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QUIVER GRASSMANNIANS ASSOCIATED TO NILPOTENT CYCLIC REPRESENTATIONS DEFINED BY A SINGLE MATRIX

BY

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Abstract. We study the geometry of the closed Białynicki-Birula cells of the quiver Grassmannians associated to a nilpotent representation of a cyclic quiver defined by a single matrix. The main result of this paper is that for the special case where we choose subrepresentations of dimension $\mathbf{1} = (1, \ldots, 1)$, the closed Białynicki-Birula cells are smooth. We also discuss their cohomology ring. Namely, we describe the Knutson–Tao basis of equivariant cohomology that is dual to fundamental classes in equivariant homology.

Introduction. The theory of quiver Grassmannians, first introduced by Schofield [14], has its roots in representation theory, but it found its way to other fields, such as algebraic geometry. This theory, from the perspective of algebraic geometry, is a rich one – we know that these spaces describe all projective varieties as was shown in [13], so they are an attractive object to study. For acyclic quivers many properties such as smoothness, irreducibility or dimension have known criteria; more information can be found in [6].

More recently, M. Lanini and A. Pütz [10] define torus actions on the quiver Grassmannians associated to the nilpotent representations of cyclic quivers, which endows the quiver Grassmannians with a GKM-variety structure and lets us compute their equivariant cohomology. They were further analyzed in [11], [12] in the context of group actions and resolution of singularities respectively.

This paper delves into the GKM-variety structure and the Białynicki-Birula decomposition of these quiver Grassmannians in the following special case:

Definition 1.2. Let Δ_n be the equioriented cycle with n vertices, $J : \mathbb{C}^N \to \mathbb{C}^N$ be a nilpotent matrix and $\mathcal{M}^{n,J}$ be the Δ_n-representation that has $\mathbb{C}^N$ over each vertex and J over each arrow in Δ_n. We define $X_{n,J}$

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to be

$$X_{n,J} := \mathrm{Gr}_{\mathbf{1}}(\mathcal{M}^{n,J}) \subseteq \prod_{i=1}^{n} \mathbb{P}^{N-1},$$

where $\mathbf{1} \in \mathbb{Z}^n$ is the n-tuple of 1's.

Without loss of generality, J can be assumed to be a Jordan matrix with M Jordan blocks. Following [10], we introduce an action of a torus $T = (\mathbb{C}^\times)^{nM+1}$ on $X_{n,J}$ and study its T-equivariant cohomology. The authors of [7] tackled a similar problem – they considered the quiver Grassmannian $\mathrm{Gr}_{\mathbf{k}}(\mathcal{M}^{n,J})$, where $\mathbf{k} = (k, \dots, k)$, but $\mathcal{M}^{n,J}$ was defined by J which is a single Jordan block of size n. In that paper one can find, for example, the Poincaré polynomials of such quiver Grassmannians, resolution of singularities and the connection between these quiver Grassmannians and totally nonnegative Grassmannians.

The Białynicki-Birula cells form a stratification of the quiver Grassmannian into affine cells, as shown in [10]. In our case we prove the following two results concerning the structure of closed Białynicki-Birula cells.

THEOREM 3.7. *Let $X_{n,J}$ be a quiver Grassmannian as in Definition* 1.2 *and $p \in X_{n,J}^T$ be a fixed point. Then the closure of the BB-cell $\mathcal{C}_p$ is a product of smooth quiver Grassmannians.*

The nilpotent Jordan matrix J corresponds to a partition of N and so we can visualize coordinates of $\mathbb{C}^N$ using the Young tableau $\mathcal{J}$ of shape associated to J. Appropriately identifying the basis vectors of $\mathbb{C}^N$ with boxes of $\mathcal{J}$ leads to the definition of twisted lexicographic order on this basis. More precisely, suppose that J has M blocks and the lth block is of size j_l. Let v_i^l be the ith Jordan basis vector of the lth block of J. We then set $v_i^l < v_k^s$ if either $j_s - k < j_l - i$ or $j_s - k = j_l - i$ and $l < k$. The closed BB-cells form a poset with respect to inclusion and we prove that these two relations are inverse to each other.

THEOREM 3.9. *For two fixed points $p, q \in X_{n,J}^T$ we have $\overline{\mathcal{C}_p} \subseteq \overline{\mathcal{C}_q}$ if and only if $q_i \leq p_i$ for every $i \in \mathbb{Z}/n\mathbb{Z}$.*

We can use the visual nature of coordinates to describe possible edges in the GKM graph of $X_{n,J}$. Namely, a fixed point $p \in X_{n,J}^T$ is determined by a choice of a basis vector v_i^l for each vertex. Thus, we can divide p into *movable parts* ([1]), i.e. segments of these lines in $\mathbb{C}^N$ that do not go to 0 under J.

([1]) Here we use the terms 'movable part' and 'mutation' appearing in [10], where these notions were introduced to describe the GKM graph of quiver Grassmannians associated to any nilpotent representation of an equioriented cycle, but our notions of movable parts and fundamental mutations are not exactly taken from [10]. The difference between our definitions and those found in [10, Definitions 6.8 and 6.9] is described in Remark 4.4.

Each movable part can be seen as a coloring of the first couple of boxes in a fixed row in $\mathcal{J}$, and a *mutation* of a movable part is defined to be either a coloring of the same number of boxes, but in a lower row, or two colorings – we split the first coloring in two. In our case we can draw a picture to illustrate further what is happening. We show in the following proposition that these constructions define edges in the GKM graph of $X_{n,J}$.

PROPOSITION 4.6 ([10, Theorem 6.15]). *Let $p, q \in X_{n,J}^T$ be two fixed points. Then an arrow $p \to q$ in the GKM graph of $X_{n,J}$ exists if and only if q is obtained from p by a mutation of exactly one movable part in p.*

We also give a way to get the label of such an edge in Proposition 4.8.

The closed Białynicki-Birula cells form a basis of equivariant homology and since they are smooth, integration over these cells can be done using the Atiyah–Bott and Berline–Vergne Theorem, which heavily simplifies actual computations. We can construct the dual basis $\{p^x \in H_T^*(X_{n,J}) : x \in X_{n,J}^T\}$ and this basis is immediately seen to be a Knutson–Tao basis of $H_T^*(X_{n,J})$, which was defined in [15]. Again, as the cells are smooth, computing things such as structure constants $c_{x,y}^z$ defined by

$$p^x \cdot p^y = \sum_{z \in X_{n,J}^T} c_{x,y}^z p^z$$

is immediate, since this comes down to linear algebra (possibly over a large vector space).

This paper is organized as follows: In Section 1 we fix notation and recall the quiver Grassmannians and nilpotent representations. In Section 2 we recall the construction of actions of two tori on quiver Grassmannians for nilpotent Δ_n-representations from [10], but only for our case. Section 3 is about the Białynicki-Birula decomposition of this particular family of quiver Grassmannians. Namely, we recall a result from [10], that the induced Białynicki-Birula decomposition is a cellular decomposition into affine cells. We further investigate what these cells look like after taking the closure in the ambient space of the quiver Grassmannian, which is a product of projective spaces. This section contains the proof that closed Białynicki-Birula cells are smooth. In Section 4 we describe the GKM graph of these quiver Grassmannians, which was also computed in [10], but we provide a description using colorings of Young tableaux. In Section 5 we define the basis dual to the equivariant fundamental classes in equivariant homology and discuss computing structure constants of this basis.

1. Basic definitions and notation. Let $T = (\mathbb{C}^\times)^r$ be an algebraic torus. For a T-space X we define its equivariant cohomology via the Borel construction. We denote by $H_T^*(X)$ the equivariant cohomology ring with coefficients in $\mathbb{Q}$. For a proper introduction into equivariant cohomology see [1].

Now we suppose that X is a $\mathbb{C}^\times$-space and we decompose the fixed point set into connected components as follows:

$$X^{\mathbb{C}^\times} = \bigcup_{k\geq 0} X_k.$$

Then we define the *Białynicki-Birula cells* or the *BB-cells* by

$$\mathcal{C}_k = \Big\{x \in X : \lim_{\lambda\to 0} \lambda \cdot x \in X_k\Big\}.$$

Together these cells form the *BB-decomposition* of X. If these cells are affine spaces, X is an equivariantly formal $\mathbb{C}^\times$-variety.

Let $Q = (Q_0, Q_1)$ be a finite and connected quiver and $M = (M_i, M_\alpha)_{i\in Q_0,\alpha\in Q_1}$ be a representation of Q over $\mathbb{C}$ (equivalently, a $\mathbb{C}Q$-module). We denote by $\dim M \in \mathbb{Z}_{\geq 0}^{|Q_0|}$ the dimension vector of M, i.e. $(\dim M)_i = \dim_{\mathbb{C}} M_i$. A proper introduction to quiver representations can be found in [2]. Given a vector $\mathbf{e} \leq \dim M$ we denote by $\mathrm{Gr}_{\mathbf{e}}(M)$ the quiver Grassmannian of subrepresentations of M of dimension $\mathbf{e}$. These spaces are known to have a (projective) scheme structure given by their embedding into a product of usual Grassmannians, since a subrepresentation of M is determined by a choice of subspaces for all M_i's. Note that these spaces in general are not varieties as they need not be irreducible or even reduced. Of course, the notion of a 'variety' in algebraic geometry is sometimes not consistent. For the authors of [10] a projective variety is just a closed subscheme of a projective space, so in that sense quiver Grassmannians are projective varieties.

