COLLOQUIUM MATHEMATICUM

Online First version

ON EUCLIDEAN ALGORITHMS FOR ORIENTED LINEAR GRASSMANNIANS

BY

MAXIM ARNOLD and OLEG KARPENKOV

Abstract. We study Euclidean algorithms and the corresponding continued fractions for oriented linear Grassmannians $G(d,n)$. We propose two algorithms: Maximal Element Elimination and Minimal Element Elimination. The first algorithm reduces the absolute maximal value of the Plücker coordinates; the algorithm works only in $G(2,n)$. The second algorithm eliminates the Plücker coordinate with the smallest absolute values, while all other coordinates may increase; the algorithm works for arbitrary $G(d,n)$. We discuss basic features of these algorithms and formulate several natural open questions for further studies.

1. Introduction. Continued fractions and their multidimensional generalizations play a central role in number theory, Diophantine approximation, and dynamical systems. Classical subtractive algorithms, such as the Euclidean algorithm, provide an effective method for computing greatest common divisors and generating continued fraction expansions in one dimension. Multidimensional generalizations have been extensively studied for vectors in $\mathbb{R}^n$, leading to algorithms such as the Jacobi–Perron algorithm [14, 6], used to attack the cubic periodicity problem by C. Hermite [5], Brun's algorithm [2], Selmer's algorithm [19], fully subtractive algorithms [16, 17], and other generalizations [21, 8, 9, 1, 7, 20]. Geometric perspectives on multidimensional continued fractions, initiated by F. Klein [10, 11], H. Minkowski [12], and G. Voronoi [22], have provided further insight into their structure.

While the case of Grassmannians $G(1,n)$, which can be canonically identified with $\mathbb{RP}^n$, is well understood, higher-dimensional Grassmannians $G(d,n)$ with $d \geq 2$ and $n-d \geq 2$ present significant challenges. Standard vector subtraction does not preserve the Grassmannian structure: the difference of two elements in $G(d,n)$ generally lies outside the Grassmannian. Conse-

2020 *Mathematics Subject Classification*: Primary 11J70; Secondary 11A05, 15A21.
Key words and phrases: Euclidean algorithm, Grassmannian, canonical form, continued fraction.
Received 8 October 2025; revised 3 April 2026.
Published online 25 October 2026.

 © Instytut Matematyczny PAN, 2026

quently, direct extensions of classical subtractive or geometric algorithms are not applicable, limiting their utility in higher dimensions.

In this paper, we introduce *generalized subtractive Euclidean algorithms* for $G(d,n)$ that respect the Plücker embedding into $\mathbb{R}^{\binom{n}{d}}$. Our approach iteratively reduces a d-dimensional plane to the first coordinate plane, so that ultimately only one Plücker coordinate remains non-zero. Starting from a collection of Plücker coordinates, the algorithms apply subtraction-type reductions via $\mathrm{GL}(n,\mathbb{Z})$ transformations, producing a single non-zero coordinate equal in absolute value to the greatest common divisor of the original coordinates.

We propose two algorithms:

(1) *Maximal Element Elimination* (Section 3.2), for $G(2,n)$, which reduces the Plücker coordinate of maximal absolute value at each step, ensuring uniform descent toward a reduced form.
(2) *Minimal Element Elimination* (Section 3.3), applicable to all $G(d,n)$, which reduces the coordinate of minimal absolute value. When a coordinate reaches zero, the dimension is decreased, allowing continuation in $G(d,n-1)$.

Both algorithms terminate in $G(d,d)$, producing a d-tuple of integer vectors whose Plücker coordinates match the original ones. Analogous to the classical Euclidean algorithm, the sequences of reductions define *generalized continued fraction expansions* for elements of Grassmannians.

Our work provides the first systematic framework for subtractive algorithms in higher Grassmannians, extending multidimensional continued fraction theory and opening the door to applications in computational geometry, algebraic periodicity detection, and higher-dimensional number theory.

Organization of paper. In Section 2 we start with the classical definition of Plücker coordinates and discuss the problem of reduction to positive Plücker coordinates. We also recall the definitions of subtractive Euclidean algorithms and study elementary subtractions in Grassmannians. Further, in Section 3 we introduce both the Maximal Element Elimination algorithm for $G(2,n)$ and the Minimal Element Elimination algorithm for $G(d,n)$ followed by discussion of their basic properties and an example. We conclude by indicating a few directions for further studies in Section 4.

