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CONSTRUCTIONS OF SYMMETRICAL SEPARABLE EQUIVALENCES AND THEIR APPLICATIONS

BY

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Abstract. Let Λ and Γ be symmetrically separably equivalent Artin algebras. We prove that there exist symmetrical separable equivalences between certain endomorphism algebras. As applications, we provide several methods to construct symmetrical separable equivalences from given ones and discuss when the rigidity dimension is invariant under symmetrical separable equivalence. Moreover, we show that symmetrical separable equivalence preserves the Frobenius-finite type, an Auslander-type condition, the (strong) Nakayama conjecture and the Auslander–Gorenstein conjecture.

1. Introduction. As a weakening of Morita equivalence, the notion of (symmetrical) separable equivalence was introduced in [20, 22], with many interesting applications (see [3, 4, 5, 20, 21, 22, 23, 36] for details). The fundamental examples are separable split Frobenius ring extensions and stable equivalences of adjoint type [34]. It was proved in [21] that two rings Λ and Γ are symmetrically separably equivalent if and only if they are linked by a biseparable Frobenius bimodule. It has been known that two symmetrically separably equivalent Artin algebras share many important properties of homology and representation, such as global dimension, representation dimension, finitely generated cohomology, and dominant dimension (see [3, 20, 23, 36] for details).

The first part of the present paper is devoted to providing several methods to produce a new symmetrical separable equivalence from a given one. Our main construction is the following result.

THEOREM A (Theorem 3.1). *Let Λ and Γ be Artin algebras symmetrically separably equivalent by means of bimodules ${}_{\Lambda}M_{\Gamma}$ and ${}_{\Gamma}N_{\Lambda}$. Let U be a finitely generated Λ-module and V a finitely generated Γ-module with $N \otimes_{\Lambda} U \in \operatorname{add}{}_{\Gamma}V$ and $M \otimes_{\Gamma} V \in \operatorname{add}{}_{\Lambda}U$. Then the endomorphism algebras $\operatorname{End}{}_{\Lambda}U$ and $\operatorname{End}{}_{\Gamma}V$ are symmetrically separably equivalent.*

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As applications of Theorem A, we have the following constructions which show that there do exist algebras and modules satisfying all conditions in Theorem A.

THEOREM B (Theorems 3.2 and 3.4). *Let Λ and Γ be symmetrically separably equivalent Artin algebras.*

(1) *Assume that Λ and Γ are representation-finite. Then their Auslander algebras are also symmetrically separably equivalent.*
(2) *The Frobenius parts of Λ and of Γ are symmetrically separably equivalent.*

As another consequence of Theorem A, we construct a new symmetrical separable equivalence from a given stable equivalence of adjoint type and a tilting module.

THEOREM C (Theorem 3.7). *Let Λ and Γ be stably equivalent of adjoint type by means of ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$, and let m be a positive integer. If T is an m-tilting Λ-module, then the endomorphism algebras* $\operatorname{End}_\Lambda T$ *and* $\operatorname{End}_\Gamma(N \otimes_\Lambda T)$ *are symmetrically separably equivalent.*

As applications of the theorems above, the invariant of the rigidity dimension, the Frobenius-finite type and the tilting property of algebras under symmetrical separable equivalence are investigated.

The second part of this paper is devoted to investigating invariant properties of the Auslander-type condition and we obtain the following result.

THEOREM D (Theorem 4.5). *Let Λ and Γ be left and right noetherian rings. Suppose that Λ and Γ are symmetrically separably equivalent. Then Λ satisfies the Auslander-type condition if and only if Γ does.*

All proofs of the results above will be given in Sections 3 and 4 after we recall some necessary notations and results in Section 2.

2. Preliminaries. Throughout this paper, all rings are left and right noetherian rings, and all modules are left modules unless stated otherwise. For a ring Λ, we denote by $\operatorname{Mod}\Lambda$ (resp. $\operatorname{mod}\Lambda$) the category consisting of all (resp. finitely generated) left Λ-modules. We write $\operatorname{pd}_\Lambda M$ (resp. $\operatorname{id}_\Lambda M$, $\operatorname{fd}_\Lambda M$) for the projective dimension (resp. injective dimension, flat dimension) of M, and the global dimension of Λ is denoted by $\operatorname{gl.dim}\Lambda$. Let M, N be finitely generated Λ-modules. We denote by $\operatorname{add} M$ the subcategory of $\operatorname{mod}\Lambda$ consisting of all direct summands of finite copies of M, and we write $N \mid M$ if N is isomorphic to a direct summand of M.

DEFINITION 2.1 ([20, 22]). Let Λ and Γ be two rings, and let ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$ be bimodules, finitely generated projective as one-sided modules. Then

Λ and Γ are said to be *separably equivalent by means of* ${}_\Lambda M_\Gamma$ *and* ${}_\Gamma N_\Lambda$ if there exist bimodules ${}_\Lambda X_\Lambda$ and ${}_\Gamma Y_\Gamma$ and bimodule isomorphisms

$$ {}_\Lambda M \otimes_\Gamma N_\Lambda \cong_\Lambda \Lambda_\Lambda \oplus_\Lambda X_\Lambda \quad \text{and} \quad {}_\Gamma N \otimes_\Lambda M_\Gamma \cong_\Gamma \Gamma_\Gamma \oplus_\Gamma Y_\Gamma. $$

If furthermore $(M \otimes_\Gamma -, N \otimes_\Lambda -)$ and $(N \otimes_\Lambda -, M \otimes_\Gamma -)$ are adjoint pairs, then Λ and Γ are said to be *symmetrically separably equivalent.*

Remark 2.2. Let Λ and Γ be symmetrically separably equivalent as in Definition 2.1.

(1) If the bimodules ${}_\Lambda X_\Lambda$ and ${}_\Gamma Y_\Gamma$ are the zero modules, then Λ is Morita equivalent to Γ.

(2) If ${}_\Lambda X_\Lambda$ and ${}_\Gamma Y_\Gamma$ are projective as bimodules, then Λ and Γ are stably equivalent of adjoint type (see [34]).

Example 2.3. (1) ([20]) For a ring Λ, $M_n(\Lambda)$ (the matrix ring of Λ degree in n) and Λ are symmetrically separably equivalent.

(2) ([27, Proposition 2]) Let k be a field of characteristic $p > 0$, and let G be a finite group. If H is a Sylow p-subgroup of G, then kG and kH are symmetrically separably equivalent.

(3) ([5, Lemma 1]) Let Λ be a ring and G be a finite group. If the order of G is invertible in Λ, then the skew group ring ΛG and Λ are symmetrically separably equivalent.

(4) ([29, Lemma 4.6]) Let Λ be a finite-dimensional algebra over an algebraically closed field k, and F be a finite separable field extension of k. Then Λ and $\Lambda \otimes_k F$ are symmetrically separably equivalent.

More examples of symmetrically separable equivalences can be found in [27].

We need the following fact.

Lemma 2.4 (see [20, 29]). *Let Λ and Γ be symmetrically separably equivalent as in Definition* 2.1. *Then*

1. ${}_\Lambda M_\Gamma$ *and* ${}_\Gamma N_\Lambda$ *are projective generators as one-sided modules;*
2. $N \otimes_\Lambda -$ *and* $M \otimes_\Gamma -$ *are exact functors and take projective modules to projective modules;*
3. *there are natural isomorphisms* $(M \otimes_\Gamma -) \circ (N \otimes_\Lambda -) \to \mathrm{Id}_{\mathrm{Mod}\,\Lambda} \oplus (X \otimes_\Lambda -)$ *and* $(N \otimes_\Lambda -) \circ (M \otimes_\Gamma -) \to \mathrm{Id}_{\mathrm{Mod}\,\Gamma} \oplus (Y \otimes_\Gamma -)$*;*
4. $\mathrm{gl.dim}\,\Lambda = \mathrm{gl.dim}\,\Gamma$.

Let Λ be an Artin algebra, and let $\mathcal{C}$ be a subcategory of $\mathrm{mod}\,\Lambda$. The morphism category [1], denoted by $\mathrm{Morph}(\mathcal{C})$, is defined as follows. The objects in $\mathrm{Morph}(\mathcal{C})$ are all the morphisms $f : C_1 \to C_0$ in $\mathcal{C}$, denoted by (C_1, C_0, f). The morphism from an object (C_1, C_0, f) to another object (C_1', C_0', f') is a pair (α_1, α_0), where $\alpha_i : C_i \to C_i'$ for $i = 0, 1$, such that $\alpha_0 f = f' \alpha_1$. For two objects $f : C_1 \to C_0$ and $f' : C_1' \to C_0'$, let $\mathcal{R}_\mathcal{C}(f, f') =$

$\{(\alpha_1, \alpha_0) : (C_1, C_0, f) \to (C_1', C_0', f')$ such that there is some $\beta : C_0 \to C_1'$ with $\alpha_0 = f'\beta\}$. It is not hard to check that $\mathcal{R}_{\mathcal{C}}$ gives an additive equivalence relation on $\mathrm{Morph}(\mathcal{C})$. The following lemma is due to [24, Lemma 2.4].

