

CONSTRUCTION OF IDEAL COTORSION PAIRS VIA
RECOLLEMENTS OF TRIANGULATED CATEGORIES

BY

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Abstract. Let $(\mathcal{T}', \mathcal{T}, \mathcal{T}'')$ be a recollement of triangulated categories. A complete ideal cotorsion pair in $\mathcal{T}$ induces complete ideal cotorsion pairs in $\mathcal{T}'$ and $\mathcal{T}''$. In addition, if $(\mathcal{I}, \mathcal{I}^\perp)$ and $(\mathcal{J}, \mathcal{J}^\perp)$ are two complete ideal cotorsion pairs in a triangulated category, then $(\mathcal{I} \cap \mathcal{J}, \langle \mathcal{I}^\perp, \mathcal{J}^\perp \rangle)$ is also a complete ideal cotorsion pair. By this method, starting from two complete ideal cotorsion pairs in $\mathcal{T}'$ and $\mathcal{T}''$, one can induce a family of complete ideal cotorsion pairs in $\mathcal{T}$.

1. Introduction. The central role in approximation theory for the case of modules, or more general abelian or exact categories, is played by the notion of cotorsion pair; cf. [7]. Fu, Herzog et al. [5, 6] extended the study of classical cotorsion pairs to ideal cotorsion pairs, in exact categories. Their construction relies on ME-extensions, which are specific to exact categories. Furthermore, Breaz and Modoi [3] extended ideal approximation theory to triangulated categories, who extended the ME-extensions to construct ideal cotorsion pairs in the triangulated category using the Toda bracket.

The construction of (ideal) cotorsion pairs is a recurring theme in relative homological algebra. In exact categories, intersection methods have been used to produce classical and ideal cotorsion pairs; see, for instance, [2, 10]. Similar ideas also appear in the triangulated setting, e.g. in [8]. Building on the fact [3] that every special precovering ideal in a triangulated category induces an ideal cotorsion pair, we develop an intersection technique for complete ideal cotorsion pairs in triangulated categories.

THEOREM 1.1. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be special precovering ideals in a subcategory $\mathcal{A}$ of a triangulated category with respect to an almost exact structure $\mathfrak{C}$ for $\mathcal{A}$. Then $\mathcal{I}_1 \cap \mathcal{I}_2$ is a special precovering ideal in $\mathcal{A}$.*

Recollements of triangulated categories, introduced by Beilinson, Bernstein and Deligne [2], provide a powerful mechanism for transferring struc-

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tures between categories. In the classical case, Chen [4] showed that cotorsion pairs can be constructed along a recollement. It is therefore natural to ask whether recollements can also be used to lift (or glue) ideal cotorsion pairs. The purpose of this paper is to give an affirmative answer and to describe explicit constructions.

THEOREM 1.2. *Consider the following recollement of triangulated categories:*

$$\begin{array}{ccccc}
 & \overset{i^*}{\curvearrowright} & & \overset{j^!}{\curvearrowright} & \\
 \mathcal{T}' & \xrightarrow{i_*} & \mathcal{T} & \xrightarrow{j^*} & \mathcal{T}'' \\
 & \underset{i_!}{\curvearrowright} & & \underset{j_*}{\curvearrowright} &
 \end{array}$$

Then $(\mathcal{I}', \mathcal{J}')$ is a complete ideal cotorsion pair in $\mathcal{T}'$ and $(\mathcal{I}'', \mathcal{J}'')$ is a complete ideal cotorsion pair in $\mathcal{T}''$ if and only if there exists a complete ideal cotorsion pair $(\langle j_! \mathcal{I}'', i_* \mathcal{I}' \rangle, \langle i_* \mathcal{J}', j_* \mathcal{J}'' \rangle)$ in $\mathcal{T}$.

The notations $\langle j_! \mathcal{I}'', i_* \mathcal{I}' \rangle$ and $\langle i_* \mathcal{J}', j_* \mathcal{J}'' \rangle$ denote the Toda brackets (see Definition 2.13) of $j_! \mathcal{I}''$, $i_* \mathcal{I}'$ and of $i_* \mathcal{J}'$, $j_* \mathcal{J}''$.

In addition, starting from $(\mathcal{I}', \mathcal{J}')$ and $(\mathcal{I}'', \mathcal{J}'')$ we obtain further complete ideal cotorsion pairs in $\mathcal{T}$. For the reader's convenience, the relationships between these constructions are summarized in a diagram in the Appendix.

The paper is organized as follows. In Section 2 we collect the necessary background on ideal cotorsion pairs in triangulated categories, including almost exact structures and phantom ideals, (special) precovering/preenveloping ideals, and Toda brackets, together with several auxiliary results. In Section 3 we develop an intersection method for ideals and prove Theorem 1.1. In Section 4 we apply these preparations to recollements of triangulated categories: we first study the transport of ideal cotorsion pairs along the recollement functors and then establish the gluing result of Theorem 1.2.

2. Preliminaries. Throughout, $\mathcal{T}, \mathcal{T}', \mathcal{T}''$ denote triangulated categories. We recall some standard notation.

In order to define the ideal cotorsion pair in a triangulated category, we recall some fundamental definitions below.

DEFINITION 2.1 ([3, Section 2.1]). Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ be a full subcategory (closed under isomorphisms) of $\mathcal{T}$. We say $\mathcal{A}$ is *closed under extensions* if for every triangle $B \rightarrow C \rightarrow A \rightarrow B[1]$ in $\mathcal{T}$ such that A and B are objects in $\mathcal{A}$, also C is an object in $\mathcal{A}$. We will denote by $\mathfrak{D}_{\mathcal{A}}$ the class of all exact triangles $\mathfrak{d} : B \rightarrow C \rightarrow A \rightarrow B[1]$ such that $A, B, C \in \mathcal{A}$. Note that every extension closed subcategory $\mathcal{A}$ of $\mathcal{T}$ is closed under finite

coproducts. For a subcategory $\mathcal{A}$ of $\mathcal{T}$, we denote by $\mathcal{A}^\rightarrow$ the class of all morphisms in $\mathcal{A}$.

DEFINITION 2.2 ([3, Section 2.1]). Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ be a full subcategory of $\mathcal{T}$ which is closed under extensions. A class $\mathfrak{C} \subseteq \mathfrak{D}_{\mathcal{A}}$ of triangles is an *almost exact structure* for $\mathcal{A}$ if

(1) $\mathfrak{C}$ is closed under isomorphisms of triangles and under finite coproducts, and contains all split triangles;

(2) $\mathfrak{C}$ is closed under base changes (homotopy pullbacks) constructed along morphisms from $\mathcal{A}$, i.e. if $\mathfrak{d} : C \rightarrow B \rightarrow A \rightarrow C[1]$ is a triangle in $\mathfrak{C}$ and $\alpha : X \rightarrow A$ is a morphism in $\mathcal{A}$ then the top triangle $\mathfrak{d}\alpha$ in every homotopy cartesian diagram

$$\begin{array}{ccccccc} \mathfrak{d}\alpha : & C & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & C[1] \\ & \parallel & & \downarrow & & \downarrow \alpha & & \downarrow \\ \mathfrak{d} : & C & \longrightarrow & B & \longrightarrow & A & \longrightarrow & C[1] \end{array}$$

is in $\mathfrak{C}$;

(3) $\mathfrak{C}$ is closed under cobase changes (homotopy pushouts) constructed along morphisms from $\mathcal{A}$, i.e. if $\mathfrak{d} : C \rightarrow B \rightarrow A \rightarrow C[1]$ is a triangle in $\mathfrak{C}$ and $\beta : C \rightarrow Z$ is a morphism in $\mathcal{A}$ then the top triangle $\beta\mathfrak{d}$ in every homotopy cartesian diagram

$$\begin{array}{ccccccc} \mathfrak{d} : & C & \longrightarrow & B & \longrightarrow & A & \longrightarrow & C[1] \\ & \downarrow \beta & & \downarrow & & \parallel & & \downarrow \\ \beta\mathfrak{d} : & Z & \longrightarrow & Y & \longrightarrow & A & \longrightarrow & Z[1] \end{array}$$

is in $\mathfrak{C}$.

If $\mathfrak{C}$ is an almost exact structure then the triangle

$$\mathfrak{d} : C \xrightarrow{f} B \xrightarrow{g} A \xrightarrow{\varphi} C[1],$$

which lies in $\mathfrak{C}$, will be called a $\mathfrak{C}$ -triangle. Moreover, we will say that f is a $\mathfrak{C}$ -inflation, g is a $\mathfrak{C}$ -deflation, and φ is a $\mathfrak{C}$ -phantom.

The class of all $\mathfrak{C}$ -phantoms is denoted by $\Phi(\mathfrak{C})$.

Note that an almost exact structure $\mathfrak{C}$ as in the definition above depends on both the triangulated category $\mathcal{T}$ and the full subcategory $\mathcal{A}$. Therefore, whenever we refer to an almost exact structure, we assume that it is constructed in an extension-closed subcategory $\mathcal{A}$ of a triangulated category $\mathcal{T}$.

