_{r\geq 1}(\lfloor Z\rfloor Y)^r\Big) \cup \Big(\bigcup_{r\geq 0}(\lfloor Z\rfloor Y)^r \lfloor Z\rfloor\Big).\end{aligned}$$

We next construct a sequence $\mathfrak{X}_n$ by recursion on $n \geq 0$. When $n = 0$, we define $\mathfrak{X}_0 := X \cup \{\mathbf{1}\}$. Assume that $\mathfrak{X}_n$ has been defined for $n \geq 0$. Then define

$$\mathfrak{X}_{n+1} := \Lambda(X, \mathfrak{X}_n) \cup \{\mathbf{1}\}.$$

For example, we have

$$\begin{aligned}\mathfrak{X}_1 = \Big(\bigcup_{r\geq 1}(X\lfloor \mathfrak{X}_0\rfloor)^r\Big) \cup \Big(\bigcup_{r\geq 0}(X\lfloor \mathfrak{X}_0\rfloor)^r X\Big) \cup \Big(\bigcup_{r\geq 1}(\lfloor \mathfrak{X}_0\rfloor X)^r\Big)\\ \cup \Big(\bigcup_{r\geq 0}(\lfloor \mathfrak{X}_0\rfloor X)^r \lfloor \mathfrak{X}_0\rfloor\Big) \cup \{\mathbf{1}\}.\end{aligned}$$

Then $X \subseteq \bigcup_{r\geq 0}(X\lfloor \mathfrak{X}_0\rfloor)^r X$, and so $\mathfrak{X}_0 \subseteq \mathfrak{X}_1$. Assume $\mathfrak{X}_{n-1} \subseteq \mathfrak{X}_n$ for $n \geq 1$. Then $\lfloor \mathfrak{X}_{n-1}\rfloor \subseteq \lfloor \mathfrak{X}_n\rfloor$. By the definition of the alternating product, we have

$$\mathfrak{X}_n = \Lambda(X, \mathfrak{X}_{n-1}) \subseteq \Lambda(X, \mathfrak{X}_n) = \mathfrak{X}_{n+1}.$$

Let

$$\mathfrak{X}_\infty := \varinjlim \mathfrak{X}_n = \bigcup_{n\geq 0}\mathfrak{X}_n.$$

An element of $\mathfrak{X}_\infty$ is what we call a *Rota–Baxter* (*bracketed*) *word* (*RBW*) [15, 24].

By [15, Lemma 2.3], every RBW $w \neq \mathbf{1}$ admits a unique *standard decomposition*

$$w = w_1 \cdots w_m, \tag{2.1}$$

in which each factor w_i alternately belongs to X or $\lfloor \mathfrak{X}_\infty\rfloor$ for $1 \leq i \leq m$ with $m \geq 1$. When $w_1 \in X$ (resp. $w_1 \in \lfloor \mathfrak{X}_\infty\rfloor$), we define the *head* $h(w)$ of w to be 0 (resp. 1). Analogously, if $w_m \in X$ (resp. $w_m \in \lfloor \mathfrak{X}_\infty\rfloor$), the *tail* $t(w)$ of w is defined to be 0 (resp. 1). The positive integer m is called the *breadth* of w, denoted by $\mathrm{bre}(w)$. By convention, we set $\mathrm{bre}(\mathbf{1}) = 0$. We further define the *depth* of w by

$$\mathrm{dep}(w) := \min\{n \mid w \in \mathfrak{X}_n\}.$$

2.2. A construction of free unitary extended Rota–Baxter algebras. Basic concepts for algebras, such as homomorphisms, subalgebras and ideals, can be similarly defined for extended Rota–Baxter algebras. Thus the class of extended Rota–Baxter algebras of weight (λ, κ) forms a category together with extended Rota–Baxter algebra homomorphisms. We now present the definition of free extended Rota–Baxter algebras of weight (λ, κ) on a unitary algebra A.

DEFINITION 2.2. Let A be a unitary algebra. A *free extended Rota–Baxter algebra* of weight (λ, κ) on A is an extended Rota–Baxter algebra $(F(A), P_A)$ together with an algebra homomorphism $j_A : A \to F(A)$ such that, for any extended Rota–Baxter algebra (R, P) of weight (λ, κ) and any algebra homomorphism $f : A \to R$, there exists a unique extended Rota–Baxter algebra homomorphism $\bar{f} : F(A) \to R$ such that $f = \bar{f} \circ j_A$, that is, the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "j_A"] \arrow[dr, "f"'] & F(A) \arrow[d, "\bar{f}"] \\ & R \end{tikzcd}$$

Denote by

$$Ш_e^{NC}(A) := \mathbf{k}\mathfrak{X}_\infty$$

the free module with basis $\mathfrak{X}_\infty$. We now define a product $\diamond_e$ on $Ш_e^{NC}(A)$ by defining $w \diamond_e w' \in Ш_e^{NC}(A)$ for $w, w' \in \mathfrak{X}_\infty$. We first define $\mathbf{1}$ to be the identity of $\diamond_e$. Now suppose that $w, w' \neq \mathbf{1}$. We use induction on $n := \mathrm{dep}(w) + \mathrm{dep}(w') \geq 0$. For the base case $n = 0$, we have $\mathrm{dep}(w) = \mathrm{dep}(w') = 0$, and so w, w' are in X. Then we define

$$w \diamond_e w' := w \cdot w'. \tag{2.2}$$

Here $\cdot$ is the product of A. When there is no danger of confusion, we may omit the symbol "$\cdot$". For the inductive step, assume that for a given $k \geq 0$, $w \diamond_e w'$ has been defined for all $w, w' \in \mathfrak{X}_\infty$ with $0 \leq n \leq k$. Consider the case of $n = k + 1$. First, we take into account the case when $\mathrm{bre}(w) = 1$ and $\mathrm{bre}(w') = 1$, that is, w and w' are in X or in $\lfloor \mathfrak{X}_\infty \rfloor$. Since $n = k + 1 \geq 1$, w and w' cannot be both in X. Then define

$$w \diamond_e w' = \begin{cases} \lfloor \bar{w} \diamond_e \lfloor \bar{w}' \rfloor \rfloor + \lfloor \lfloor \bar{w} \rfloor \diamond_e \bar{w}' \rfloor & \\ \qquad + \lambda \lfloor \bar{w} \diamond_e \bar{w}' \rfloor + \kappa \bar{w} \diamond_e \bar{w}' & \text{if } w = \lfloor \bar{w} \rfloor \text{ and } w' = \lfloor \bar{w}' \rfloor, \\ ww' & \text{if } w \in X \text{ and } w' \in \lfloor \mathfrak{X}_\infty \rfloor, \\ ww' & \text{if } w \in \lfloor \mathfrak{X}_\infty \rfloor \text{ and } w' \in X. \end{cases} \tag{2.3}$$

Here the product in the first case is well-defined by the induction hypothesis, since

$$\operatorname{dep}(\bar{w}) + \operatorname{dep}(\lfloor \bar{w}' \rfloor) = \operatorname{dep}(\lfloor \bar{w} \rfloor) + \operatorname{dep}(\bar{w}') = k, \quad \operatorname{dep}(\bar{w}) + \operatorname{dep}(\bar{w}') = k - 1,$$

both $\leq k$. The product in the second and third cases is concatenation.

Secondly, we consider the case when $\operatorname{bre}(w) > 1$ or $\operatorname{bre}(w') > 1$. By (2.1), we can write $w = w_1 \cdots w_m$ and $w' = w'_1 \cdots w'_{m'}$. Define

$$w \diamond_e w' = w_1 \cdots w_{m-1}(w_m \diamond_e w'_1) w'_2 \cdots w'_{m'}, \tag{2.4}$$

where $w_m \diamond_e w'_1$ is defined by (2.3) and the other products are given by concatenation. Extending $\diamond_e$ bilinearly gives a binary operation

$$\diamond_e : Ш_e^{\mathrm{NC}}(A) \otimes Ш_e^{\mathrm{NC}}(A) \to Ш_e^{\mathrm{NC}}(A).$$

We give some properties of $\diamond_e$ for later applications.