LEMMA 2.5. *Let Λ and Γ be Artin algebras and $U \in \mathrm{mod}\,\Lambda$. Then there is an equivalence $\mathrm{H}_A : \mathrm{Morph}(\mathrm{add}\,_\Lambda U)/\mathcal{R}_U \to \mathrm{mod}\,A$, where $A = \mathrm{End}\,_\Lambda U$ and $\mathcal{R}_U = \mathcal{R}_{\mathrm{add}\,_\Lambda U}$, given by $\mathrm{H}_A((U_1, U_0, f)) = \mathrm{Coker}(\mathrm{Hom}_\Lambda(U, f))$ for any object (U_1, U_0, f) in $\mathrm{Morph}(\mathrm{add}\,_\Lambda U)$, and the unique morphism*

$$\mathrm{H}_A(\alpha_1, \alpha_0) : \mathrm{H}_A((U_1, U_0, f)) \to \mathrm{H}_A((U_1', U_0', f'))$$

that makes the diagram

$$\begin{tikzcd}
\mathrm{Hom}_\Lambda(U, U_1) \arrow[r, "{(U,f)}"] \arrow[d, "{(U,\alpha_1)}"'] & \mathrm{Hom}_\Lambda(U, U_0) \arrow[r] \arrow[d, "{(U,\alpha_0)}"'] & \mathrm{H}_A((U_1, U_0, f)) \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}_\Lambda(U, U_1') \arrow[r, "{(U,f')}"] & \mathrm{Hom}_\Lambda(U, U_0') \arrow[r] & \mathrm{H}_A((U_1', U_0', f')) \arrow[r] & 0
\end{tikzcd}$$

commute, for any morphism (α_1, α_0) from an object (U_1, U_0, f) to another object (U_1', U_0', f').

Let K be a finitely generated Λ-module. Recall that an exact sequence

$$0 \to K \to C^0 \to C^1 \to \cdots \to C^n \to 0$$

in $\mathrm{mod}\,\Lambda$ with all C^i in $\mathcal{C}$ is called an *n-$\mathcal{C}$-coresolution* of K. An *n-$\mathcal{C}$-coresolution* of K is said to be *proper* if it is also $\mathrm{Hom}_\Lambda(-, \mathcal{C})$-exact. Dual to [15, Lemma 4.2], we have the following result.

LEMMA 2.6. *Let Λ be an Artin algebra, and let X, X_1 and T be Λ-modules such that $X_1 \in \mathrm{add}\,X$. If X has a proper n-$\mathrm{add}\,T$-coresolution*

$$0 \to X \xrightarrow{f_0} T_0 \to \cdots \to T_{n-1} \xrightarrow{f_n} T_n \to 0,$$

then X_1 also has a proper n-$\mathrm{add}\,T$-coresolution

$$0 \to X_1 \xrightarrow{f_0'} T_0' \to \cdots \to T_{n-1}' \xrightarrow{f_{n-1}'} T_n' \to 0.$$

Let Λ be a left and right noetherian ring and let

$$0 \to \Lambda \to I^0(\Lambda) \to I^1(\Lambda) \to I^2(\Lambda) \to \cdots \to I^n(\Lambda) \to \cdots$$

be the minimal injective coresolution of $_\Lambda\Lambda$. Following [32], the *dominant dimension* of Λ, denoted by $\mathrm{dom.dim}\,\Lambda$, is defined to be the minimal number n such that $I^n(\Lambda)$ is not projective. If the $I^i(\Lambda)$ are injective-projective for all $i \geq 0$, then $\mathrm{dom.dim}\,\Lambda = \infty$. For a positive integer n, recall from [8] that Λ is said to be an *n-Gorenstein ring* if $\mathrm{fd}\,_\Lambda I^i(\Lambda) \leq i$ for any $0 \leq i \leq n-1$, and Λ is said to satisfy the *Auslander condition* if Λ is an n-Gorenstein ring for all n. As a nontrivial generalization of the notion of the Auslander condition, Huang and Iyama [14] introduced an Auslander-type condition as follows. For any $n, k \geq 0$, Λ is said to be $G_n(k)$ if $\operatorname{fd}{}_\Lambda I^i(\Lambda) \leq k+i$ for all $0 \leq i \leq n-1$. It is easy to see that Λ is an n-Gorenstein ring if and only if Λ is $G_n(0)$. Auslander-type conditions play a significant role in the representation theory of algebras, homological algebra and noncommutative algebraic geometry (see [13, 14, 16, 17, 18] for details).

Finally, let us list some homological conjectures, which are still open.

Auslander–Gorenstein conjecture. An Artin algebra satisfying the Auslander condition is a Gorenstein algebra (that is, the left and right self-injective dimensions of Λ are finite).

Nakayama conjecture. An Artin algebra Λ with $\operatorname{dom.dim} \Lambda = \infty$ is self-injective.

Gorenstein symmetric conjecture. Let Λ be an Artin algebra. Then $\operatorname{id}{}_\Lambda \Lambda = \operatorname{id} \Lambda_\Lambda$.

Strong Nakayama conjecture. Any finitely generated module M over an Artin algebra Λ with $\operatorname{Ext}_\Lambda^{\geq 0}(M, \Lambda) = 0$ is zero.

3. Construction of symmetrical separable equivalences. In this section, all rings are Artin R-algebras, where R is a commutative Artin ring, and all modules are finitely generated left modules. We assume that Λ and Γ are symmetrically separably equivalent by means of bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$. That is, there exist bimodules ${}_\Lambda X_\Lambda$ and ${}_\Gamma Y_\Gamma$ and bimodule isomorphisms ${}_\Lambda M \otimes_\Gamma N_\Lambda \cong {}_\Lambda \Lambda_\Lambda \oplus {}_\Lambda X_\Lambda$ and ${}_\Gamma N \otimes_\Lambda M_\Gamma \cong {}_\Gamma \Gamma_\Gamma \oplus {}_\Gamma Y_\Gamma$. We define functors $\mathrm{T}_N = N \otimes_\Lambda - : \operatorname{Mod} \Lambda \to \operatorname{Mod} \Gamma$ and $\mathrm{T}_M = M \otimes_\Gamma - : \operatorname{Mod} \Gamma \to \operatorname{Mod} \Lambda$, which are exact functors such that $(\mathrm{T}_M, \mathrm{T}_N)$ and $(\mathrm{T}_N, \mathrm{T}_M)$ are adjoint pairs by Lemma 2.4(2) and our assumptions.

Theorem 3.1. *Let Λ and Γ be symmetrically separably equivalent by means of ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$. Let U be a Λ-module and V a Γ-module such that $N \otimes_\Lambda U \in \operatorname{add}{}_\Gamma V$ and $M \otimes_\Gamma V \in \operatorname{add}{}_\Lambda U$. Then the endomorphism algebras $\operatorname{End}{}_\Lambda U$ and $\operatorname{End}{}_\Gamma V$ are symmetrically separably equivalent.*

Proof. We use some ideas of [24, Theorem 1.1].

Let $A = \operatorname{End}{}_\Lambda U$ and $B = \operatorname{End}{}_\Gamma V$. Clearly, A and B are also Artin algebras. By Lemma 2.5, there exist equivalences of categories

$$\mathrm{H}_A : \operatorname{Morph}(\operatorname{add}{}_\Lambda U)/\mathcal{R}_U \to \operatorname{mod} A,$$
$$\mathrm{H}_B : \operatorname{Morph}(\operatorname{add}{}_\Gamma V)/\mathcal{R}_V \to \operatorname{mod} B.$$

Since $N \otimes_\Lambda U \in \operatorname{add}{}_\Gamma V$, we define a functor

$$\widetilde{\mathrm{T}}_N : \operatorname{Morph}(\operatorname{add}{}_\Lambda U)/\mathcal{R}_U \to \operatorname{Morph}(\operatorname{add}{}_\Gamma V)/\mathcal{R}_V$$

by

$$\widetilde{\mathrm{T}}_N((U_1, U_0, f)) = (\mathrm{T}_N(U_1), \mathrm{T}_N(U_0), \mathrm{T}_N(f)),$$

for any object (U_1, U_0, f) in $\mathrm{Morph}(\mathrm{add}\,_\Lambda U)/\mathcal{R}_U$. And for a morphism $(\alpha_1, \alpha_0) + \mathcal{R}_U(f, f')$ from (U_1, U_0, f) to (U_1', U_0', f'), we define

$$\widetilde{\mathrm{T}}_N((\alpha_1, \alpha_0) + \mathcal{R}_U(f, f')) = (\mathrm{T}_N(\alpha_1), \mathrm{T}_N(\alpha_0)) + \mathcal{R}_V(\mathrm{T}_N(f), \mathrm{T}_N(f')).$$

Noting that $\mathrm{T}_N(\mathcal{R}_U(f, f')) \subseteq \mathcal{R}_V(\mathrm{T}_N(f), \mathrm{T}_N(f'))$, it is not hard to check that $\widetilde{\mathrm{T}}_N$ is well defined. Similarly, we have a functor $\widetilde{\mathrm{T}}_M : \mathrm{Morph}(\mathrm{add}\,_\Gamma V)/\mathcal{R}_V \to \mathrm{Morph}(\mathrm{add}\,_\Lambda U)/\mathcal{R}_U$, because $M \otimes_\Gamma V \in \mathrm{add}\,_\Lambda U$.