DEFINITION 2.3 ([3, Definition 2.2.2]). A class $\mathcal{E}$ of morphisms in $\mathcal{T}$ is called a *phantom $\mathcal{A}$ -ideal* if

- (1) $\mathcal{E} \subseteq \mathcal{T}(\mathcal{A}, \mathcal{A}[1]) = \bigcup_{A, B \in \mathcal{A}} \mathcal{T}(A, B[1])$;
- (2) $\mathcal{E}$ is closed under sums of morphisms;

(3) $\mathcal{A}^\rightarrow[1]\mathcal{E}\mathcal{A}^\rightarrow \subseteq \mathcal{E}$, i.e. for every chain of composable morphisms

$$B \xrightarrow{g} A \xrightarrow{i} C[1] \xrightarrow{f[1]} D[1]$$

in $\mathcal{T}$ such that $i \in \mathcal{E}$ and $f, g \in \mathcal{A}^\rightarrow$ we have $f[1]ig \in \mathcal{E}$.

We will frequently use the base change (homotopy pullback) and cobase change (homotopy pushout) closure in the definition above. In particular, the phrase “base change” always refers to the homotopy cartesian construction inside $\mathcal{T}$ along a morphism in $\mathcal{A}$.

LEMMA 2.4 ([1]). *Let $\mathcal{T}$ be a triangulated category, and $\mathcal{A}$ a full subcategory which is closed under extensions. The following are equivalent for a class $\mathfrak{C} \subseteq \mathfrak{D}_{\mathcal{A}}$ of triangles in $\mathcal{A}$ which is closed under isomorphism:*

- (1) $\mathfrak{C}$ is an almost exact structure;
- (2) $\mathcal{A}^\rightarrow[1]\Phi(\mathfrak{C})\mathcal{A}^\rightarrow \subseteq \Phi(\mathfrak{C})$ and $\Phi(\mathfrak{C})$ is closed under (finite direct) sums of morphisms.

Consequently,

- (1') if $\mathfrak{C}$ is an almost exact structure from $\mathcal{A}$, then $\Phi(\mathfrak{C})$ is a phantom $\mathcal{A}$ -ideal;
- (2') for every phantom $\mathcal{A}$ -ideal $\mathcal{I}$, the class

$$\mathfrak{D}(\mathcal{I}) = \{\mathfrak{d} \in \mathfrak{D} \mid \text{the phantom of } \mathfrak{d} \text{ is in } \mathcal{I}\}$$

is an almost exact structure;

- (3') the correspondences from (1') and (2') above are inverse to each other.

There exists a one-to-one correspondence between almost exact structures and phantom $\mathcal{A}$ -ideals.

DEFINITION 2.5 ([3, Section 2.2]). A class $\mathcal{I} \subseteq \mathcal{A}^\rightarrow$ of morphisms is called an *ideal* in the category $\mathcal{A}$ if for any morphisms $f : A \rightarrow B$, $g : C \rightarrow D$ in $\mathcal{A}$ and $i : B \rightarrow C$ in $\mathcal{I}$, we have $g \circ i \circ f \in \mathcal{I}$.

DEFINITION 2.6. Let $\Phi(\mathfrak{C})$ denote the set of all $\mathfrak{C}$ -phantoms, which is a phantom $\mathcal{A}$ -ideal in a triangulated category $\mathcal{T}$. Let i, j be morphisms in $\mathcal{A}$. We say that i and j are *orthogonal*, denoted $i \perp j$, if $j[1] \circ \phi \circ i = 0$ for any $\phi \in \Phi(\mathfrak{C})$. Let $\mathcal{I}, \mathcal{J}$ be ideals in $\mathcal{A}$. The set $\mathcal{I}^\perp$ is the set of all morphisms j with $i \perp j$ for any $i \in \mathcal{I}$. Similarly, ${}^\perp\mathcal{J}$ is the set of all morphisms i with $i \perp j$ for any $j \in \mathcal{J}$. The pair $(\mathcal{I}, \mathcal{J})$ is an *ideal cotorsion pair* if $\mathcal{I}^\perp = \mathcal{J}$ and ${}^\perp\mathcal{J} = \mathcal{I}$.

DEFINITION 2.7 ([3, Section 3.1]). Let $\mathcal{A}$ be an extension closed full subcategory of $\mathcal{T}$ with an almost exact structure $\mathfrak{C}$. Let $\mathcal{I}$ be an ideal in $\mathcal{A}$. A morphism $i : X \rightarrow A$ from $\mathcal{I}$ is an $\mathcal{I}$ -*precover* for A if all morphisms $i' : X' \rightarrow A$ from $\mathcal{I}$ factor through i . The ideal $\mathcal{I}$ is a *precovering ideal* if every object in $\mathcal{A}$ has an $\mathcal{I}$ -precover.

A morphism $i : X \rightarrow A$ in $\mathcal{I}$ is a *special $\mathcal{I}$ -precover* (with respect to $\mathfrak{C}$) if in the corresponding triangle

$$B \rightarrow X \xrightarrow{i} A \xrightarrow{k} B[1]$$

we have $k \in (\mathcal{I}^\perp)[1]\Phi(\mathfrak{C})$. We say $\mathcal{I}$ is a *special precovering ideal* if every object in $\mathcal{A}$ has a special $\mathcal{I}$ -precover.

Dually, the definitions of *$\mathcal{I}$ -preenvelope* and *special $\mathcal{I}$ -preenveloping* can be given.

We note that a morphism $i : X \rightarrow A$ is a special $\mathcal{I}$ -precover with respect to $\mathfrak{C}$ if there exists a homotopy pushout diagram

$$\begin{array}{ccccccc} Y & \longrightarrow & Z & \longrightarrow & A & \xrightarrow{\phi} & Y[1] \\ \downarrow j & & \downarrow & & \parallel & & \downarrow j[1] \\ B & \longrightarrow & X & \xrightarrow{i} & A & \xrightarrow{\varphi} & B[1] \end{array}$$

such that $j \in \mathcal{I}^\perp$, and the top triangle is in $\mathfrak{C}$.

DEFINITION 2.8. Let $F : \mathcal{A} \rightarrow \mathcal{A}'$ be a functor. Let $\mathcal{I}$ and $\mathcal{I}'$ be ideals of $\mathcal{A}$ and $\mathcal{A}'$, respectively. The *image ideal* $F\mathcal{I}$ is the ideal of $\mathcal{A}'$ generated by the image set $F(\mathcal{I})$. Explicitly,

$$F\mathcal{I} = \{f' \circ F(i) \circ g' \mid i \in \mathcal{I}, f', g' \in \mathcal{A}'^{\rightarrow}\}.$$

The *inverse image ideal* $F^{-1}\mathcal{I}'$ is the ideal of $\mathcal{A}$ defined by

$$F^{-1}\mathcal{I}' = \{\alpha \in \mathcal{A}^{\rightarrow} \mid F(\alpha) \in \mathcal{I}'\}.$$

LEMMA 2.9. Let $F : \mathcal{A} \rightleftarrows \mathcal{A}' : G$ be an adjoint pair. Let $\mathcal{I} \subseteq \mathcal{A}^{\rightarrow}$ and $\mathcal{I}' \subseteq \mathcal{A}'^{\rightarrow}$ be ideals.

If G is fully faithful, then the following are equivalent:

- (1) $GFI \subseteq \mathcal{I}$;
- (2) $F\mathcal{I} = G^{-1}\mathcal{I}$.

If F is fully faithful, then the following are equivalent:

- (3) $FG\mathcal{I}' \subseteq \mathcal{I}'$;
- (4) $G\mathcal{I}' = F^{-1}\mathcal{I}'$.

Proof. We only show (1) $\Rightarrow$ (2). Let $\alpha' \in F\mathcal{I}$. By definition, there exist $\alpha \in \mathcal{I}$ and morphisms f', g' in $\mathcal{A}'$ such that $\alpha' = g' \circ F(\alpha) \circ f'$. Applying G , we get $G(\alpha') = G(g') \circ GF(\alpha) \circ G(f')$. By (1), $GF(\alpha) \in \mathcal{I}$. Since $\mathcal{I}$ is an ideal, the composition $G(\alpha')$ remains in $\mathcal{I}$. Thus, $\alpha' \in G^{-1}\mathcal{I}$, proving $F\mathcal{I} \subseteq G^{-1}\mathcal{I}$.