Let $M \in \mathrm{rep}_{\mathbb{C}}(Q)$ be a representation over $\mathbb{C}$ and B be the set of basis elements of each vector space in M. We divide B into subsets corresponding to each vertex. Let $B^i \subset B$ be the subset of B that is the $\mathbb{C}$-basis of M_i. *The coefficient quiver of* M with respect to the basis B is defined to be the following quiver:

- The set of vertices is B.
- For $v_k^i \in B^i, v_l^j \in B^j$ an arrow $v_k^i \to v_l^j$ exists if and only if there exists an arrow $\alpha : i \to j$ such that the lth coordinate of $M_\alpha(v_k^i)$, written in the $\mathbb{C}$-basis that is B^j, is nonzero.

We denote this coefficient quiver by $Q(M, B)$. This tool was used to find well-suited torus actions on quiver Grassmannians, for example in [10], and this paper follows this construction.

We will be interested in a special family of representations of a certain family of quivers. Namely, let Δ_n be the oriented cycle quiver, that is, an equioriented n-gon. We label both the vertices and edges using the set $\mathbb{Z}/n\mathbb{Z}$.

A Δ_n-representation M is called *nilpotent* if there exists a number $K > 0$ such that M, considered as a $\mathbb{C}\Delta_n$-module, is annihilated by the ideal of paths of length at least K. Explicitly, this means that all compositions of K consecutive maps from M vanish.

EXAMPLE 1.1. A basic example of a nilpotent representation is a representation that has the same vector space, say of dimension N, over each vertex and the same nilpotent map over each arrow. Since we can simultaneously change basis, we assume without loss of generality that over each vertex we have $\mathbb{C}^N$ and our map is a Jordan matrix $J : \mathbb{C}^N \to \mathbb{C}^N$, whose Jordan block decomposition is given by

$$J = J_1 \oplus \cdots \oplus J_M, \qquad \text{where } J_i \in M_{j_i \times j_i}(\{0,1\}),\ j_1 \geq \cdots \geq j_M.$$

As J must be nilpotent, each Jordan block has zeroes on the diagonal. We will denote this representation by $\mathcal{M}^{n,J} = (\mathcal{M}_i^{n,J} = \mathbb{C}^N, \mathcal{M}_\alpha^{n,J} = J)_{i,\alpha \in \mathbb{Z}/n\mathbb{Z}}$.

Following [10], we construct a GKM-variety structure on any quiver Grassmannian associated to any nilpotent Δ_n-representation and we will use this result to study the geometry of $\mathrm{Gr}_{\mathbf{1}}(\mathcal{M}^{n,J})$, where $\mathbf{1} = (1, \ldots, 1)$.

DEFINITION 1.2. Let Δ_n be the equioriented cycle with $n > 0$ vertices, $J : \mathbb{C}^N \to \mathbb{C}^N$ be a nilpotent matrix and $\mathcal{M}^{n,J}$ be the Δ_n-representation as in Example 1.1. We define $X_{n,J}$ to be

$$X_{n,J} := \mathrm{Gr}_{\mathbf{1}}(\mathcal{M}^{n,J}) \subseteq \prod_{i=1}^{n} \mathbb{P}^{N-1},$$

where $\mathbf{1} \in \mathbb{Z}^n$ is an n-tuple of 1's.

Throughout, we use $\langle v_1, \ldots, v_n \rangle$ as shorthand for $\mathrm{span}(v_1, \ldots, v_n)$. We denote the set $\{1, \ldots, M\}$ by $[M]$.

2. Torus actions on the main object. Fix a nilpotent endomorphism $J : \mathbb{C}^N \to \mathbb{C}^N$. In this section we recall known results from [10] applied to our example of $X_{n,J} = \mathrm{Gr}_{\mathbf{1}}(\mathcal{M}^{n,J})$ from Definition 1.2 and $\mathbf{1} = (1, \ldots, 1)$. This space clearly embeds into $\mathcal{P} := (\mathbb{P}^{N-1})^n$ and the authors of [10] in fact define actions of two tori on $\mathcal{P}$ that fix $X_{n,J}$.

Let us start with the $\mathbb{C}^\times$-action. Fix $l \in \{1, \ldots, M\}$ and let $V_l \subseteq \mathbb{C}^N$ be the subspace fixed by the lth Jordan block in J. We denote the Jordan basis of V_l by $v_1^l, \ldots, v_{j_l}^l$. Consider a linear action of $\mathbb{C}^\times$ on $\mathbb{C}^N$, defined on the basis vectors $v_i^l \in \mathbb{C}^N$ as follows:

$$\lambda \cdot v_i^l := \lambda^{l - M(j_l - i)} v_i^l \quad \text{for } \lambda \in \mathbb{C}^\times. \tag{2.1}$$

In other words, the weight of $v_{j_l}^l$ is l and as we decrease the index of the basis vector we decrease the weight by M (so long as the vectors lie in the same V_l). Note that we act on each coordinate by different weights. After we pass to the projective space $\mathbb{P}^{N-1}$, the fixed point set under the induced torus action is the set of (linear spans of) standard basis vectors. We extend this action to the diagonal action on $\mathcal{P}$. This requires a check that $X_{n,J}$ is $\mathbb{C}^\times$-invariant under this action; we refer the reader to [5, Lemma 1.1] for the proof.

Next, we describe an action of $T = (\mathbb{C}^\times)^{nM+1}$ on $\mathcal{P}$ that fixes $X_{n,J}$, but omit describing the whole construction here and state the resulting torus action; the readers interested in where this action comes from are referred to [10, Section 5.2]. We start with the 0th vertex and define the following torus action on basis vectors:

$$(t_0, \{t_{r,s}\}_{r\in\mathbb{Z}/n\mathbb{Z}, s\in[M]}) \cdot v_i^l := t_0^{i-1} t_{j_l-i,l} v_i^l \quad \text{for } (t_0, \{t_{r,s}\}_{r\in\mathbb{Z}/n\mathbb{Z}, s\in[M]}) \in T.$$

This naturally extends to $\mathbb{P}^{N-1}$, and to extend this to $\mathcal{P}$ we use an automorphism $\rho : T \to T$ defined as follows:

$$\rho(t_0) := t_0, \quad \rho(t_{j,k}) := t_{j+1,k}. \tag{2.2}$$

Now, if v_i^l lies over the jth vertex, then we set

$$(t_0, \{t_{r,s}\}_{r\in\mathbb{Z}/n\mathbb{Z}, s\in[M]}) \cdot v_i^l := \rho^j(t_0, \{t_{r,s}\}_{r\in\mathbb{Z}/n\mathbb{Z}, s\in[M]}) \cdot v_i^l = t_0^{i-1} t_{j_l-i+j,l} v_i^l.$$

This torus action makes $X_{n,J}$ a T-equivariant subvariety of its ambient space, by [10, Lemma 5.10].

Example 2.1. Consider the case of a Jordan matrix with two blocks $J = J_3 \oplus J_2$ and three vertices. Then the $\mathbb{C}^\times$-action over each vertex is given by

$$\lambda \cdot (a_1^1, a_2^1, a_3^1, a_1^2, a_2^2) = (\lambda^{-3} a_1^1, \lambda^{-1} a_2^1, \lambda^1 a_3^1, a_1^2, \lambda^2 a_2^2)$$

and the $T = (\mathbb{C}^\times)^7$-action on the zeroth, first and second vertices is given by

$$\begin{aligned}
t \cdot (a_1^1, a_2^1, a_3^1, a_1^2, a_2^2) &= (t_{2,1} a_1^1, t_0 t_{1,1} a_2^1, t_0^2 t_{0,1} a_3^1, t_{1,2} a_1^2, t_0 t_{0,2} a_2^2),\\
t \cdot (a_1^1, a_2^1, a_3^1, a_1^2, a_2^2) &= (t_{0,1} a_1^1, t_0 t_{2,1} a_2^1, t_0^2 t_{1,1} a_3^1, t_{2,2} a_1^2, t_0 t_{1,2} a_2^2),\\
t \cdot (a_1^1, a_2^1, a_3^1, a_1^2, a_2^2) &= (t_{1,1} a_1^1, t_0 t_{0,1} a_2^1, t_0^2 t_{2,1} a_3^1, t_{0,2} a_1^2, t_0 t_{2,2} a_2^2).
\end{aligned}$$

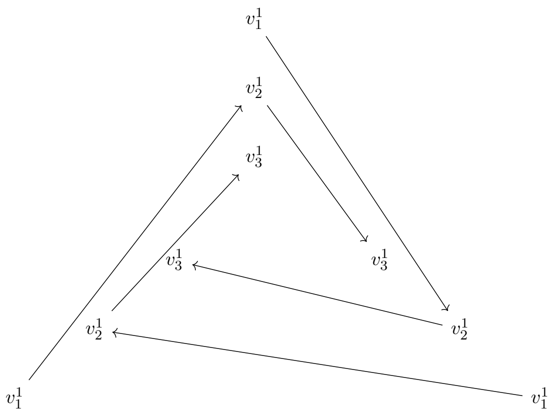
There is a beautiful interplay between these two actions – the $\mathbb{C}^\times$-action induces a BB-decomposition of $X_{n,J}$ that turns out to be a cellular decomposition by [10, Theorem 5.7], which implies that $X_{n,J}$ is equivariantly formal. The fixed point sets $X_{n,J}^{\mathbb{C}^\times}$ and $X_{n,J}^T$ are equal and the $\mathbb{C}^\times$-action can be recovered from the T-action via a cocharacter $\chi : \mathbb{C}^\times \to T$ defined by

$$\chi(\lambda) := (\lambda^M, \{\lambda^{k-Mj_k+M}\}_{j\in\mathbb{Z}/n\mathbb{Z}, k\in[M]}).$$

The T-variety $X_{n,J}$ is a GKM-variety by [10, Theorem 6.6] and its GKM graph can be described using the coefficient quiver of $\mathcal{M}^{n,J}$, but in our specific case we can give descriptions of the BB-cells and the GKM of $X_{n,J}$ in a simpler language.