We denote by $\mathbb{F}$ the composition

$$\mathrm{mod}\, A \xrightarrow{\mathrm{H}_A^{-1}} \mathrm{Morph}(\mathrm{add}\,_\Lambda U)/\mathcal{R}_U \xrightarrow{\widetilde{\mathrm{T}}_N} \mathrm{Morph}(\mathrm{add}\,_\Gamma V)/\mathcal{R}_V \xrightarrow{\mathrm{H}_B} \mathrm{mod}\, B,$$

and by $\mathbb{G}$ the composition

$$\mathrm{mod}\, B \xrightarrow{\mathrm{H}_B^{-1}} \mathrm{Morph}(\mathrm{add}\,_\Gamma V)/\mathcal{R}_V \xrightarrow{\widetilde{\mathrm{T}}_M} \mathrm{Morph}(\mathrm{add}\,_\Lambda U)/\mathcal{R}_U \xrightarrow{H_A} \mathrm{mod}\, A,$$

where H_A^{-1} (resp. H_B^{-1}) is the inverse of H_A (resp. H_B). Clearly, $\mathbb{F}$ and $\mathbb{G}$ are additive functors.

We divide the following proof into four steps.

STEP 1: We show that $\mathbb{F}$ and $\mathbb{G}$ are exact functors.

Let $0 \to D_1 \to D_2 \to D_3 \to 0$ be a short exact sequence in $\mathrm{mod}\, A$. By the definition of $\mathrm{H}_A : \mathrm{Morph}(\mathrm{add}\,_\Lambda U)/\mathcal{R}_U \to \mathrm{mod}\, A$, one has $H_A^{-1}(D_1) = (U_1, U_0, f)$ and $H_A^{-1}(D_3) = (U_1', U_0', f_3)$, where $U_i, U_i' \in \mathrm{add}\,_\Lambda U$, for $i = 0, 1$. Then there are projective presentations

$$\mathrm{Hom}_\Lambda(U, U_1) \xrightarrow{(U, f_1)} \mathrm{Hom}_\Lambda(U, U_0) \to D_1 \to 0 \quad \text{of } D_1,$$

$$\mathrm{Hom}_\Lambda(U, U_1') \xrightarrow{(U, f_3)} \mathrm{Hom}_\Lambda(U, U_0') \to D_3 \to 0 \quad \text{of } D_3.$$

Note that there are isomorphisms $\mathrm{Hom}_\Lambda(U, U_i) \oplus \mathrm{Hom}_\Lambda(U, U_i') \cong \mathrm{Hom}_\Lambda(U,\ U_i \oplus U_i')$ for $i = 0, 1$. By the horseshoe lemma and Yoneda's lemma, there exist $\alpha_i : U_i \to U_i \oplus U_i'$ and $\beta_i : U_i \oplus U_i' \to U_i'$ for $i = 0, 1$ and $f_2 : U_1 \oplus U_1' \to U_0 \oplus U_0'$ such that the following exact diagram is commutative:

(3.1)
$$\begin{tikzcd}
\varepsilon_1 & 0 \arrow[r] & \mathrm{Hom}_\Lambda(U, U_1) \arrow[r, "(U,\alpha_1)"] \arrow[d, "(U,f_1)"'] & \mathrm{Hom}_\Lambda(U, U_1 \oplus U_1') \arrow[r, "(U,\beta_1)"] \arrow[d, "(U,f_2)"'] & \mathrm{Hom}_\Lambda(U, U_1') \arrow[r] \arrow[d, "(U,f_3)"'] & 0 \\
\varepsilon_0 & 0 \arrow[r] & \mathrm{Hom}_\Lambda(U, U_0) \arrow[r, "(U,\alpha_0)"] \arrow[d] & \mathrm{Hom}_\Lambda(U, U_0 \oplus U_0') \arrow[r, "(U,\beta_0)"] \arrow[d] & \mathrm{Hom}_\Lambda(U, U_1') \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & D_1 \arrow[r] \arrow[d] & D_2 \arrow[r] \arrow[d] & D_3 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

where ε_0 and ε_1 are canonical split exact sequences.

Let $K_i = \operatorname{Ker} f_i$ for $i = 1, 2, 3$. By Yoneda's lemma and the snake lemma, there exists an exact commutative digram in $\operatorname{mod} \Lambda$ as follows:

(3.2)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] & K_1 \arrow[r, "\alpha_2"] \arrow[d] & K_2 \arrow[r, "\beta_2"] \arrow[d] & K_3 \arrow[d] & \\
\varepsilon_3 & 0 \arrow[r] & U_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & U_1 \oplus U_1' \arrow[r, "\beta_1"] \arrow[d, "f_2"'] & U_1' \arrow[r] \arrow[d, "f_3"'] & 0 \\
\varepsilon_2 & 0 \arrow[r] & U_0 \arrow[r, "\alpha_0"] & U_0 \oplus U_0' \arrow[r, "\beta_0"] & U_0' \arrow[r] & 0
\end{tikzcd}$$

where the exact short sequences ε_2 and ε_3 are split. In particular, we have a short sequence in $\operatorname{mod} \Lambda$,

(3.3)
$$0 \to K_1 \xrightarrow{\alpha_2} K_2 \xrightarrow{\beta_2} K_3.$$

Since the functor $\operatorname{Hom}_\Lambda(U, -)$ is left exact and $D_i = \operatorname{Coker} \operatorname{Hom}_\Lambda(U, f_i)$ for all $i = 1, 2, 3$, by applying the functor $\operatorname{Hom}_\Lambda(U, -)$ to the commutative exact diagram (3.2) and taking the cokernels, we have the following exact commutative diagram:

(3.4)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] & \operatorname{Hom}_\Lambda(U, K_1) \arrow[r, "{(U,\alpha_2)}"] \arrow[d] & \operatorname{Hom}_\Lambda(U, K_2) \arrow[r, "{(U,\beta_3)}"] \arrow[d] & \operatorname{Hom}_\Lambda(U, K_3) \arrow[d] & \\
\varepsilon_1 & 0 \arrow[r] & \operatorname{Hom}_\Lambda(U, U_1) \arrow[r, "{(U,\alpha_1)}"] \arrow[d, "{(U,f_1)}"'] & \operatorname{Hom}_\Lambda(U, U_1 \oplus U_1') \arrow[r, "{(U,\beta_1)}"] \arrow[d, "{(U,f_2)}"'] & \operatorname{Hom}_\Lambda(U, U_1') \arrow[r] \arrow[d, "{(U,f_3)}"'] & 0 \\
\varepsilon_0 & 0 \arrow[r] & \operatorname{Hom}_\Lambda(U, U_0) \arrow[r, "{(U,\alpha_0)}"] \arrow[d] & \operatorname{Hom}_\Lambda(U, U_0 \oplus U_0') \arrow[r, "{(U,\beta_0)}"] \arrow[d] & \operatorname{Hom}_\Lambda(U, U_1') \arrow[r] \arrow[d] & 0 \\
 & & D_1 \arrow[d] & D_2 \arrow[d] & D_3 \arrow[d] & \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

Comparing the diagram (3.1) with (3.4), we see that there exists an exact sequence

(3.5)
$$0 \to \operatorname{Hom}_\Lambda(U, K_1) \xrightarrow{(U,\alpha_2)} \operatorname{Hom}_\Lambda(U, K_2) \xrightarrow{(U,\beta_2)} \operatorname{Hom}_\Lambda(U, K_3) \to 0$$

by the snake lemma.

On the other hand, noting that the functor T_N is exact, applying T_N to the diagram (3.2) gives rise to the following commutative exact diagram in $\operatorname{mod}\mathit{\Gamma}$:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] & \mathrm{T}_N(K_1) \arrow[r, "\mathrm{T}_N(\alpha_2)"] \arrow[d] & \mathrm{T}_N(K_2) \arrow[r, "\mathrm{T}_N(\beta_2)"] \arrow[d] & \mathrm{T}_N(K_3) \arrow[d] & \\
\varepsilon_5 & 0 \arrow[r] & \mathrm{T}_N(U_1) \arrow[r, "\mathrm{T}_N(\alpha_1)"] \arrow[d, "\mathrm{T}_N(f_1)"'] & \mathrm{T}_N(U_1\oplus U_1') \arrow[r, "\mathrm{T}_N(\beta_1)"] \arrow[d, "\mathrm{T}_N(f_2)"'] & \mathrm{T}_N(V_1) \arrow[r] \arrow[d, "\mathrm{T}_N(f_3)"'] & 0 \\
\varepsilon_4 & 0 \arrow[r] & \mathrm{T}_N(U_0) \arrow[r, "\mathrm{T}_N(\alpha_0)"] & \mathrm{T}_N(U_0\oplus U_0') \arrow[r, "\mathrm{T}_N(\beta_0)"] & \mathrm{T}_N(V_0) \arrow[r] & 0
\end{tikzcd} \tag{3.6}$$

where the short exact sequences ε_4 and ε_5 are split.