Conversely, let $\alpha' \in G^{-1}\mathcal{I}$. By definition, $G(\alpha') \in \mathcal{I}$. Since G is fully faithful, we have an isomorphism $\alpha' \cong FG(\alpha')$. Note that $FG(\alpha') = F(G(\alpha'))$ is the image of an element in $\mathcal{I}$ under F , so it belongs to $F\mathcal{I}$. Since $F\mathcal{I}$ is an ideal closed under isomorphism, $\alpha' \in F\mathcal{I}$. Therefore, $G^{-1}\mathcal{I} \subseteq F\mathcal{I}$. ■

LEMMA 2.10. *Let (F, G) be an adjoint pair with $F : \mathcal{A} \rightarrow \mathcal{A}'$, and $\mathcal{I}$ be an ideal in $\mathcal{A}$. If $\mathcal{I}$ is a precovering ideal, then so is $F\mathcal{I}$.*

Proof. For any object A' in $\mathcal{A}'$, since $\mathcal{I}$ is a precovering ideal in $\mathcal{A}$, there exists an $\mathcal{I}$ -precover $\alpha : X \rightarrow G(A')$ with $\alpha \in \mathcal{I}$. Using the adjoint pair (F, G) , we obtain a morphism $\phi : F(X) \rightarrow A'$ given by the composition, $\phi = \epsilon_{A'} \circ F(\alpha)$, where $\epsilon : FG \rightarrow \text{Id}_{\mathcal{A}'}$ is the counit of the adjunction. Since $\alpha \in \mathcal{I}$, we have $\phi \in F\mathcal{I}$.

Next, we show that ϕ is an $F\mathcal{I}$ -precover. Let $\psi : Y \rightarrow A'$ be any morphism in the ideal $F\mathcal{I}$. By the definition of the image ideal, ψ factors through the image of a morphism in $\mathcal{I}$. Specifically, ψ can be written as $\psi = \beta \circ F(\gamma) \circ \delta$, where $\gamma : C \rightarrow B$ is in $\mathcal{I}$. Consider the adjoint morphism

$$\theta = G(\beta) \circ \eta_B \circ \gamma : C \rightarrow G(A')$$

corresponding to the part $\beta \circ F(\gamma)$, where $\eta : \text{Id}_{\mathcal{A}} \rightarrow GF$ is the unit. Since $\gamma \in \mathcal{I}$, the composition θ belongs to $\mathcal{I}$. Because $\alpha : X \rightarrow G(A')$ is an $\mathcal{I}$ -precover, there exists a morphism $h : C \rightarrow X$ such that $\theta = \alpha \circ h$. Applying F and composing with $\epsilon_{A'}$, we get

$$\epsilon_{A'} \circ F(\theta) = \epsilon_{A'} \circ F(\alpha) \circ F(h) = \phi \circ F(h).$$

On the other hand, by the naturality of the adjunction, $\epsilon_{A'} \circ F(\theta)$ recovers the morphism $\beta \circ F(\gamma)$. Therefore, $\beta \circ F(\gamma) = \phi \circ F(h)$, which implies

$$\psi = \beta \circ F(\gamma) \circ \delta = \phi \circ (F(h) \circ \delta).$$

This shows that ψ factors through ϕ . Thus, $F\mathcal{I}$ is a precovering ideal. ■

We next recall recollements, which will be used in the construction of ideal cotorsion pairs via Toda brackets.

DEFINITION 2.11 ([2]). Let $\mathcal{T}, \mathcal{T}', \mathcal{T}''$ be triangulated categories. Then $\mathcal{T}$ admits a *recollement* relative to $\mathcal{T}'$ and $\mathcal{T}''$, diagrammatically expressed by

$$\begin{array}{ccccc} & i^* & & j_! & \\ & \curvearrowright & & \curvearrowright & \\ \mathcal{T}' & \xrightarrow{i_*} & \mathcal{T} & \xrightarrow{j^*} & \mathcal{T}'' \\ & \curvearrowleft & & \curvearrowleft & \\ & i^! & & j_* & \end{array}$$

if the six exact functors satisfy the following conditions:

- (1) each functor is left adjoint to the one below it;
- (2) $i_*, j_!, j_*$ are fully faithful;
- (3) $j^* i_* = 0$ (and also $i^* j_! = 0$ and $i^! j_* = 0$);
- (4) each object T in $\mathcal{T}$ determines the exact triangles

$$\begin{aligned} i_* i^! T &\rightarrow T \rightarrow j_* j^* T \rightarrow i_* i^! T[1], \\ j_! j^* T &\rightarrow T \rightarrow i_* i^* T \rightarrow j_! j^* T[1]. \end{aligned}$$

LEMMA 2.12 ([9]). *If $F : \mathcal{T}' \rightarrow \mathcal{T}$ is left or right adjoint to G , then F is a fully faithful functor if and only if there exists an isomorphism $\text{id}_{\mathcal{T}'} \cong GF$.*

Therefore, $\text{id}_{\mathcal{T}'} \cong i^! i_*$, $\text{id}_{\mathcal{T}''} \cong j^* j_!$ and $i^* i_* \cong \text{id}_{\mathcal{T}'}$, $j^* j_* \cong \text{id}_{\mathcal{T}''}$ when i^* , i_* , $i^!$, $j_!$, j^* , j_* are the functors in Definition 2.11.

Recall that a zero morphism is characterized by its factorization through the zero object. By the naturality of the adjunction, the unique morphism $F(A) \rightarrow 0_{\mathcal{B}} \rightarrow B$ corresponds to $A \rightarrow G(0_{\mathcal{B}}) \rightarrow G(B)$. Since $G(0_{\mathcal{B}}) \cong 0_{\mathcal{A}}$, the latter represents the zero morphism in $\mathcal{A}$. Thus, the adjunction isomorphism preserves zero morphisms.

DEFINITION 2.13. Let $\mathfrak{d} : Y \xrightarrow{f} Z \xrightarrow{g} X \xrightarrow{\varphi} Y[1]$ be a triangle in $\mathcal{T}$. If $i : Y \rightarrow U$ and $j : V \rightarrow X$ are morphisms, then the *Toda bracket* $\langle i, j \rangle_{\mathfrak{d}}$ is the set of all morphisms $\xi : V \rightarrow U$ such that $\xi = \xi'' \circ \xi'$, where $\xi' : V \rightarrow Z$ and $\xi'' : Z \rightarrow U$ are morphisms which make the following diagram commutative:

$$\begin{array}{ccccccc}
 & & & & V & & \\
 & & & & \downarrow j & & \\
 & & \xi' & & & & \\
 & & \swarrow & & & & \\
 \mathfrak{d} : Y & \xrightarrow{f} & Z & \xrightarrow{g} & X & \longrightarrow & Y[1] \\
 & \searrow \xi'' & & & & & \\
 & & U & & & &
 \end{array}$$

If $\mathcal{I}$ and $\mathcal{J}$ are two classes of morphisms then the union of all Toda brackets $\langle i, j \rangle_{\mathfrak{d}}$ with $i \in \mathcal{I}$, $j \in \mathcal{J}$ and $\mathfrak{d} \in \mathfrak{C}$ is denoted by $\langle \mathcal{I}, \mathcal{J} \rangle_{\mathfrak{C}}$, and it is called the *Toda bracket* of $\mathcal{I}$ and $\mathcal{J}$ induced by $\mathfrak{C}$.

LEMMA 2.14 ([3, Theorem 4.4.3]). *Suppose that there are enough $\mathfrak{C}$ -injective morphisms. If $\mathcal{I}$ and $\mathcal{J}$ are special precovering ideals in $\mathcal{A}$, then*

$$(\mathcal{I}\mathcal{J})^{\perp} = \langle \mathcal{J}^{\perp}, \mathcal{I}^{\perp} \rangle_{\mathfrak{C}}.$$

3. Construct Ideal cotorsion pair by intersection. In this section, we show that the intersection of two special precovering ideals is also a special precovering ideal. We start with the following lemmas.

LEMMA 3.1. *If $\mathcal{I}_1$ and $\mathcal{I}_2$ are ideals in $\mathcal{A}$, then $\mathcal{I}_1 \cap \mathcal{I}_2$ is also an ideal in $\mathcal{A}$.*

Proof. It follows immediately from the definition that $0 \in \mathcal{I}_1 \cap \mathcal{I}_2$. Since an ideal is closed under compositions with arbitrary morphisms and finite direct sums, the intersection of ideals is also closed under compositions with arbitrary morphisms and finite direct sums. ■

Next, we consider the intersection of object ideals. This is also an important reason for studying the intersection of ideals.

LEMMA 3.2. *Let $\mathcal{A}$ be an extension-closed subcategory of $\mathcal{T}$ and $\mathfrak{C}$ an almost exact structure in $\mathcal{A}$. If $\mathcal{D}$ and $\mathcal{E}$ are two classes of objects in $\mathcal{A}$ closed under finite direct sums, then $\text{Ideal}(\mathcal{E} \cap \mathcal{D}) \subseteq \text{Ideal}(\mathcal{E}) \cap \text{Ideal}(\mathcal{D})$.*

Proof. For any morphism $i \in \text{Ideal}(\mathcal{E} \cap \mathcal{D})$, there exists an object $X \in \mathcal{E} \cap \mathcal{D}$ such that 1_X factors through i . Then $i \in \text{Ideal}(\mathcal{E})$ and $i \in \text{Ideal}(\mathcal{D})$, that is, $i \in \text{Ideal}(\mathcal{E}) \cap \text{Ideal}(\mathcal{D})$. ■

EXAMPLE 3.3. Let $\mathcal{D}$ be the class generated by an object D , and $\mathcal{E}$ be the class generated by an object E with $D \neq E$. Suppose that there exists a nonzero morphism in $\text{Hom}(D, E)$ or $\text{Hom}(E, D)$. Then $\text{Ideal}(\mathcal{E} \cap \mathcal{D})$ is empty, but $\text{Ideal}(\mathcal{E}) \cap \text{Ideal}(\mathcal{D})$ is not.