Notation 2.2. After fixing the basis of the vector space $\mathcal{M}^{n,J}$ to be the sets $\{v_i^l\}$ over every vertex, the coefficient quiver of $\mathcal{M}^{n,J}$ with respect to this basis can be drawn as M spirals starting at each vertex of Δ_n, where the lth spiral spans j_l vertices. We adopt the notation from [10] and denote this quiver by $Q(\mathcal{M}^{n,J})$.

Example 2.3. Take for example $n = 3$ and $J = J_3$ to be a single Jordan block of size 3×3. Then the coefficient quiver $Q(\mathcal{M}^{n,J})$ looks as follows:



The authors of [10] describe the GKM graph of $X_{n,J}$ in [10, Theorem 6.15]. In later sections we will see how the description of the GKM graph of $X_{n,J}$ comes from the one given in [10].

Remark 2.4. There are two notable things about the action of the larger torus T. First, as observed in [10], the action of the larger torus T is *not* an action via automorphisms of $\mathcal{M}^{n,J}$ as a quiver representation. However, after restricting to the subtorus $T' \subseteq T$ defined by the equation $t_0 = 1$, we *do* get an action via quiver representation automorphisms. It might be of geometrical interest to ask whether this subtorus $T' \subseteq \mathrm{Aut}(\mathcal{M}^{n,J})$ is a maximal torus. The second thing is that the torus T, considered as a subtorus of $\mathrm{GL}(\mathcal{M}^{n,J})$, is *not* the maximal torus acting linearly on the vector space $\mathcal{M}^{n,J}$ such that $\mathrm{Gr}_{\mathbf{1}}(\mathcal{M}^{n,J})$ is stable under this action.

These two problems in the case of J consisting of one Jordan block and dimension vectors of the form $(k, \ldots, k)$ with $k \geq 1$ have been tackled in [7], more precisely, in Remark 4.1 and Section 4.1 respectively. There we may find that the torus T' *is* maximal in the automorphism group $\mathrm{Aut}(\mathcal{M}^{n,J})$ of quiver representation automorphisms.

Suppose now that $J : \mathbb{C}^N \to \mathbb{C}^N$ consists of M Jordan blocks. By definition we have

$$\mathrm{Aut}(\mathcal{M}^{n,J}) = \{(\varphi_i)_{i \in \mathbb{Z}/n\mathbb{Z}} \in (\mathrm{GL}_N(\mathbb{C}))^n : \varphi_{i+1} J = J \varphi_i \text{ for all } i\}.$$

The torus T' can be viewed as the 'diagonal' torus inside $\mathrm{Aut}(\mathcal{M}^{n,J})$ – this is the largest torus such that any $(\varphi_i) \in T'$ is diagonal. In fact, projecting T' to any $\mathrm{GL}_N(\mathbb{C})$ we observe that T' becomes the maximal diagonal torus of $\mathrm{GL}_N(\mathbb{C})$ as T' acts with different weights on each entry of the diagonal. Thus T' is in fact a maximal torus.

In [7] one can find an explicit description of $\mathrm{Aut}(\mathcal{M}^{n,J})$ in the case of J being a single Jordan block.

PROPOSITION 2.5 ([7, Proposition 3.6]). *Let $J : \mathbb{C}^n \to \mathbb{C}^n$ be a nilpotent single Jordan block of size $n \times n$. With respect to the standard basis (that is, the basis $(v_k)_{k=1}^n$ such that $Jv_k = v_{k+1}$ for $k < n$ and $Jv_n = 0$), elements of $\mathrm{End}(\mathcal{M}^{n,J})$ are tuples $(\varphi_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ such that the matrix φ_i consists of the following columns:*

$$\varphi_i = (\varphi_i(v_1), J\varphi_{i-1}(v_1), \dots, J^{n-1}\varphi_{i-n+1}(v_1)), \tag{2.3}$$

REMARK 2.6. More explicitly, the first column of φ_i is free, and the second column comes from φ_{i-1}: it is $\varphi_{i-1}(v_1)$ shifted down by one index, the rest of the columns follow an analogous construction. Such a tuple is invertible, i.e. lies in $\mathrm{Aut}(\mathcal{M}^{n,J})$, if and only if for every i, the first entry in the vector $\varphi_i(v_1)$ is nonzero.

We apply this to arrive at a description of $\mathrm{End}(\mathcal{M}^{n,J})$, where $J : \mathbb{C}^N \to \mathbb{C}^N$ consists of M Jordan blocks. More precisely, let us decompose $\mathbb{C}^N = V_1 \oplus \cdots \oplus V_M$ into Jordan subspaces of each of the M Jordan blocks of J and let $J = J_1 \oplus \cdots \oplus J_M$ be the corresponding decomposition into Jordan blocks. Suppose that $(\varphi_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ lies in $\mathrm{End}(\mathcal{M}^{n,J})$ and let $\varphi_i^{l,k}\colon V_k \to V_l$ be the restriction of φ_i to a map between the Jordan subspaces V_k and V_l. We call such a restriction a *block* of size $j_l \times j_k$ inside φ_i, where j_l and j_k are the dimensions of V_l and V_k respectively. Then the equation

$$\varphi_{i+1}J = J\varphi_i$$

translates to a system of equations between blocks:

$$\varphi_{i+1}^{l,k}J_k = J_l\varphi_i^{l,k}.$$

The same argument as in the proof of [7, Proposition 3.6] applies to this situation as well, meaning that each block $\varphi_i^{l,k}$, regarded as a matrix in the standard basis, is of the form analogous to (2.3), i.e. the first column is free, the second column is the shifted version of the first column of $\varphi_{i-1}^{l,k}$, and so on.

The description of $\mathrm{End}(\mathcal{M}^{n,J})$ is therefore pretty easy: it is a grid of 'lower triangular' matrices. The description of $\mathrm{Aut}(\mathcal{M}^{n,J})$ becomes much more complex than in the one block scenario.

3. Białynicki-Birula decomposition. Keeping the notation from the previous section, we focus on the $\mathbb{C}^\times$-action on $X_{n,J}$ given by (2.1) and the induced BB-decomposition of $X_{n,J}$. We start with the known description of the fixed point set.

Proposition 3.1 ([5, Theorem 1, Proposition 1]). *Fixed points of $X_{n,J}$ under both torus actions consist of representations that have the same vector spaces spanned by the fixed basis $\{v_i^l : i = 1,\dots,j_l,\ l = 1,\dots,M\}$ over all vertices. These subrepresentations are in bijective correspondence with those successor closed subquivers $S \subseteq Q(\mathcal{M}^{n,J})$ such that for any vertex $i \in \Delta_n$ there is exactly one basis element of the vector space over i that is a vertex of S. Here, $Q(\mathcal{M}^{n,J})$ is the coefficient quiver of $\mathcal{M}^{n,J}$ defined in Notation 2.2.*

In particular, any fixed point $p \in X^T_{n,J}$, considered as a subquiver of $Q(\mathcal{M}^{n,J})$ decomposes into connected components.

Definition 3.2. Let $p \in X^T_{n,J}$ be a fixed point and $S_p \subseteq Q(\mathcal{M}^{n,J})$ be the subquiver defined by p. Then S_p decomposes into connected successor-closed subquivers

$$S_p = (\sigma_1, \dots, \sigma_k),$$

where $\sigma_i \subseteq S_p$ are the maximal connected subquivers of S_p. Each σ_i is called a *movable part in p* and any such σ_i defines a representation of the A-type equioriented quiver

$$\sigma_i \rightsquigarrow (\langle v_{j_l-m}\rangle, \langle v_{j_l-m+1}\rangle, \dots, \langle v_{j_l}\rangle)$$

for some $m \geq 0$ and $l = 1,\dots,M$. We shall say that σ_i is of length m and comes from the lth block. To fully define σ_i we also need either the starting vertex or the terminal vertex, since σ_i corresponds to a part of p.

Here we turn to Young tableaux. Our fixed Jordan matrix J corresponds to a Young tableau of shape J, which we denote by $\mathcal{J}$. Each box corresponds to a basis vector v_i^l in $\mathbb{C}^N$. We can fill $\mathcal{J}$ with $\mathbb{C}^\times$-weights that act on these vectors, getting a nice table of weights.