Recall that the functor $\operatorname{Hom}_\Gamma(V,-)$ is left exact and we have $\mathbb{F}(D_i) = \operatorname{Coker}(\operatorname{Hom}_\Gamma(V, \mathrm{T}_N(f_i))$ for $i = 1, 2, 3$. By applying the functor $\operatorname{Hom}_\Gamma(V,-)$ to the diagram (3.6) and taking cokernels, one obtains the following exact commutative diagram in $\operatorname{mod} B$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Hom}_\Gamma(V,\mathrm{T}_N(K_1)) \arrow[r, "(\mathrm{T}_N(\alpha_2))_*"] \arrow[d] & \operatorname{Hom}_\Gamma(V,\mathrm{T}_N(K_2)) \arrow[r, "(\mathrm{T}_N(\beta_2))_*"] \arrow[d] & \operatorname{Hom}_\Gamma(V,\mathrm{T}_N(K_3)) \arrow[d] & \\
0 \arrow[r] & \operatorname{Hom}_\Gamma(V,\mathrm{T}_N(U_1)) \arrow[r, "(\mathrm{T}_N(\alpha_1))_*"] \arrow[d, "(\mathrm{T}_N(f_1))_*"'] & \operatorname{Hom}_\Gamma(V,\mathrm{T}_N(U_1\oplus U_1')) \arrow[r, "(\mathrm{T}_N(\beta_1))_*"] \arrow[d, "(\mathrm{T}_N(f_2))_*"'] & \operatorname{Hom}_\Gamma(V,\mathrm{T}_N(U_1')) \arrow[r] \arrow[d, "(\mathrm{T}_N(f_3))_*"'] & 0 \\
0 \arrow[r] & \operatorname{Hom}_\Gamma(V,\mathrm{T}_N(U_0)) \arrow[r, "(\mathrm{T}_N(\alpha_0))_*"] \arrow[d] & \operatorname{Hom}_\Gamma(V,\mathrm{T}_N(U_0\oplus U_0')) \arrow[r, "(\mathrm{T}_N(\beta_0))_*"] \arrow[d] & \operatorname{Hom}_\Gamma(V,\mathrm{T}_N(V_0)) \arrow[r] \arrow[d] & 0 \\
 & \mathbb{F}(D_1) \arrow[d] & \mathbb{F}(D_2) \arrow[d] & \mathbb{F}(D_3) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $(-)_*$ stands for $\operatorname{Hom}_\Gamma(V,-)$.

We claim that $(\mathrm{T}_N(\beta_2))_*$ in the diagram above is epic. Indeed, applying the exact functor T_N to the short exact sequence (3.3) yields the following exact sequence in $\operatorname{mod}\mathit{\Gamma}$:

$$0 \to \mathrm{T}_N(K_1) \xrightarrow{\mathrm{T}_N(\alpha_2)} \mathrm{T}_N(K_2) \xrightarrow{\mathrm{T}_N(\beta_2)} \mathrm{T}_N(K_3). \tag{3.7}$$

Since $(\mathrm{T}_M, \mathrm{T}_N)$ is an adjoint pair and the functors $\operatorname{Hom}_\Lambda(\mathrm{T}_M(V),-)$ and $\operatorname{Hom}_\Gamma(V,-)$ are left exact, by applying the functor $\operatorname{Hom}_\Lambda(\mathrm{T}_M(V),-)$

to (3.3) and $\mathrm{Hom}_\Gamma(V,-)$ to (3.7), respectively, we obtain the commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_\Lambda(\mathrm{T}_M(V),K_1) \arrow[r, "(\alpha_2)_\dagger"] \arrow[d, "\cong"'] & \mathrm{Hom}_\Lambda(\mathrm{T}_M(V),K_2) \arrow[r, "(\beta_2)_\dagger"] \arrow[d, "\cong"'] & \mathrm{Hom}_\Lambda(\mathrm{T}_M(V),K_3) \arrow[d, "\cong"'] \\
0 \arrow[r] & \mathrm{Hom}_\Gamma(V,\mathrm{T}_N(K_1)) \arrow[r, "(\mathrm{T}_N(\alpha_2))_*"] & \mathrm{Hom}_\Gamma(V,\mathrm{T}_N(K_2)) \arrow[r, "((\mathrm{T}_N(\beta_2))_*"] & \mathrm{Hom}_\Gamma(V,\mathrm{T}_N(K_3))
\end{tikzcd}$$

where $(-)_\dagger$ stands for $\mathrm{Hom}_\Lambda(\mathrm{T}_M(V),-)$. Because of $\mathrm{T}_M(V)\in \operatorname{add} U$ and the exactness of the sequence (3.5), we see that $(\beta_2)_\dagger$ in the diagram above is epic, and hence so is $((\mathrm{T}_N(\beta_2))_*$, as claimed.

By the snake lemma again, there is an exact sequence $0\to \mathbb{F}(D_1)\to \mathbb{F}(D_2)\to \mathbb{F}(D_3)\to 0$ in $\operatorname{mod} B$, which means that the functor $\mathbb{F}$ is exact.

Similarly, we find that the functor $\mathbb{G}$ is exact.

Step 2: We verify that $\mathbb{F}$ and $\mathbb{G}$ take projective modules to projective modules.

Let P_0 be a projective A-module. By Lemma 2.5, there exists a Λ-module $U_0\in \operatorname{add}{}_\Lambda U$ such that $P_0=\mathrm{Hom}_\Lambda(U,U_0)$, which implies $\mathrm{H}_A^{-1}(P_0)=(0,U_0,0)$. Notice that $\mathrm{T}_N(U_0)\in \operatorname{add}(\mathrm{T}_N(U))\subseteq \operatorname{add} V$, and $\widetilde{\mathrm{T}}_N\mathrm{H}_A^{-1}(P_0)=(0,\mathrm{T}_N(U_0),0)$. Hence $\mathbb{F}(P_0)=\mathrm{Hom}_\Gamma(V,\mathrm{T}_N(U_0))$ is a projective B-module.

The case of $\mathbb{G}$ can be proved in a similar way.

Step 3: We verify that $\mathbb{F}$ and $\mathbb{G}$ induce a separable equivalence between A and B.

Since $\mathbb{F}$ and $\mathbb{G}$ are additive exact functors, there exist natural isomorphisms $\mathbb{F}\cong \mathbb{F}(A)\otimes_A - : \operatorname{mod} A\to \operatorname{mod} B$ and $\mathbb{G}\cong \mathbb{G}(B)\otimes_B - : \operatorname{mod} B\to \operatorname{mod} A$, where $\mathbb{F}(A)=\mathrm{Hom}_\Gamma({}_\Gamma V_B, {}_\Gamma N\otimes_\Lambda U_A)$ is a (B,A)-bimodule and $\mathbb{G}(B)=\mathrm{Hom}_\Lambda({}_\Lambda U_A, {}_\Lambda M\otimes_\Gamma V_B)$ is an (A,B)-bimodule, by Watt's theorem (see [28, Theorem 3.33]). Noting that $\mathbb{F}$ takes projective modules to projective modules, we see that ${}_B\mathbb{F}(A)\cong {}_B\mathbb{F}(A)\otimes_A A$ is a projective B-module. On the other hand, since $\mathbb{F}\cong {}_B\mathbb{F}(A)\otimes_A -$ is an exact functor, we find that $\mathbb{F}(A)_A$ is also a projective right A-module. Similarly, we deduce that ${}_A\mathbb{G}(B)_B$ is projective as a one-sided module.

Since $\mathbb{F}$ and $\mathbb{G}$ are additive exact functors, so is the composition of $\mathbb{F}$ and $\mathbb{G}$. By Watt's theorem again, we have $\mathbb{G}\circ\mathbb{F}\cong \mathbb{G}\mathbb{F}(A)\otimes_A - : \operatorname{mod} A\to \operatorname{mod} A$, where $\mathbb{G}\mathbb{F}(A)=\mathrm{Hom}_\Lambda({}_\Lambda U_A, {}_\Lambda M\otimes_\Gamma N\otimes_\Lambda U_A)$ is an A-bimodule. On the other hand, there are natural isomorphisms

$$\mathbb{G}\circ\mathbb{F}\cong (\mathbb{G}(B)\otimes_B -)\circ(\mathbb{F}(A)\otimes_A -)\cong (\mathbb{G}(B)\otimes_B \mathbb{F}(A))\otimes_A -.$$

Thus, we have an A-bimodule isomorphism ${}_A\mathbb{G}(B)\otimes_B\mathbb{F}(A)_A\cong_A \mathbb{G}\mathbb{F}(A)_A=\mathrm{Hom}_\Lambda({}_\Lambda U_A, M\otimes_\Gamma N\otimes_\Lambda U_A)$.

Since Λ and Γ are symmetrically separably equivalent by assumption, there exists a Λ-bimodule isomorphism $\rho=(\rho_1,\rho_2): M\otimes_\Gamma N\to \Lambda\oplus X$.

Note that $A = \mathrm{Hom}_\Lambda(U_A, U_A)$ and $X' =: \mathrm{Hom}_\Lambda({}_\Lambda U_A, {}_\Lambda X \otimes_\Lambda U_A)$ are A-bimodules. It is not hard to check that the natural isomorphism

$$\bar{\rho} : \mathrm{Hom}_\Lambda(U, M \otimes_\Gamma N \otimes_\Lambda U_A) \to \mathrm{Hom}_\Lambda(U, U) \oplus \mathrm{Hom}_\Lambda({}_\Lambda U_A, {}_\Lambda X \otimes_\Lambda U_A)$$

given by $\bar{\rho}(f) = (\mu(\rho_1 \otimes \mathrm{Id}_U)f, (\rho_2 \otimes \mathrm{Id}_U)f)$, where $\mu : \Lambda \otimes_\Lambda U \to U$ is the multiplication map, is an A-bimodule isomorphism. Therefore, ${}_A\mathbb{G}(B) \otimes_B \mathbb{F}(A)_A \cong A \oplus X'$, as desired.

Similarly, there exists a B-bimodule isomorphism $\mathbb{F}(A) \otimes_A \mathbb{G}(B) \cong B \oplus Y'$, where $Y' = \mathrm{Hom}_\Gamma({}_\Gamma V_B, {}_\Gamma Y \otimes_\Gamma V_B)$ is a B-module.

Step 4: We show $(\mathbb{F}(A) \otimes_A -, \mathbb{G}(B) \otimes_B -)$ and $(\mathbb{G}(B) \otimes_B -, \mathbb{F}(A) \otimes_A -)$ are adjoint pairs.

