

UNIVERSAL MAPPINGS WITH FIXED RANGE IN  $\mathbb{R}^n$ 

BY

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**Abstract.** We prove the existence of universal elements in classes of continuous and semi-open mappings having the same range  $Y$ . Our study is based on the so-called saturated classes of topological spaces. Based on this result, we also prove the existence of universal elements in the class  $C^{\text{soP}}(\mathbb{E}, \mathbb{R}^n)$  of all continuous and semi-open mappings with domains in a saturated class  $\mathbb{E}$  of topological spaces and range  $\mathbb{R}^n$ . We apply our results to various saturated classes  $\mathbb{E}$ .

**1. Introduction.** The continuous mappings are studied in various mathematical branches. For example, we can find relations of continuous mappings with dimensions of spaces and convergences of sequences/nets [Ch, En, En1, Ke, N1, N2, P]. Among related studies, we can find types of continuous mappings such as open and semi-open mappings [AC, B, BMT, C, GMS, Kim, M].

In this connection, a natural question arises: *Given a fixed space  $Y$  and a class of continuous mappings with range  $Y$ , does there exist a universal mapping in this class?* That is, *does there exist a continuous mapping the study of which can give information for all other mappings belonging to the same class?* Here by a *universal mapping* for a class  $\mathbb{F}$  of continuous  $Y$ -mappings (i.e. continuous mappings with the same range  $Y$ ) we mean a mapping  $f$  belonging to this class such that for every  $g \in \mathbb{F}$  there exists a  $Y$ -embedding of  $g$  into  $f$  (see [I] and Definitions 2.4 and 2.5).

In the book [I], we can find the first answer, where the *saturated class* of topological spaces was defined. Saturated classes of  $T_0$ -topological spaces of weight less than or equal to a fixed infinite cardinal  $\tau$  have universal elements. In particular, saturated classes of spaces give the way to construct so-called containing spaces. These spaces yield the existence of universal spaces and of universal mappings with domains these containing spaces.

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**THEOREM 1.1** (Universal mappings [I]). *Let  $\tau$  be a fixed infinite cardinal. Having a saturated class  $\mathbb{E}$  of  $T_0$ -spaces of weight less than or equal to  $\tau$ , in the class  $C(\mathbb{E}, Y)$  of all continuous mappings with domain in  $\mathbb{E}$  and range the same  $T_0$ -space  $Y$  of weight less than or equal to  $\tau$ , there exist universal mappings.*

More specifically, we get a positive answer to the above question in classes of continuous and open mappings:

**THEOREM 1.2** (Universal open mappings [II]). *Let  $\tau$  be a fixed infinite cardinal. Having a saturated class  $\mathbb{E}$  of  $T_0$ -spaces of weight less than or equal to  $\tau$ , in the class  $C^{op}(\mathbb{E}, Y)$  of all continuous and open mappings with domain in  $\mathbb{E}$  and range the same  $T_0$ -space  $Y$  of weight less than or equal to  $\tau$ , there exist universal mappings.*

**2. The universality problem.** The study of universal mappings and saturated classes of spaces begins with the universality problem for topological spaces.

**DEFINITION 2.1** ([I]). If  $\mathbb{L}$  is a class of topological spaces, then a topological space  $T$  is called *universal* in  $\mathbb{L}$  if the following conditions are satisfied:

- (i)  $T \in \mathbb{L}$ ,
- (ii) for every  $X \in \mathbb{L}$ , there exists an embedding of  $X$  into  $T$ .

A topological space  $T$  which satisfies the second condition only is called a *containing space* for the class  $\mathbb{L}$ .

Problems concerning universal objects appeared in topology in its early development, when special classes of separable metrizable spaces were considered. Firstly, for a given class of spaces one sought to find a universal element with no general method for constructing such elements. Then, in [I] the author presented such a method. In particular, given an indexed collection  $\mathbf{S}$  of  $T_0$ -spaces of weight less than or equal to a given infinite cardinal  $\tau$ , a containing space  $T$  for  $\mathbf{S}$  was constructed. Based on this method, many results related to universal elements were proved. For example, for a fixed infinite cardinal  $\tau$ , there exist universal elements (see [I1, I, MS])

- (i) in the class of all  $T_0$ -spaces of weight less than or equal to  $\tau$ ,
- (ii) in the class of all regular  $T_0$ -spaces of weight less than or equal to  $\tau$ ,
- (iii) in the class of all completely regular  $T_0$ -spaces of weight less than or equal to  $\tau$ ,
- (iv) in the class of all almost regular  $T_0$ -spaces of weight less than or equal to  $\tau$ .

Studying the universality problem, the notion of saturated class of spaces can give an answer. The existence of universal elements is not sufficient for a

class of spaces to be saturated. However, saturated classes of spaces certainly have universal elements. In particular, the following main result was proved.