Example 3.3. Consider the case $J = J_4 \oplus J_3 \oplus J_3 \oplus J_2$. Then we get the following Young tableaux:

$$J \rightsquigarrow \begin{array}{|c|c|c|c|}\hline v_4^1 & v_3^1 & v_2^1 & v_1^1 \\ \hline v_3^2 & v_2^2 & v_1^2 & \\ \cline{1-3} v_3^3 & v_2^3 & v_1^3 & \\ \cline{1-3} v_2^4 & v_1^4 & & \\ \cline{1-2} \end{array} \qquad \mathbb{C}^\times\text{-weights} = \begin{array}{|c|c|c|c|}\hline 1 & -3 & -7 & -11 \\ \hline 2 & -2 & -6 & \\ \cline{1-3} 3 & -1 & -5 & \\ \cline{1-3} 4 & 0 & & \\ \cline{1-2} \end{array}$$

We can endow the set of basis elements with a ‘twisted’ lexicographic order. We state the exact definition below.

Definition 3.4. On the set of basis elements $\{v_i^l : i = 1, \ldots, j_l,\, l = 1, \ldots, M\}$ consider the twisted lexicographic order

$$v_i^l < v_j^s \iff \text{either } j_s - j < j_l - i, \text{ or } j_s - j = j_l - i \text{ and } l < s.$$

This order is of course equivalent to the order of $\mathbb{C}^\times$-weights, i.e.

$$\mathrm{weight}(v_i^l) < \mathrm{weight}(v_j^s) \iff v_i^l < v_j^s.$$

We also extend this order to a partial order on fixed points: For two fixed points $p, q \in X_{n,J}^T$ we say that $p < q$ if $p_i \leq q_i$ for all $i \in \mathbb{Z}/n\mathbb{Z}$ and $p_j < q_j$ for at least one $j \in \mathbb{Z}/n\mathbb{Z}$.

Remark 3.5. Note that we can check the inequality $p < q$ for two fixed points $p, q \in X_{n,J}^T$ only at vertices either ending or starting movable parts in p. As we are working with subrepresentations of dimension $\mathbf{1}$, these vertices uniquely determine fixed points. This is special only to subrepresentations of this particular dimension vector.

With this we can parametrize the BB-cells.

Proposition 3.6. *Let $p \in X_{n,J}^T$ be a fixed point. Then the BB-cell $\mathcal{C}_p \subset X_{n,J}$ is an affine space.*

This was already proven in [10, Theorem 5.7] in a much more general setting, but for our special case we will present a proof which will result in an explicit and visual description of these cells. Note that in [10, Theorem 5.7] one can also find explicit equations defining these cells.

Proof of Proposition 3.6. Fix $p \in X_{n,J}^T$. First let us consider $i \in \mathbb{Z}/n\mathbb{Z}$ and we shall study

$$(\mathcal{C}_p)_i = \{\nu_i \in \mathbb{P}^{N-1} : \nu \in \mathcal{C}_p\}.$$

Suppose that $p_i = \langle v_j^l \rangle$. Then $(\mathcal{C}_p)_i$ lies in the BB-cell of the ambient variety $\mathcal{C}_i \subseteq \mathbb{P}^{N-1}$ associated to $p_i \in \mathbb{P}^{N-1}$. This space is easy to describe:

$$\mathcal{C}_i = \{u \in \mathbb{C}^N : u_j^l \neq 0, u_k^s = 0 \text{ for all } (k,s) \text{ such that } v_k^s < v_j^l\}.$$

Visually, the points of $\mathcal{C}_i$ look as follows:

$$\begin{array}{|c|c|ccc|c|c|c}
\cline{1-2}\cline{4-6}
u_{j_1}^1 & u_{j_1-1}^1 & \cdots & \multicolumn{1}{|c|}{u_{j+1}^1} & 0 & 0 & \cdots \\
\cline{1-2}\cline{4-6}
u_{j_2}^2 & u_{j_1-1}^1 & \cdots & \multicolumn{1}{|c|}{u_{j+1}^2} & 0 & 0 & \cdots \\
\cline{1-2}\cline{4-6}
\multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{\vdots} & & \multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{\vdots} & \\
\cline{1-2}\cline{4-6}
u_{j_l}^l & u_{j_l-1}^l & \cdots & \multicolumn{1}{|c|}{u_{j+1}^l} & 1 & 0 & \cdots \\
\cline{1-2}\cline{4-6}
\multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{\vdots} & & \multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{\vdots} & \\
\cline{1-2}\cline{4-6}
u_{j_M}^M & u_{j_M-1}^M & \cdots & \multicolumn{1}{|c|}{u_{j+1}^M} & u_j^M & 0 & \cdots \\
\cline{1-2}\cline{4-6}
\end{array}$$

We can choose $u_j^l = 1$ to get a parametrization of $\mathcal{C}_i$ as an affine space. The BB-cell of the quiver Grassmannian is the intersection

$$\mathcal{C}_p = \Big(\prod_{i=0}^{n-1} \mathcal{C}_i\Big) \cap X_{n,J}.$$

To pass to another vertex we distinguish two cases:

(1) $Jv_i^l = 0$. Then for any $u \in \mathcal{C}_i$ we have $Ju = 0$ since $v_k^s \geq v_j^l$ implies $Jv_k^s = 0$. Thus vectors from $(\mathcal{C}_p)_i$ do not impose any conditions on $(\mathcal{C}_p)_{i+1}$.
(2) $Jv_i^l = v_{i+1}^l$ is nonzero. Then for any $\nu \in \mathcal{C}_p$ we have

$$J\nu_i = \nu_{i+1}.$$

This simply follows from the fact that ν_i must be spanned by a vector which has a nonzero coordinate next to v_i^l, which does not go to 0 under J. So vectors in $(\mathcal{C}_p)_{i+1}$ are completely determined by vectors in $(\mathcal{C}_p)_i$.

Therefore if we have a movable part in p running through vertices i through m, then $(\mathcal{C}_p)_i = \mathcal{C}_i$ is an affine space and $(\mathcal{C}_p)_{i+k} = J^k(\mathcal{C}_i)$ for all $k = 0, \dots, m$. The dimension of $\mathcal{C}_i$ is given by

$$\dim(\mathcal{C}_i) = |\{v_j^s : v_j^s > p_i\}|.$$

The whole cell $\mathcal{C}_p$ is the product of either $\mathcal{C}_i$'s, in case of i being a starting vertex of a movable part in p, or images of elements of $\mathcal{C}_i$ for the other vertices. Thus the dimension of $\mathcal{C}_p$ is the sum of the dimensions of $\mathcal{C}_i$'s for all i that start a movable part in p. ∎

Next, we shall focus on the closures of cells. One of the more crucial results regarding closed cells is that they are smooth, which is exactly what we shall prove now. This is a miracle of the subrepresentation vector being $\mathbf{1} = (1, \dots, 1)$. Even for J being a single Jordan block, if we choose other dimension vectors we get singular closed BB-cells, as observed in [7, Remark 3.15].

Theorem 3.7. *Let $p \in X_{n,J}^T$ be a fixed point. Then the closure of the BB-cell $\mathcal{C}_p$ is a product of smooth quiver Grassmannians.*

Proof. We will provide an explicit description of these subvarieties. First consider the case where $p \in X_{n,J}^T$ is a single movable part. Again, denote by $\overline{\mathcal{C}_i} \subseteq \mathbb{P}^{N-1}$ the closed BB-cell associated to the fixed point p_i for $i \in \mathbb{Z}/n\mathbb{Z}$. Each $\overline{\mathcal{C}_i}$ is isomorphic to a projective space $\mathbb{P}^{d_i-1}$ by forgetting all zeros imposed on vectors spanning lines in $\overline{\mathcal{C}_i}$, where

$$d_i = |\{v_k^s : v_k^s \geq p_i\}|.$$

Let $R_p \in \operatorname{rep}_{\mathbb{C}} A_n$ be the representation of the equioriented A_n-quiver defined by

$$R_p : \mathbb{C}^{d_0} \xrightarrow{\pi_0} \mathbb{C}^{d_1} \xrightarrow{\pi_1} \mathbb{C}^{d_2} \to \cdots \to \mathbb{C}^{d_n},$$

where $\pi_i : \mathbb{C}^{d_i} \to \mathbb{C}^{d_{i+1}}$ is the map induced from J after forgetting those basis vectors v_j^s such that $v_j^s < p_i$. In particular, the π_i's are epimorphisms! By forgetting all zeros in the description of $\overline{\mathcal{C}_p}$ we get an isomorphism

$$\overline{\mathcal{C}_p} \cong \mathrm{Gr}_{\mathbf{1}}(R_p).$$

Since all arrows in R_p are epimorphisms, it follows that the decomposition of R_p into irreducible components is a decomposition into components that are all supported at the first vertex. As these are all injective, we find that $\mathrm{Ext}^1(R_p, R_p) = 0$. From the known theory of acyclic quiver Grassmannians [4, Corollary 4] we deduce that $\mathrm{Gr}_{\mathbf{1}}(R_p)$ is smooth, irreducible and of dimension

$$\dim \mathrm{Gr}_{\mathbf{1}}(R_p) = \langle \mathbf{1}, \mathbf{d} - \mathbf{1} \rangle_{A_{n+1}} = \sum_{i=0}^{n} (d_i - 1) - \sum_{i=0}^{n-1} (d_i - 1) = d_n - 1.$$

If p is comprised of many movable parts, then since each movable part ends at a vector space p_i such that $Jp_i = 0$, we know that $\overline{\mathcal{C}_p}$ is the product of quiver Grassmannians associated to each movable part. ∎

Another crucial property of this BB-decomposition is that the closed cells form a stratification. Since the quiver Grassmannian $X_{n,J}$ has a stratification by affine cells, it is equivariantly formal, which is one of the ingredients baked into the definition of GKM spaces. This was proven in [10].