Since $N \otimes_\Lambda U \in \mathrm{add}\, V$ and $B = \mathrm{Hom}_\Gamma(V, V)$, we have the isomorphisms

$$\begin{aligned}
&\mathrm{Hom}_B(\mathbb{F}(A), B)\\
&\qquad = \mathrm{Hom}_B(\mathrm{Hom}_\Gamma(V, N \otimes_\Lambda U), \mathrm{Hom}_\Gamma(V, V))\\
&\qquad \cong \mathrm{Hom}_\Gamma(N \otimes_\Lambda U, V) \quad \text{(by Yoneda's lemma)}\\
&\qquad \cong \mathrm{Hom}_\Lambda(U, M \otimes_\Gamma V) \quad \text{(by the adjoint pair } (N \otimes_\Lambda -, M \otimes_\Gamma -))\\
&\qquad = \mathbb{G}(B).
\end{aligned}$$

Thus, for any B-module L,

$$\mathbb{G}(B) \otimes_B L \cong \mathrm{Hom}_B(\mathbb{F}(A), B) \otimes_B L \cong \mathrm{Hom}_B(\mathbb{F}(A), L),$$

since $\mathbb{F}(A)$ is a projective B-module. This implies that $(\mathbb{F}(A) \otimes_A -, \mathbb{G}(B) \otimes_B -)$ is an adjoint pair. In a similar way, we see that $(\mathbb{G}(B) \otimes_B -, \mathbb{F}(A) \otimes_A -)$ is also an adjoint pair. The proof of Theorem 3.1 is complete. ∎

As applications of Theorem 3.1, we shall construct new symmetrical separable equivalences from given ones.

Recall that an Artin algebra Λ is said to be *representation-finite*, if there exist finitely many nonisomorphic indecomposable Λ-modules. Recall that a Λ-module U is called an *additive generator* if $\mathrm{add}\, {}_\Lambda U = \mathrm{mod}\, \Lambda$. It is clear that an Artin algebra is representation-finite if and only of it has an additive generator. Let Λ be a representation-finite Artin algebra. The endomorphism algebra of an additive generator for Λ is called the *Auslander algebra* of Λ, which is unique up to Morita equivalence. According to Theorem 3.1, we have the following result.

Theorem 3.2. *Let Λ and Γ be representation-finite. If Λ and Γ are symmetrically separably equivalent, then so are their Auslander algebras.*

Proof. Let ${}_\Lambda U$ and ${}_\Gamma V$ be additive generators for Λ and Γ, respectively. Then U and V satisfy the conditions in Theorem 3.1, which yields the assertion. ∎

Let Λ be an Artin algebra. Recall from [11] that a projective Λ-module P is said to be *ν-stable projective* if $\nu^i(P)$ is projective for all $i \geq 1$, where $\nu = \mathrm{D}\,\mathrm{Hom}_\Lambda(-,\Lambda)$ is the Nakayama functor of Λ. We denote by Λ-stp the subcategory of $\mathrm{mod}\,\Lambda$ consisting of all nonisomorphic ν-stable projective Λ-modules. Following [11], we define the *Frobenius part* of Λ, which is self-injective and unique up to Morita equivalence, to be the endomorphism algebra of a projective Λ-module V with $\mathrm{add}\,{}_\Lambda V = \Lambda\text{-stp}$. Moreover, Λ is said to be *Frobenius-finite* if its Frobenius part is representation-finite. Frobenius parts of Artin algebras and Frobenius-finite algebras play crucial roles in the study of the Auslander–Reiten conjecture, dominant dimensions and stable equivalences (see [7, 11, 25, 35, 37] for details).

LEMMA 3.3. *Let Λ and Γ be symmetrically separably equivalent by means of bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$, and let K be a Λ-module and L a Γ-module.*

(1) *For any $i \geq 1$,*

$$\nu_\Gamma^i(N \otimes_\Lambda K) \cong N \otimes_\Lambda \nu_\Lambda^i(K) \quad \text{and} \quad \nu_\Lambda^i(M \otimes_\Gamma L) \cong M \otimes_\Gamma \nu_\Gamma^i(L).$$

(2) *If $\Lambda\text{-stp} = \mathrm{add}\,{}_\Lambda K$, then $\Gamma\text{-stp} = \mathrm{add}\,{}_\Gamma(N \otimes_\Lambda K)$,*

(3) *If $\Gamma\text{-stp} = \mathrm{add}\,{}_\Gamma L$, then $\Lambda\text{-stp} = \mathrm{add}\,{}_\Lambda(M \otimes_\Gamma L)$.*

Proof. (1) We only prove $\nu_\Gamma^i(N \otimes_\Lambda K) \cong N \otimes_\Lambda \nu_\Lambda^i(K)$.

Note that $(M \otimes_\Gamma -, N \otimes_\Lambda -)$ and $(N \otimes_\Lambda -, M \otimes_\Gamma -)$ are adjoint pairs by assumption. Thus

$$\begin{aligned}
\nu_\Gamma(N \otimes_\Lambda K) &= \mathrm{D}\,\mathrm{Hom}_\Gamma(N \otimes_\Lambda K, \Gamma) \cong \mathrm{D}\,\mathrm{Hom}_\Lambda(K, M) \\
&\cong \mathrm{D}(\mathrm{Hom}_\Lambda(K,\Lambda) \otimes_\Lambda M) \quad (\text{because } {}_\Lambda M \text{ is projective}) \\
&\cong \mathrm{Hom}_\Lambda(M, \nu_\Lambda(K)) \quad (\text{by the adjoint isomorphism}) \\
&\cong \mathrm{Hom}_\Lambda(M \otimes_\Gamma \Gamma, \nu_\Lambda(K)) \cong \mathrm{Hom}_\Gamma(\Gamma, N \otimes_\Lambda \nu_\Lambda(K)) \\
&\cong N \otimes_\Lambda \nu_\Lambda(K).
\end{aligned}$$

Hence $\nu_\Gamma^2(N \otimes_\Lambda K) \cong \nu_\Gamma(\nu_\Gamma(N \otimes_\Lambda K)) \cong \nu_\Gamma(N \otimes \nu_\Lambda(K)) \cong N \otimes_\Lambda \nu_\Lambda^2(K)$. The assertion is obtained by induction.

(2) For any $i \geq 1$, we know from (1) that $\nu_\Gamma^i(N \otimes_\Lambda K) \cong N \otimes_\Lambda \nu_\Lambda^i(K)$. Since $K \in \Lambda\text{-stp}$, $\nu_\Lambda^i(K)$ is projective, so is $N \otimes_\Lambda \nu_\Lambda^i(K)$ by Lemma 2.4(2), which yields $N \otimes_\Lambda K \in \Gamma\text{-stp}$. For any $L \in \Gamma\text{-stp}$, it follows from (1) that $\nu_\Lambda^i(M \otimes_\Gamma L) \cong M \otimes_\Gamma \nu_\Gamma^i(L)$ is projective. Thus $M \otimes_\Gamma L \in \Lambda\text{-stp} = \mathrm{add}\,{}_\Lambda K$, and hence $N \otimes_\Lambda M \otimes_\Gamma L \in \mathrm{add}(N \otimes_\Lambda K)$. Noticing that ${}_\Gamma L|_\Gamma(N \otimes_\Lambda M \otimes_\Gamma L)$ by Lemma 2.4(3), we have $\Gamma\text{-stp} = \mathrm{add}(N \otimes_\Lambda K)$, as desired.

(3) The proof is similar to that of (2). ∎

THEOREM 3.4. *Let Λ and Γ be symmetrically separably equivalent. Then*

(1) *the Frobenius parts of Λ and of Γ are symmetrically separably equivalent;*

(2) *Λ is Frobenius-finite if and only if Γ is.*

Proof. Assume that there exist a Λ-module K and a Γ-module L such that Λ-stp $= \operatorname{add} K$ and Γ-stp $= \operatorname{add} L$, respectively.

(1) From Lemma 3.3(2)(3), it follows that $N \otimes_\Lambda K \in \Gamma$-stp $= \operatorname{add} L$ and $M \otimes_\Gamma L \in \operatorname{add} K$. Hence $\operatorname{End}_\Lambda K$ and $\operatorname{End}_\Gamma L$ are symmetrically separably equivalent by Theorem 3.1.

(2) Assume that Λ is Frobenius-finite. Then $\operatorname{End}_\Lambda K$ is representation-finite. Noting that $\operatorname{End}_\Lambda K$ and $\operatorname{End}_\Gamma L$ are symmetrically separably equivalent by (1), we see that $\operatorname{End}_\Gamma L$ is representation-finite by [27, Theorem 6], which means that Γ is Frobenius-finite, as desired.

Analogously, we can prove that Λ is Frobenius-finite when Γ is. ■

As a consequence of Theorem 3.4, we get the following corollary, in which the second result is [11, Corollary 5.4].

Corollary 3.5. *Let Λ be an Artin algebra and G be a finite group such that the order of G is invertible in Λ. Suppose that ΛG is the skew group algebra of Λ by G. Then*

(1) *the Frobenius parts of ΛG and of Λ are symmetrically separably equivalent;*
(2) *Λ is Frobenius-finite if and only if ΛG is.*

Proof. According to [5, Lemma 1], Λ and ΛG are symmetrically separably equivalent, and the corollary follows from Theorem 3.4 immediately. ■

Let Λ be an Artin algebra. Recall from [26] that a finitely generated Λ-module T is said to be an *m-tilting module* if the following conditions are satisfied: (1) $\operatorname{pd}_\Lambda T \leq m$; (2) $\operatorname{Ext}_\Lambda^{\geq 1}(T,T) = 0$; (3) ${}_\Lambda\Lambda$ has an m-$\operatorname{add} T$-coresolution $0 \to_\Lambda \Lambda \to T_0 \to T_1 \to \cdots \to T_m \to 0$. Recall from [9] that an Artin algebra is called a *tilted algebra* if it is the endomorphism algebra of a 1-tilting module over a hereditary algebra. It is clear that an Artin algebra Λ is a tilted algebra if and only if there is a 1-tilting Λ-module such that its endomorphism algebra is hereditary.