**THEOREM 2.2** (Universal spaces [I]). *In any saturated class of  $T_0$ -spaces of weight less than or equal to a fixed infinite cardinal  $\tau$  there exist universal elements.*

Moreover, in [I2, I] classes of mappings and their universal elements have been studied as well. As in [I], all mappings considered in this paper are assumed to be continuous. We denote by  $D_f$  and  $R_f$  the domain and the range of the mapping  $f$ , respectively. Also, in the rest of the paper, we suppose that a fixed  $T_0$ -space of weight less than or equal to a fixed infinite cardinal  $\tau$ , denoted by  $Y$ , is given.

**DEFINITION 2.3.** A mapping  $f$  is said to be a  $Y$ -mapping if  $R_f = Y$ .

**DEFINITION 2.4.** Let  $f$  and  $g$  be two  $Y$ -mappings. An embedding  $i$  of  $D_g$  into  $D_f$  is said to be a  $Y$ -embedding of  $g$  into  $f$  if  $f \circ i = g$ , that is, the diagram of Figure 1 is commutative.

$$\begin{array}{ccc} D_f & \xrightarrow{f} & Y \\ i \uparrow & \nearrow g & \\ D_g & & \end{array}$$

Fig. 1. Embedding of  $g$  into  $f$

**DEFINITION 2.5.** An element  $f$  of a class  $\mathbb{F}$  of  $Y$ -mappings is said to be *universal* in  $\mathbb{F}$  if for every  $g \in \mathbb{F}$ , there exists a  $Y$ -embedding of  $g$  into  $f$ .

**DEFINITION 2.6.** A class  $\mathbb{F}$  of  $Y$ -mappings is said to be *topological* if for any  $Y$ -mappings  $f$  and  $g$  and for every homeomorphism  $i$  of  $D_g$  onto  $D_f$  such that  $f \circ i = g$ , the condition  $f \in \mathbb{F}$  implies  $g \in \mathbb{F}$ .

In what follows all the classes of  $Y$ -mappings under consideration are assumed to be topological. Also,  $\{D_f : f \in \mathbb{F}\}$  is called the *domain-class*.

In this paper, we study the universality problem for classes of semi-open mappings with the same range  $Y$ , that is, for semi-open  $Y$ -mappings. We prove that in the class of semi-open  $Y$ -mappings, where each semi-open mapping of this class has its domain in a saturated class of spaces, there exist universal elements. Based on this main result, we give positive answers to the universality problem in various classes of  $Y$ -mappings determined by the semi-open property, verifying the existence of corresponding universal elements.

**DEFINITION 2.7.** Given two topological spaces  $X$  and  $Y$ , a mapping  $f : X \rightarrow Y$  is called *semi-open* if  $\text{Int}_Y(f(U)) \neq \emptyset$  for every non-empty open

subset  $U$  of  $X$ , where  $\text{Int}_Y(f(U))$  denotes the interior of the set  $f(U)$  in the space  $Y$ .

A bibliographic investigation of semi-open mappings shows that such mappings were considered under other names: “feebly open mapping” [F], “somewhat open mapping” [GH], “quasi-open mapping” [Kao] and finally, “semi-open mapping” [HZZ]. Also, semi-open mappings are studied for example in [AC, B, BMT, C, GMS, Kim, M]. Each open mapping is semi-open, but the converse does not hold, as shown by the following example.

EXAMPLE 2.8. We consider the continuous mapping  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$f(x) = \begin{cases} 2x & \text{if } x \leq 1/3, \\ 1 - x & \text{if } 1/3 \leq x \leq 2/3, \\ 2x - 1 & \text{if } 2/3 \leq x. \end{cases}$$

It is easy to see that  $f$  is semi-open (see [HZZ]). Since  $f((1/4, 3/4)) = [1/3, 2/3]$ ,  $f$  is not open. Now, let  $f_i : \mathbb{R} \rightarrow \mathbb{R}$ ,  $i = 1, \dots, n$ , be  $n$  copies of  $f$ . We consider the product  $g \equiv \prod_{i=1}^n f_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . The mapping  $g$  is not open [En, Proposition 2.3.29]. However,  $g$  is semi-open [Kim, Proposition 7].

In addition, in [GMS], the universality problem for classes  $C^{\text{sop}}(\mathbb{E}, \mathbb{L})$  of continuous and semi-open mappings were studied. In particular, the following result was proved.

THEOREM 2.9 (Universal semi-open mappings [GMS]). *Let  $\tau$  be a fixed infinite cardinal. Having two topological saturated classes  $\mathbb{E}$  and  $\mathbb{L}$  of  $T_0$ -spaces of weight less than or equal to  $\tau$ , in the class  $C^{\text{sop}}(\mathbb{E}, \mathbb{L})$  of all continuous and semi-open mappings with domain in  $\mathbb{E}$  and range in  $\mathbb{L}$  there exist universal mappings.*

In Theorem 2.9 we consider classes  $C^{\text{sop}}(\mathbb{E}, \mathbb{L})$  of continuous and semi-open mappings with different ranges but in the present paper we study classes  $C^{\text{sop}}(\mathbb{E}, Y)$  of continuous and semi-open mappings with the same range  $Y$ . The main result of this article is not a special case of the above theorem, as considering  $\mathbb{L} = \{Y\}$ , we fail to have a saturated, or even a topological, class of spaces. Thus, we apply another method to prove the existence of universal mappings in such classes of continuous and semi-open mappings with the same range.