The above lemma shows that an intersection of object ideals can contain more elements than the ideal of the intersection of objects, which enables an intersection of ideals to have properties that the ideal of the intersection of objects does not possess.

Next, we show that the intersection of two special precovering ideals is also a special precovering ideal.

Proof of Theorem 1.1. Let A be any object in $\mathcal{A}$. Since $\mathcal{I}_1$ and $\mathcal{I}_2$ are special precovering ideals, there exist the following distinguished triangles for $k = 1, 2$:

$$K_k \rightarrow B_k \xrightarrow{i_k} A \xrightarrow{g_k} K_k[1]$$

such that $i_k \in \mathcal{I}_k$, and the morphisms g_k decompose as $g_k = j_k[1] \circ \phi_k$, where $j_k \in \mathcal{I}_k^\perp$ and $\phi_k \in \Phi(\mathfrak{C})$.

Consider the morphisms $\phi_1 : A \rightarrow D_1[1]$ and $\phi_2 : A \rightarrow D_2[1]$ in $\Phi(\mathfrak{C})$. Since $\Phi(\mathfrak{C})$ is closed under finite coproducts, the direct sum morphism $\phi_1 \oplus \phi_2 : A \oplus A \rightarrow D_1[1] \oplus D_2[1]$ belongs to $\Phi(\mathfrak{C})$. Since $\Phi(\mathfrak{C})$ is closed under base change, we can pull back along the diagonal morphism $\Delta = (1_A, 1_A)^T : A \rightarrow A \oplus A$. Define $\Psi : A \rightarrow D_1[1] \oplus D_2[1]$ by

$$\Psi := (\phi_1 \oplus \phi_2) \circ \Delta = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Thus, $\Psi \in \Phi(\mathfrak{C})$. Define $J : D_1 \oplus D_2 \rightarrow K_1 \oplus K_2$ to be the diagonal matrix

$$J := \begin{pmatrix} j_1 & 0 \\ 0 & j_2 \end{pmatrix}.$$

Since $j_1 \in \mathcal{I}_1^\perp$ and $j_2 \in \mathcal{I}_2^\perp$, by the properties of orthogonal ideals (specifically that $(\mathcal{I}_1 \cap \mathcal{I}_2)^\perp$ contains sums of morphisms from $\mathcal{I}_1^\perp$ and $\mathcal{I}_2^\perp$), we have $J \in (\mathcal{I}_1 \cap \mathcal{I}_2)^\perp$.

Define the morphism $G : A \rightarrow K_1[1] \oplus K_2[1]$ by

$$G := J[1] \circ \Psi = \begin{pmatrix} j_1[1] \circ \phi_1 \\ j_2[1] \circ \phi_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

Complete G to a distinguished triangle in $\mathfrak{C}$ as follows:

$$K_1 \oplus K_2 \rightarrow P \xrightarrow{f} A \xrightarrow{G} K_1[1] \oplus K_2[1].$$

It remains to show $f \in \mathcal{I}$. Using the projections $\pi_k : K_1[1] \oplus K_2[1] \rightarrow K_k[1]$ for $k = 1, 2$, we have

$$g_k \circ f = \pi_k \circ G \circ f = 0.$$

Since $g_k \circ f = 0$, f factors through $i_k \in \mathcal{I}_k$. Since $\mathcal{I}_k$ is an ideal, this implies $f \in \mathcal{I}_k$. Therefore, $f \in \mathcal{I}_1 \cap \mathcal{I}_2$. $\square$

Note that when the almost exact structure is given by all triangles, the right orthogonal of $\mathcal{I}_1 \cap \mathcal{I}_2$ coincides with the sum of the right orthogonals of $\mathcal{I}_1$ and $\mathcal{I}_2$, that is, $(\mathcal{I}_1 \cap \mathcal{I}_2)^\perp = \mathcal{I}_1^\perp + \mathcal{I}_2^\perp$.

4. Ideal cotorsion pair in recollement of triangulated categories.

Note that if $\mathcal{A} = \mathcal{T}$ and $\mathfrak{C}$ is the class of all triangles then every precovering ideal is special. Dually, every preenveloping ideal in $\mathcal{T}$ is special with respect to the class of all triangles in $\mathcal{T}$. Thus, next we only consider the case where $\mathcal{A} = \mathcal{T}$ and $\mathfrak{C}$ is the class of all triangles. In this case, the Toda bracket $\langle \mathcal{I}, \mathcal{J} \rangle_{\mathfrak{C}}$ is denoted by $\langle \mathcal{I}, \mathcal{J} \rangle$ for simplicity. And the condition of having enough $\mathfrak{C}$ -injective morphisms and having enough $\mathfrak{C}$ -projective morphisms are satisfied in our setting, since the $\mathfrak{C}$ is the class of all triangles in $\mathcal{A}$.

4.1. Functorial transport along a recollement. In this subsection, we show that an ideal cotorsion pair in $\mathcal{T}$ can induce ideal cotorsion pairs in $\mathcal{T}'$ and $\mathcal{T}''$ under some conditions.

LEMMA 4.1. *Consider a recollement as in Definition 2.11. Let $(\mathcal{I}, \mathcal{J})$ be a (complete) ideal cotorsion pair in $\mathcal{T}$.*

- (1) *If $j_! j^* \mathcal{I} \subseteq \mathcal{I}$, then $(j^* \mathcal{I}, j^* \mathcal{J})$ is a (complete) ideal cotorsion pair in $\mathcal{T}''$.*
- (2) *If $i_* i^*(\mathcal{I}) \subseteq \mathcal{I}$, then $(i^* \mathcal{I}, i^* \mathcal{J})$ is a (complete) ideal cotorsion pair in $\mathcal{T}'$.*

Proof. We prove the case for j^* ; the other case follows by duality.

STEP 1: We deduce the stability of $\mathcal{J}$ from the stability of $\mathcal{I}$. We show that $j_* j^*(\mathcal{J}) \subseteq \mathcal{J}$. Let $\beta : A \rightarrow B$ be a morphism in $\mathcal{J}$. To show $j_* j^*(\beta) \in \mathcal{J}$, we consider the composition $\xi = j_* j^*(\beta)[1] \circ \phi \circ \alpha$ with any $\alpha \in \mathcal{I}$ and any phantom $\phi \in \Phi(\mathfrak{C})$. Consider the adjoint pair (j^*, j_*) . The morphism ξ corresponds to the morphism $\xi' = j^*(\beta)[1] \circ \epsilon_{j^* A} \circ j^*(\phi) \circ j^*(\alpha)$, where $\epsilon_{j^* A} : j^* j_*(j^* A) \rightarrow j^* A$ is the counit of the adjoint pair (j^*, j_*) . Furthermore, consider the adjoint pair $(j_!, j^*)$. The morphism ξ' corresponds to the morphism $\tilde{\xi} = \beta[1] \circ \epsilon_A \circ j_!(\epsilon_{j^* A}) \circ j_! j^*(\phi) \circ j_! j^*(\alpha)$. Since $j_! j^*(\mathcal{I}) \subseteq \mathcal{I}$, and

$(\mathcal{I}, \mathcal{J})$ is an ideal cotorsion pair, such a composition must vanish, so $\tilde{\xi} = 0$. By the bijection of the adjunction, $\xi = 0$. Therefore, $j_*j^*(\beta) \in \mathcal{J}$.

STEP 2: We show that $j^*\mathcal{I} \perp j^*\mathcal{J}$ in $\mathcal{T}''$. Let α'' be a morphism in $j^*\mathcal{I}$, β'' be a morphism in $j^*\mathcal{J}$ and φ'' be a phantom morphism in $\mathcal{T}''$. By the definition of the image ideal, α'' can be written as the following composition: $\alpha'' = f'' \circ j^*(\alpha) \circ g''$ where $\alpha : A \rightarrow B$ is a morphism in $\mathcal{I}$, and f'', g'' are morphisms in $\mathcal{T}''$.

To show $\beta'' \circ \varphi'' \circ \alpha'' = 0$, it suffices to show $\beta'' \circ \varphi'' \circ f'' \circ j^*(\alpha) = 0$. Consider the adjoint pair (j^*, j_*) . The composition $\Psi'' = \beta'' \circ \varphi'' \circ f'' \circ j^*(\alpha)$ in $\mathcal{T}''$ corresponds to the following morphism in $\mathcal{T}$:

$$\Psi = j_*(\beta'') \circ j_*(\varphi'') \circ j_*(f'') \circ \eta_B \circ \alpha,$$

where $\eta_B : B \rightarrow j_*j^*(B)$ is the unit morphism.