Lemma 2.3. *Let $w, w' \in \mathfrak{X}_\infty \setminus \{\mathbf{1}\}$. If $t(w) \neq h(w')$, then for every $w'' \in \mathfrak{X}_\infty$,*

$$(ww') \diamond_e w'' = w(w' \diamond_e w''), \quad w'' \diamond_e (ww') = (w'' \diamond_e w)w'. \tag{2.5}$$

Proof. This follows directly from the definition of $\diamond_e$ in (2.4). ∎

Lemma 2.4. *Let $u, u', u'' \in \mathfrak{X}_\infty$. If the associativity*

$$(u \diamond_e u') \diamond_e u'' = u \diamond_e (u' \diamond_e u'')$$

holds when $\operatorname{bre}(u) = \operatorname{bre}(u') = \operatorname{bre}(u'') = 1$, *then it holds for all RBWs.*

Proof. If any of $\operatorname{bre}(w)$, $\operatorname{bre}(w')$, or $\operatorname{bre}(w'')$ is zero, then the corresponding element is the identity $\mathbf{1}$, and the claim follows from the definition of $\diamond_e$. Otherwise, we have $\operatorname{bre}(w)$,$\operatorname{bre}(w')$,$\operatorname{bre}(w'') \geq 1$, and we proceed by induction on $m := \operatorname{bre}(w) + \operatorname{bre}(w') + \operatorname{bre}(w'')$, where $m \geq 3$. For the detailed inductive steps, we refer to [46, Lemma 3.8], which is analogous to [15, Lemma 2.8]. ∎

Finally, for all $w \in \mathfrak{X}_\infty$, we have $\lfloor w \rfloor \in \mathfrak{X}_\infty$. We then define a linear operator

$$P_e : Ш_e^{\mathrm{NC}}(A) \to Ш_e^{\mathrm{NC}}(A), \quad w \mapsto \lfloor w \rfloor.$$

For the rest of this paper, unless alternative notations are specifically given, we will use the notation $\lfloor\ \rfloor$ interchangeably with P_e. Let

$$j_X : X \cup \{\mathbf{1}\} \hookrightarrow \mathfrak{X}_\infty$$

be the natural injection which extends to an algebra homomorphism

$$j_A : A \to Ш_e^{\mathrm{NC}}(A).$$

With these preparations in place, we may now state the main result of this section.

THEOREM 2.5. *Let A be a unitary algebra with a basis $X \cup \{\mathbf{1}\}$.*

(1) *The triple $(Ш_e^{NC}(A), \diamond_e, \mathbf{1})$ is a unitary algebra.*
(2) *The triple $(Ш_e^{NC}(A), \diamond_e, P_e)$ is a unitary extended Rota–Baxter algebra of weight (λ, κ).*
(3) *The quintuple $Ш_e^{NC}(A) := (Ш_e^{NC}(A), \diamond_e, \mathbf{1}, P_e, j_A)$ is a free unitary extended Rota–Baxter algebra of weight (λ, κ) on A.*

Proof. (1) By the definition of $\diamond_e$, $\mathbf{1}$ is the identity of $Ш_e^{NC}(A)$. To prove that $(Ш_e^{NC}(A), \diamond_e)$ is an algebra, it suffices to verify the associativity for basis elements $w, w', w'' \in \mathfrak{X}_\infty$, that is,

$$(w \diamond_e w') \diamond_e w'' = w \diamond_e (w' \diamond_e w''). \tag{2.6}$$

We use induction on $n := \mathrm{dep}(w) + \mathrm{dep}(w') + \mathrm{dep}(w'') \geq 0$. If $n = 0$, then $\mathrm{dep}(w) = \mathrm{dep}(w') = \mathrm{dep}(w'') = 0$, and so $w, w', w'' \in \mathfrak{X}_0$. By (2.2), we have

$$(w \diamond_e w') \diamond_e w'' = ww'w' = w \diamond_e (w' \diamond_e w'').$$

For a given $k \geq 0$, assume that (2.6) holds for $0 \leq n \leq k$, and consider the case of $n = k + 1 \geq 1$. If $t(w) \neq h(w')$, then by (2.5), we have

$$(w \diamond_e w') \diamond_e w'' = (ww') \diamond_e w'' = w(w' \diamond_e w'') = w \diamond_e (w' \diamond_e w'').$$

Similarly, if $t(w') \neq h(w'')$, then by (2.5) again,

$$(w \diamond_e w') \diamond_e w'' = (w \diamond_e w')w'' = w \diamond_e (w'w'') = w \diamond_e (w' \diamond_e w'').$$

It remains to treat the case where $t(w) = h(w')$ and $t(w') = h(w'')$. By Lemma 2.4, the verification of (2.6) reduces to the case $\mathrm{bre}(w) = \mathrm{bre}(w') = \mathrm{bre}(w'') = 1$. In this reduction, each of w, w', w'' lies in $X \cup \lfloor \mathfrak{X}_\infty \rfloor$. If any one of them belongs to X, the head-tail equalities force all three to be in X, and associativity follows directly from that of the algebra A. Otherwise, all three lie in $\lfloor \mathfrak{X}_\infty \rfloor$; we may write $w = \lfloor \bar{w} \rfloor$, $w' = \lfloor \bar{w}' \rfloor$, and $w'' = \lfloor \bar{w}'' \rfloor$ with $\bar{w}, \bar{w}', \bar{w}'' \in \mathfrak{X}_\infty$. For this final case, the remaining verification is detailed in [46, Theorem 3.5(a)] and is analogous to the proof of [15, Theorem 2.6(a)].

(2) By the definition of P_e and (2.3), we obtain

$$P_e(w) \diamond_e P_e(w') = P_e\big(P_e(w) \diamond_e w'\big) + P_e\big(w \diamond_e P_e(w')\big) + \lambda P_e(w \diamond_e w') + \kappa w \diamond_e w'.$$

Thus, P_e is an extended Rota–Baxter operator of weight (λ, κ) on $Ш_e^{NC}(A)$.

(3) Let $(R, *, P, 1_R)$ be a unitary extended Rota–Baxter algebra of weight (λ, κ) and let $f : A \to R$ be a $\mathbf{k}$-algebra homomorphism. We will extend f to an extended Rota–Baxter algebra homomorphism $\bar{f} : Ш_e^{NC}(A) \to R$ by induction on $\mathrm{dep}(w) \geq 0$ for $w \in \mathfrak{X}_\infty$. If $\mathrm{dep}(w) = 0$, then $w \in \mathfrak{X}_0 = X \cup \{\mathbf{1}\}$. Define

$$\bar{f}(w) = f(w), \quad \bar{f}(\mathbf{1}) = 1_R. \tag{2.7}$$

For a given $k \geq 0$, assume that $\bar{f}$ has been defined for all $w \in \mathfrak{X}_\infty$ with $\mathrm{dep}(w) \leq k$, and consider the case of $\mathrm{dep}(w) = k + 1 \geq 1$ for $w \in \mathfrak{X}_\infty$. Since

$\mathrm{dep}(w) = k+1$, we have $w \in \mathfrak{X}_{k+1} \setminus \mathfrak{X}_k$. By the definition of $\mathfrak{X}_{k+1}$, we have

$$\begin{aligned}\mathfrak{X}_{k+1} = \Big(\bigcup_{r\geq 1} (X\lfloor \mathfrak{X}_k \rfloor)^r\Big) \cup \Big(\bigcup_{r\geq 0} (X\lfloor \mathfrak{X}_k \rfloor)^r X\Big) \\ \cup \Big(\bigcup_{r\geq 1} (\lfloor \mathfrak{X}_k \rfloor X)^r\Big) \cup \Big(\bigcup_{r\geq 0} (\lfloor \mathfrak{X}_k \rfloor X)^r \lfloor \mathfrak{X}_k \rfloor\Big) \cup \{\mathbf{1}\}.\end{aligned}$$

Assume w is in the first union component $\bigcup_{r\geq 1}(X\lfloor \mathfrak{X}_n \rfloor)^r$ above. Then we can write

$$w = \prod_{i=1}^{r} (w_{2i-1}\lfloor w_{2i} \rfloor),$$

where $w_{2i-1} \in X$ and $w_{2i} \in \mathfrak{X}_k, 1 \leq i \leq r$. By the definitions of $\diamond_e$ and P_e, we get

$$w = \mathop{\diamond_e}_{i=1}^{r} (w_{2i-1} \diamond_e P_e(w_{2i})).$$

Define

$$\bar{f}(w) = \mathop{*}_{i=1}^{r} (\bar{f}(w_{2i-1}) * P(\bar{f}(w_{2i}))), \tag{2.8}$$

where $\bar{f}(w_{2i-1})$ and $\bar{f}(w_{2i})$ are well-defined by the induction hypothesis, since $w_{2i-1} \in X$ and $\mathrm{dep}(w_{2i}) \leq k$. Similarly, we can define $\bar{f}(w)$ if w is in the other components. This completes the induction and so the definition of $\bar{f}$. Extending by $\mathbf{k}$-linearity, we obtain a linear map $\bar{f} : Ш_e^{NC}(A) \to R$, which satisfies $\bar{f} \circ j_A = f$ by (2.7).