The paper is organized as follows: In Section 3, we present the basic notions and notations related to containing spaces and saturated class of spaces. In Section 4, we prove our main result, solving the universality problem in a class of semi-open  $Y$ -mappings, having their domains in a saturated class of spaces. This result also answers the universality problem in other

classes of  $Y$ -mappings determined by the semi-open property. We apply the main theorem of this section to the  $n$ -Euclidean space  $\mathbb{R}^n$ ,  $n = 1, 2, \dots$ .

**3. Containing spaces.** In this section we present the method of constructing containing spaces (which lead to the existence of universal spaces) in a given indexed collection  $\mathbf{S}$  of  $T_0$ -spaces of weight less than or equal to a given infinite cardinal  $\tau$ . The method presented below is given in [I1] and in more details in [I2]. Throughout, we denote by  $\tau$  a fixed infinite cardinal, by  $\tau^+$  the least cardinal greater than  $\tau$ , by  $\mathcal{F}$  the set of all finite subsets of  $\tau$ , and we assume that all spaces are  $T_0$ -spaces of weight less than or equal to  $\tau$ .

Moreover, we shall use indexed sets. An *indexed set* is a mapping  $F$  of a set  $\Lambda$  into a set  $Y$ . Hence,  $F$  is a subset of  $\Lambda \times Y$  and its elements have the form  $(\lambda, y_\lambda)$ , where  $\lambda \in \Lambda$  and  $y_\lambda = F(\lambda)$ . However, for simplicity each element  $(\lambda, y_\lambda)$  of  $F$  is identified with the element  $y_\lambda$  of  $Y$  and the indexed set  $F$  is denoted by  $\{y_\lambda : \lambda \in \Lambda\}$ .

An indexed set  $F : \Lambda \rightarrow Y$  is said to be an *indexation* of  $Y$  if  $F(\Lambda) = Y$  and the set  $\Lambda$  is called the *indexing set* of  $F$ . Usually, in order to emphasize the use of  $\Lambda$ ,  $F$  is called a  $\Lambda$ -indexed set.

For every  $X \in \mathbf{S}$  we fix an indexed base  $M^X \equiv \{U_\delta^X : \delta \in \tau\}$  and consider the indexed collection  $\mathbf{M} \equiv \{M^X : X \in \mathbf{S}\}$ , which is called a *co-mark* of  $\mathbf{S}$ . Then we consider an indexed family  $\mathbf{R} \equiv \{\sim^s : s \in \mathcal{F}\}$  of equivalence relations on  $\mathbf{S}$ . On the set of all pairs  $(x, X)$ , where  $x \in X \in \mathbf{S}$ , we consider a new equivalence relation, denoted by  $\sim_{\mathbf{R}}^{\mathbf{M}}$ . The set of all equivalence classes of  $\sim_{\mathbf{R}}^{\mathbf{M}}$  is denoted by  $T(\mathbf{M}, \mathbf{R})$ . Under some conditions on  $\mathbf{R}$ , we define a base  $\mathbf{B}^T$  for a topology on the set  $T(\mathbf{M}, \mathbf{R})$  such that the corresponding space is a  $T_0$ -space of weight less than or equal to  $\tau$ . This space will be the required containing space.

**Marks and co-marks for spaces.** Any  $\tau$ -indexed base

$$M \equiv \{U_\delta : \delta \in \tau\}$$

of  $X$  is called a *mark* of  $X$  and a space  $X$  is called *marked* if a mark of  $X$  is chosen. An  $\mathbf{S}$ -indexed collection

$$\mathbf{M} \equiv \{\{U_\delta^X : \delta \in \tau\} : X \in \mathbf{S}\},$$

where  $\{U_\delta^X : \delta \in \tau\}$  is a  $\tau$ -indexed base for  $X$ , is called a *co-mark* of  $\mathbf{S}$ . The co-mark  $\mathbf{M}$  of  $\mathbf{S}$  is said to be a *co-extension* of a co-mark

$$\mathbf{M}^+ \equiv \{\{V_\delta^X : \delta \in \tau\} : X \in \mathbf{S}\}$$

of  $\mathbf{S}$  if there exists a one-to-one mapping  $\theta$  of  $\tau$  into itself such that for every  $X \in \mathbf{S}$  and for every  $\delta \in \tau$ ,  $V_\delta^X = U_{\theta(\delta)}^X$ . The corresponding mapping  $\theta$  is called an *indicial mapping* from  $\mathbf{M}^+$  to  $\mathbf{M}$ .