From Step 1, we established that $j_*(\beta'') \in \mathcal{J}$.

Since $(\mathcal{I}, \mathcal{J})$ is a cotorsion pair in $\mathcal{T}$, we have $\Psi = 0$. By the isomorphism of the adjunction, $\Psi = 0$ implies $\Psi'' = 0$. Therefore, $\beta'' \circ \varphi'' \circ \alpha'' = \Psi'' \circ g'' = 0$. Thus, $j^*\mathcal{I} \perp j^*\mathcal{J}$.

STEP 3:

(i) We show that $(j^*\mathcal{I})^\perp \subseteq j^*\mathcal{J}$. Let $\beta'' : A'' \rightarrow B''$ be a morphism in $(j^*\mathcal{I})^\perp$. Note that we have an isomorphism $\beta'' \cong j^*j_*(\beta'')$. Thus, to show $\beta'' \in j^*\mathcal{J}$, it suffices to show $j_*(\beta'') \in \mathcal{J}$. Since $\mathcal{J} = \mathcal{I}^\perp$, we verify this by testing against any $\alpha \in \mathcal{I}$ and any phantom $\phi \in \Phi(\mathfrak{C})$. Consider the composition $\xi = j_*(\beta'')[1] \circ \phi \circ \alpha$. Using the adjoint pair (j^*, j_*) , we find that the morphism ξ corresponds to $\xi'' = \beta''[1] \circ \epsilon_{A''} \circ j^*(\phi) \circ j^*(\alpha)$, where $\epsilon_{A''} : j^*j_*A'' \rightarrow A''$ is the counit morphism. Since $\alpha \in \mathcal{I}$, we have $j^*(\alpha) \in j^*\mathcal{I}$. Since $\beta'' \in (j^*\mathcal{I})^\perp$, we have $\xi'' = 0$. Therefore, the composition ξ vanishes. Thus $j_*(\beta'') \in \mathcal{I}^\perp = \mathcal{J}$.

(ii) We show that ${}^\perp(j^*\mathcal{J}) \subseteq j^*\mathcal{I}$. Let $\alpha'' : A'' \rightarrow B''$ be a morphism in ${}^\perp(j^*\mathcal{J})$. Note that we have an isomorphism $\alpha'' \cong j^*j_!(\alpha'')$. Thus, to show $\alpha'' \in j^*\mathcal{I}$, it suffices to show $j_!(\alpha'') \in \mathcal{I}$. Since $\mathcal{I} = {}^\perp\mathcal{J}$, we verify this by testing against any $\beta \in \mathcal{J}$ and any phantom $\phi \in \Phi(\mathfrak{C})$. Consider the composition $\xi = \beta[1] \circ \phi \circ j_!(\alpha'')$. Using the adjoint pair $(j_!, j^*)$, we find that the morphism ξ corresponds to $\xi'' = j^*(\beta)[1] \circ j^*(\phi) \circ \eta_{B''} \circ \alpha''$, where $\eta_{B''} : B'' \rightarrow j^*j_!B''$ is the unit morphism. Since $\beta \in \mathcal{J}$, we have $j^*(\beta) \in j^*\mathcal{J}$. Since $\alpha'' \in {}^\perp(j^*\mathcal{J})$, we have $\xi'' = 0$. Therefore, the composition ξ vanishes. Thus $j_!(\alpha'') \in {}^\perp\mathcal{J} = \mathcal{I}$.

Therefore, we conclude that $(j^*\mathcal{I}, j^*\mathcal{J})$ is an ideal cotorsion pair in $\mathcal{T}''$.

Due to Lemma 2.10, if $\mathcal{I}$ is precovering in $\mathcal{T}$, then $j^*\mathcal{I}$ is precovering in $\mathcal{T}''$. Note that every precovering ideal is special. Therefore, $(j^*\mathcal{I}, j^*\mathcal{J})$ is a complete ideal cotorsion pair in $\mathcal{T}''$. ■

4.2. Gluing from $\mathcal{T}'$, $\mathcal{T}''$ to $\mathcal{T}$

LEMMA 4.2. *Consider a recollement as in Definition 2.11. Then $(j_! \mathcal{T}, i_* \mathcal{T})$, $(i_* \mathcal{T}, j_* \mathcal{T})$ are complete ideal cotorsion pairs in $\mathcal{T}$.*

Proof. It is a well-known fact that a recollement induces a TTF triple $(\text{Im } j_!, \text{Im } i_*, \text{Im } j_*)$. This follows from the adjunctions and the canonical triangles associated with the recollement. And cotorsion pairs in $\mathcal{T}$ can induce complete ideal cotorsion pairs with respect to the proper class of all triangles in $\mathcal{T}$. ■

LEMMA 4.3. *Consider a recollement as in Definition 2.11. Let $\mathcal{I}'$ be an ideal in $\mathcal{T}'$ and $\mathcal{I}''$ be an ideal in $\mathcal{T}''$. Then*

- (1) $\langle i_* \mathcal{I}', j_* \mathcal{T}'' \rangle = \{\xi \in \mathcal{T} \mid i^!(\xi) \in \mathcal{I}'\}$;
- (2) $\langle i_* \mathcal{T}', j_* \mathcal{I}'' \rangle = \{\xi \in \mathcal{T} \mid j^*(\xi) \in \mathcal{I}''\}$;
- (3) $\langle j_! \mathcal{I}'', i_* \mathcal{T}' \rangle = \{\xi \in \mathcal{T} \mid j^*(\xi) \in \mathcal{I}''\}$;
- (4) $\langle j_! \mathcal{T}'', i_* \mathcal{I}' \rangle = \{\xi \in \mathcal{T} \mid i^*(\xi) \in \mathcal{I}'\}$.

Proof. We only prove (1); the other cases are similar.

STEP 1: $\langle i_* \mathcal{I}', j_* \mathcal{T}'' \rangle \subseteq \{\xi \in \mathcal{T} \mid i^!(\xi) \in \mathcal{I}'\}$. Let $\xi : F \rightarrow K$ be a morphism in the Toda bracket $\langle i_* \mathcal{I}', j_* \mathcal{T}'' \rangle$. By definition, there exists a triangle $A \xrightarrow{a} C \xrightarrow{b} B \rightarrow A[1]$ and morphisms $\xi' : F \rightarrow C$, $\xi'' : C \rightarrow K$ such that $\xi = \xi'' \circ \xi'$, $f := \xi'' \circ a \in i_* \mathcal{I}'$ and $g := b \circ \xi' \in j_* \mathcal{T}''$:

$$\begin{array}{ccccc}
 & & & F & \\
 & & & \swarrow \xi' & \downarrow g \\
 A & \xrightarrow{a} & C & \xleftarrow{b} & B \\
 \downarrow f & & \swarrow \xi'' & & \\
 & & K & &
 \end{array}$$

We need to show $i^!(\xi) \in \mathcal{I}'$. Applying the functor $i^!$ to the triangle, we get the following triangle in $\mathcal{T}'$:

$$i^! A \xrightarrow{i^!(a)} i^! C \xrightarrow{i^!(b)} i^! B \rightarrow i^! A[1].$$

Since $g \in j_* \mathcal{T}''$, g factors through an object in the image of j_* . Since $i^! j_* = 0$, we have $i^!(g) = 0$. Thus, $i^!(b) \circ i^!(\xi') = i^!(g) = 0$. There exists a morphism $h' : i^! F \rightarrow i^! A$ such that $i^!(\xi') = i^!(a) \circ h'$. Substitute this into $i^!(\xi)$: $i^!(\xi) = i^!(\xi'') \circ i^!(\xi') = i^!(\xi'') \circ i^!(a) \circ h' = i^!(f) \circ h'$. Note that $f \in i_* \mathcal{I}'$, which implies $i^!(f) \in \mathcal{I}'$ (via the isomorphism $i^! i_* \cong \text{Id}$). Since $\mathcal{I}'$ is an ideal, the composition $i^!(f) \circ h'$ remains in $\mathcal{I}'$. Thus, $i^!(\xi) \in \mathcal{I}'$.