Note that for all $w \in \mathfrak{X}_\infty$, $P_e(w) = \lfloor w \rfloor \in \mathfrak{X}_\infty$. Then by the definition of $\bar{f}$, we have

$$\bar{f}(P_e(w)) = P(\bar{f}(w)). \tag{2.9}$$

Further, $\bar{f}$ is an algebra homomorphism, as follows immediately from the proof of [46, Theorem 3.5(c)].

Finally, we prove the uniqueness of $\bar{f}$. Suppose $\tilde{f}$ is another extended Rota–Baxter algebra homomorphism satisfying $\tilde{f} \circ j_A = f$. Then

$$\bar{f}(w) = \tilde{f}(w), \qquad w \in \mathfrak{X}_\infty,$$

which follows by induction on the depth $n := \mathrm{dep}(w) \geq 0$. ∎

Remark 2.6. As mentioned above, the free unitary extended Rota–Baxter algebra is constructed on the unitary algebra $A = \mathbf{k}(X \cup \{\mathbf{1}\})$, where $\mathbf{1}$ serves as the identity element of the algebra $Ш_e^{NC}(A)$. In contrast, for the construction of free nonunitary extended Rota–Baxter algebras on an algebra B, we do not require B to possess an identity element (see [46, Theorem 3.5]).

Accordingly, compared with the nonunitary case, the proof for the unitary case $Ш_e^{NC}(A)$ entails two additional verifications: first, that $\mathbf{1}$ acts as the identity of $Ш_e^{NC}(A)$; and second, that associativity holds in all instances involving $\mathbf{1}$. Furthermore, when verifying the universal property for $Ш_e^{\mathrm{N}C}(A)$, the unique extended Rota–Baxter algebra homomorphism is required to preserve the identities. All the other steps of the verification are identical in both cases.

3. Hopf algebra structures on free unitary extended Rota–Baxter algebras. In this section we prove the existence of a bialgebra structure on the free extended Rota–Baxter algebra $Ш_e^{\mathrm{N}C}(A)$ by applying its universal property. Furthermore, a Hopf algebra structure on $Ш_e^{\mathrm{N}C}(A)$ is provided.

3.1. The bialgebra structure by the universal property

LEMMA 3.1. *Let $A := (A, m_A, u_A, \mathit{\Delta}_A, \varepsilon_A)$ be a bialgebra. Let $Ш_e^{\mathrm{N}C}(A)$ be the free extended Rota–Baxter algebra of weight (λ, κ). Let $\mu \in \mathbf{k}$ be a root of $t^2 - \lambda t + \kappa$. Then there exists a unique extended Rota–Baxter algebra homomorphism $\varepsilon_e : Ш_e^{\mathrm{N}C}(A) \to \mathbf{k}$ such that*

$$\varepsilon_e \circ j_A = \varepsilon_A \quad \textit{and} \quad \varepsilon_e \circ P_e = -\mu\, \mathrm{id}_{\mathbf{k}} \circ \varepsilon_e. \tag{3.1}$$

Proof. By a direct computation, $-\mu\, \mathrm{id}_{\mathbf{k}}$ is an extended Rota–Baxter operator of weight (λ, κ) on $\mathbf{k}$. Thus $(\mathbf{k}, -\mu\, \mathrm{id}_{\mathbf{k}})$ forms an extended Rota–Baxter algebra. By the universal property of the free extended Rota–Baxter algebra $Ш_e^{\mathrm{N}C}(A)$, there exists a unique extended Rota–Baxter algebra homomorphism $\varepsilon_e : Ш_e^{\mathrm{N}C}(A) \to \mathbf{k}$ satisfying (3.1). ∎

In the following, for a given weight (λ, κ), assume that $\mu \in \mathbf{k}$ is a root of $t^2 - \lambda t + \kappa$. We now construct an extended Rota–Baxter operator on $Ш_e^{\mathrm{N}C}(A) \otimes Ш_e^{\mathrm{N}C}(A)$. For this, the symbol $\blacksquare$ is employed to denote the binary operation in the tensor product algebra $Ш_e^{\mathrm{N}C}(A) \otimes Ш_e^{\mathrm{N}C}(A)$. Moreover, we define a linear map

$$Q : Ш_e^{\mathrm{N}C}(A) \otimes Ш_e^{\mathrm{N}C}(A) \to Ш_e^{\mathrm{N}C}(A) \otimes Ш_e^{\mathrm{N}C}(A)$$

by

$$Q(w \otimes w') := (P_e(w) + \mu w) \otimes \varepsilon_e(w')1 + w \otimes P_e(w'), \quad w, w' \in Ш_e^{\mathrm{N}C}(A). \tag{3.2}$$

LEMMA 3.2. *The linear map Q is an extended Rota–Baxter operator of weight (λ, κ) on $Ш_e^{\mathrm{N}C}(A) \otimes Ш_e^{\mathrm{N}C}(A)$.*

Proof. Let $w_1, w_2, w_1', w_2' \in Ш_e^{\mathrm{N}C}(A)$. By (3.2), we have

$$\begin{aligned}
&Q(w_1 \otimes w_1') \blacksquare Q(w_2 \otimes w_2') \\
&= \big((P_e(w_1) + \mu w_1) \otimes \varepsilon_e(w_1')1 + w_1 \otimes P_e(w_1')\big) \\
&\quad \blacksquare \big((P_e(w_2) + \mu w_2) \otimes \varepsilon_e(w_2')1 + w_2 \otimes P_e(w_2')\big) \\
&= \big((P_e(w_1) + \mu w_1) \diamond_e (P_e(w_2) + \mu w_2)\big) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 \\
&\quad + \big((P_e(w_1) + \mu w_1) \diamond_e w_2\big) \otimes \varepsilon_e(w_1')P_e(w_2') \\
&\quad + \big(w_1 \diamond_e (P_e(w_2) + \mu w_2)\big) \otimes P_e(w_1')\varepsilon_e(w_2') + (w_1 \diamond_e w_2) \otimes (P_e(w_1') \diamond_e P_e(w_2'))
\end{aligned}$$