**Indexed families of equivalence relations.** We shall deal with  $\mathcal{F}$ -indexed families of equivalence relations on  $\mathbf{S}$ . For such a family

$$\mathbf{R} \equiv \{\sim^s : s \in \mathcal{F}\}$$

we shall write

$$\mathbf{C}(\mathbf{R}) = \bigcup \{\mathbf{C}(\sim^s) : s \in \mathcal{F}\},$$

where  $\mathbf{C}(\sim^s)$  denotes the set of all equivalence classes of  $\sim^s$ . We will frequently refer to the minimal ring of subsets of  $\mathbf{S}$  containing  $\mathbf{C}(\mathbf{R})$ , which is denoted by  $\mathbf{C}^\diamond(\mathbf{R})$ .

Let  $\mathbf{R}_1 \equiv \{\sim_1^s : s \in \mathcal{F}\}$  and  $\mathbf{R}_0 \equiv \{\sim_0^s : s \in \mathcal{F}\}$  be two  $\mathcal{F}$ -indexed families of equivalence relations on  $\mathbf{S}$ . We say that  $\mathbf{R}_1$  is a *final refinement* of  $\mathbf{R}_0$  if for every  $s \in \mathcal{F}$ , there exists  $t \in \mathcal{F}$  such that  $\sim_1^t \subset \sim_0^s$ .

An  $\mathcal{F}$ -indexed family  $\mathbf{R} \equiv \{\sim^s : s \in \mathcal{F}\}$  of equivalence relations on  $\mathbf{S}$  is said to be *admissible* if the following conditions are satisfied:

- (i)  $\sim^\emptyset = \mathbf{S} \times \mathbf{S}$ ,
- (ii) for every  $s \in \mathcal{F}$  the number of  $\sim^s$ -equivalence classes is finite,
- (iii)  $\sim^s \subset \sim^t$  if  $t \subset s$ .

**$\mathbf{M}$ -standard family of equivalence relations.** Let  $\mathbf{S}$  be an indexed collection of spaces and  $\mathbf{M} \equiv \{\{U_\delta^X : \delta \in \tau\} : X \in \mathbf{S}\}$  be a co-mark of  $\mathbf{S}$ . For every  $s \in \mathcal{F} \setminus \{\emptyset\}$ , an equivalence relation  $\sim_{\mathbf{M}}^s$  on  $\mathbf{S}$  is defined as follows:

Two elements  $X$  and  $Y$  of  $\mathbf{S}$  are  $\sim_{\mathbf{M}}^s$ -equivalent if there exists an isomorphism  $i$  of the algebra of subsets of  $X$  generated by  $\{U_\delta^X : \delta \in s\}$  onto the algebra of subsets of  $Y$  generated by  $\{U_\delta^Y : \delta \in s\}$  such that  $i(U_\delta^X) = U_\delta^Y$  for every  $\delta \in s$ . We also set  $\sim_{\mathbf{M}}^\emptyset = \mathbf{S} \times \mathbf{S}$ .

We denote by

$$\mathbf{R}_{\mathbf{M}} \equiv \{\sim_{\mathbf{M}}^s : s \in \mathcal{F}\}$$

the indexed admissible family of equivalence relations  $\sim_{\mathbf{M}}^s$  on  $\mathbf{S}$ , which is called  **$\mathbf{M}$ -standard**. An admissible family  $\mathbf{R}$  of equivalence relations on  $\mathbf{S}$  is said to be  **$\mathbf{M}$ -admissible** if  $\mathbf{R}$  is a final refinement of  $\mathbf{R}_{\mathbf{M}}$ .

**The containing space**  $\mathbf{T} \equiv \mathbf{T}(\mathbf{M}, \mathbf{R})$ . Let  $\mathbf{R} \equiv \{\sim^s : s \in \mathcal{F}\}$  be an  $\mathbf{M}$ -admissible family of equivalence relations on  $\mathbf{S}$ . On the set of all pairs  $(x, X)$ , where  $X \in \mathbf{S}$  and  $x \in X$ , we consider an equivalence relation defined  $\sim_{\mathbf{R}}^{\mathbf{M}}$  as follows:  $(x, X) \sim_{\mathbf{R}}^{\mathbf{M}} (y, Y)$  if and only if  $X \sim^s Y$  for every  $s \in \mathcal{F}$ , and either  $x \in U_\delta^X$  and  $y \in U_\delta^Y$  for every  $\delta \in \tau$ , or  $x \notin U_\delta^X$  and  $y \notin U_\delta^Y$  for every  $\delta \in \tau$ .

The set of all equivalence classes of  $\sim_{\mathbf{R}}^{\mathbf{M}}$  is denoted by  $\mathbf{T}(\mathbf{M}, \mathbf{R})$  or simply by  $\mathbf{T}$ , that is,  $\mathbf{T} = \mathbf{C}(\sim_{\mathbf{R}}^{\mathbf{M}})$ . (It is assumed that  $\mathbf{T}(\mathbf{M}, \mathbf{R}) = \emptyset$  if all elements of  $\mathbf{S}$  are empty.)