Proposition 3.8. *The BB-decomposition of $X_{n,J}$ forms a stratification, i.e. if for $p, q \in X_{n,J}^T$ we have $\mathcal{C}_p \cap \overline{\mathcal{C}_q} \neq \emptyset$, then $\mathcal{C}_p \subseteq \overline{\mathcal{C}_q}$.*

Proof. This follows immediately from the fact that if we fix $i \in \mathbb{Z}/n\mathbb{Z}$ and pick two fixed points $p, q \in X_{n,J}^T$, then both p_i and q_i are fixed points in $\mathbb{P}^{N-1}$ under the $\mathbb{C}^\times$-action defined over the ith vertex of Δ_n. In particular, if $\mathcal{C}, \mathcal{D} \subseteq \mathbb{P}^{N-1}$ are the BB-cells corresponding to p_i and q_i respectively in the ambient space, then

$$\mathcal{C} \cap X_{n,J} = (\mathcal{C}_p)_i, \quad \mathcal{D} \cap X_{n,J} = (\mathcal{C}_q)_i.$$

These larger BB-cells do form a stratification, so if $\mathcal{C}_p \cap \overline{\mathcal{C}_q} \neq \emptyset$, then $\mathcal{C}_p \subseteq \overline{\mathcal{C}_q}$, since we are actually comparing the larger BB-cells. ∎

The partial order on fixed points defined in Definition 3.4 lets us neatly describe the partial order of the closed BB-cells.

Theorem 3.9. *For two fixed points $p, q \in X_{n,J}^T$ we have $\overline{\mathcal{C}_p} \subseteq \overline{\mathcal{C}_q}$ if and only if $q \leq p$.*

Proof. This is true for the BB-cells of all $\mathbb{P}^{N-1}$ as the lexicographic order is the order of weights acting on this space. ∎

REMARK 3.10. Theorem 3.9 therefore implies that our poset structure on $X^T_{n,J}$, as defined in Definition 3.4, mimics the Bruhat order on, e.g., flag manifolds. The authors of [8] studied different combinatorial models of fixed points of quiver Grassmannians associated to nilpotent representations defined only using a single Jordan block, but with a different dimension vector of subrepresentations. Our model is closest to the model of juggling patterns, but as we have proven in Theorem 3.9, this model is isomorphic (as a poset) to the poset of closed BB-cells, which was the second model considered in [8]. In our case of dimension of subrepresentations being the vector $\mathbf{1}$ everything degenerates, so our 'juggling patterns' are fully determined by their endpoints. On the other hand, here we also need to keep track of the particular Jordan block where the given part of a juggling pattern lives in, which is something that the authors of [8] did not have to keep track of.

When it comes to calculations in cohomology, this becomes combinatorially intense, even if we restrict ourselves to just a single Jordan block. This particular case has been studied in [7, Appendix B]. We shall give a few observations about this case below.

EXAMPLE 3.11. Suppose that J is a single Jordan block of size N. In this case the BB-decomposition simplifies greatly as we do not have to worry about other Jordan blocks. Our one-dimensional torus acts on each $\mathbb{C}^N$ in $\mathcal{M}^{n,J}$ in the standard way.

In this case, a fixed point $p \in X^T_{n,J}$ corresponds uniquely to a nonempty subset $I_p \subseteq \mathbb{Z}/n\mathbb{Z}$, i.e. the subset of those vertices where a movable part in p ends. In other words, we have

$$I_p = \{i \in \mathbb{Z}/n\mathbb{Z} : Jp_i = 0\}.$$

Subsets that define a fixed point must be nonempty and the difference of two consecutive points in such a subset must be at most N. The partial order from Definition 3.4 simplifies to the usual order on the power set of $\mathbb{Z}/n\mathbb{Z}$.

Open BB-cells are, of course, affine spaces; their dimension obtained from Proposition 3.6 simplifies to

$$\dim \mathcal{C}_p = |I^c_p|.$$

Movable parts in fixed points are parametrized by fixing a vertex (for example fixing the ending vertex of a segment) and length.

Observe that in this case we can restrict ourselves to the case $N \leq n$.

PROPOSITION 3.12. *Let $\mathcal{M}^{n,J}_N$ be the representation of Δ_n which has the same matrix over each arrow J that is a single Jordan block of size N. If*

$N \geq n$, then

$$\mathrm{Gr}_{\mathbf{1}}(\mathcal{M}_N^{n,J}) \cong \mathrm{Gr}_{\mathbf{1}}(\mathcal{M}_n^{n,J}).$$

If $N \leq N'$, then there exists an embedding

$$\mathrm{Gr}_{\mathbf{1}}(\mathcal{M}_N^{n,J}) \hookrightarrow \mathrm{Gr}_{\mathbf{1}}(\mathcal{M}_{N'}^{n,J}).$$

Proof. To see the first part let us assume that $M \in \mathrm{Gr}_{\mathbf{1}}(\mathcal{M}_N^{n,J})$ with $N \geq n$ and fix a vertex $i \in \mathbb{Z}/n\mathbb{Z}$. Let $M_i = \langle u \rangle$ and define

$$j = \min\,(k : u_k \neq 0).$$

After applying J to u n times we arrive at the condition $J^n u \in \langle u \rangle$. Note that either $j + n > N$, in which case $J^n u = 0$, or $j + n \leq N$, and then

$$\min\,(k : (J^n u)_k \neq 0) = j + n.$$

For the condition $J^n u \in \langle u \rangle$ to be met we must thus have $J^n u = 0$ and this is equivalent to u having 0 as its first $N - n$ coordinates. Thus any element of $\mathrm{Gr}_{\mathbf{1}}(\mathcal{M}_N^{n,J})$ is in fact a subrepresentation of $\mathcal{M}_n^{n,J}$.

For $N \leq N'$ the embedding is simply given by the map $\mathcal{M}_N^{n,J} \to \mathcal{M}_{N'}^{n,J}$, which is induced by the map $\mathbb{C}^N \to \mathbb{C}^{N'}$ that embeds $\mathbb{C}^N$ into the higher-dimensional space by putting an appropriate string of zeroes at the start. ∎

We can endow $X_{n,J}^T$ with a (commutative) semigroup structure by letting $p \cap q$ be the fixed point defined by the union of subsets

$$I_{p\cap q} := I_p \cup I_q \subseteq \mathbb{Z}/n\mathbb{Z},$$

where $I_p, I_q \subseteq \mathbb{Z}/n\mathbb{Z}$ are the subsets corresponding to p and q respectively. Observe that for all $i \in \mathbb{Z}/n\mathbb{Z}$ we have

$$(p \cap q)_i = \min(p_i, q_i), \tag{3.1}$$

where $\min(p_i, q_i)$ is the point in $\mathbb{P}^{N-1}$ spanned by the vector that is the lesser of vectors spanning p_i and q_i.

This construction lets us describe the intersections of closed BB-cells.

PROPOSITION 3.13. *Let $p, q \in X_{n,J}^T$ be two fixed points. Then*

$$\overline{\mathcal{C}_p} \cap \overline{\mathcal{C}_q} = \overline{\mathcal{C}_{p\cap q}}.$$

Proof. This is immediate when we consider BB-cells on $\mathbb{P}^{N-1}$ for fixed $i \in \mathbb{Z}$, since we have (3.1). ∎

4. GKM structure. As we have already recalled in Section 3, the T-variety $X_{n,J}$ is a GKM variety. A priori, a GKM graph is a nonoriented graph whose set of vertices is the set $X_{n,J}^T$ of fixed points and the set of edges is the set of one-dimensional orbits. We can orient it in our case using the $\mathbb{C}^\times$-action on $X_{n,J}$, as in [10]. We give a precise definition below.

Definition 4.1. Let X be a GKM variety with respect to a fixed T-action and $\chi : \mathbb{C}^\times \to T$ be a cocharacter such that the induced $\mathbb{C}^\times$-action on X via χ has the same set of fixed points

$$X^T = X^{\mathbb{C}^\times}.$$

Let $x, y \in X^T$ be two fixed points connected by an edge, i.e. we have a one-dimensional orbit O such that $x, y \in \overline{O}$. We orient this edge as $x \to y$ if for any (and thus all) $\nu \in O$ we have

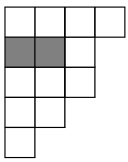
$$\lim_{\lambda \to 0} \lambda \cdot \nu = x,$$

where $\lambda \in \mathbb{C}^\times$ acts on ν via χ.

In particular, if an arrow $p \to q$ exists, then for the orbit O that induces this arrow we have $O \subseteq \mathcal{C}_p$. In [10] there is a description of one-dimensional orbits of any quiver Grassmannian endowed with their torus action and in our case this description becomes simpler.

We can encode movable parts using Young tableaux. If $\mathcal{J}$ is the Young tableau of shape corresponding to the partition corresponding to J, then a movable part S of length m and from the lth block is a coloring of $\mathcal{J}$, where we color the first $m + 1$ boxes in the lth row in $\mathcal{J}$ and leave the rest blank.