In the following, we shall construct a new symmetrical separable equivalence from a given stable equivalence of adjoint type and a tilting module.

Lemma 3.6. *Let Λ and Γ be stably equivalent of adjoint type via ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$, and let T be an m-tilting Λ-module. Then*

(1) *$N \otimes_\Lambda T$ is an m-tilting Γ-module;*
(2) *$M \otimes_\Gamma N \otimes_\Lambda T \in \operatorname{add}_\Lambda T$.*

Proof. By assumption, there exist projective bimodules ${}_\Lambda P_\Lambda$ and ${}_\Gamma Q_\Gamma$, and bimodule isomorphisms ${}_\Lambda M \otimes_\Gamma N_\Lambda \cong \Lambda \oplus P$ and ${}_\Gamma N \otimes_\Lambda M_\Gamma \cong \Gamma \oplus Q$, such that $(N \otimes_\Lambda -, M \otimes_\Gamma -)$ and $(M \otimes_\Gamma -, N \otimes_\Lambda -)$ are adjoint pairs.

(1) Note that the exact functor $N \otimes_\Lambda -$ takes projective Λ-modules to projective Γ-modules by Lemma 2.4(2). One has $\operatorname{pd}_\Gamma(N \otimes_\Lambda T) \leq \operatorname{pd}_\Lambda T = m$.

On the other hand, since $(M \otimes_\Gamma -, N \otimes_\Lambda -)$ is an adjoint pair and M_Γ is projective, for any $i \geq 1$ we have

$$\begin{aligned}\operatorname{Ext}^i_\Gamma(N \otimes_\Lambda T, N \otimes_\Lambda T) &\cong \operatorname{Ext}^i_\Lambda(M \otimes_\Gamma N \otimes_\Lambda T, T)\\ &\cong \operatorname{Ext}^i_\Lambda((\Lambda \oplus P) \otimes_\Lambda T, T)\\ &\cong \operatorname{Ext}^i_\Lambda(T \oplus (P \otimes_\Lambda T), T)\\ &\cong \operatorname{Ext}^i_\Lambda(T, T) \oplus \operatorname{Ext}^i_\Lambda(P \otimes_\Lambda T, T) = 0\end{aligned}$$

from the fact that $P \otimes_\Lambda T$ is a projective Λ-module.

Since T is an m-tilting Λ-module, there exists an m-$\operatorname{add} T$-coresolution

$$0 \to \Lambda \to T_0 \to T_1 \to \cdots \to T_m \to 0.$$

Applying the functor $N \otimes_\Lambda -$ to the sequence above gives rise to the following exact sequence in $\operatorname{mod} \Gamma$:

$$0 \to N \to N \otimes_\Lambda T_0 \to N \otimes_\Lambda T_1 \to \cdots \to N \otimes_\Lambda T_m \to 0,$$

which is a proper m-add$(N \otimes_\Lambda T)$-coresolution, because $\operatorname{Ext}^{\geq 1}_\Gamma(N \otimes_\Lambda T, N \otimes_\Lambda T) = 0$. Note that ${}_\Gamma\Gamma | {}_\Gamma N$ by Lemma 2.4(1). It follows from Lemma 2.6 that there is a proper m-add$(N \otimes_\Lambda T)$-coresolution of ${}_\Gamma\Gamma : 0 \to \Gamma \to T_0' \to T_1' \to \cdots \to T_m' \to 0$. Thus, we have proved that $N \otimes_\Lambda T$ is an m-tilting Λ-module.

(2) By assumption, there is a Λ-module isomorphism $M \otimes_\Gamma N \otimes_\Lambda T \cong T \oplus (P \otimes_\Lambda T)$. We need to prove $P \otimes_\Lambda T \in \operatorname{add} T$. In fact, since $N \otimes_\Lambda T$ is an m-tilting module, and $(N \otimes_\Lambda -, M \otimes_\Gamma -)$ is an adjoint pair, there exist isomorphisms

$$\begin{aligned}0 = \operatorname{Ext}^1_\Gamma(N \otimes_\Lambda T, N \otimes_\Lambda T) &\cong \operatorname{Ext}^1_\Lambda(T, M \otimes_\Gamma N \otimes_\Lambda T)\\ \cong \operatorname{Ext}^1_\Lambda(T, T \oplus P \otimes_\Lambda T) &\cong \operatorname{Ext}^1_\Lambda(T, T) \oplus \operatorname{Ext}^1_\Lambda(T, P \otimes_\Lambda T),\end{aligned}$$

which means $\operatorname{Ext}^1_\Lambda(T, P \otimes_\Lambda T) = 0$. According to [33, Lemma 3.3], it follows that $P \otimes_\Lambda T \in \operatorname{gen} T$, and hence $P \otimes_\Lambda T \in \operatorname{add} T$ by the projectivity of the Λ-module $P \otimes_\Lambda T$, as desired. ∎

THEOREM 3.7. *Let Λ and Γ be stably equivalent of adjoint type by means of ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$. If T is an m-tilting Λ-module, then the algebras* $\operatorname{End}_\Lambda T$ *and* $\operatorname{End}_\Gamma(N \otimes_\Lambda T)$ *are symmetrically separably equivalent.*

Proof. Since T is an m-tilting Λ-module, one has $M \otimes_\Gamma N \otimes_\Lambda T \in \operatorname{add}{}_\Lambda T$ by Lemma 3.6. The assertion follows from Theorem 3.1. ∎

As a consequence of Theorem 3.7, a result of [31] is obtained.

COROLLARY 3.8. *Let Λ and Γ be stably equivalent of adjoint type. Then Λ is a tilted algebra if and only if Γ is.*

Proof. Assume that Λ is a tilted algebra. Then there is a 1-tilting Λ-module T such that $\operatorname{End}_\Lambda T$ is a hereditary algebra. Due to Lemma 3.6, it follows that $N \otimes_\Lambda T$ is a 1-tilting Γ-module. Because $\mathrm{End}_\Lambda T$ and $\mathrm{End}_\Gamma(N \otimes_\Lambda T)$ are symmetrically separably equivalent by Theorem 3.7, one gets

$$\mathrm{gl.dim}\,\mathrm{End}_\Gamma(N \otimes_\Lambda T) = \mathrm{gl.dim}\,\mathrm{End}_\Lambda T \leq 1$$

by Lemma 2.4(4), which implies that Γ is a tilted algebra. ∎

As another application of Theorem 3.1, we obtain the following result, which slightly improves [23, Theorem 1.1].

COROLLARY 3.9. *Let Λ and Γ be symmetric k-algebras and let M be a (Γ, Λ)-bimodule inducing separable equivalence between Λ and Γ. Suppose that U is a Λ-module and V a Γ-module such that $M^* \otimes_\Lambda U \in \mathrm{add}_\Gamma V$ and $M \otimes_\Gamma V \in \mathrm{add}_\Lambda U$. Then the dominant dimensions of $\mathrm{End}_\Lambda U$ and of $\mathrm{End}_\Gamma V$ are equal.*

Proof. According to [21], Λ and Γ are symmetrically separably equivalent by means of M and its dual module $M^* = \mathrm{Hom}_\Lambda(M, \Lambda)$. By Theorem 3.1, $\mathrm{End}_\Lambda U$ and $\mathrm{End}_\Gamma V$ are symmetrically separably equivalent. The assertion follows directly from [36, Remark 2.4(1)]. ∎

Let Λ be an Artin algebra. Recall from [1] that $V \in \mathrm{mod}\,\Lambda$ is said to be an *Auslander generator* if V is a generator-cogenerator for Λ-modules. Recall from [6] that the *rigidity dimension of* Λ, denoted by $\mathrm{rig.dim}\,\Lambda$, is defined to be

$$\mathrm{rig.dim}(\Lambda) = \sup\{\mathrm{dom.dim}(\mathrm{End}_\Lambda(V)) : \\ V \text{ is an Auslander generator in } \mathrm{mod}\,\Lambda\}$$

and the *representation dimension* of Λ, denoted by $\mathrm{rep.dim}\,\Lambda$, is defined to be

$$\mathrm{rep.dim}(\Lambda) = \inf\{\mathrm{gl.dim}(\mathrm{End}_\Lambda(V)) : \\ V \text{ is an Auslander generator in } \mathrm{mod}\,\Lambda\}.$$

The notion of rigidity dimension was introduced by Chen et al. [6] to measure homologically the quality of the best resolutions of Artin algebras, and it was proved that it is an invariant of stable equivalence of adjoint type. The notion of representation dimension was introduced by Auslander [1] to measure how far away an Artin algebra is from being representation-finite type in the homological sense.

In the following, we shall discuss the invariance properties of the rigidity dimension and the representation dimension of Artin algebras under symmetrical separable equivalences, and recover a result about the representation dimension in [36].