For every  $\mathbf{H} \in \mathbf{C}^\diamond(\mathbf{R})$  the set of all  $\mathbf{a} \in \mathbf{T}(\mathbf{M}, \mathbf{R})$  for which there exists an element  $(x, X) \in \mathbf{a}$  such that  $X \in \mathbf{H}$  is denoted by  $\mathbf{T}(\mathbf{M}, \mathbf{R}, \mathbf{H}) \equiv \mathbf{T}(\mathbf{H})$ . These subsets  $\mathbf{T}(\mathbf{H})$ ,  $\mathbf{H} \in \mathbf{C}^\diamond(\mathbf{R})$ , of  $\mathbf{T}$  are called *primary*. For every  $\delta \in \tau$

and  $\mathbf{H} \in C^\diamond(\mathbf{R})$  we denote by  $U_\delta^T(\mathbf{H})$  the set of all  $\mathbf{a} \in T(\mathbf{M}, \mathbf{R})$  for which there exists an element  $(x, X) \in \mathbf{a}$  such that  $X \in \mathbf{H}$  and  $x \in U_\delta^X$ .

We denote by  $B_*^T$  the family of all sets of the form  $U_\delta^T(\mathbf{H})$ , where  $\delta \in \tau$ ,  $\mathbf{H} \in C(\sim^t)$  and  $\sim^t \subset \sim_{\mathbf{M}}^{\{\delta\}}$ . The set  $B_*^T$  is a base for a topology on  $T$ . The set  $T$  equipped with this topology will be called the *containing space for the indexed collection  $\mathbf{S}$  corresponding to the co-mark  $\mathbf{M}$  and the family  $\mathbf{R}$* . Since  $|B_*^T| \leq \tau$ , the weight of  $T$  is less than or equal to  $\tau$ . Also,  $T$  is a  $T_0$ -space [I, Proposition 1.2.9].

For every subset  $\kappa$  of  $\tau$  and  $\mathbf{L} \in C^\diamond(\mathbf{R})$  we set

$$\begin{aligned} B_\kappa^T &\equiv \{U_\delta^T(\mathbf{H}) : \delta \in \kappa \text{ and } \mathbf{H} \in C(\mathbf{R})\}, \\ B_{\diamond, \kappa}^T &\equiv \{U_\delta^T(\mathbf{H}) : \delta \in \kappa \text{ and } \mathbf{H} \in C^\diamond(\mathbf{R})\}, \\ B_\kappa^{\mathbf{L}} &\equiv \{U_\delta^T(\mathbf{H}) \in B_\kappa^T : \mathbf{H} \subseteq \mathbf{L}\}, \\ B_{\diamond, \kappa}^{\mathbf{L}} &\equiv \{U_\delta^T(\mathbf{H}) \in B_{\diamond, \kappa}^T : \mathbf{H} \subseteq \mathbf{L}\}. \end{aligned}$$

If for every  $X \in \mathbf{S}$  the family  $\{U_\delta^X : \delta \in \kappa\}$  is a base for  $X$ , then the families  $B_\kappa^T$  and  $B_{\diamond, \kappa}^T$  are bases for  $T$  and the families  $B_\kappa^{\mathbf{L}}$  and  $B_{\diamond, \kappa}^{\mathbf{L}}$  are bases for  $T(\mathbf{L})$ . The families  $B_\kappa^T$  and  $B_\kappa^{\mathbf{L}}$  (respectively,  $B_{\diamond, \kappa}^T$  and  $B_{\diamond, \kappa}^{\mathbf{L}}$ ) are called  $\kappa$ -*standard bases* (respectively,  $(\diamond, \kappa)$ -*standard bases*) for  $T$  and  $T(\mathbf{L})$ , respectively.

**The natural embedding.** Let  $\mathbf{S}$  be an indexed collection of  $T_0$ -spaces of weight less than or equal to  $\tau$  and let  $X \in \mathbf{S}$ . Then for every  $x \in X$ , there exists a unique element  $\mathbf{a}$  of the containing space  $T$  which contains the pair  $(x, X)$ . We define a mapping  $i_T^X$  of  $X$  into  $T$  setting

$$i_T^X(x) = \mathbf{a},$$

which is called *natural*.

It was proved that for  $X \in \mathbf{S}$ , the mapping  $i_T^X$  is an embedding of  $X$  into  $T$  [I, Proposition 1.2.10].

The above construction of containing spaces leads to the following definition of a saturated class of spaces, which is very useful for the universality property as in any saturated class of spaces there exist universal elements.

**DEFINITION 3.1 ([I]).** A class  $\mathbb{P}$  of spaces is said to be *saturated* if for every indexed collection  $\mathbf{S}$  of elements of  $\mathbb{P}$ , there exists a co-mark  $\mathbf{M}^+$  of  $\mathbf{S}$  satisfying the following condition: for every co-extension  $\mathbf{M}$  of  $\mathbf{M}^+$  there exists an  $\mathbf{M}$ -admissible family  $\mathbf{R}^+$  of equivalence relations on  $\mathbf{S}$  such that for every admissible family  $\mathbf{R}$  of equivalence relations on  $\mathbf{S}$ , which is a final refinement of  $\mathbf{R}^+$ , and for every element  $\mathbf{L} \in C^\diamond(\mathbf{R})$ , the space  $T(\mathbf{L})$  belongs to  $\mathbb{P}$ .