STEP 2: $\{\xi \in \mathcal{T} \mid i^!(\xi) \in \mathcal{I}'\} \subseteq \langle i_* \mathcal{I}', j_* \mathcal{T}'' \rangle$. Let $\xi : F \rightarrow K$ be a morphism such that $i^!(\xi) \in \mathcal{I}'$. Consider the canonical gluing triangle for the object F :

$$i_* i^! F \xrightarrow{\epsilon_F} F \xrightarrow{\eta_F} j_* j^* F \xrightarrow{\delta} i_* i^! F[1]$$

(here we use ϵ_F and η_F as the counit and unit maps). We construct a Toda bracket diagram for ξ :

$$\begin{array}{ccccc}
 & & & & F \\
 & & & & \downarrow \eta_F \\
 & & & & \parallel \\
 i_* i^! F & \xrightarrow{\epsilon_F} & F & \xrightarrow{\eta_F} & j_* j^* F \\
 \downarrow i_* i^! (\xi) & & \nearrow \xi & & \\
 i_* i^! K & & & & \\
 \downarrow \epsilon_K & & & & \\
 K & & & &
 \end{array}$$

Now we check the conditions for the class memberships. The compositions $\xi \circ \epsilon_F$ and $\epsilon_K \circ i_* i^! (\xi)$ both correspond to $i^! (\xi)$ via the adjoint pair $(i_*, i^!)$, that is, $\xi \circ \epsilon_F = \epsilon_K \circ i_* i^! (\xi)$. Since $F \in \mathcal{T}$ and $j^* \mathcal{T} \subseteq \mathcal{T}''$, we have $j^* F \in \mathcal{T}''$. Then $\eta_F = j_*(1_{j^* F}) \circ \eta_F \in j_* \mathcal{T}''$. Due to $i^! (\xi) \in \mathcal{I}'$, we have $i_* i^! (\xi) \in i_* \mathcal{I}'$. Since $i_* \mathcal{I}'$ is an ideal in $\mathcal{T}$, the composition $\epsilon_K \circ i_* i^! (\xi)$ belongs to $i_* \mathcal{I}'$. Thus, $\xi \in \langle i_* \mathcal{I}', j_* \mathcal{T}'' \rangle$. ■

LEMMA 4.4. *Consider a recollement as in Definition 2.11. Let $\mathcal{I}'$ be an ideal in $\mathcal{T}'$ and $\mathcal{I}''$ be an ideal in $\mathcal{T}''$. Then*

- (1) $(i_* \mathcal{I}')^\perp = \{\beta \in \mathcal{T} \mid i^! (\beta) \in \mathcal{I}'^\perp\};$
- (2) $(j_* \mathcal{I}'')^\perp = \{\beta \in \mathcal{T} \mid j^* (\beta) \in (\mathcal{I}'')^\perp\};$
- (3) ${}^\perp(i_* \mathcal{I}') = \{\alpha \in \mathcal{T} \mid i^* (\alpha) \in {}^\perp \mathcal{I}'\};$
- (4) ${}^\perp(j_* \mathcal{I}'') = \{\alpha \in \mathcal{T} \mid j^* (\alpha) \in {}^\perp \mathcal{I}''\}.$

Proof. We only prove (1); the other cases are similar.

STEP 1: $\{\beta \mid i^! (\beta) \in \mathcal{I}'^\perp\} \subseteq (i_* \mathcal{I}')^\perp$. Let $\beta \in \{\beta \mid i^! (\beta) \in \mathcal{I}'^\perp\}$. We need to show that for any $\alpha \in i_* \mathcal{I}'$ and any $\phi \in \Phi(\mathfrak{C})$, the composition $\beta[1] \circ \phi \circ \alpha$ is 0. By definition, α factors as $\alpha = f \circ i_*(\alpha') \circ g$ for some $\alpha' : A' \rightarrow B' \in \mathcal{I}'$ and morphisms f, g in $\mathcal{T}$. Consider the composition $\xi = \beta[1] \circ \phi \circ f \circ i_*(\alpha')$. Using the adjoint pair $(i_*, i^!)$, we see that the morphism ξ corresponds to the composition $\xi' = i^! (\beta)[1] \circ i^! (\phi) \circ i^! (f) \circ \eta_{B'} \circ \alpha'$, where η is the unit of the adjunction. Since $i^! (\beta) \in \mathcal{I}'^\perp$, $\alpha' \in \mathcal{I}'$ and $i^! (\phi) \in \Phi(\mathfrak{C}')$, the composition ξ' vanishes. Thus, $\xi = 0$. Hence $\beta \in (i_* \mathcal{I}')^\perp$.

STEP 2: $(i_* \mathcal{I}')^\perp \subseteq \{\beta \mid i^! (\beta) \in (\mathcal{I}')^\perp\}$. Let $\beta : A \rightarrow B \in (i_* \mathcal{I}')^\perp$. We need to show that $i^! (\beta) \in (\mathcal{I}')^\perp$. Let α' be any morphism in $\mathcal{I}'$ and ϕ' be any phantom morphism in $\Phi(\mathfrak{C}')$. It suffices to show that the composition $\xi' = i^! (\beta)[1] \circ \phi' \circ \alpha'$ vanishes. Using the adjoint pair $(i_*, i^!)$, we find that the

morphism ξ' corresponds to the composition $\xi = \beta[1] \circ \epsilon_{A[1]} \circ i_*(\phi') \circ i_*(\alpha')$ in $\mathcal{T}$, where ϵ is the counit of the adjunction. Since $i_*(\alpha') \in i_*\mathcal{I}'$ and $i_*(\phi') \in \Phi(\mathfrak{C})$, the composition ξ vanishes. Thus, $\xi' = 0$ for any $\alpha' \in \mathcal{I}'$ and $\phi' \in \Phi(\mathfrak{C})$. That is, $i^!(\beta) \in (\mathcal{I}')^\perp$. Therefore, $(i_*\mathcal{I}')^\perp = \{\beta \in \mathcal{T} \mid i^!(\beta) \in (\mathcal{I}')^\perp\}$. ■

PROPOSITION 4.5. *Consider a recollement as in Definition 2.11. If $(\mathcal{I}', \mathcal{J}')$ is a complete ideal cotorsion pair in $\mathcal{T}'$, then $(i_*\mathcal{I}', \langle i_*\mathcal{J}', j_*\mathcal{T}'' \rangle)$ and $(\langle j_!\mathcal{T}'', i_*\mathcal{I}' \rangle, i_*\mathcal{J}')$ are complete ideal cotorsion pairs in $\mathcal{T}$. If $(\mathcal{I}'', \mathcal{J}'')$ is a complete ideal cotorsion pair in $\mathcal{T}''$, then $(j_!\mathcal{I}'', \langle i_*\mathcal{T}', j_*\mathcal{J}'' \rangle)$ and $(\langle j_!\mathcal{I}'', i_*\mathcal{T}' \rangle, j_*\mathcal{J}'')$ are complete ideal cotorsion pairs in $\mathcal{T}$.*

Proof. We only show that $(i_*\mathcal{I}', \langle i_*\mathcal{J}', j_*\mathcal{T}'' \rangle)$ is a complete ideal cotorsion pair; the other cases are similar. Recall that $\mathcal{I}'$ is a special precovering ideal in $\mathcal{T}'$. By Lemma 2.10, $i_*\mathcal{I}'$ is a precovering ideal in $\mathcal{T}$. Since $\mathfrak{C}$ is assumed to be the class of all triangles in $\mathcal{T}$, every precovering ideal is special, ensuring that $i_*\mathcal{I}'$ is a special precovering ideal in $\mathcal{T}$. Then $(i_*\mathcal{I}', (i_*\mathcal{I}')^\perp)$ is a complete ideal cotorsion pair. By Lemmas 4.3 and 4.4, we have $(i_*\mathcal{I}')^\perp = \langle i_*\mathcal{J}', j_*\mathcal{T}'' \rangle$. Therefore, $(i_*\mathcal{I}', \langle i_*\mathcal{J}', j_*\mathcal{T}'' \rangle)$ is a complete ideal cotorsion pair in $\mathcal{T}$. ■

LEMMA 4.6. *Consider a recollement as in Definition 2.11. Let $\mathcal{I}'$ be an ideal in $\mathcal{T}'$ and $\mathcal{I}''$ be an ideal in $\mathcal{T}''$. Define a subset $\mathcal{I}$ of $\mathcal{T}$ by*

$$\mathcal{I} = \{\alpha \in \mathcal{T} \mid i^*(\alpha) \in \mathcal{I}', j^*(\alpha) \in \mathcal{I}''\}.$$

Then $\mathcal{I}$ is an ideal in $\mathcal{T}$. Moreover,

- (1) $i^*\mathcal{I} = \mathcal{I}' = \{\alpha \in \mathcal{T}' \mid i_*(\alpha) \in \mathcal{I}\};$
- (2) $j^*\mathcal{I} = \mathcal{I}'' = \{\alpha \in \mathcal{T}'' \mid j_!(\alpha) \in \mathcal{I}\}.$

Proof. Since i^* and j^* are additive functors, and moreover $\mathcal{I}'$ and $\mathcal{I}''$ are ideals, $\mathcal{I}$ is closed under addition and composition by construction. It follows that $\mathcal{I}$ is an ideal in $\mathcal{T}$.