2. Preliminary notions and definitions. A point or vector is said to be *integer* if all its coordinates are integers. We say that a linear space is *integer* if it contains a full-rank integer sublattice.

From now on we fix a basis of integer vectors and consider each vector as a collection of coordinates in this basis.

2.1. Plücker coordinates of linear Grassmannians. First of all, denote by $G(d,n)$ the *linear Grassmannian* of d-spaces in $\mathbb{R}^n$.

For a given d-dimensional space π we consider its basis $(v_1,\dots,v_d)$, and write it as a single $d\times n$-matrix: the rows of this matrix are the coordinates of $\{v_j\}_{j=1,\dots,d}$ in the fixed basis of $\mathbb{R}^n$. This matrix has precisely $N=\binom{n}{d}$ minors of dimension d. The collection of such minors ordered lexicographically is said to be the *Plücker coordinates* of π. For a d-dimensional space $\pi\subset\mathbb{R}^n$ we denote them by $P=(p_{i_1,\dots,i_d})$, where the indices $(i_1,\dots,i_d)$ satisfy $1\le i_1<\dots<i_d\le n$.

The mapping of the elements of $G(d,n)$ to their Plücker coordinates is called the *Plücker embedding* of $G(d,n)$ into $\mathbb{RP}^N$ with $N=\binom{n}{d}$. Note that Plücker coordinates are uniquely defined up to a common scalar factor.

REMARK 2.1. For $d=2$ Plücker coordinates are naturally identified with the skew-symmetric $n\times n$-matrix P where $P_{i,j}$ equals the determinant of the i-th and j-th columns. Here diagonal elements are all zeros; the upper-triangular elements are precisely the Plücker coordinates, and the lower-triangular elements are the negatives of the Plücker coordinates.

2.2. Reduction to positive Plücker coordinates. Let us start with the following natural definition.

DEFINITION 2.2. We say that a d-tuple of linearly independent vectors in $\mathbb{R}^n$ is *totally positive/non-negative* if the corresponding Plücker coordinates are all positive/non-negative.

Let us address the following question.

QUESTION 1. Does there exist an integer basis transformation sending a given collection of non-zero Plücker coordinates to positive ones?

The answer is affirmative for $G(1,n)$ and for $G(2,n)$. In the case of $G(1,n)$, one just swaps the signs of the basis vectors corresponding to the negative Plücker coordinates. The case $d=2$ is a bit more complicated. We start with the following proposition.

PROPOSITION 2.3. *Consider a linear space π spanned by two vectors $v_1=(a_1,\dots,a_n)$ and $v_2=(b_1,\dots,b_n)$. Then the pair (v_1,v_2) is totally positive if and only if there exists a line $\ell\subset\mathbb{R}^2$ passing through the origin such that the two-dimensional column vectors (a_i,b_i), $i=1,\dots,n$, are distinct and are contained in one of the half-planes of $\mathbb{R}^2\setminus\ell$; and they are ordered counterclockwise in this half-plane.*

Proof. The Plücker coordinate $p_{i,j}$ is equal to the oriented volume of the parallelogram spanned by the vectors (a_i,b_i) and (a_j,b_j), which is positive by construction. ∎

REMARK 2.4. Note that every $2 \times n$-matrix can be transferred to a non-negative form by changing the signs of vectors (a_i, b_i) for some i and reordering these vectors afterwards. Furthermore, one can send all the vectors to the half-plane $x_1 \geq 0$.

COROLLARY 2.5. *There exists an integer basis transformation sending a given collection of non-zero Plücker coordinates in $G(2, n)$ to positive ones.*

Proof. The proof is constructive, and it is provided by the following algorithm.

Input. We are given Plücker coordinates P, none of them zero.

- *Stage I.* First of all, we make all coordinates $p_{1,n}$ positive by changing the signs of the corresponding basis vector when $p_{1,n}$ is negative. Note that all the resulting vectors will be in one half-plane with respect to the line passing through (a_1, b_1) (bear in mind that we do not know the coordinates of the vector (a_1, b_1), but only Plücker coordinates).
- *Stage II.* It remains to swap the order of basis vectors according to the following rule: if $p_{i,j} < 0$ then we swap the i-th and j-th vectors. We do this iteratively, taking always the first $p_{i,j}$ in the lexicographic order.
  In the worst case when all $p_{i,j}$ are negative (i.e. the vectors (a_i, b_i) are all in the wrong order, cf. Proposition 2.3), we need to swap $(n-1)(n-2)/2$ times.