The co-mark  $\mathbf{M}^+$  is called an *initial co-mark of  $\mathbf{S}$  corresponding to the class  $\mathbb{P}$* , and the family  $\mathbf{R}^+$  is called an *initial family of  $\mathbf{S}$  corresponding to the co-mark  $\mathbf{M}$  and the class  $\mathbb{P}$* .

**4. Universal semi-open mappings with fixed range.** In this section we give our main result, proving that in classes of semi-open  $Y$ -mappings there exist universal elements. Corollaries of this result prove the existence of universal semi-open  $Y$ -mappings in various other classes of continuous mappings and the existence of universal semi-open mappings with fixed range  $\mathbb{R}^n$ ,  $n = 1, 2, \dots$ .

The following lemma enables studying semi-open mappings by investigating sets of bases.

**LEMMA 4.1** ([GMS]). *Let  $X$  be a topological space of weight less than or equal to  $\tau$ . A mapping  $f : X \rightarrow Y$  is semi-open if and only if there exists a base  $\mathcal{B}$  of  $X$  with  $|\mathcal{B}| \leq \tau$  such that  $\text{Int}_Y(f(U)) \neq \emptyset$  for every non-empty element  $U$  of  $\mathcal{B}$ .*

**LEMMA 4.2.** *Let  $\mathbb{F}$  be a topological class of mappings. Then there exists an indexed collection  $\mathbf{F}$  of elements of  $\mathbb{F}$  such that for every  $Y$ -mapping  $g \in \mathbb{F}$  there exist a  $Y$ -mapping  $f \in \mathbf{F}$  and a homeomorphism  $h$  of  $D_g$  onto  $D_f$  such that  $f \circ h = g$ .*

*Proof.* All spaces in this paper are  $T_0$ -spaces of weight less than or equal to  $\tau$  and hence of cardinality less than or equal to  $2^\tau$  (see [En, Theorem 1.5.1]). Let  $Z$  be a fixed set with  $|Z| = 2^\tau$ , where  $|Z|$  denotes the cardinality of  $Z$ . We define

$$\mathbf{F} = \{f \in \mathbb{F} : \text{the domain } D_f \text{ of } f \text{ is contained in } Z\}.$$

For each  $Y$ -mapping  $g \in \mathbb{F}$ , we consider a “topological copy”  $Z(g)$  of  $D_g$  as a subset of  $Z$ . Let  $i : Z(g) \rightarrow D_g$  be a homeomorphism. Put  $f = g \circ i$ . Then  $D_f = Z(g)$ . Since we have  $\mathbb{F}$  is topological and  $g \in \mathbb{F}$ ,  $f \in \mathbb{F}$ . Hence  $f \in \mathbf{F}$ . Moreover,  $f \circ i^{-1} = g$ , where  $i^{-1} : D_g \rightarrow D_f$  is a homeomorphism. ■

**THEOREM 4.3.** *Let  $\mathbb{E}$  be a saturated class of spaces such that the class  $C^{\text{sop}}(\mathbb{E}, Y)$  of all continuous and semi-open mappings with domain in  $\mathbb{E}$  and range  $Y$  is topological. Then in  $C^{\text{sop}}(\mathbb{E}, Y)$  there exist universal elements.*

*Proof.* Since  $C^{\text{sop}}(\mathbb{E}, Y)$  is topological, by Lemma 4.2 there exists an indexed collection  $\mathbf{F} \subset C^{\text{sop}}(\mathbb{E}, Y)$  such that for every  $g \in C^{\text{sop}}(\mathbb{E}, Y)$  there exist  $f \in \mathbf{F}$  and a homeomorphism  $h$  of  $D_g$  onto  $D_f$  such that  $f \circ h = g$ , that is, the diagram

$$\begin{array}{ccc} D_f & \xrightarrow{f} & Y \\ h \uparrow & \nearrow g & \\ D_g & & \end{array}$$

is commutative.

Let

$$\mathbf{S}_d \equiv \{D_f : f \in \mathbf{F}\},$$

and let  $M \equiv \{V_\delta : \delta \in \tau\}$  be a mark of  $Y$ . Let also

$$\mathbf{M} \equiv \{\{U_\delta^f : \delta \in \tau\} : f \in \mathbf{F}\}$$

be an initial co-mark of  $\mathbf{S}_d$  corresponding to the class  $\mathbb{E}$  (it exists since  $\mathbb{E}$  is saturated). Without loss of generality we can suppose that  $\mathbf{M}$  is a co-extension of

$$\mathbf{M}^{-r} \equiv \{\{f^{-1}(V_\delta) : \delta \in \tau\} : f \in \mathbf{F}\}.$$

Also, we consider an initial family  $R$  of equivalence relations on  $\mathbf{S}_d$  corresponding to the co-mark  $\mathbf{M}$  and the class  $\mathbb{E}$  (existing since  $\mathbb{E}$  is saturated). Then according to the construction of Section 3, we have the containing space  $T(\mathbf{M}, R)$ .