$$
\begin{aligned}
&= \big(P_e(w_1 \diamond_e P_e(w_2)) + P_e(P_e(w_1) \diamond_e w_2) + \lambda P_e(w_1 \diamond_e w_2) + \kappa w_1 \diamond_e w_2\big) \\
&\quad \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 \\
&\quad + (\mu P_e(w_1) \diamond_e w_2 + \mu w_1 \diamond_e P_e(w_2) + (\lambda\mu - \kappa)(w_1 \diamond_e w_2)) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 \\
&\quad + (P_e(w_1) \diamond_e w_2) \otimes \varepsilon_e(w_1')P_e(w_2') + \mu(w_1 \diamond_e w_2) \otimes \varepsilon_e(w_1')P_e(w_2') \\
&\quad + (w_1 \diamond_e P_e(w_2)) \otimes P_e(w_1')\varepsilon_e(w_2') + \mu(w_1 \diamond_e w_2) \otimes P_e(w_1')\varepsilon_e(w_2') \\
&\quad + (w_1 \diamond_e w_2) \otimes \big(P_e(w_1' \diamond_e P_e(w_2')) + P_e(P_e(w_1') \diamond_e w_2') \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \lambda P_e(w_1' \diamond_e w_2') + \kappa w_1' \diamond_e w_2'\big) \\
&\quad \text{(by replacing } \mu^2 \text{ by } \lambda\mu - \kappa, \text{ since } \mu^2 - \lambda\mu + \kappa = 0) \\
&= P_e(w_1 \diamond_e P_e(w_2)) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 + P_e(P_e(w_1) \diamond_e w_2) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 \\
&\quad + \lambda P_e(w_1 \diamond_e w_2) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 + \mu(P_e(w_1) \diamond_e w_2) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 \\
&\quad + \mu(w_1 \diamond_e P_e(w_2)) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 + \lambda\mu(w_1 \diamond_e w_2) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 \\
&\quad + (P_e(w_1) \diamond_e w_2) \otimes \varepsilon_e(w_1')P_e(w_2') + \mu(w_1 \diamond_e w_2) \otimes \varepsilon_e(w_1')P_e(w_2') \\
&\quad + (w_1 \diamond_e P_e(w_2)) \otimes P_e(w_1')\varepsilon_e(w_2') + \mu(w_1 \diamond_e w_2) \otimes P_e(w_1')\varepsilon_e(w_2') \\
&\quad + (w_1 \diamond_e w_2) \otimes P_e(w_1' \diamond_e P_e(w_2')) + (w_1 \diamond_e w_2) \otimes P_e(P_e(w_1') \diamond_e w_2') \\
&\quad + \lambda(w_1 \diamond_e w_2) \otimes P_e(w_1' \diamond_e w_2') + \kappa(w_1 \diamond_e w_2) \otimes (w_1' \diamond_e w_2') \\
&\quad \text{(by collecting the terms with } \kappa(w_1 \diamond_e w_2) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&Q((w_1 \otimes w_1') \blacksquare Q(w_2 \otimes w_2')) + Q(Q(w_1 \otimes w_1') \blacksquare (w_2 \otimes w_2')) \\
&\quad + \lambda Q((w_1 \otimes w_1') \blacksquare (w_2 \otimes w_2')) + \kappa(w_1 \otimes w_1') \blacksquare (w_2 \otimes w_2') \\
&= Q\Big((w_1 \otimes w_1') \blacksquare \big((P_e(w_2) + \mu w_2) \otimes \varepsilon_e(w_2')1 + w_2 \otimes P_e(w_2')\big)\Big) \\
&\quad + Q\Big(\big((P_e(w_1) + \mu w_1) \otimes \varepsilon_e(w_1')1 + w_1 \otimes P_e(w_1')\big) \blacksquare (w_2 \otimes w_2')\Big) \\
&\quad + \lambda Q((w_1 \diamond_e w_2) \otimes (w_1' \diamond_e w_2')) + \kappa(w_1 \diamond_e w_2) \otimes (w_1' \diamond_e w_2') \\
&= Q\big((w_1 \diamond_e (P_e(w_2) + \mu w_2)) \otimes (w_1' \diamond_e \varepsilon_e(w_2')1) + (w_1 \diamond_e w_2) \otimes (w_1' \diamond_e P_e(w_2'))\big) \\
&\quad + Q\big(((P_e(w_1) + \mu w_1) \diamond_e w_2) \otimes (\varepsilon_e(w_1')1 \diamond_e w_2') + (w_1 \diamond_e w_2) \otimes (P_e(w_1') \diamond_e w_2')\big) \\
&\quad + \lambda Q((w_1 \diamond_e w_2) \otimes (w_1' \diamond_e w_2')) + \kappa(w_1 \diamond_e w_2) \otimes (w_1' \diamond_e w_2') \\
&= Q\big((w_1 \diamond_e P_e(w_2)) \otimes w_1'\varepsilon_e(w_2') + \mu(w_1 \diamond_e w_2) \otimes w_1'\varepsilon_e(w_2') \\
&\quad + (w_1 \diamond_e w_2) \otimes (w_1' \diamond_e P_e(w_2'))\big) + Q\big((P_e(w_1) \diamond_e w_2) \otimes \varepsilon_e(w_1')w_2' \\
&\quad + \mu(w_1 \diamond_e w_2) \otimes \varepsilon_e(w_1')w_2' + (w_1 \diamond_e w_2) \otimes (P_e(w_1') \diamond_e w_2')\big) \\
&\quad + \lambda P_e(w_1 \diamond_e w_2) \otimes \varepsilon_e(w_1' \diamond_e w_2')1 + \lambda\mu(w_1 \diamond_e w_2) \otimes \varepsilon_e(w_1' \diamond_e w_2')1 \\
&\quad + \lambda(w_1 \diamond_e w_2) \otimes P_e(w_1' \diamond_e w_2') + \kappa(w_1 \diamond_e w_2) \otimes (w_1' \diamond_e w_2') \\
&= \big(P_e(w_1 \diamond_e P_e(w_2)) + \mu(w_1 \diamond_e P_e(w_2))\big) \otimes \varepsilon_e(w_1'\varepsilon_e(w_2'))1 \\
&\quad + (w_1 \diamond_e P_e(w_2)) \otimes P_e(w_1'\varepsilon_e(w_2')) + \mu(P_e(w_1 \diamond_e w_2) + \mu(w_1 \diamond_e w_2)) \\
&\quad \otimes \varepsilon_e(w_1'\varepsilon_e(w_2'))1 + \mu(w_1 \diamond_e w_2) \otimes P_e(w_1'\varepsilon_e(w_2'))
\end{aligned}
$$

$$
\begin{aligned}
&+ (P_e(w_1 \diamond_e w_2) + \mu(w_1 \diamond_e w_2)) \otimes \varepsilon_e(w_1' \diamond_e P_e(w_2'))1 \\
&+ (w_1 \diamond_e w_2) \otimes P_e(w_1' \diamond_e P_e(w_2')) \\
&+ (P_e(P_e(w_1) \diamond_e w_2) + \mu(P_e(w_1) \diamond_e w_2)) \otimes \varepsilon_e(\varepsilon_e(w_1')w_2')1 \\
&+ (P_e(w_1) \diamond_e w_2) \otimes P_e(\varepsilon_e(w_1')w_2') + \mu(P_e(w_1 \diamond_e w_2) + \mu(w_1 \diamond_e w_2)) \\
&\qquad\qquad \otimes \varepsilon_e(\varepsilon_e(w_1')w_2')1 + \mu(w_1 \diamond_e w_2) \otimes P_e(\varepsilon_e(w_1')w_2') \\
&+ (P_e(w_1 \diamond_e w_2) + \mu(w_1 \diamond_e w_2)) \otimes \varepsilon_e(P_e(w_1') \diamond_e w_2')1 \\
&+ (w_1 \diamond_e w_2) \otimes P_e(P_e(w_1') \diamond_e w_2') + \lambda P_e(w_1 \diamond_e w_2) \otimes \varepsilon_e(w_1' \diamond_e w_2')1 \\
&+ \lambda\mu(w_1 \diamond_e w_2) \otimes \varepsilon_e(w_1' \diamond_e w_2')1 + \lambda(w_1 \diamond_e w_2) \otimes P_e(w_1' \diamond_e w_2') \\
&+ \kappa(w_1 \diamond_e w_2) \otimes (w_1' \diamond_e w_2') \\
=\ & (P_e(w_1 \diamond_e P_e(w_2)) + \mu(w_1 \diamond_e P_e(w_2))) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 \\
&+ (w_1 \diamond_e P_e(w_2)) \otimes P_e(w_1')\varepsilon_e(w_2') \\
&+ \mu(P_e(w_1 \diamond_e w_2) + \mu(w_1 \diamond_e w_2)) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 \\
&+ \mu(w_1 \diamond_e w_2) \otimes P_e(w_1')\varepsilon_e(w_2') \\
&+ (P_e(w_1 \diamond_e w_2) + \mu(w_1 \diamond_e w_2)) \otimes \underline{\varepsilon_e(w_1')\varepsilon_e(P_e(w_2'))1} \\
&+ (w_1 \diamond_e w_2) \otimes P_e(w_1' \diamond_e P_e(w_2')) \\
&+ (P_e(P_e(w_1) \diamond_e w_2) + \mu(P_e(w_1) \diamond_e w_2)) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 \\
&+ (P_e(w_1) \diamond_e w_2) \otimes \varepsilon_e(w_1')P_e(w_2') \\
&+ \mu(P_e(w_1 \diamond_e w_2) + \mu(w_1 \diamond_e w_2)) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 \\
&+ \mu(w_1 \diamond_e w_2) \otimes \varepsilon_e(w_1')P_e(w_2') \\
&+ (P_e(w_1 \diamond_e w_2) + \mu(w_1 \diamond_e w_2)) \otimes \underline{\varepsilon_e(P_e(w_1'))\varepsilon_e(w_2')1} \\
&+ (w_1 \diamond_e w_2) \otimes P_e(P_e(w_1') \diamond_e w_2') \\
&+ \lambda P_e(w_1 \diamond_e w_2) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 + \lambda\mu(w_1 \diamond_e w_2) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 \\
&+ \lambda(w_1 \diamond_e w_2) \otimes P_e(w_1' \diamond_e w_2') + \kappa(w_1 \diamond_e w_2) \otimes (w_1' \diamond_e w_2') \\
&\text{(by (3.2) and } \varepsilon_e \text{ being an algebra homomorphism)} \\
=\ & \big(P_e(w_1 \diamond_e P_e(w_2)) + \mu(w_1 \diamond_e P_e(w_2))\big) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 \\
&+ (w_1 \diamond_e P_e(w_2)) \otimes P_e(w_1')\varepsilon_e(w_2') \\
&+ \mu(P_e(w_1 \diamond_e w_2) + \mu(w_1 \diamond_e w_2)) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 \\
&+ \mu(w_1 \diamond_e w_2) \otimes P_e(w_1')\varepsilon_e(w_2') - \mu(P_e(w_1 \diamond_e w_2) \\
&+ \mu(w_1 \diamond_e w_2)) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 + (w_1 \diamond_e w_2) \otimes P_e(w_1' \diamond_e P_e(w_2')) \\
&+ \big(P_e(P_e(w_1) \diamond_e w_2) + \mu(P_e(w_1) \diamond_e w_2)\big) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 \\
&+ (P_e(w_1) \diamond_e w_2) \otimes \varepsilon_e(w_1')P_e(w_2') \\
&+ \mu(P_e(w_1 \diamond_e w_2) + \mu(w_1 \diamond_e w_2)) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 \\
&+ \mu(w_1 \diamond_e w_2) \otimes \varepsilon_e(w_1')P_e(w_2') \\
&- \mu(P_e(w_1 \diamond_e w_2) + \mu(w_1 \diamond_e w_2)) \otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 \\
&+ (w_1 \diamond_e w_2) \otimes P_e(P_e(w_1') \diamond_e w_2')
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda P_e(w_1 \diamond_e w_2)\otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 + \lambda\mu(w_1 \diamond_e w_2)\otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1\\
&+\lambda(w_1 \diamond_e w_2)\otimes P_e(w_1' \diamond_e w_2') + \kappa(w_1 \diamond_e w_2)\otimes (w_1' \diamond_e w_2')\\
&\text{(by applying } \varepsilon_e \circ P_e = -\mu\,\mathrm{id}_{\mathbf{k}} \circ \varepsilon_e \text{ to the two underlined factors)}\\
=\ & P_e(w_1 \diamond_e P_e(w_2))\otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 + \mu(w_1 \diamond_e P_e(w_2))\otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1\\
&+(w_1 \diamond_e P_e(w_2))\otimes P_e(w_1')\varepsilon_e(w_2') + \mu(w_1 \diamond_e w_2)\otimes P_e(w_1')\varepsilon_e(w_2')\\
&+(w_1 \diamond_e w_2)\otimes P_e(w_1' \diamond_e P_e(w_2')) + P_e(P_e(w_1)\diamond_e w_2)\otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1\\
&+\mu(P_e(w_1)\diamond_e w_2)\otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 + (P_e(w_1)\diamond_e w_2)\otimes \varepsilon_e(w_1')P_e(w_2')\\
&+\mu(w_1 \diamond_e w_2)\otimes \varepsilon_e(w_1')P_e(w_2') + (w_1 \diamond_e w_2)\otimes P_e(P_e(w_1')\diamond_e w_2')\\
&+\lambda P_e(w_1 \diamond_e w_2)\otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1 + \lambda\mu(w_1 \diamond_e w_2)\otimes \varepsilon_e(w_1')\varepsilon_e(w_2')1\\
&+\lambda(w_1 \diamond_e w_2)\otimes P_e(w_1' \diamond_e w_2') + \kappa(w_1 \diamond_e w_2)\otimes (w_1' \diamond_e w_2')\\
&\text{(by collecting terms with } \mu(P_e(w_1\diamond_e w_2)+\mu(w_1\diamond_e w_2))\otimes\varepsilon_e(w_1')\varepsilon_e(w_2')1).
\end{aligned}
$$