Example 4.2. Consider $J = J_4 \oplus J_3 \oplus J_3 \oplus J_2 \oplus J_1$ and a movable part of length 1 in the second block. Its coloring is



These colorings let us visualize fundamental mutations (defined in [10, Definition 6.9]), i.e. edges in the GKM graph of $X_{n,J}$.

Definition 4.3. Let S be a movable part of length m in some fixed point and λ be the coloring of $\mathcal{J}$ associated with S. A *mutation* of S is either

(1) a single coloring λ', whose colored segment is the same length as in λ, but in a lower row; or
(2) two colorings λ_1, λ_2, where λ_1 is a coloring of the same row and length m' and λ_2 is a coloring of any row and first m'' elements, so that

$$m' + m'' = m + 1.$$

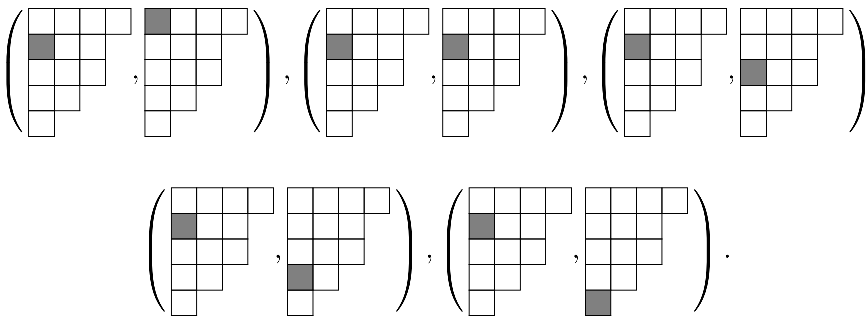
In other words, we split the colored boxes in λ into two colorings that correspond to movable parts, with the restriction that at least one movable part must be from the same block as λ.

Remark 4.4. Now is the perfect time to note that we do not use the same definitions of a movable part and mutations as in [10, Definitions 6.8 and 6.9].

Let $S \subseteq Q(\mathcal{M}^{n,J})$ be a successor-closed subquiver. Then movable parts of S, according to [10], are those connected subquivers of S that start at a vertex where a segment of S starts. Movable parts for us are just segments. The difference also lies in mutations – we have two different kinds of mutations that correspond to either moving a whole segment to a lower Jordan block or splitting a segment into two segments. In [10] a fundamental mutation of a movable part is an operation that moves the whole movable part to a lower part of the coefficient quiver. In other words, our movable parts parametrize many movable parts as in [10], but we have different mutations, whereas there is only one kind of mutation in [10].

Example 4.5. Consider the movable part from Example 4.2. Then there are two possible mutations of type 1:

We can move both gray boxes together only until we hit the second to last row – in the last row we have no room to fit two boxes horizontally. There are also five possible mutations of type 2:



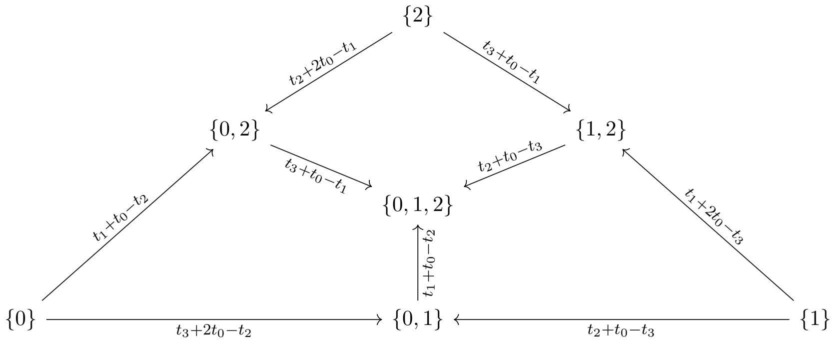
Fundamental mutations of type 2 will always leave one of the diagrams unchanged, namely, in all above examples the diagram on the left stays fixed and we can put the one remaining gray box wherever we want, which results in five possible mutations.

One can imagine the colored boxes as, for example, a stack of cards in a game of solitaire. Mutations correspond to either moving the whole stack 'downwards' or making two stacks, but we can put the new stack wherever we want (we can put the stacks on top of each other in this scenario). These actions characterize exactly the edges of the GKM graph of $X_{n,J}$, as is illustrated by the following proposition. The general version of the proposition below was proven in [10, 11].

Proposition 4.6 ([10, Theorem 6.15]). *Let $p, q \in X_{n,J}^T$ be two fixed points. Then an arrow $p \to q$ in the GKM graph of $X_{n,J}$ exists if and only if q is obtained from p by a mutation of exactly one movable part in p.*

Before the proof we shall give an example of a GKM graph.

Example 4.7. Let J be a single Jordan block of size 3 and suppose we have $n = 3$ vertices. We will denote the fixed points by the subsets of $\{0, 1, 2\}$ that they define. Mutations in this case can only correspond to the action of dividing movable parts into two, which on the level of subsets correspond to adding a single element to the subset defining a fixed point. The GKM graph of $X_{3,J}$ in this case turns out to be



Proof of Proposition 4.6. For two fixed points $p, q \in X_{n,J}^T$ let $p + q$ be the subrepresentation given by

$$(p + q)_i = \langle v_i^l + v_j^s \rangle,$$

where $i \in \mathbb{Z}/n\mathbb{Z}, p_i = \langle v_i^l \rangle$ and $q_i = \langle v_j^s \rangle$.

It is easy to see that if we assume that q is obtained from p by a mutation of exactly one movable part in p, then $p_i = q_i$ for all vertices except those that correspond to the movable part in p that gets mutated to create q. On this movable part we have $p_i < q_i$. To see the arrow $p \to q$, we just take the orbit $T \cdot (p + q)$. The representation $p + q$ has one nonzero coordinate everywhere outside one movable part in p, so the orbit of this element is one-dimensional and

$$\lambda \cdot (p + q) \xrightarrow{\lambda \to 0} p, \quad \lambda \cdot (p + q) \xrightarrow{\lambda \to \infty} q.$$

Now suppose that there exists an orbit $O \subseteq X_{n,J}$ such that $p, q \in \overline{O}$ and $O \subseteq \mathcal{C}_p$. Since for any $\nu \in O$ we have $\lambda \cdot \nu \xrightarrow{\lambda \to \infty} q$, it follows that we must have $\overline{\mathcal{C}_q} \subseteq \overline{\mathcal{C}_p}$ and by Theorem 3.9 we have $p_i \leq q_i$ for all i. Let us pick an element $\nu \in O \subseteq \mathcal{C}_p$. By the explicit description of the BB-cells in Proposition 3.6 we know that ν is determined by vector spaces put over the vertices which are starting movable parts in p. Since the orbit $O = T \cdot \nu$ has to be one-dimensional, we can only have one movable part in p over which q

differs from p, since we would have an orbit of larger dimension otherwise. We can therefore assume that p is a single movable part, say starting at the ith vertex. The whole T-orbit is purely determined by what happens at the ith vertex, since lines in other vertices are just images under J. This reduces the case to $\mathbb{P}^{N-1}$. We know the one-dimensional orbits there: in the GKM graph of $\mathbb{P}^{N-1}$ under the induced torus action an edge $p_i \to q_i$ exists if and only if $p_i < q_i$. Let $p_i = \langle v_j^l \rangle, q_i = \langle v_k^s \rangle$ and λ_p, λ_q be colorings of $\mathcal{J}$ associated to p and q respectively. We can distinguish two cases:

(1) If $j = k$, then we must have $l < s$. In this case λ_q came from λ_p by moving the whole colored row in λ_p down.
(2) If $j < k$, then λ_q came about from dividing the colored row in λ_p into two parts. We split this row at the $(i + j_s - j)$th vertex. ∎

We also need to describe the labels of the edges of the GKM graph, which is done below. The general formula can be found in [10, Theorem 6.15] or [11, Therem 2.20].

Proposition 4.8 ([10, Theorem 6.15]). *Suppose that in the GKM graph of $X_{n,J}$ there exists an edge $p \to q$ between two fixed points $p, q \in X_{n,J}^T$. Then the label of this edge can be chosen to be the following polynomial:*

$$w(p, q) = t_{m+j,l} + (j_l - j_k + i - m)t_0 - t_{i+j,l},$$

where q is created by mutating a movable part in p that starts at the jth vertex, is of length m and comes from the lth block. The augmentation occurs at the $(i + j)$th vertex.

Proof. We need the weight of the T-action on which we act on the $\mathbb{P}^1$ that is equiavariantly isomorphic to the closed orbit $\overline{T \cdot (p + q)}$. The isomorphism $\overline{T \cdot (p + q)} \to \mathbb{P}^1$ is given by looking at the vector space lying over the starting point of the movable part in p which we augment to obtain q.