We consider the mapping  $f_Y^T : T(\mathbf{M}, R) \rightarrow Y$  defined by

$$f_Y^T(\mathbf{a}) = f(x), \quad \text{where } (x, D_f) \in \mathbf{a} \in T(\mathbf{M}, R).$$

First, we prove that  $f_Y^T$  is well-defined. Let  $\mathbf{a} \in T(\mathbf{M}, R)$ . We suppose that  $(x, D_f), (y, D_g) \in \mathbf{a}$ . Then, for every  $\delta \in \tau$  either  $x \in U_\delta^f$  and  $y \in U_\delta^g$  or  $x \notin U_\delta^f$  and  $y \notin U_\delta^g$ . In particular, for every  $\delta \in \tau$  either  $x \in f^{-1}(V_\delta)$  and  $y \in g^{-1}(V_\delta)$  or  $x \notin f^{-1}(V_\delta)$  and  $y \notin g^{-1}(V_\delta)$ , which means that for every  $\delta \in \tau$  either  $f(x) \in V_\delta$  and  $g(y) \in V_\delta$  or  $f(x) \notin V_\delta$  and  $g(y) \notin V_\delta$ . Since  $Y$  is a  $T_0$ -space, we have  $f(x) = g(y)$ .

Secondly, we prove that  $f_Y^T$  is continuous. Let  $\theta$  be an indicial mapping from  $\mathbf{M}^{-r}$  to  $\mathbf{M}$ . We consider the  $\tau$ -standard base  $B_\tau^T$  for  $T(\mathbf{M}, R)$  and the element  $U_\varepsilon^T(\mathbf{S}_d)$  of  $B_\tau^T$ . We consider here  $\mathbf{S}_d$  as an element of  $C(R)$ . It suffices to prove that for every  $\delta \in \tau$ ,

$$(f_Y^T)^{-1}(V_\delta) = U_\varepsilon^T(\mathbf{S}_d),$$

where  $\theta(\delta) = \varepsilon$ . Let  $\delta \in \tau$  and  $\mathbf{a} \in (f_Y^T)^{-1}(V_\delta)$ . Let also  $(x, D_f) \in \mathbf{a}$ . Then  $f_Y^T(\mathbf{a}) = f(x) \in V_\delta$ . Hence  $x \in f^{-1}(V_\delta) = U_\varepsilon^f$  and therefore  $\mathbf{a} \in U_\varepsilon^T(\mathbf{S}_d)$ . Conversely, let  $\mathbf{a} \in U_\varepsilon^T(\mathbf{S}_d)$  and  $(x, D_f) \in \mathbf{a}$ . Then  $x \in U_\varepsilon^f = f^{-1}(V_\delta)$ . Therefore  $f(x) = f_Y^T(\mathbf{a}) \in V_\delta$ , that is,  $\mathbf{a} \in (f_Y^T)^{-1}(V_\delta)$ .

Now, we show that  $f_Y^T \in C^{\text{sop}}(\mathbb{E}, Y)$ , that is, the mapping  $f_Y^T$  of  $T(\mathbf{M}, R)$  into  $Y$  is semi-open. We note that since  $\mathbb{E}$  is saturated, the domain of this mapping belongs to  $\mathbb{E}$ . Then, by Lemma 4.1, we consider the  $\tau$ -standard base

$$B_\tau^T \equiv \{U_\delta^T(\mathbf{L}) : \delta \in \tau \text{ and } \mathbf{L} \in C(R)\}$$

for  $T \equiv T(\mathbf{M}, R)$  and prove that for every non-empty element  $U$  of this base we have

$$\text{Int}_Y(f_Y^T(U)) \neq \emptyset.$$

Let  $U \equiv U_\delta^T(\mathbf{L})$ . First, we shall prove that

$$(4.1) \quad f_Y^T(U_\delta^T(\mathbf{L})) = \bigcup \{f(U_\delta^f) : D_f \in \mathbf{L}\}.$$

Let  $\mathbf{a} \in U_\delta^T(\mathbf{L})$  and  $(x, D_f) \in \mathbf{a}$ . Then  $x \in U_\delta^f$  and  $D_f \in \mathbf{L}$ . Therefore,  $f(x) \in f(U_\delta^f)$ , proving the  $\subseteq$  inclusion of (4.1). Conversely, let  $y \in \bigcup \{f(U_\delta^f) : D_f \in \mathbf{L}\}$ . Then there exists  $f \in \mathbf{F}$  such that  $D_f \in \mathbf{L}$  and  $y \in f(U_\delta^f)$ . Therefore, there exists  $x \in U_\delta^f$  (and thus  $x \in D_f$ ) such that  $f(x) = y$ . We denote by  $\mathbf{a}$  the point of  $\mathbf{T}$  containing the pair  $(x, D_f)$ . Since  $D_f \in \mathbf{L}$ , we have  $\mathbf{a} \in U_\delta^T(\mathbf{L})$ . Moreover,  $f_Y^T(\mathbf{a}) = f(x) = y$ , proving the  $\supseteq$  inclusion of (4.1).