Then the ith term in the former expansion of $Q(w_1 \otimes w_1') \blacksquare Q(w_2 \otimes w_2')$ coincides with the $\sigma(i)$th term in the latter expansion of

$$
Q((w_1 \otimes w_1')\blacksquare Q(w_2 \otimes w_2')) + Q(Q(w_1 \otimes w_1')\blacksquare (w_2 \otimes w_2')) + \lambda Q((w_1 \otimes w_1')\blacksquare (w_2 \otimes w_2')).
$$

Here $\sigma \in \Sigma_{14}$ is the permutation

$$
\begin{pmatrix} i \\ \sigma(i) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 1 & 6 & 11 & 7 & 2 & 12 & 8 & 9 & 3 & 4 & 5 & 10 & 13 & 14 \end{pmatrix}.
$$

Thus, Q is an extended Rota–Baxter operator of weight (λ,κ). ∎

By Lemma 3.2, $(Ш_e^{NC}(A)\otimes Ш_e^{NC}(A), \blacksquare, Q)$ becomes an extended Rota–Baxter algebra of weight (λ,κ). By the universal property of $Ш_e^{NC}(A)$, there exists a unique extended Rota–Baxter algebra homomorphism

$$
\Delta_e : Ш_e^{NC}(A) \to Ш_e^{NC}(A)\otimes Ш_e^{NC}(A)
$$

such that

$$
\Delta_e \circ j_A = \Delta_A \quad \text{and} \quad \Delta_e \circ P_e = Q \circ \Delta_e. \tag{3.3}
$$

Lemma 3.3. *Let ε_e and Δ_e be as above. Define a linear operator on $Ш_e^{NC}(A)\otimes Ш_e^{NC}(A)\otimes Ш_e^{NC}(A)$ by*

$$
\tilde{Q}(u\otimes v\otimes w) = (P_e(u)+\mu u)\otimes \varepsilon_e(v)1\otimes \varepsilon_e(w)1 + u\otimes (P_e(v)+\mu v)\otimes \varepsilon_e(w)1 + u\otimes v\otimes P_e(w). \tag{3.4}
$$

Then

(1) *$\tilde{Q}$ is an extended Rota–Baxter operator of weight (λ,κ),*
(2) *$(\mathrm{id}\otimes\Delta_e)\Delta_e$ and $(\Delta_e\otimes\mathrm{id})\Delta_e$ are extended Rota–Baxter algebras homomorphisms from $(Ш_e^{NC}(A), P_e)$ to $(Ш_e^{NC}(A)\otimes Ш_e^{NC}(A)\otimes Ш_e^{NC}(A), \tilde{Q})$.*

Proof. The proof is similar to that of [48, Lemma 4.5]. ∎

By direct calculation, we obtain

LEMMA 3.4. *If P is an extended Rota–Baxter operator on an algebra R, then both $P \otimes \mathrm{id}$ and $\mathrm{id} \otimes P$ are extended Rota–Baxter operators on $R \otimes R$.*

Define a linear map $u_e : \mathbf{k} \to Ш_e^{\mathrm{NC}}(A)$ by $u_e(1_{\mathbf{k}}) = \mathbf{1}$. Then u_e is the unit map of $Ш_e^{\mathrm{NC}}(A)$.

THEOREM 3.5. *With the notations above, $(Ш_e^{\mathrm{NC}}(A), \diamond_e, u_e, \Delta_e, \varepsilon_e)$ forms a bialgebra.*

Proof. By Theorem 2.5, $\big(Ш_e^{\mathrm{NC}}(A), \diamond_e, u_e)$ is a unitary algebra. Since Δ_e and ε_e are algebra homomorphisms, it remains to verify the coassociativity of Δ_e and the counicity of ε_e. For the coassociativity of Δ_e, we shall show that

$$(\Delta_e \otimes \mathrm{id})\Delta_e = (\mathrm{id} \otimes \Delta_e)\Delta_e.$$

By Lemma 3.3(2), both $(\Delta_e \otimes \mathrm{id})\Delta_e$ and $(\mathrm{id} \otimes \Delta_e)\Delta_e$ are extended Rota–Baxter algebra homomorphisms from $(Ш_e^{\mathrm{NC}}(A), P_e)$ to $(Ш_e^{\mathrm{NC}}(A) \otimes Ш_e^{\mathrm{NC}}(A) \otimes Ш_e^{\mathrm{NC}}(A), \tilde{Q})$. Furthermore, by (3.3) and $\Delta_e|_A = \Delta_A$, we obtain

$$\begin{aligned}(\Delta_e \otimes \mathrm{id})\Delta_e \circ j_A &= (\Delta_e \otimes \mathrm{id})\Delta_A = (\Delta_A \otimes \mathrm{id})\Delta_A,\\ (\mathrm{id} \otimes \Delta_e)\Delta_e \circ j_A &= (\mathrm{id} \otimes \Delta_e)\Delta_A = (\mathrm{id} \otimes \Delta_A)\Delta_A.\end{aligned}$$

Since Δ_A is coassociative, we have

$$(\Delta_e \otimes \mathrm{id})\Delta_e \circ j_A = (\Delta_A \otimes \mathrm{id})\Delta_A = (\mathrm{id} \otimes \Delta_A)\Delta_A = (\mathrm{id} \otimes \Delta_e)\Delta_e \circ j_A.$$

By the universal property of $Ш_e^{\mathrm{NC}}(A)$,

$$(\mathrm{id} \otimes \Delta_e)\Delta_e = (\Delta_e \otimes \mathrm{id})\Delta_e,$$

as desired.