LEMMA 3.10. *Let Λ and Γ be separably equivalent by means of bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$, and let V be a Λ-module. Suppose that V is an Auslander generator for Λ-modules. Then so is $N \otimes_\Lambda V$ as a Γ-module.*

Proof. Since ${}_\Lambda\Lambda \in \operatorname{add}{}_\Lambda V$ and N is a projective generator for Γ-modules by assumption, one has $\Gamma \in \operatorname{add}{}_\Gamma(N\otimes_\Lambda V)$. On the other hand, let A be a Γ-module. Then $M \otimes_\Gamma A \in \operatorname{mod}\Lambda$, and so there is a Λ-module monomorphism $f : M \otimes_\Gamma A \to V^{(n)}$ for some positive integer n, because V is a cogenerator for Λ-modules. Since the functor $N \otimes_\Lambda -$ is exact, $N\otimes_\Lambda f : N\otimes_\Lambda M\otimes_\Gamma A \to N \otimes_\Lambda V^{(n)}$ is a Γ-module monomorphism. Notice that ${}_\Gamma A|_\Gamma(N \otimes_\Lambda M \otimes_\Gamma A)$ by Lemma 2.4(3). Hence, there exists a split monomorphism $i : A \to N \otimes_\Lambda M\otimes_\Gamma A$. So the composition $g = (N\otimes_\Lambda f)i : A \to (N\otimes_\Lambda V)^{(n)}$ is a Γ-module monomorphism, which means that $N\otimes_\Lambda V$ is a cogenerator for Γ-modules. ∎

For a ring Λ, let T be a finitely generated Λ-bimodule. Following Hirata [10], T is said to be *centrally projective* over Λ if ${}_\Lambda T_\Lambda \in \operatorname{add}({}_\Lambda\Lambda_\Lambda)$.

THEOREM 3.11. *Let Λ and Γ be symmetrically separably equivalent via ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$. Assume that $M \otimes_\Gamma N$ is centrally projective over Λ. Then*

(1) $\operatorname{rig.dim}(\Lambda) \le \operatorname{rig.dim}(\Gamma)$;
(2) $\operatorname{rep.dim}(\Lambda) \ge \operatorname{rep.dim}(\Gamma)$.

Proof. (1) If $\operatorname{rig.dim}(\Lambda) = \infty$, there is nothing to prove.

Assume that $\operatorname{rig.dim}(\Lambda) = m<\infty$. Then there exists an Auslander generator V for Λ-modules with $\operatorname{gl.dim}(\operatorname{End}{}_\Lambda V)<\infty$ such that $\operatorname{dom.dim}(\operatorname{End}{}_\Lambda V) = m$. It follows from Lemma 3.10 that $N \otimes_\Lambda V$ is an Auslander generator for Γ-modules.

On the other hand, since ${}_\Lambda M \otimes_\Gamma N_\Lambda \in \operatorname{add}{}_\Lambda\Lambda_\Lambda$ by assumption, we have ${}_\Lambda M\otimes_\Gamma N\otimes_\Lambda V \in \operatorname{add}{}_\Lambda V$. According to Theorem 3.1, it follows that $\operatorname{End}{}_\Lambda V$ and $\operatorname{End}{}_\Gamma(N \otimes_\Lambda V)$ are symmetrically separably equivalent. Hence,

$$\operatorname{gl.dim}\operatorname{End}{}_\Gamma(N \otimes_\Lambda V) = \operatorname{gl.dim}\operatorname{End}{}_\Lambda V < \infty,$$
$$\operatorname{dom.dim}\operatorname{End}{}_\Gamma(N \otimes_\Lambda V) = \operatorname{dom.dim}\operatorname{End}{}_\Lambda V = m,$$

by Lemma 2.4(4) and [36, Remark 2.4(1)]. This shows

$$\operatorname{rig.dim}(\Lambda) \le \operatorname{rig.dim}(\Gamma).$$

(2) Assume that $\operatorname{rep.dim}\Lambda = m$. Then there exists an Auslander generator V for Λ-modules such that $\operatorname{gl.dim}\operatorname{End}{}_\Lambda V = m$. By Lemma 3.10, $N\otimes_\Lambda V$ is an Auslander generator. Since ${}_\Gamma M \otimes_\Lambda N_\Gamma$ is centrally projective by assumption, it follows from Theorem 3.1 that $\operatorname{End}{}_\Lambda V$ and $\operatorname{End}{}_\Gamma(N\otimes_\Lambda V)$ are symmetrically separably equivalent. Hence, $\operatorname{gl.dim}\operatorname{End}{}_\Gamma(N \otimes_\Lambda V) = m$ by Lemma 2.4(4), which implies $\operatorname{rep.dim}\Gamma \le m$, as desired. ∎

COROLLARY 3.12. *Let Λ and Γ be symmetrically separably equivalent by means of bimodules ${}_\Gamma M_\Lambda$ and ${}_\Lambda N_\Gamma$. Suppose that $M \otimes_\Gamma N$ and $N\otimes_\Lambda M$ are centrally projective over Λ and Γ, respectively. Then*

(1) $\operatorname{rig.dim}(\Lambda) = \operatorname{rig.dim}(\Gamma)$*;*
(2) ([36, Theorem 1.2]) $\operatorname{rep.dim}(\Gamma) = \operatorname{rep.dim}(\Lambda)$.

There are examples of separable equivalences satisfying all conditions of Corollary 3.12.

(1) Let Λ be a finite-dimensional algebra over an algebraically closed field k, and F a separable field extension of k. Following [29, Lemma 4.6], $\Lambda \otimes_k F$ and Λ are symmetrically separably equivalent via ${}_{\Lambda\otimes_k F}(\Lambda \otimes_k F)_\Lambda$ and ${}_\Lambda(\Lambda \otimes_k F)_{\Lambda\otimes_k F}$. Also ${}_\Lambda(\Lambda \otimes_k F) \otimes_{\Lambda\otimes_k F} (\Lambda \otimes_k F)$ is centrally projective over Λ, and $(\Lambda \otimes_k F) \otimes_\Lambda (\Lambda \otimes_k F)$ is centrally projective over $\Lambda \otimes_k F$.

(2) Let Λ be an Artin R-algebra, where R is a commutative Artin ring. If Λ is an excellent extension of R, it follows from [30, Lemma 3.11] that Λ and R are symmetrically separably equivalent by means of ${}_\Lambda\Lambda_R$ and ${}_R\Lambda_\Lambda$. Moreover, $\Lambda \otimes_R \Lambda$ and $\Lambda \otimes_\Lambda \Lambda$ are centrally projective over Λ and R, respectively.

4. The Auslander-type condition and homological conjectures. In this section, we shall prove that the Auslander-type condition and homological conjectures, including the Gorenstein symmetric conjecture, the (strong) Nakayama conjecture and the Auslander–Gorenstein conjecture, are preserved under symmetrical separable equivalence. We begin with the following observation.

LEMMA 4.1. *Let Λ and Γ be two rings and let I be a Λ-module and J a right Λ-module. Suppose that Λ and Γ are symmetrically separably equivalent bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$. Then*

(1) *I is an injective Λ-module if and only if so is $N \otimes_\Lambda I$ as a Γ-module;*
(2) *J is an injective right Λ-module if and only if so is $J \otimes_\Lambda M$ as a right Γ-module;*
(3) *F is a flat Λ-module if and only if so is $N \otimes_\Lambda F$ as a Γ-module.*

Proof. (1) Since M_Γ is projective and $(M \otimes_\Gamma -, N \otimes_\Lambda -)$ is an adjoint pair, for any $i \geq 1$ and any Γ-module K there is an isomorphism $\operatorname{Ext}^i_\Gamma(K, N \otimes_\Lambda I) \cong \operatorname{Ext}^i_\Lambda(M \otimes_\Gamma K, I) = 0$, which shows that $N \otimes_\Lambda I$ is an injective Γ-module, as desired.

Conversely, assume that $N \otimes_\Lambda I$ is an injective Γ-module. Then so is the Λ-module $M \otimes_\Gamma N \otimes_\Lambda I$. Noticing that $I \mid M \otimes_\Gamma N \otimes_\Lambda I$, we get the assertion.

(2) The proof is similar to (1).

(3) Let $0 \to A \to B \to C \to 0$ be an exact sequence of right Γ-modules. Since $- \otimes_\Gamma N$ is an exact functor and F is flat, the sequence

$$0 \to A \otimes_\Gamma N \otimes_\Lambda F \to B \otimes_\Gamma N \otimes_\Lambda F \to C \otimes_\Gamma N \otimes_\Lambda F \to 0$$

is exact, which implies that $N \otimes_\Lambda F$ is flat.

Conversely, assume that $N \otimes_\Lambda F$ is flat. Then so is $M \otimes_\Gamma N \otimes_\Lambda F$ as a Λ-module. Note that $F \mid M \otimes_\Gamma N \otimes_\Lambda F$. Hence F is flat. ∎

COROLLARY 4.2. *Let Λ and Γ be symmetrically separably equivalent by means of bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$, and let V be a Λ-module and W a right Λ-module. Then*

(1) $\operatorname{id}_\Lambda V = \operatorname{id}_\Gamma(N \otimes_\Lambda V)$;
(2) $\operatorname{id} W_\Lambda = \operatorname{id}(W \otimes_\Lambda M)_\Gamma$;
(3) $\operatorname{fd}_\Lambda V = \operatorname{fd}_\Gamma(N \otimes_\Lambda V)$;
(4) $\operatorname{fd} W_\Lambda = \operatorname{fd}(W \otimes_\Lambda M)_\Gamma$;
(5) $\operatorname{pd}_\Lambda V = \operatorname{pd}_\Gamma(N \otimes_\Lambda V)$;
(6) $\operatorname{pd} W_\Lambda = \operatorname{pd}_\Gamma(W \otimes_\Lambda M)$.