Then, by (4.1), we have

$$(4.2) \quad \text{Int}_Y(f_Y^T(U_\delta^T(\mathbf{L}))) \supseteq \bigcup \{\text{Int}_Y(f(U_\delta^f)) : D_f \in \mathbf{L}\}.$$

Since  $U_\delta^T(\mathbf{L})$  is a non-empty set, there exist  $\mathbf{a} \in U_\delta^T(\mathbf{L})$  and  $(x, D_f) \in \mathbf{a}$  such that  $x \in U_\delta^f$  and  $D_f \in \mathbf{L}$ , that is,  $U_\delta^f$  is a non-empty open set. Since  $f$  is a semi-open mapping, we have

$$\text{Int}_Y(f(U_\delta^f)) \neq \emptyset$$

and hence

$$\bigcup \{\text{Int}_Y(f(U_\delta^f)) : D_f \in \mathbf{L}\} \neq \emptyset.$$

Therefore, by (4.2), we have

$$\text{Int}_Y(f_Y^T(U_\delta^T(\mathbf{L}))) \neq \emptyset$$

as desired.

Now, to prove that  $f_Y^T$  is universal in the class  $\mathbf{C}^{\text{sup}}(\mathbb{E}, Y)$  it is enough to show that for each  $f \in \mathbf{F}$  there exists an embedding  $i_f : D_f \rightarrow \mathbf{T}(\mathbf{M}, \mathbf{R})$  such that  $f_Y^T \circ i_f = f$  (see Figure 2). Indeed, we take for  $i_f$  the natural embedding of  $D_f$  into  $\mathbf{T}(\mathbf{M}, \mathbf{R})$ . Then for  $x \in D_f$ ,  $(x, D_f) \in \mathbf{a} \in \mathbf{T}(\mathbf{M}, \mathbf{R})$  we have  $i_f(x) = \mathbf{a} \in \mathbf{T}(\mathbf{M}, \mathbf{R})$  and thus

$$(f_Y^T \circ i_f)(x) = f_Y^T(i_f(x)) = f_Y^T(\mathbf{a}) = f(x). \blacksquare$$

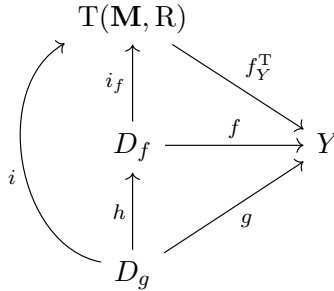


Fig. 2. The universal mapping  $f_Y^T$

COROLLARY 4.4. *Let  $\mathbb{E}$  be any of the following classes of spaces:*

- (i) *the class of regular spaces,*
- (ii) *the class of almost regular spaces,*

- (iii) the class of regular spaces of small inductive dimension  $\text{ind} \leq \alpha$ ,  $\alpha \in \tau^+$ ,
- (iv) the class of regular countable-dimensional spaces (in the sense of  $\text{ind}$ ),
- (v) the class of regular strongly countable-dimensional spaces (in the sense of  $\text{ind}$ ),
- (vi) the class of regular locally finite-dimensional spaces (in the sense of  $\text{ind}$ ).

Then in the class  $C^{\text{top}}(\mathbb{E}, Y)$  of all continuous and semi-open  $Y$ -mappings with domain in  $\mathbb{E}$  and range  $Y$  there exist universal elements.

*Proof.* Since the above classes of spaces are topological, the class  $C^{\text{top}}(\mathbb{E}, Y)$  is also topological. So, considering that all classes given in this corollary are saturated [I, MS] by Theorem 4.3 we have the desired fact. ■

COROLLARY 4.5. *Let  $\mathbb{E}$  be any of the following classes of spaces:*

- (i) the class of separable metrizable spaces,
- (ii) the class of separable metrizable spaces of small inductive dimension  $\text{ind} \leq \alpha$ ,  $\alpha \in \tau^+$ ,
- (iii) the class of separable metrizable countable-dimensional spaces (in the sense of small inductive dimension  $\text{ind}$ ),
- (iv) the class of separable metrizable strongly countable-dimensional spaces (in the sense of  $\text{ind}$ ),
- (v) the class of separable metrizable locally finite-dimensional spaces (in the sense of  $\text{ind}$ ).

Then in the class  $C^{\text{top}}(\mathbb{E}, Y)$  of all continuous and semi-open  $Y$ -mappings with the domain in  $\mathbb{E}$  and range  $Y$  there exist universal elements.

*Proof.* Since the above classes of spaces are topological, the class  $C^{\text{top}}(\mathbb{E}, Y)$  is also topological. So, considering that all classes given in this corollary are saturated [I], by Theorem 4.3, we have the desired fact. ■

Concluding the section, we note that specific results can be obtained for classes of continuous and semi-open mappings with range the  $n$ -Euclidean space  $\mathbb{R}^n$ ,  $n = 1, 2, \dots$ .

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