Output. After the transformations of Stages I and II we have all vectors in one half-plane ordered counterclockwise. According to Proposition 2.3 all the Plücker coordinates will be positive. ∎

Note that the input of the above algorithm does not contain vectors (a_i, b_i), but only the corresponding Plücker coordinates P.

The answer to Question 1 is affirmative for $G(3, 5)$ as well, since $G(3, 5)$ is naturally isomorphic to $G(2, 5)$. However, for $G(3, 6)$ the positivity cannot be achieved by reflections and reorderings only, due to the following counterexample.

EXAMPLE 2.6. Recall that each element of $G(3, 6)$ has $20 = \binom{6}{3}$ Plücker coordinates p_{i_1,i_2,i_3}; they correspond to all possible ordered triplets $\mathbf{i} = (i_1 < i_2 < i_3)$. Note that

- reflecting the j-th basis vector results in the sign change of all $p_{\mathbf{i}}$ with $j \in \mathbf{i}$;
- swapping the j-th and k-th basis vectors results in the sign change of all $p_{\mathbf{i}}$ with $(j, k) \in \mathbf{i}$.

Hence the former operation provides 10 sign changes and the latter 4, none of which affects the parity of the number of negative coordinates.

Consider the element of $G(3,6)$ defined by the following three vectors:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -3 & -2 & -1 \\ 0 & 0 & 1 & 8 & 5 & 1 \end{bmatrix}$$

Its Plücker coordinates are all positive except for $p_{4,5,6}$:

$$P = (1 : 8 : 5 : 1 : 3 : 2 : 1 : 1 : 5 : 3 : 1 : 1 : 1 : 3 : 7 : 4 : 1 : 2 : 1 : -1).$$

Hence this element cannot be made totally positive by reordering or reflecting the vectors.

REMARK 2.7. Note that one can bring a given element of the Grassmannian $G(d,n)$ to a positive form by the following iterative algorithm:

(i) Fix the first $d-1$ columns and reflect the columns u_{m_1} with $m_1 > d-1$ and $p_{1,\dots,d-1,m_1} < 0$.
(ii) Next, for any $m_1 > m_2 > d-2$, if $p_{1,\dots,d-2,m_2,m_1} < 0$ the operation $u_{m_2} \mapsto u_{m_2} + u_{d-1}$ does not affect the coordinates from Step (i), increasing the coordinate $p_{1,\dots,d-2,m_2,m_1}$.

Then continue for $k = 3, \dots, d-1$ as follows:

(k) For any $m_1 > \dots > m_k > d-k$ if $p_{1,\dots,d-k-1,m_k,\dots,m_1} < 0$ the operation $u_{m_k} \mapsto u_{m_k} + u_{d-k+1}$ does not affect the coordinates from the previous steps, increasing the coordinate $p_{1,\dots,m_k,\dots,m_1}$.

In this method we have used rather arbitrary $\mathrm{SL}(n,\mathbb{Z})$ transformations. In contrast to the case of symmetries about coordinate planes and swapping coordinates, the new ones can change the maximal absolute value of the Plücker coordinates. For that reason this method cannot be used in Maximal Element Elimination. Thus, the question which remains open to our knowledge can be stated as

PROBLEM 2.8. *Is there some natural way to make all Plücker coordinates positive without changing their maximal absolute value?*

2.3. Integer linear Grassmannians. Let $\mathbb{Z}G(d,n) \subset G(d,n)$ denote the subset of all integer linear spaces. Note that $\mathbb{Z}G(d,n)$ is everywhere dense in $G(d,n)$ in the standard topology.

PROPOSITION 2.9. *A linear space $\pi \in G(d,n)$ is integer if and only if its Plücker coordinates define a rational point in $\mathbb{RP}^{N-1}$.*

Recall that the *integer volume* of the sublattice generated by vectors $v_1, \dots, v_d \in \pi$ is the index of this sublattice in the integer lattice of π.

REMARK 2.10. Consider linearly independent integer vectors $v_1, \dots, v_d$ spanning π. Then the greatest common divisor of the Plücker coordinates of

these vectors coincides with the integer volume of the sublattice generated by $v_1, \ldots, v_d$ (see, e.g., [7]).