By definition, for any $\alpha \in \mathcal{I}$, $i^*(\alpha) \in \mathcal{I}'$, so $i^*\mathcal{I} \subseteq \mathcal{I}'$. Conversely, let $\alpha \in \mathcal{I}'$. Since $j^*i_* = 0$ implies $j^*(i_*(\alpha)) = 0 \in \mathcal{I}''$, and $i^*(i_*(\alpha)) \cong \alpha \in \mathcal{I}'$, we have $i_*(\alpha) \in \mathcal{I}$. That is, $\mathcal{I}' \subseteq \{\alpha \in \mathcal{T}' \mid i_*(\alpha) \in \mathcal{I}\}$. Thus, $\alpha \cong i^*(i_*(\alpha)) \in i^*\mathcal{I}$. This shows $i^*\mathcal{I} = \mathcal{I}'$. If $\alpha \in \{\alpha \in \mathcal{T}' \mid i_*(\alpha) \in \mathcal{I}\}$, then $i_*(\alpha) \in \mathcal{I}$. By the definition of $\mathcal{I}$, this implies $i^*(i_*(\alpha)) \in \mathcal{I}'$. Since $i^*i_* \cong \text{Id}_{\mathcal{T}'}$, we have $\alpha \in \mathcal{I}'$, so $\{\alpha \in \mathcal{T}' \mid i_*(\alpha) \in \mathcal{I}\} \subseteq \mathcal{I}'$.

Similarly, by definition, $j^*\mathcal{I} \subseteq \mathcal{I}''$. Conversely, let $\alpha \in \mathcal{I}''$. Since $i^*j_! = 0$ implies $i^*(j_!(\alpha)) = 0 \in \mathcal{I}'$, and $j^*(j_!(\alpha)) \cong \alpha \in \mathcal{I}''$, we have $j_!(\alpha) \in \mathcal{I}$. This means $\mathcal{I}'' \subseteq \{\alpha \in \mathcal{T}'' \mid j_!(\alpha) \in \mathcal{I}\}$. Thus, $\alpha \cong j^*(j_!(\alpha)) \in j^*\mathcal{I}$. This shows $j^*\mathcal{I} = \mathcal{I}''$. If $\alpha \in \{\alpha \in \mathcal{T}'' \mid j_!(\alpha) \in \mathcal{I}\}$, then $j_!(\alpha) \in \mathcal{I}$. By the definition of $\mathcal{I}$, $j^*(j_!(\alpha)) \in \mathcal{I}''$. Since $j^*j_! \cong \text{Id}_{\mathcal{T}''}$, we have $\alpha \in \mathcal{I}''$, so $\{\alpha \in \mathcal{T}'' \mid j_!(\alpha) \in \mathcal{I}\} \subseteq \mathcal{I}''$. ■

Similarly, we state the following result without proof.

LEMMA 4.7. *Let $\mathcal{J}'$ be an ideal in $\mathcal{T}'$, and $\mathcal{J}''$ be an ideal in $\mathcal{T}''$. Define a subset $\mathcal{J}$ of $\mathcal{T}$ by*

$$\mathcal{J} = \{\beta \in \mathcal{T} \mid i^!(\beta) \in \mathcal{J}', j^*(\beta) \in \mathcal{J}''\}.$$

Then $\mathcal{J}$ is an ideal in $\mathcal{T}$. Moreover,

- (1) $i^!\mathcal{J} = \mathcal{J}' = \{\beta \in \mathcal{T}' \mid i_*(\beta) \in \mathcal{J}\};$
- (2) $j^*\mathcal{J} = \mathcal{J}'' = \{\beta \in \mathcal{T}'' \mid j_*(\beta) \in \mathcal{J}\}.$

Proof of Theorem 1.2. Assume that $(\mathcal{I}', \mathcal{J}')$ is a complete ideal cotorsion pair in $\mathcal{T}'$ and $(\mathcal{I}'', \mathcal{J}'')$ is a complete ideal cotorsion pair in $\mathcal{T}''$. According to Proposition 4.5, we have complete ideal cotorsion pairs $(i_*\mathcal{I}', \langle i_*\mathcal{J}', j_*\mathcal{T}'' \rangle)$ and $(j_!\mathcal{I}'', \langle i_*\mathcal{T}', j_*\mathcal{J}'' \rangle)$ in $\mathcal{T}$. In addition, Lemma 2.14 guarantees the existence of a complete ideal cotorsion pair

$$(\langle j_!\mathcal{I}'', i_*\mathcal{I}' \rangle, \langle i_*\mathcal{J}', j_*\mathcal{T}'' \rangle \langle i_*\mathcal{T}', j_*\mathcal{J}'' \rangle).$$

Next, we claim that the product $\langle i_*\mathcal{J}', j_*\mathcal{T}'' \rangle \langle i_*\mathcal{T}', j_*\mathcal{J}'' \rangle$ of ideals is equal to $\langle i_*\mathcal{J}', j_*\mathcal{J}'' \rangle$.

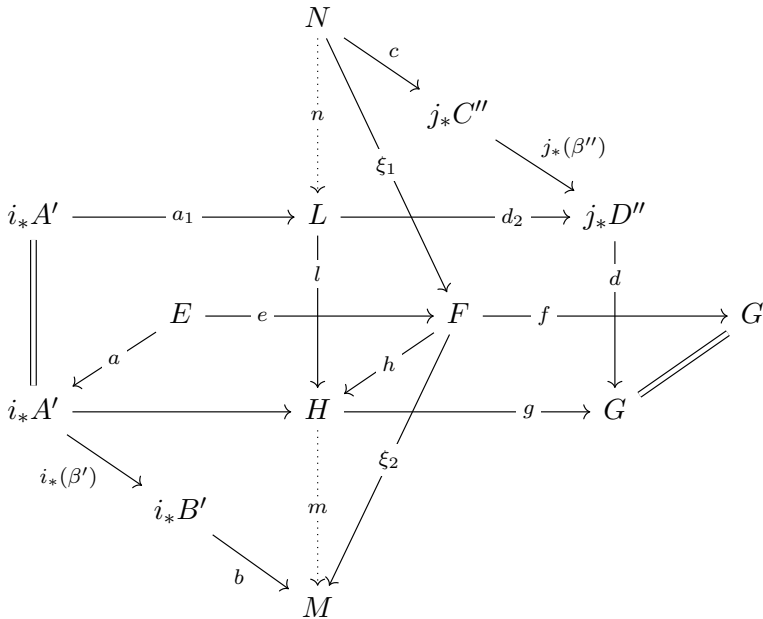
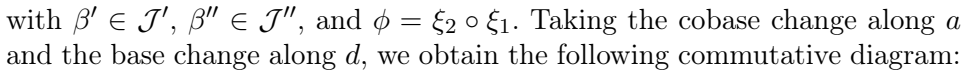
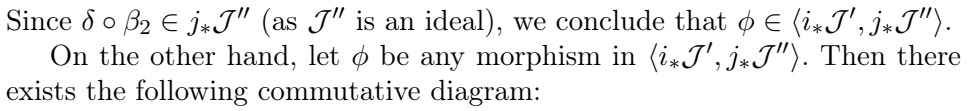
First, let ϕ be any morphism in $\langle i_*\mathcal{J}', j_*\mathcal{T}'' \rangle \langle i_*\mathcal{T}', j_*\mathcal{J}'' \rangle$. By definition, there exists the following commutative diagram:

$$\begin{array}{ccccc}
 & & D & \xrightarrow{d} & E & \xleftarrow{\xi_1} & H & \downarrow \beta_2 & F \\
 & & \downarrow f_1 & & \swarrow \xi_2 & & & & \\
 & & K & & & & & & \\
 & & \downarrow f_2 & & & & & & \\
 C & \xrightarrow{\quad} & B & \xleftarrow{\xi_3} & A & & & & \\
 \downarrow \beta_1 & \swarrow \xi_4 & & & & & & & \\
 M & & & & & & & &
 \end{array}$$

where $\beta_1 \in i_*\mathcal{J}'$, $f_2 \in j_*\mathcal{T}''$, $f_1 \in i_*\mathcal{T}'$, $\beta_2 \in j_*\mathcal{J}''$. The morphism ϕ is given by the composition $\phi = \xi_4 \circ \xi_3 \circ \xi_2 \circ \xi_1$. Consider the composition $f_2 \circ f_1$. Since f_1 factors through $i_*\mathcal{T}'$ and f_2 factors through $j_*\mathcal{T}''$, and moreover $j^*i_* = 0$, the orthogonality of the recollement implies that

$$\mathrm{Hom}_{\mathcal{T}}(i_*X', j_*Y'') \cong \mathrm{Hom}_{\mathcal{T}''}(j^*i_*X', Y'') = 0.$$

Thus, we have $f_2 \circ f_1 = 0$. Consequently, $f_2 \circ \xi_2 \circ d = f_2 \circ f_1 = 0$. The vanishing of this composition implies the existence of a morphism $\delta : F \rightarrow A$ such that $f_2 \circ \xi_2 = \delta \circ f$. This yields the following commutative diagram:



Since the diagram commutes, specifically $\xi_2 \circ e = b \circ i_*(\beta') \circ a$, there exists a morphism $m : H \rightarrow M$ such that $\xi_2 = m \circ h$. Similarly, $d \circ j_*(\beta'') \circ c = f \circ \xi_1 = g \circ h \circ \xi_1$ implies that there exists a morphism $n : N \rightarrow L$ such that $h \circ \xi_1 = l \circ n$. Thus, we have

$$\phi = \xi_2 \circ \xi_1 = m \circ l \circ n = (m \circ l) \circ (\text{id}_L \circ n).$$

This factorization implies that $\phi \in \langle i_*\mathcal{J}', j_*\mathcal{T}'' \rangle \langle i_*\mathcal{T}', j_*\mathcal{J}'' \rangle$. Thus, we have established the equality

$$\langle i_*\mathcal{J}', j_*\mathcal{T}'' \rangle \langle i_*\mathcal{T}', j_*\mathcal{J}'' \rangle = \langle i_*\mathcal{J}', j_*\mathcal{J}'' \rangle.$$

It follows that $(\langle j_!\mathcal{I}'', i_*\mathcal{I}' \rangle, \langle i_*\mathcal{J}', j_*\mathcal{J}'' \rangle)$ is a complete ideal cotorsion pair in $\mathcal{T}$.