Suppose first that this movable part in p starts at the 0th vertex, is of length m and comes from the lth block; in particular $p_i = \langle v_{j_l - m}^l \rangle$. Suppose further that $q_i = \langle v_{j_k - i}^k \rangle$, i.e. the augmentation occurs at the ith vertex and at one of the movable parts from the kth block. The torus acts on these vectors with weights

$$w_1 = t_0^{j_l - m - 1} t_{m,l}, \quad w_2 = t_0^{j_k - i - 1} t_{i,l}.$$

So the torus acts on the closed orbit identified with $\mathbb{P}^1$ with weight

$$w = t_{m,l} + (j_l - j_k + i - m)t_0 - t_{i,l}.$$

The above polynomial is also a weight in the torus representation $T_q \overline{\mathcal{C}_p}$. Now, if the movable part in p that we are augmenting starts at the jth vertex, then we have to translate everything by j:

$$w(p, q) = t_{m+j,l} + (j_l - j_k + i - m)t_0 - t_{i+j,l}.$$

This is the label of the edge $p \to q$ in the GKM graph of $X_{n,J}$ and a weight in the torus representation $T_q\overline{\mathcal{C}_p}$. ∎

REMARK 4.9. There is a natural $(\mathbb{Z}/n\mathbb{Z})$-action on $X_{n,J}$, which comes from rotating the quiver in the direction given by the arrows in Δ_n. Since the $\mathbb{C}^\times$-action defined by formula (2.1) is independent of the vertex over which we define this action, it follows that the $(\mathbb{Z}/n\mathbb{Z})$-action is immediately seen to lift to the set of fixed point $X_{n,J}^T$ as well as the set of (closed) BB-cells. The T-action on $X_{n,J}$ commutes with the $(\mathbb{Z}/n\mathbb{Z})$-action up to a twist by $\rho : T \to T$ and so, similarly to the case of Schubert classes in the usual Grassmannians, this gives a particular symmetry on the GKM graph of $X_{n,J}$. Namely, if $\alpha \in H_T^*(X_{n,J})$ is a class, then for the generator $\tau \in \mathbb{Z}/n\mathbb{Z}$ we can set

$$(\tau \cdot \alpha)|_p = \alpha|_{\tau(p)} \circ \rho, \tag{4.1}$$

where $\rho : H_T^*(*) \to H_T^*(*)$ is the map induced by $\rho : T \to T$ defined by (2.2).

Such group actions were considered in [8, Section 3.3]. The authors of [8] considered the case of quiver Grassmannians made up of nilpotent representations constructed from a single Jordan block, similarly to [7], and more general dimension vectors.

We are now ready to delve into the analysis of the multiplicative structure of $H_T^*(X_{n,J})$.

5. Multiplicative structure of equivariant cohomology. This last section is devoted to analyzing the multiplicative structure of the equivariant cohomology ring $H_T^*(X_{n,J})$. M. Lanini and A. Pütz provide a construction of the Knutson–Tao basis for equivariant cohomology [11, Theorem 3.9] for certain quiver Grassmannians. This basis depends on a chosen orientation of the GKM graph of $X_{n,J}$ using cocharacters, as we recalled in Definition 4.1. We give a precise definition below:

DEFINITION 5.1 ([15, Definition 2.12]). Let X be any GKM variety under a fixed T-action and $\chi : \mathbb{C}^\times \to T$ be a cocharacter such that the induced $\mathbb{C}^\times$-action by χ preserves fixed points $X^T = X^{\mathbb{C}^\times}$. This induced $\mathbb{C}^\times$-action determines an orientation on the GKM graph of X as in Definition 4.1. A set $\{q^x : x \in X^T\}$ is called a *Knutson–Tao basis* if its elements satisfy the following:

- For each $y \in X^T$ the polynomial $q^x|_y$ is either 0 in the case that there is no oriented path from y to x or is homogeneous of degree $\deg(q^x|_x)$ in the case where there is such a path.
- The restriction $q^x|_x$ is the product of weights associated to edges going from x in the GKM graph of X.

If a Knutson–Tao basis exists, then it is a basis of $H^*_T(X)$ as an $H^*_T(*)$-module [15, Proposition 2.13]. If our GKM variety X is Palais–Smale, meaning that X satisfies the hypothesis of Definition 5.1 and for each edge $x \to y$ in the GKM graph of X the number of edges going out of x is strictly greater than the amount of edges going out of y, then this basis is unique [15, Lemma 2.16].

One other remark is that the orientation of the GKM graph needs to have no oriented cycles. For us this is true as our order on fixed points agrees with the (inverted) order of the poset of closed BB-cells, according to Theorem 3.9. Picking such an orientation that agrees with the poset of closed BB-cells was also done in [11] for a certain family of quiver Grassmannians, called homogeneous representations. Note that quiver Grassmannians constructed with Jordan matrices built with more than one Jordan block do not fit into this family.

This basis behaves well with respect to the order on X^T given by paths in the graph. In our case this order coincides with the inverted lexicographic order. These bases arose from the Schubert basis for equivariant cohomology of the usual Grassmannians in [9]. Similarly to flag varieties and usual Grassmannians, a central question regarding these basis is how they multiply. Since $\{q^x : x \in X^T\}$ is a basis of $H^*_T(X)$ as an $H^*_T(*)$-module, it follows that the computation of the structure constants

$$q^x \cdot q^y = \sum_{z \in X^T} c^z_{x,y} q^z$$

determines the multiplicative structure of $H^*_T(X)$, considered as an $H^*_T(*)$-algebra.

In our case $X = X_{n,J}$, each cell $\overline{\mathcal{C}_x}$ gives an equivariant fundamental class $[\overline{\mathcal{C}_x}] \in H^T_*(X_{n,J})$ and the set of fundamental classes is a basis of equivariant homology. By Theorem 3.7 the closed BB-cells are smooth and so integration over such a cell is much simpler to grasp. Using this pairing we can get the dual basis $\{p^x \in H^*_T(X_{n,J}) : x \in X^T_{n,J}\}$, characterized by

$$\int_{\overline{\mathcal{C}_y}} p^x|_{\overline{\mathcal{C}_y}} = \delta_{x,y}.$$

Using the Atiyah–Bott and Berline–Vergne formula we can further rewrite the above formula as

$$\int_{\overline{\mathcal{C}_y}} p^x|_{\overline{\mathcal{C}_y}} = \sum_{z \in X^T_{n,J} : z \in \overline{\mathcal{C}_y}} \frac{p^x|_z}{e(T_z\overline{\mathcal{C}_y})} = \delta_{x,y},$$

where $e(T_z\overline{\mathcal{C}_y})$ is the product of weights of the torus representation $T_z\overline{\mathcal{C}_y}$. The above formula was originally proven in [3], but one can find more modern approaches in [1]. These products of weights can be read off directly from the GKM graph of $X_{n,J}$. Namely, for a fixed point $x \in X_{n,J}^T$ we first consider the subgraph of $X_{n,J}$ spanned by all oriented paths starting from x. For any $y \in X_{n,J}^T$ such that $y \in \overline{\mathcal{C}_x}$ we find that the vertex corresponding to y lies in this subgraph. To get the Euler class of the torus representation $T_y\overline{\mathcal{C}_x}$ we take the product of the labels of all edges either from or to y in this subgraph. However, one has to be careful with the sign of these labels – normally, labels of edges in the GKM graph are defined up to a scalar multiple and since we need weights of the torus representation $T_y\overline{\mathcal{C}_x}$, one has to take the appropriately scaled generator of the ideal that generates labels of edges in the GKM graph of $X_{n,J}$. In Proposition 4.8 we give the appropriately scaled labels.

Example 5.2 ([7, Example 5.6]). Take $n = 3$ and $J = J_3$. Then there are 7 fixed points associated to nonempty subsets of $\{0, 1, 2\}$. The GKM graph of X_{3,J_3} in this case is given in Example 4.7. Let us order the coordinates of $H_T^*(X_{n,J}^T) \cong H_T^*(*)^7$ according to the following order of fixed points:

$$(\{0\}, \{1\}, \{2\}, \{0, 1\}, \{1, 2\}, \{0, 2\}, \{1, 2, 3\}).$$

The Knutson–Tao basis of $H_T^*(X_{3,J_3})$ consists of three types of elements:

(1) The identity $1 \in H_T^*(X_{3,J_3})$, dual to the smallest cell $\overline{\mathcal{C}_{\{0,1,2\}}}$.
(2) Three elements dual to the medium-sized cells, i.e. those corresponding to two-element subsets

$$p^{\{0,1\}} = (t_1 + t_0 - t_2, t_1 + 2t_0 - t_3, 0, t_1 + t_0 - t_2, 0, 0, 0),$$
$$p^{\{1,2\}} = (0, t_2 + t_0 - t_3, t_2 + 2t_0 - t_1, 0, t_2 + t_0 - t_3, 0, 0),$$
$$p^{\{0,2\}} = (t_3 + 2t_0 - t_2, 0, t_3 + t_0 - t_1, 0, 0, t_3 + t_0 - t_1, 0).$$

(3) Three elements dual to the largest cells:

$$p^{\{0\}} = ((t_2 - 2t_0 - t_3)(t_2 - t_0 - t_1), 0, \dots, 0),$$
$$p^{\{1\}} = (0, (t_3 - 2t_0 - t_1)(t_3 - t_0 - t_2), 0, \dots, 0),$$
$$p^{\{2\}} = (0, 0, (t_1 - 2t_0 - t_2)(t_1 - t_0 - t_3), 0, \dots, 0).$$

From simple linear algebra one can immediately observe that this dual basis is always a Knutson–Tao basis. The authors of [11, Theorem 3.22] define a family of quiver Grassmannians for which a Knutson–Tao basis exists (and is unique); our GKM space $X_{n,J}$ is a member of this family.