We say that a vector of Plücker coordinates is *primitive integer* if all its entries are integers and their greatest common divisor equals 1. Integer Plücker coordinates are defined up to multiplication by ± 1 of all elements simultaneously.

REMARK 2.11. In the simplest case of lines through the origin in the plane, if a line ℓ contains some vector (a, b) then its Plücker coordinates give a point $(a : b) \in \mathbb{RP}^1$. Here the ratio b/a determines the tangent of the slope of ℓ. In the case of an integer line, b/a is rational.

2.4. Classical subtractive algorithms in $\mathbb{R}^n$. We start with a brief description of classical subtractive algorithms in $\mathbb{R}^n$. They will be substantially used in the algorithms for Grassmannians below.

2.4.1. *Euclidean algorithm.* Given two integers (p_0, q_0) we perform the following operation:

$$(p_i, q_i) \mapsto (p_{i+1}, q_{i+1}) = (q_i, p_i - a_i q_i), \quad \text{where } a_i = \lfloor p_i/q_i \rfloor.$$

After finitely many (say $N + 1$) iterations, the algorithm will arrive at the pair of integers $(\gcd(p_0, q_0), 0)$ where it terminates.

The output $(a_0, \ldots, a_N)$ of this algorithm gives a *regular continued fraction* of p_0/q_0:

$$\frac{p_0}{q_0} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_N}}}}.$$

It is usually denoted by $[a_0; a_1 : \ldots : a_N]$.

Note that the Euclidean algorithm can be represented in terms of $\mathrm{GL}(2, \mathbb{Z})$ matrix multiplication as follows:

$$\begin{pmatrix} \gcd(p_0, q_0) \\ 0 \end{pmatrix} = \left(\prod_{i=1}^{N} \begin{pmatrix} 0 & 1 \\ 1 & -a_{N-i} \end{pmatrix} \right) \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}.$$

The product of matrices on the right-hand side corresponds to an elementary lattice basis transformation reducing the coordinates of the starting vector to $\gcd(p_0, q_0)$.

2.4.2. *Multidimensional generalizations.* Assume that we have constructed a vector $(x_{1,i}, \ldots, x_{n,i})$ and let $x_{2,i} \neq 0$. Then we define the next vector $(x_{1,i+1}, \ldots, x_{n,i+1})$ to be

$$\left(x_{2,i}, x_{3,i} - \left\lfloor \frac{x_{3,i}}{x_{2,i}} \right\rfloor x_{2,i}, \ldots, x_{n,i} - \left\lfloor \frac{x_{n,i}}{x_{2,i}} \right\rfloor x_{2,i}, x_{1,i} - \left\lfloor \frac{x_{1,i}}{x_{2,i}} \right\rfloor x_{2,i} \right).$$

As an output of this step we have the collection of integers

$$\left(\left\lfloor\frac{x_{3,i}}{x_{2,i}}\right\rfloor,\dots,\left\lfloor\frac{x_{n,i}}{x_{2,i}}\right\rfloor,\left\lfloor\frac{x_{1,i}}{x_{2,i}}\right\rfloor\right).$$

It is the ith element of the *Jacobi–Perron multidimensional continued fraction algorithm*.

Once $x_{2,N}=0$, one swaps the second and the last coordinates and continues the process with one coordinate less. The algorithm terminates when there is only one non-zero coordinate left.

REMARK 2.12. Note that the i-th step of the Jacobi–Perron multidimensional continued fraction algorithm is described by multiplication of vectors by the following $\mathrm{GL}(n,\mathbb{Z})$ matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0\\ 0 & -\frac{x_{3,i}}{x_{2,i}} & 1 & 0 & \dots & 0\\ 0 & -\frac{x_{4,i}}{x_{2,i}} & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & -\frac{x_{n,i}}{x_{2,i}} & 0 & 0 & \dots & 1\\ 1 & -\frac{x_{1,i}}{x_{2,i}} & 0 & 0 & \dots & 0 \end{pmatrix}$$

REMARK 2.13. In the multidimensional case ($n\geq 3$) there is no unique choice of a subtractive algorithm. In this paper we consider the subtractive algorithm that in the case $n=3$ coincides with one of the oldest subtractive algorithms, the Jacobi–Perron algorithm (see [14, 6]). Instead of the Jacobi–Perron algorithm one can use any other subtractive algorithm (see, e.g., [2, 19, 16, 17, 15, 21, 8, 9]).