Conversely, assume that $(\langle j_!\mathcal{I}'', i_*\mathcal{I}' \rangle, \langle i_*\mathcal{J}', j_*\mathcal{J}'' \rangle)$ is a complete ideal cotorsion pair in $\mathcal{T}$. It follows from Lemma 4.1 that the pairs obtained by applying the functors $i^*, i^!$ and j^* preserve the structure. Specifically,

- (1) $(i^*\langle j_!\mathcal{I}'', i_*\mathcal{I}' \rangle, i^!\langle i_*\mathcal{J}', j_*\mathcal{J}'' \rangle)$ is a pair in $\mathcal{T}'$;
- (2) $(j^*\langle j_!\mathcal{I}'', i_*\mathcal{I}' \rangle, j^*\langle i_*\mathcal{J}', j_*\mathcal{J}'' \rangle)$ is a pair in $\mathcal{T}''$.

Using the properties of the Toda brackets and the functors (noting that $i^*i_* \cong \text{Id}$, $j^*j_! \cong \text{Id}$, etc., and the orthogonality relations), we calculate

$$\begin{aligned} i^*\langle j_!\mathcal{I}'', i_*\mathcal{I}' \rangle &= \langle i^*j_!\mathcal{I}'', i^*i_*\mathcal{I}' \rangle = \langle 0, \mathcal{I}' \rangle = \mathcal{I}', \\ i^!\langle i_*\mathcal{J}', j_*\mathcal{J}'' \rangle &= \langle i^!i_*\mathcal{J}', i^!j_*\mathcal{J}'' \rangle = \langle \mathcal{J}', 0 \rangle = \mathcal{J}', \\ j^*\langle j_!\mathcal{I}'', i_*\mathcal{I}' \rangle &= \langle j^*j_!\mathcal{I}'', j^*i_*\mathcal{I}' \rangle = \langle \mathcal{I}'', 0 \rangle = \mathcal{I}'', \\ j^*\langle i_*\mathcal{J}', j_*\mathcal{J}'' \rangle &= \langle j^*i_*\mathcal{J}', j^*j_*\mathcal{J}'' \rangle = \langle 0, \mathcal{J}'' \rangle = \mathcal{J}''. \end{aligned}$$

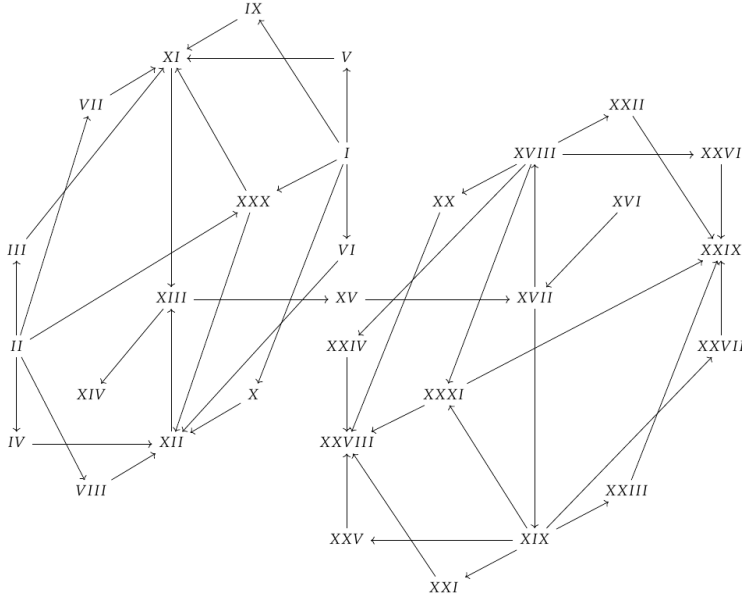
Therefore, $(\mathcal{I}', \mathcal{J}')$ and $(\mathcal{I}'', \mathcal{J}'')$ are complete ideal cotorsion pairs. ■

REMARK. By using the Toda bracket and intersection, one gets many ideal cotorsion pairs. In order to show the relationship between different ideal cotorsion pairs constructed from $(\mathcal{I}', \mathcal{J}')$ and $(\mathcal{I}'', \mathcal{J}'')$ more directly, we give a diagram in the Appendix.

5. Appendix

I: $i_*\mathcal{I}'j_!\mathcal{I}''$	II: $j_!\mathcal{I}''i_*\mathcal{I}'$
III: $\langle j_!\mathcal{T}'', i_*\mathcal{I}' \rangle i_*\mathcal{I}'$	IV: $i_*\mathcal{I}'\langle j_!\mathcal{I}'', i_*\mathcal{T}' \rangle$
V: $i_*\mathcal{I}'\langle j_!\mathcal{T}'', i_*\mathcal{I}' \rangle$	VI: $\langle j_!\mathcal{I}'', i_*\mathcal{T}' \rangle j_!\mathcal{I}''$
VII: $\langle j_!\mathcal{I}'', i_*\mathcal{T}' \rangle i_*\mathcal{I}'$	VIII: $j_!\mathcal{I}''\langle j_!\mathcal{I}'', i_*\mathcal{T}' \rangle$
IX: $j_!\mathcal{I}''\langle j_!\mathcal{T}'', i_*\mathcal{I}' \rangle$	X: $\langle j_!\mathcal{T}'', i_*\mathcal{I}' \rangle j_!\mathcal{I}''$
XI: $i_*\mathcal{I}'$	XII: $j_!\mathcal{I}''$
XIII: $i_*\mathcal{I}' \oplus j_!\mathcal{I}''$	XIV: $\langle i_*\mathcal{I}', j_!\mathcal{I}'' \rangle$
XV: $\langle j_!\mathcal{I}'', i_*\mathcal{I}' \rangle$	XVI: $\langle j_!\mathcal{T}'', i_*\mathcal{I}' \rangle \langle j_!\mathcal{I}'', i_*\mathcal{T}' \rangle$

XVII:	$\langle j_! \mathcal{T}'', i_* \mathcal{I}' \rangle \cap \langle j_! \mathcal{I}'', i_* \mathcal{T}' \rangle$	XVIII:	$\langle j_! \mathcal{T}'', i_* \mathcal{I}' \rangle$
XIX:	$\langle j_! \mathcal{I}'', i_* \mathcal{T}' \rangle$	XX:	$\langle j_! \mathcal{I}'', \langle j_! \mathcal{T}'', i_* \mathcal{I}' \rangle \rangle$
XXI:	$\langle \langle j_! \mathcal{I}'', i_* \mathcal{T}' \rangle, i_* \mathcal{I}' \rangle$	XXII:	$\langle \langle j_! \mathcal{T}'', i_* \mathcal{I}' \rangle, i_* \mathcal{I}' \rangle$
XXIII:	$\langle j_! \mathcal{I}'', \langle j_! \mathcal{I}'', i_* \mathcal{T}' \rangle \rangle$	XXIV:	$\langle i_* \mathcal{I}', \langle j_! \mathcal{T}'', i_* \mathcal{I}' \rangle \rangle$
XXV:	$\langle \langle j_! \mathcal{I}'', i_* \mathcal{T}' \rangle, j_! \mathcal{I}'' \rangle$	XXVI:	$\langle \langle j_! \mathcal{T}'', i_* \mathcal{I}' \rangle, j_! \mathcal{I}'' \rangle$
XXVII:	$\langle i_* \mathcal{I}', \langle j_! \mathcal{I}'', i_* \mathcal{T}' \rangle \rangle$	XXVIII:	$\langle \langle j_! \mathcal{I}'', i_* \mathcal{T}' \rangle, \langle j_! \mathcal{T}'', i_* \mathcal{I}' \rangle \rangle$
XXIX:	$\langle \langle j_! \mathcal{T}'', i_* \mathcal{I}' \rangle, \langle j_! \mathcal{I}'', i_* \mathcal{T}' \rangle \rangle$	XXX:	$i_* \mathcal{I}' \cap j_! \mathcal{I}''$
XXXI:	$\langle j_! \mathcal{T}'', i_* \mathcal{I}' \rangle \oplus \langle j_! \mathcal{I}'', i_* \mathcal{T}' \rangle$		



The arrows in the above diagram represent embedded maps. All ideals in the diagram are ideal cotorsion pairs with their right orthogonal classes.

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