For the counicity of ε_e, it suffices to prove that

$$(\varepsilon_e \otimes \mathrm{id})\Delta_e = \beta_\ell \quad \text{and} \quad (\mathrm{id} \otimes \varepsilon_e)\Delta_e = \beta_r, \tag{3.5}$$

where β_ℓ and β_r are algebra isomorphisms given by

$$\begin{aligned}\beta_\ell &: Ш_e^{\mathrm{NC}}(A) \to \mathbf{k} \otimes Ш_e^{\mathrm{NC}}(A), \qquad u \mapsto 1_{\mathbf{k}} \otimes u,\\ \beta_r &: Ш_e^{\mathrm{NC}}(A) \to Ш_e^{\mathrm{NC}}(A) \otimes \mathbf{k}, \qquad u \mapsto u \otimes 1_{\mathbf{k}}.\end{aligned} \tag{3.6}$$

Set

$$\varphi := \beta_\ell^{-1}(\varepsilon_e \otimes \mathrm{id})\Delta_e \quad \text{and} \quad \psi := \beta_r^{-1}(\mathrm{id} \otimes \varepsilon_e)\Delta_e.$$

Since Δ_e and ε_e are both algebra homomorphisms, φ and ψ are also algebra homomorphisms on $\text{Ш}_e^{NC}(A)$. For any $w \in \text{Ш}_e^{NC}(A)$, we have

$$\begin{aligned}
\varphi\circ P_e(w) &= (\beta_\ell^{-1}(\varepsilon_e\otimes\mathrm{id})\varDelta_e)P_e(w) = \big(\beta_\ell^{-1}(\varepsilon_e\otimes\mathrm{id})\big)\big(\varDelta_e P_e\big)(w)\\
&= (\beta_\ell^{-1}(\varepsilon_e\otimes\mathrm{id}))\big(Q\varDelta_e\big)(w) \quad \text{(by (3.3))}\\
&= (\beta_\ell^{-1}(\varepsilon_e\otimes\mathrm{id}))Q\Big(\sum_{(w)} w_1\otimes w_2\Big) \quad \text{(in Sweedler's notation)}\\
&= \sum_{(w)}\Big(\beta_\ell^{-1}(\varepsilon_e\otimes\mathrm{id})\big((P_e(w_1)+\mu w_1)\otimes\varepsilon_e(w_2)1+w_1\otimes P_e(w_2)\big)\Big) \quad \text{(by (3.2))}\\
&= \sum_{(w)}\Big(\beta_\ell^{-1}\big(\varepsilon_e(P_e(w_1))\otimes\varepsilon_e(w_2)1+\mu\varepsilon_e(w_1)\otimes\varepsilon_e(w_2)1+\varepsilon_e(w_1)\otimes P_e(w_2)\big)\Big)\\
&= \sum_{(w)}\Big(\beta_\ell^{-1}\big(-\mu\varepsilon_e(w_1)\otimes\varepsilon_e(w_2)1+\mu\varepsilon_e(w_1)\otimes\varepsilon_e(w_2)1+\varepsilon_e(w_1)\otimes P_e(w_2)\big)\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (3.1))}\\
&= \sum_{(w)}\big(\beta_\ell^{-1}(\varepsilon_e(w_1)\otimes P_e(w_2))\big)\\
&= \sum_{(w)}\varepsilon_e(w_1)P_e(w_2) = P_e\Big(\sum_{(w)}\varepsilon_e(w_1)w_2\Big) = P_e\Big(\sum_{(w)}\beta_\ell^{-1}(\varepsilon_e(w_1)\otimes w_2)\Big)\\
&= P_e\big(\beta_\ell^{-1}(\varepsilon_e\otimes\mathrm{id})\varDelta_e(w)\big) = P_e\circ\varphi(w).
\end{aligned}$$

Thus φ is an extended Rota–Baxter algebra homomorphism on $\text{Ш}_e^{NC}(A)$. By (3.3) and $\varepsilon_e|_A = \varepsilon_A$,

$$\begin{aligned}
\varphi\circ j_A &= \beta_\ell^{-1}(\varepsilon_e\otimes\mathrm{id})\varDelta_e\circ j_A = \beta_\ell^{-1}(\varepsilon_e\otimes\mathrm{id})\varDelta_A = \beta_\ell^{-1}(\varepsilon_A\otimes\mathrm{id})\varDelta_A\\
&= \mathrm{id}_A \quad \text{(by the left counicity of } \varDelta_A).
\end{aligned}$$

By the universal property of $\text{Ш}_e^{NC}(A)$ again, we infer that $\varphi = \mathrm{id}$, that is, $(\varepsilon_e\otimes\mathrm{id})\varDelta_e = \beta_\ell$. The right counicity of $\varDelta_e$ can be checked by similar arguments. ■

3.2. The Hopf algebra structure by connected filtered conditions. This section shows that the bialgebra $\big(\text{Ш}_e^{NC}(A), \diamond_e, u_e, \varDelta_e, \varepsilon_e\big)$ is a connected filtered bialgebra, and so is a Hopf algebra.

DEFINITION 3.6. A bialgebra $H := (H, m_H, u_H, \varDelta_H, \varepsilon_H)$ is called an (increasing) *filtered bialgebra* if there are $\mathbf{k}$-submodules $H_n, n\geq 0$, of H such that

(a) $H_n\subseteq H_{n+1}$;
(b) $H = \bigcup_{n\geq 0} H_n$;
(c) $H_pH_q\subseteq H_{p+q}$ for all $p, q\geq 0$;
(d) $\varDelta(H_n)\subseteq \sum_{p+q=n} H_p\otimes H_q$ for all $p, q, n\geq 0$;
(e) $H_n = \operatorname{im} u_H\oplus (H_n\cap\ker\varepsilon_H)$ for all $n\geq 0$.

A filtered bialgebra H is called *connected* if $H_0 = \operatorname{im} u_H$ $(=\mathbf{k}1_H)$.

Then by [19, Theorem 3.4], we have

THEOREM 3.7. *If H is a connected filtered bialgebra, then H is a Hopf algebra.*

DEFINITION 3.8. Let $A := (A, m_A, u_A, \Delta_A, \varepsilon_A)$ be a connected filtered bialgebra with basis $\tilde{X} := X \cup \{\mathbf{1}\}$. Then $\tilde{X}$ is called *filtered* if there exists an (increasing) filtration $\tilde{X} = \bigcup_{n\geq 0} X_n$ satisfying

$$A_n = \mathbf{k}X_n, \qquad X \subseteq \ker \varepsilon_A, \qquad X_0 = \{\mathbf{1}\}.$$

Furthermore, if $x \in X_n \backslash X_{n-1}$, then x is said to have *degree* n in A, denoted by $\deg_A(x)$.

We now give the main result of this section.

THEOREM 3.9. *Let $A := (A, m_A, u_A, \Delta_A, \varepsilon_A)$ be a connected filtered bialgebra with a filtered basis $\tilde{X}$. Then the quintuple $(Ш_e^{\mathrm{NC}}(A), \diamond_e, u_e, \Delta_e, \varepsilon_e)$ is a Hopf algebra.*

Proof. By Theorem 3.5, $Ш_e^{\mathrm{NC}}(A) := (Ш_e^{\mathrm{NC}}(A), \diamond_e, u_e, \Delta_e, \varepsilon_e)$ is a bialgebra. To prove that $Ш_e^{NC}(A)$ is a Hopf algebra, it suffices to prove that $Ш_e^{NC}(A)$ is a connected filtered bialgebra by Theorem 3.7. For this, we first define an increasing filtration of $Ш_e^{NC}(A)$ $(= \mathbf{k}\mathfrak{X}_\infty)$. For $u \in \mathfrak{X}_\infty$, we use induction on $\mathrm{dep}(u)$ to define the degree of u, denoted by $\deg(u)$. If $\mathrm{dep}(u) = 0$, then $u \in \tilde{X}$ and we define

$$\deg(u) := \deg_A(u). \tag{3.7}$$

For a given $k \geq 0$, assume that $\deg(u)$ has been defined for all $u \in \mathfrak{X}_\infty$ with $\mathrm{dep}(u) \leq k$, and consider the case of $\mathrm{dep}(u) = k + 1 \geq 1$. If $\mathrm{bre}(u) = 1$, we can write $u = \lfloor \bar{u} \rfloor$ for some $\bar{u} \in \mathfrak{X}_\infty$, since $\mathrm{dep}(u) \geq 1$. Define

$$\deg(u) := \deg(\bar{u}) + 1, \tag{3.8}$$

where $\deg(\bar{u})$ is well-defined by the induction hypothesis. If $m := \mathrm{bre}(u) \geq 2$, let $u = u_1 \cdots u_m$ be the standard decomposition of u. Then define

$$\deg(u) := \deg(u_1) + \cdots + \deg(u_m). \tag{3.9}$$

For simplicity, set $H := Ш_e^{NC}(A)$ and $Y_n := \{y \in \mathfrak{X}_\infty \mid \deg(y) \leq n\}$. In particular, $Y_0 = \{\mathbf{1}\}$. Denote

$$H_n := \mathbf{k}Y_n, \qquad n \geq 0.$$

Thus, we have $H_0 = \mathbf{k1} = \operatorname{im} u_e$, proving the connectedness of H.