Proof. We only prove (1); the proofs of (2)–(6) are similar.

Without loss of generality, we may consider $\operatorname{id}_\Lambda V = n < \infty$. Then there is the following exact sequence in $\operatorname{Mod}\Lambda$:

$$0 \to_\Lambda V \to E_0 \to E_1 \to \cdots \to E_n \to 0,$$

with each E_i injective. Since N is a projective right Λ-module, we get a new exact sequence

$$0 \to N \otimes_\Lambda V \to N \otimes_\Lambda E_0 \to N \otimes_\Lambda E_1 \to \cdots \to N \otimes_\Lambda E_n \to 0,$$

where $N \otimes_\Lambda E_i$ is an injective Γ-module for each $0 \le i \le n$, by Lemma 4.1. This means $\operatorname{id}_\Gamma(N \otimes_\Lambda V) \le n$.

Conversely, assume $\operatorname{id}_\Gamma(N \otimes_\Lambda V) = m$. Then $\operatorname{id}_\Lambda(M \otimes_\Gamma N \otimes_\Lambda V) \le m$. Since ${}_\Lambda V|_\Lambda(M \otimes_\Gamma N \otimes_\Lambda V)$ by Lemma 2.4(3), it follows that $\operatorname{id}_\Lambda V \le m$. The assertion follows immediately. ∎

COROLLARY 4.3. *Let Λ and Γ be symmetrically separably equivalent. Then*

(1) $\operatorname{id}_\Lambda \Lambda = \operatorname{id}_\Gamma \Gamma$;
(2) $\operatorname{id} \Lambda_\Lambda = \operatorname{id} \Gamma_\Gamma$.

Proof. (1) According to Corollary 4.2(1), it follows that $\operatorname{id}_\Lambda \Lambda = \operatorname{id}_\Gamma N$. Combining this with the fact that ${}_\Gamma N$ is a projective generator, one gets $\operatorname{id}_\Lambda \Lambda = \operatorname{id}_\Gamma \Gamma$.

(2) is dual to (1). ∎

LEMMA 4.4. *Let*

$$0 \to {}_\Lambda\Lambda \to I^0(\Lambda) \to I^1(\Lambda) \to \cdots \to I^n(\Lambda) \to \cdots \tag{4.1}$$

and

$$0 \to {}_\Gamma\Gamma \to I^0(\Gamma) \to I^1(\Gamma) \to \cdots \to I^n(\Gamma) \to \cdots \tag{4.2}$$

be the minimal injective resolutions of ${}_\Lambda\Lambda$ and of ${}_\Gamma\Gamma$, respectively. For any $i \ge 0$, we have

(1) $\operatorname{fd}_\Lambda I^i(\Lambda) = \operatorname{fd}_\Gamma I^i(\Gamma)$;
(2) $\operatorname{pd}_\Lambda I^i(\Lambda) = \operatorname{pd}_\Lambda I^i(\Gamma)$.

Proof. We only prove (1); the proof of (2) is similar.

Applying the exact functor $N \otimes_\Lambda -$ to the injective resolution (4.1) induces an exact sequence

$$0 \to_\Gamma N \to N \otimes_\Lambda I^0(\Lambda) \to N \otimes_\Lambda I^1(\Lambda) \to \cdots \to N \otimes_\Lambda I^n(\Lambda) \to \cdots,$$

where $N \otimes_\Lambda I^i(\Lambda)$ is an injective Γ-module for all $i \geq 0$ by Lemma 4.1. Noting that (4.2) is the minimal injective resolution of ${}_\Gamma\Gamma$ and N is a projective generator for Γ-modules by Lemma 2.4(1), it is not hard to check that $I^i(\Gamma) \mid N \otimes_\Lambda I^i(\Lambda)$ for all $i \geq 0$. Hence, for each i, we have $\mathrm{fd}\,_\Lambda I^i(\Gamma) \leq \mathrm{fd}\,_\Gamma(N \otimes_\Lambda I^i(\Lambda)) = \mathrm{fd}\,_\Lambda I^i(\Lambda)$ by Corollary 4.2.

Dually, we have $\mathrm{fd}\,_\Lambda I^i(\Lambda) \leq \mathrm{fd}\,_\Gamma I^i(\Gamma)$ for any $i \geq 0$. ∎

THEOREM 4.5. *Let Λ and Γ be symmetrically separably equivalent and let n, k be integers. Then Λ is $G_n(k)$ if and only if Γ is.*

Proof. Suppose that Λ satisfies $G_n(k)$. Then we have $\mathrm{fd}\, I^i(\Lambda) \leq i+k$ for any $0 \leq i \leq n-1$. It follows from Lemma 4.4 that $\mathrm{fd}\, I^i(\Gamma) \leq i+k$ for any $0 \leq i \leq n-1$. Thus, Γ satisfies $G_n(k)$.

Similarly, we verify that Λ is $G_n(k)$ when Γ is. ∎

Let n be a positive integer. Following [12], a left and right noetherian ring Λ is *left quasi-n-Gorenstein* if the left flat dimension of the ith term in the minimal injective resolution of Λ as a left Λ-module is less than or equal to $i+1$ for any $0 \leq i \leq n-1$. This is a nontrivial generalization of n-Gorenstein rings. It is easy to see that a left and right noetherian ring is left quasi-n-Gorenstein if and only if it is $G_k(1)$. The left quasi-n-Gorenstein rings play an important role in the study of the Auslander–Gorenstein conjecture and the Nakayama conjecture (see [12, 13] for details).

According to Theorem 4.5, we have the following corollary.

COROLLARY 4.6. *Let Λ and Γ be symmetrically separably equivalent, and let n be a nonnegative integer. Then*

(1) *Λ is n-Gorenstein if and only if Γ is;*
(2) *Λ is quasi n-Gorenstein if and only if Γ is;*
(3) *Λ satisfies the Auslander condition if and only if Γ does.*

Proof. Note that Λ is n-Gorenstein if and only if Λ is $G_n(0)$, and that Λ is quasi-n-Gorenstein if and only if Λ is $G_n(1)$. Thus (1) and (2) are obtained from Theorem 4.5 directly. (3) follows from (1) directly. ∎

Finally, we shall show that symmetrical separable equivalence preserves the Gorenstein symmetric conjecture, the (strong) Nakayama conjecture and the Auslander–Gorenstein conjecture.

THEOREM 4.7. *Let Λ and Γ be symmetrically stably equivalent Artin R-algebras. Then*

(1) *Λ satisfies the Gorenstein symmetric conjecture if and only if Γ does;*
(2) *Λ satisfies the strong Nakayama conjecture if and only if Γ does;*
(3) *Λ satisfies the Auslander–Gorenstein conjecture if and only if Γ does;*
(4) *Λ satisfies the Nakayama conjecture if and only if Γ does.*

Proof. (1) According to Corollary 4.3, $\mathrm{id}\,{}_\Lambda\Lambda = \mathrm{id}\,\Lambda_\Lambda$ if and only if $\mathrm{id}\,{}_\Gamma\Gamma = \mathrm{id}\,\Gamma_\Gamma$, and the assertion follows.

(2) Assume that Γ satisfies the strong Nakayama conjecture. Let V be a Λ-module with $\mathrm{Ext}_\Lambda^{\geq 0}(V,\Lambda) = 0$. It is not hard to check that $\mathrm{Ext}_\Lambda^{\geq 0}(V,P) = 0$ for any finitely generated projective Λ-module P. Since N_Λ is projective and $(N\otimes_\Lambda -, M\otimes_\Gamma -)$ is an adjoint pair, it follows that for any $i \geq 0$, $\mathrm{Ext}_\Gamma^i(N\otimes_\Lambda V,\Gamma) \cong \mathrm{Ext}_\Lambda^i(V,M) = 0$. Therefore, $N\otimes_\Lambda V = 0$ by assumption, which implies $V = 0$, because N_Λ is a projective generator for right Λ-modules. Thus Λ satisfies the strong Nakayama conjecture.

Similarly, we can prove that Γ satisfies the strong Nakayama conjecture when Λ does.

(3) Assume that Γ satisfies the Auslander–Gorenstein conjecture and Λ satisfies the Auslander condition. From Corollary 4.6(3) it follows that Γ satisfies the Auslander condition, and hence Γ is a Gorenstein algebra (that is, $\mathrm{id}\,{}_\Gamma\Gamma < \infty$ and $\mathrm{id}\,\Gamma_\Gamma < \infty$) by assumption, which implies that Λ is a Gorenstein algebra by Corollary 4.3.

Similarly, we can prove that Γ satisfies the Auslander–Gorenstein conjecture when Λ does.

(4) Assume Γ satisfies the Nakayama conjecture and $\mathrm{dom.dim}\,\Lambda = \infty$. Then $\mathrm{dom.dim}\,\Gamma = \infty$ by Lemma 4.4(2), so that Γ is self-injective by assumption. It follows from Corollary 4.3 that Λ is self-injective, as desired.

Similarly, we can prove that Γ satisfies the Nakayama conjecture when Λ does. ∎

As a consequence of Theorem 4.7, we obtain the following corollary, which improves [38, Theorem 3.3(2)].

COROLLARY 4.8. *Let Λ be an Artin algebra, and let G be a finite group acting on Λ, whose order is invertible in Λ. Then Λ satisfies the strong Nakayama conjecture if and only if the skew group ΛG does.*

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