Elements of this dual basis are homogeneous (we grade $H_T^*(X_{n,J})$ in the usual fashion, i.e. $\deg t_0 = \deg t_{i,j} = 2$ for all i, j); the degree of p^x is given by the dimension of the cell it corresponds to:

$$\deg(p^x) = 2 \dim(\overline{\mathcal{C}_x}).$$

Let $c^z_{x,y}$ be the structure constants for the basis $\{p^x\}_{x\in X^T_{n,J}}$, i.e.

$$p^x \cdot p^y = \sum_{z\in X^T} c^z_{x,y} p^z.$$

Taking the integral on both sides over $\overline{\mathcal{C}_z}$ we find that the structure constants are given by

$$c^z_{x,y} = \int\limits_{\overline{\mathcal{C}_z}} p^x|_{\overline{\mathcal{C}_z}} \cdot p^y|_{\overline{\mathcal{C}_z}} = \sum_{s\in X^T_{n,J}:\, \overline{\mathcal{C}_x}\cup\overline{\mathcal{C}_y}\subseteq\overline{\mathcal{C}_s}\subseteq\overline{\mathcal{C}_z}} \frac{p^x|_s \cdot p^y|_s}{e(T_s\overline{\mathcal{C}_z})},$$

The latter equation is obtained by applying the Atiyah–Bott and Berline–Vergne formula for the resulting integral. We can outline that the basis $\{p^x : x \in X^T_{n,J}\}$ satisfies the same symmetry as the closed BB-cells that these classes are dual to, i.e. if $\tau \in \mathbb{Z}/n\mathbb{Z}$ is the generator, then

$$p^{\tau(x)} = \tau(p^x),$$

where τ acts on a fixed point by rotation of the quiver and on $H^*_T(X_{n,J})$ by formula (4.1).

Example 5.3. To see this symmetry we can turn to the explicitly written basis in Example 5.2. Observe that in this case τ acts on fixed points in the following way:

$$\{0\} \xrightarrow{\tau} \{1\} \xrightarrow{\tau} \{2\},$$
$$\{0,1\} \xrightarrow{\tau} \{1,2\} \xrightarrow{\tau} \{0,2\},$$
$$\{0,1,2\} \xrightarrow{\tau} \{0,1,2\}.$$

Further, $\rho : T \to T$ is the automorphism satisfying

$$\rho(t_0) = t_0, \quad \rho(t_1) = t_2, \quad \rho(t_2) = t_3, \quad \rho(t_3) = t_1.$$

One can therefore see that the formulas given in Example 5.2 agree with this rotational symmetry. For example, on the largest cells we get

$$\tau(p^{\{0\}}) = (0, (t_3 - 2t_0 - t_1)(t_3 - t_0 - t_2, 0, \dots, 0) = p^{\{1\}}.$$

6. Generalizations. The result of Theorem 3.7 is a very special case for the dimension vector being $\mathbf{1}$. In [7, Remark 3.15] one can find an example of a quiver Grassmannian of the form $\mathrm{Gr}_{\mathbf{k}}(\mathcal{M}^{n,J_n})$, where $\mathbf{k} = (k, \dots, k)$ and $J_n : \mathbb{C}^n \to \mathbb{C}^n$ is just a single Jordan block of size n, such that the Grassmannian has nonsmooth irreducible components.

One thing that one might expect is that for dimension vectors $\mathbf{k} = (k, \dots, k)$, any n and any nilpotent operator J, the quiver Grassmannian $\mathrm{Gr}_{\mathbf{k}}(\mathcal{M}^{n,J})$ should have equidimensional irreducible components – this was proved in the case of J being a single Jordan block in [7]. For general nilpotent operators J this property also fails to hold. Consider the operator

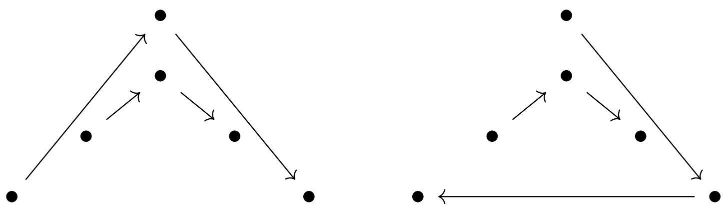
$J = J_3 \oplus J_3$ that consists of two 3×3 Jordan blocks, $n = 3$ and dimension vector $\mathbf{2} = (2, 2, 2)$. As described in [10], we can endow $\mathrm{Gr}_{\mathbf{2}}(\mathcal{M}^{3,J})$ with an action of a torus T such that $\mathrm{Gr}_{\mathbf{2}}(\mathcal{M}^{3,J})$ is a GKM variety and

- fixed points correspond to successor-closed subquivers of the coefficient quiver of $\mathcal{M}^{3,J}$ that have two basis vectors over each vertex of Δ_3,
- given two fixed points $p, q \in (\mathrm{Gr}_{\mathbf{2}}(\mathcal{M}^{3,J}))^T$, a one-dimensional orbit $p \to q$ exists if and only if q is obtained from p by moving down exactly one movable part.

Observe that 'moving down a movable part' from a fixed point p (for representations $\mathcal{M}^{n,J}$) is either

- interchanging a whole segment in p into a segment of the same length but lower with respect to ordering of Jordan blocks, or
- splitting a segment into two smaller segments; the new segment that inherited the endpoint remains in the same block, but the other new segment can be from any block.

In particular, if a one-dimensional orbit $p \to q$ exists, then q has either the same number of segments as p or one more segment, and the same holds true for the set of vertices of Δ_n over which a segment from q starts. Coming back to our example, consider the following two fixed points of $\mathrm{Gr}_{\mathbf{2}}(\mathcal{M}^{3,J_3\oplus J_3})$:



p $\qquad\qquad$ q

So p consists of two segments that both start at the 0th vertex and q consists of two segments that start at the 0th and 1st segment respectively. We pick q so that both segments come from the lower Jordan block. We claim that the closed BB-cells $\overline{\mathcal{C}_p}$ and $\overline{\mathcal{C}_q}$ form irreducible components of $\mathrm{Gr}_{\mathbf{2}}(\mathcal{M}^{3,J_3\oplus J_3})$ and that they are of different dimensions.

If $\overline{\mathcal{C}_p}$ were not an irreducible component, then it would be in some other closed BB-cell, i.e. we could find $s \in \mathrm{Gr}_{\mathbf{2}}(\mathcal{M}^{3,J_3\oplus J_3})^T$ such that there exists an edge $s \to p$ in the oriented GKM graph of $\mathrm{Gr}_{\mathbf{2}}(\mathcal{M}^{3,J_3\oplus J_3})$. But this means we could obtain p either by splitting segments at some other fixed point or by moving segments down; but this is not possible, since p is constructed by taking both segments of maximal length. The same argument works for $\overline{\mathcal{C}_q}$. Thus these two fixed points induce irreducible components of $\mathrm{Gr}_{\mathbf{2}}(\mathcal{M}^{3,J_3\oplus J_3})$.

After some simple analysis of the BB-cells one can see that $\overline{\mathcal{C}_p}$ is of dimension 8, whereas $\overline{\mathcal{C}_q}$ is of dimension 6, thus making $\mathrm{Gr}_{\mathbf{2}}(\mathcal{M}^{3,J_3\oplus J_3})$ not even equidimensional.

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REFERENCES

[1] D. Anderson and W. Fulton, *Equivariant Cohomology in Algebraic Geometry*, Cambridge Univ. Press, 2023.

[2] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras*, Cambridge Univ. Press, 2007.

[3] M. F. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology 23 (1984), 1–28.

[4] P. Caldero and M. Reineke, *On the quiver Grassmannian in the acyclic case*, J. Pure Appl. Algebra 212 (2008), 2369–2380.

[5] G. Cerulli Irelli, *Quiver Grassmannians associated with string modules*, J. Algebraic Combin. 33 (2011), 259–276.

[6] G. Cerulli Irelli, *Three lectures on quiver Grassmannians*, in: Representation Theory and Beyond, Contemp. Math. 758, Amer. Math. Soc., 2020, 57–89.

[7] E. Feigin, M. Lanini, and A. Pütz, *Totally nonnegative Grassmannians, Grassmann necklaces, and quiver Grassmannians*, Canad. J. Math. 75 (2023), 1076–1109.

[8] E. Feigin, M. Lanini, and A. Pütz, *Generalized juggling patterns, quiver grassmannians and affine flag varieties*, Math. Z. 308 (2024).

[9] A. Knutson and T. Tao, *Puzzles and equivariant cohomology of Grassmannians*, Duke Math. J. 119 (2003), 221–260.

[10] M. Lanini and A. Pütz, *GKM-theory for torus actions on cyclic quiver Grassmannians*, Algebra Number Theory 17 (2023), 2055–2096.

[11] M. Lanini and A. Pütz, *Permutation actions on quiver Grassmannians for the equioriented cycle via GKM-theory*, J. Algebraic Combin. 57 (2023), 915–956.

[12] A. Pütz and M. Reineke, *Desingularizations of quiver Grassmannians for the equioriented cycle quiver*, Pacific J. Math. 326 (2023), 109–133.

[13] M. Reineke, *Every projective variety is a quiver Grassmannian*, Algebr. Represent. Theory 16 (2012), 1313–1314.

[14] A. Schofield, *General representations of quivers*, Proc. London Math. Soc. 65 (1992), 46–64.

[15] J. S. Tymoczko, *Permutation representations on Schubert varieties*, Amer. J. Math. 130 (2008), 1171–1194.

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