We proceed to prove that H is filtered. The argument is divided into five steps (a)–(e), according to Definition 3.6.

(a) For all $n \geq 0$, we have $H_n \subseteq H_{n+1}$, since $Y_n \subseteq Y_{n+1}$.

(b) For each n, $Y_n \subseteq \mathfrak{X}_\infty$, and so $H_n \subseteq H$. For every $y \in \mathfrak{X}_\infty$, y has a degree, and so $y \in Y_m$ for some m. Thus $H = \bigcup_{n\geq 0} H_n$.

(c) For all $p, q \geq 0$, we prove that $H_p H_q \subseteq H_{p+q}$ by induction on the sum $s := p+q \geq 0$. If $s = 0$, then $p = q = 0$. Let $w, w' \in H_0 (= \mathbf{k}\mathbf{1})$. By the definition of $\diamond_e$, we have $w \diamond_e w' \in H_0$. For a given $k \geq 0$, assume that $H_p H_q \subseteq H_{p+q}$ holds for all $s \leq k$, and consider the case of $s = k+1 \geq 1$. If $p = 0$ or $q = 0$, then $w = c\mathbf{1}$ or $w' = c\mathbf{1}$ for some $c \in \mathbf{k}$, and so $w \diamond_e w' \in H_{p+q}$. Now suppose that $p, q \geq 1$. Let $u \in Y_p$ and let $v \in Y_q$. Write u, v as their standard decompositions:

$$u = u_1 \cdots u_m \quad \text{and} \quad v = v_1 \cdots v_n.$$

Let $\ell := m + n \geq 2$. There are two cases to consider.

Case 1: $\ell = 2$. Then $m = n = 1$, and so u, v are in X or $\lfloor \mathfrak{X}_\infty \rfloor$. If one of u, v is in X, then by the definition of $\diamond_e$ and (3.9), we have

$$u \diamond_e v = uv \in H_{\deg(u)+\deg(v)} \subseteq H_{p+q}.$$

If u, v are both in $\lfloor \mathfrak{X}_\infty \rfloor$, then write $u = \lfloor \bar{u} \rfloor$ and $v = \lfloor \bar{v} \rfloor$ for some $\bar{u}, \bar{v} \in \mathfrak{X}_\infty$. By (2.3), we have

$$u \diamond_e v = \lfloor \bar{u} \diamond_e v \rfloor + \lfloor u \diamond_e \bar{v} \rfloor + \lambda \lfloor \bar{u} \diamond_e \bar{v} \rfloor + \kappa \bar{u} \diamond_e \bar{v}.$$

By (3.8), we obtain

$$\begin{aligned}
&\deg(\bar{u}) + \deg(v) = \deg(u) + \deg(\bar{v}) = \deg(u) + \deg(v) - 1 \leq p + q - 1 = k, \\
&\deg(\bar{u}) + \deg(\bar{v}) = \deg(u) + \deg(v) - 2 \leq p + q - 2 = k - 1.
\end{aligned}$$

By the induction hypothesis, we get $\bar{u} \diamond_e v$, $u \diamond_e \bar{v} \in H_{\deg(u)+\deg(v)-1}$ and $\bar{u} \diamond_e \bar{v} \in H_{\deg(u)+\deg(v)-2}$. By (3.8) again, we have

$$\lfloor H_n \rfloor \subseteq H_{n+1}.$$

Hence, $\lfloor \bar{u} \diamond_e v \rfloor$, $\lfloor u \diamond_e \bar{v} \rfloor \in H_{\deg(u)+\deg(v)}$, and $\lfloor \bar{u} \diamond_e \bar{v} \rfloor \in H_{\deg(u)+\deg(v)-1}$. So $u \diamond_e v \in H_{\deg(u)+\deg(v)} \subseteq H_{p+q}$.

Case 2: $\ell > 2$. Then $m \geq 2$ or $n \geq 2$. We consider three subcases.

Subcase (i): $m \geq 2$ and $n = 1$. Then by (2.4), we have

$$u \diamond_e v = u_1 u_2 \cdots u_{m-1} (u_m \diamond_e v).$$

By (3.9), we have $\deg(u) = \deg(u_1 u_2 \cdots u_{m-1}) + \deg(u_m)$. Moreover, by the proof of Case 1, we obtain $u_m \diamond_e v \in H_{\deg(u_m)+\deg(v)}$. Then by (3.9) again,

$$\begin{aligned}
\deg(u \diamond_e v) &= \deg(u_1 u_2 \cdots u_{m-1}) + \deg(u_m \diamond_e v) \\
&\leq \deg(u_1 u_2 \cdots u_{m-1}) + \deg(u_m) + \deg(v) \\
&= \deg(u) + \deg(v),
\end{aligned}$$

proving $u \diamond_e v \in H_{\deg(u)+\deg(v)} \subseteq H_{p+q}$.

SUBCASE (ii): $m = 1$ and $n \geq 2$. The proof is similar to Subcase (i).

SUBCASE (iii): $m \geq 2$ and $n \geq 2$. Let $v' = v_2 \cdots v_n$. So $v = v_1 v'$. Then by the definition of $\diamond_e$, we get

$$u \diamond_e v = (u \diamond_e v_1) v'.$$

By Subcase (i), we get $u \diamond_e v_1 \in H_{\deg(u)+\deg(v_1)}$. Then by (3.9),

$$\begin{aligned}\deg(u \diamond_e v) &= \deg(u \diamond_e v_1) + \deg(v') \\ &\leq \deg(u) + \deg(v_1) + \deg(v') = \deg(u) + \deg(v),\end{aligned}$$

proving $u \diamond_e v \in H_{\deg(u)+\deg(v)} \subseteq H_{p+q}$. Hence, the proof is completed by induction.

(d) We now prove that

$$\Delta_e(H_n) \subseteq \sum_{p+q=n} H_p \otimes H_q, \quad n \geq 0. \tag{3.10}$$

For this, it suffices to verify

$$\Delta_e(w) = \sum_{(w)} w_1 \otimes w_2 \in \sum_{p+q=n} H_p \otimes H_q,\ w \in Y_n, \tag{3.11}$$

where w_1 and w_2 are non-zero linear multiples of elements in $\mathfrak{X}_\infty$ with $\deg(w_1) + \deg(w_2) \leq \deg(w)$. We use induction on $\deg(w) \geq 0$. When $\deg(w) = 0$, $w = \mathbf{1}$. Since $\Delta_e(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$, we have $\Delta_e(\mathbf{1}) \in \sum_{p+q=n} H_p \otimes H_q$. For a given $k \geq 0$, assume that (3.11) holds for $\deg(w) \leq k$, and consider $w \in \mathfrak{X}_\infty$ with $\deg(w) = k+1 \geq 1$. We consider two cases.