2.5. Elementary subtraction of basis vectors

DEFINITION 2.14 (Elementary subtraction of basis vectors). Let $1\leq s\neq t\leq n$. The linear map defined on basis vectors of $\mathbb{R}^n$ as

$$v_i\mapsto\begin{cases} v_i & \text{if } i\neq s,\\ v_s-v_t & \text{otherwise,}\end{cases}$$

is said to be an *elementary subtraction*. Denote it by $T_{s,t}$.

DEFINITION 2.15. Let $\mathbf{i}$ be an unordered d-tuple of distinct indices. Denote by $\sigma(\mathbf{i})$ the permutation that puts the indices in increasing order.

Let s,t be two indices such that $s\in\mathbf{i}$ and $t\notin\mathbf{i}$. We denote by $\mathbf{i}_{s\to t}$ the d-tuple of integers with s replaced by t (without changing the order).

PROPOSITION 2.16. *Consider an ordered d-tuple $\mathbf{i}$. The subtraction $T_{s,t}$ changes the coordinates $p_{\mathbf{i}}$ as follows:*

$$p_{\mathbf{i}} \mapsto \begin{cases} p_{\mathbf{i}} - \operatorname{sign}(\sigma(\mathbf{i}_{s\to t})) \cdot p_{\sigma(\mathbf{i}_{s\to t})} & \text{if } s \in \mathbf{i} \text{ and } t \notin \mathbf{i}, \\ p_{\mathbf{i}} & \text{otherwise.} \end{cases}$$

Proof. This follows from the definition of determinant. ∎

Consider the case of $G(2,n)$. According to the matrix representation from Remark 2.1 we set $p_{i,j} = -p_{j,i}$ for $j < i$. Therefore, the above expression for $T_{s,t}$ takes the form

$$\begin{cases} p_{j,s} \mapsto p_{j,s} + \operatorname{sgn}(t-j)p_{j,t} \\ p_{s,j} \mapsto p_{s,j} + \operatorname{sgn}(t-j)p_{t,j} \end{cases} \qquad \text{for } j \neq t, \tag{2.1}$$

and thus can be regarded as a column operation on the matrix of the Plücker coordinates, tweaked so as to preserve skew-symmetry.

$$P = \begin{pmatrix} 0 & p_{1,2} & \cdots & p_{1,s} & \cdots & p_{1,t} & \cdots & p_{1,n} \\ \vdots & \ddots & & \vdots & \leftarrow & \vdots & & \vdots \\ & 0 & & p_{s-1,s} & & & & \\ -p_{1,s} & \cdots & & 0 & p_{s,s+1} & p_{s,t} & \cdots & p_{s,n} \\ \vdots & & & & \ddots & \vdots & & \vdots \\ -p_{1,t} & \cdots & & & 0 & p_{t,t+1} & \cdots & p_{t,n} \\ \vdots & & & & & 0 & \ddots & \\ & & & & & \vdots & \leftarrow & p_{n-1,n} \\ -p_{1,n} & \cdots & & -p_{s,n} & \cdots & -p_{t,n} & \cdots & 0 \end{pmatrix}$$

REMARK 2.17. The subtractions $T_{s,t}$ and $T_{t,s}$ lead to the same set of absolute values of the coefficients, in particular their maximum is the same.

REMARK 2.18. We can always make our maximal element lie outside the second diagonal $(j, j+1)$. Namely, if $p_{i,i+1}$ is the maximal element for the positive Grassmannian $\{v_1, \ldots, v_n\}$, then $p_{1,n}$ is the maximal element for the positive Grassmannian $\{v_{i+1}, v_{i+2}, \ldots, v_n, -v_1, -v_2, \ldots, -v_i\}$. In such settings we can always use the subtraction $T_{j,j-1}$.

3. Continued fraction algorithms for Grassmannians. Consider a d-tuple $(v_1, \ldots, v_d)$ of vectors in $\mathbb{R}^n$. We denote by $w_i \in \mathbb{R}^d$ the vector of i-th coordinates of $v_1, \ldots, v_d$ for $i = 1, \ldots, n$. Note that the matrices of v-vectors and w-vectors are transpose to each other.

In what follows we will use the lexicographic order for the set of all indices of Plücker coordinates.