CASE 1: $\mathrm{bre}(w) = 1$. Then w is in $X \subseteq A$ or $\lfloor \mathfrak{X}_\infty \rfloor$. If $w \in X \subseteq A$, by A being a connected filtered bialgebra and (3.3), we obtain

$$\Delta_e(w) = \Delta_e \circ j_A(w) = \Delta_A(w) = \sum_{(w)} w_1 \otimes w_2,$$

where $\deg(w_1) + \deg(w_2) \leq \deg(w)$. If $w \in \lfloor \mathfrak{X}_\infty \rfloor$, we write $w = P_e(\bar{w})$ for some $\bar{w} \in \mathfrak{X}_\infty$ with $\deg(\bar{w}) = \deg(w) - 1 \leq k$. By (3.3),

$$\Delta_e(w) = \Delta_e\big(P_e(\bar{w})\big) = Q\big(\Delta_e(\bar{w})\big).$$

By the induction hypothesis on $\deg(w)$, we have $\Delta_e(\bar{w}) = \sum_{(\bar{w})} \bar{w}_1 \otimes \bar{w}_2$ with $\deg(\bar{w}_1) + \deg(\bar{w}_2) \leq \deg(\bar{w}) \leq k$. Then by (3.5) and (3.6),

$$\begin{aligned}\Delta_e(w) = Q(\Delta_e(\bar{w})) &= Q\Big(\sum_{(\bar{w})} \bar{w}_1 \otimes \bar{w}_2\Big) \\ &= \sum_{(\bar{w})} \big((P_e(\bar{w}_1) + \lambda \bar{w}_1) \otimes \varepsilon_e(\bar{w}_2) 1 + \bar{w}_1 \otimes P_e(\bar{w}_2)\big) \\ &= \sum_{(\bar{w})} \big(P_e(\bar{w}_1 \varepsilon_e(\bar{w}_2)) \otimes 1 + \lambda \bar{w}_1 \varepsilon_e(\bar{w}_2) \otimes 1 + (\mathrm{id} \otimes P_e)(\bar{w}_1 \otimes \bar{w}_2)\big)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{(\bar{w})} P_e(\beta_r^{-1}(\bar{w}_1 \otimes \varepsilon_e(\bar{w}_2))) \otimes 1 + \lambda \sum_{(\bar{w})} \beta_r^{-1}(\bar{w}_1 \otimes \varepsilon_e(\bar{w}_2)) \otimes 1 \\
&\quad + \sum_{(\bar{w})} (\mathrm{id} \otimes P_e)(\bar{w}_1 \otimes \bar{w}_2) \\
&= P_e\Big(\beta_r^{-1}\Big(\sum_{(\bar{w})} \bar{w}_1 \otimes \varepsilon_e(\bar{w}_2)\Big)\Big) \otimes 1 + \lambda\beta_r^{-1}\Big(\sum_{(\bar{w})} \bar{w}_1 \otimes \varepsilon_e(\bar{w}_2)\Big) \otimes 1 \\
&\quad + (\mathrm{id} \otimes P_e)\Big(\sum_{(\bar{w})} \bar{w}_1 \otimes \bar{w}_2\Big) \\
&= P_e(\beta_r^{-1}(\mathrm{id} \otimes \varepsilon_e)\Delta_e(\bar{w})) \otimes 1 + \lambda(\beta_r^{-1}(\mathrm{id} \otimes \varepsilon_e)\Delta_e(\bar{w})) \otimes 1 \\
&\quad + (\mathrm{id} \otimes P_e)\Delta_e(\bar{w}) \\
&= P_e(\bar{w}) \otimes 1 + \lambda\bar{w} \otimes 1 + (\mathrm{id} \otimes P_e)\Delta_e(\bar{w}) \\
&= w \otimes 1 + \lambda\bar{w} \otimes 1 + (\mathrm{id} \otimes P_e)\Delta_e(\bar{w}) \\
&= w \otimes 1 + \lambda\bar{w} \otimes 1 + \sum_{(\bar{w})}(\bar{w}_1 \otimes P_e(\bar{w}_2)),
\end{aligned}
$$

where

$$
\begin{aligned}
&\deg(w) + \deg(1) = \deg(w) \le k+1, \quad \deg(\bar{w}) + \deg(1) = \deg(\bar{w}) \le k, \\
&\deg(\bar{w}_1) + \deg(P_e(\bar{w}_2)) = \deg(\bar{w}_1) + \deg(\bar{w}_2) + 1 \le \deg(\bar{w}) + 1 \le k+1.
\end{aligned}
$$

Case 2: $\mathrm{bre}(w) \ge 2$. Then we can write $w = w_1 w_2$, where w_1, w_2 are in $\mathfrak{X}_\infty \backslash \{1\}$ with $\mathrm{bre}(w_1), \mathrm{bre}(w_2) \ge 1$. By $\deg(w_1), \deg(w_2) \le k$ and the induction hypothesis, we get

$$
\Delta_e(w_1) = \sum_{(w_1)} w_{11} \otimes w_{12} \quad \text{and} \quad \Delta_e(w_2) = \sum_{(w_2)} w_{21} \otimes w_{22},
$$

where $\deg(w_{11}) + \deg(w_{12}) \le \deg(w_1)$ and $\deg(w_{21}) + \deg(w_{22}) \le \deg(w_2)$. Then

$$
\begin{aligned}
\Delta_e(w) = \Delta_e(w_1 w_2) = \Delta_e(w_1 \diamond_e w_2) &= \Delta_e(w_1) \blacksquare \Delta_e(w_2) \\
&= \Big(\sum_{(w_1)} w_{11} \otimes w_{12}\Big) \blacksquare \Big(\sum_{(w_2)} w_{21} \otimes w_{22}\Big) \\
&= \sum_{(w_1)} \sum_{(w_2)} (w_{11} \diamond_e w_{21}) \otimes (w_{12} \diamond_e w_{22}).
\end{aligned}
$$

Since $H_p H_q \subseteq H_{p+q}$, we get

$$
w_{11} \diamond_e w_{21} \in H_{\deg(w_{11})+\deg(w_{21})} \quad \text{and} \quad w_{12} \diamond_e w_{22} \in H_{\deg(w_{12})+\deg(w_{22})}.
$$

Let $\tilde{w}_1 := w_{11} \diamond_e w_{21}$ and let $\tilde{w}_2 := w_{12} \diamond_e w_{22}$. Then

$$
\Delta_e(w) = \sum_{(w_1)} \sum_{(w_2)} \tilde{w}_1 \otimes \tilde{w}_2.
$$

So

$$\begin{aligned}\deg(\tilde{w}_1)+\deg(\tilde{w}_2) \leq\ & \deg(w_{11})+\deg(w_{21})+\deg(w_{12})+\deg(w_{22})\\ \leq\ & \deg(w_1)+\deg(w_2)=\deg(w).\end{aligned}$$

This completes the induction and thus the proof of (3.11).

(e) Finally, we shall verify that

$$H_n = \operatorname{im} u_e \oplus (H_n \cap \ker \varepsilon_e), \quad n \geq 0. \tag{3.12}$$

Since $u_e(1_{\mathbf{k}}) = \mathbf{1}$ and $\varepsilon_e(\mathbf{1}) = 1_{\mathbf{k}}$, we have $\varepsilon_e u_e = \mathrm{id}_{\mathbf{k}}$. Let $\varphi = u_e \varepsilon_e$. Then $\varphi^2 = \varphi$, proving

$$H = \operatorname{im} \varphi \oplus \ker \varphi = \operatorname{im} u_e \oplus \ker \varepsilon_e. \tag{3.13}$$

For $n = 0$, $H_0 = \mathbf{k}\mathbf{1} = \operatorname{im} u_e$ and $H_0 \cap \ker \varepsilon_e = \{0\}$, and so (3.12) holds. For $n \geq 1$, let $a \in H_n$. Then we can write $a = \varepsilon_e(a)\mathbf{1} + \big(a - \varepsilon_e(a)\mathbf{1}\big)$. Since

$$\varepsilon_e(a - \varepsilon_e(a)\mathbf{1}) = \varepsilon_e(a) - \varepsilon_e(a) = 0,$$

we have $a - \varepsilon_e(a)\mathbf{1} \in \ker \varepsilon_e$, and so $H_n = \operatorname{im} u_e + (H_n \cap \ker \varepsilon_e)$. By (3.13),

$$\operatorname{im} u_e \cap (H_n \cap \ker \varepsilon_e) = H_n \cap (\operatorname{im} u_e \cap \ker \varepsilon_e) = \{0\}.$$

This finishes the proof of (3.12). ∎

By the above proof, we obtain

COROLLARY 3.10. *With the notation above, for $w = P_e(\bar{w}) \in \lfloor \mathfrak{X}_\infty \rfloor$, let $\varDelta_e(\bar{w}) = \sum_{(\bar{w})} \bar{w}_1 \otimes \bar{w}_2$. Then*

$$\varDelta_e(w) = (w + \lambda \bar{w}) \otimes 1 + \sum_{(\bar{w})} (\bar{w}_1 \otimes P_e(\bar{w}_2)),$$

which is called a weighted Hochschild 1-cocycle condition.

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