3.1. Input and output of the algorithms. In this subsection we discuss the common items for both our algorithms: input data and their outputs.

Input. Let $d < n$ be two positive integers. We are also given a collection P of $N = \binom{n}{d}$ integers that defines an element of $G(d, n)$.

Tasks of the algorithm:

- The first task is to find a d-tuple of linearly independent integer vectors whose Plücker coordinates coincide with P.
- The second task is to build a "generalized continued fraction" that encodes the basis transformations leading to the d-tuple of vectors of the first task.

REMARK 3.1. Both tasks of the proposed algorithms are in the spirit of the classical Euclidean algorithm that finds the gcd of a pair of integers. New algorithms provide a d-tuple of integer vectors together with the index of the sublattice they generate in the d-plane they span. This sublattice index can be seen as a straightforward generalization of gcd.

The outcome of the second task is a sequence of $\mathrm{GL}(n, \mathbb{Z})$ transformations, similar to the ones generated by classical subtractive multidimensional continued fraction algorithms (see, e.g., [18] and [7, Section 27.4]).

Output. As an output we have the following:

- A *d-tuple of integer vectors* whose Plücker coordinates equal P. (In addition we will get the index of the sublattice they generate.)
- A *sequence of* $\mathrm{GL}(n, \mathbb{Z})$ *transformations*, which we treat as the *continued-fraction sequence* for the algorithm.

As in the classical Euclidean algorithm, the resulting vectors (together with the index of the sublattice they generate) are obtained at the end of the algorithm, while the elements of the continued-fraction sequence are generated at the consecutive steps of the algorithm.

REMARK 3.2. Both algorithms, applied to a non-integer input P, provide the Grassmannian analog of the classical Gauss map. We encourage the reader to study the properties of the corresponding dynamical systems.

3.2. Maximal Element Elimination for $G(2, n)$. Let us describe an algorithm that works for the case of $G(2, n)$.

3.2.1. *Zeroing Step.* The purpose of this step is to zero out one of the Plücker coordinates. In the case of the maximal element algorithm we iteratively reduce the coordinate with the maximal absolute value using integer bases transformations in $\mathbb{R}^n$ until we hit a zero Plücker coordinate.

- *Stage I.* Produce a basis where all the Plücker coordinates $p_{i,j}$ (with $i < j$) are positive. This is done by the algorithm in the proof of Corollary 2.5. The corresponding integer transformation matrix is *added to the continued-fraction sequence.*
- *Stage II.* Select a Plücker coordinate $p_{i,j}$ (with $i < j$) attaining the maximal value. In case there are several such elements, pick the one with the lexicographically smallest index, with one exception: we always prioritize $p_{1,n}$.
- *Stage III.* This stage is performed only if in Stage II the coordinate $p_{i,i+1}$ was selected, i.e., the case $j = i+1$. First, we cyclically shift the coordinate vectors in $\mathbb{R}^n$ by $n-i$ positions. Secondly, we multiply all the basis vectors of $\mathbb{R}^n$ from $i+1$ to n by -1. As a result, all the Plücker coordinates $p_{i,j}$ (with $i < j$) are positive and the coordinate $p_{1,n}$ becomes one of the maxima. We select it for the next stage.
  The corresponding basis transformation matrix is *added to the continued-fraction sequence.*
- *Stage IV.* Here we get the maximal coordinate $p_{i,j}$ with $i < j-1$. Now we perform the subtraction

$$v_j \mapsto v_j - v_{j-1}.$$

  The corresponding basis transformation matrix is *added to the continued-fraction sequence.*
  If all Plücker coordinates are non-zero, we *return to Stage I.*
- *Go to Dimension Reduction Step.* Once we get $p_{s,t} = 0$ for some $s < t$ we stop the iterations and move to Dimension Reduction Step below.

3.2.2. *Dimension Reduction Step.* After Zeroing Step, some Plücker coordinate $p_{s,t}$ is zero.

- *Stage I.* Swap the coordinates of $\mathbb{R}^n$ so that in new coordinates $p_{1,2} = 0$.
- *Stage II.* We have two cases here.

  (1) Assume that there is some i such that $p_{1,i} \neq 0$. Then the coefficient of proportionality of the vectors w_1 and w_2 is precisely $p_{2,i} : p_{1,i}$. We apply the usual two-dimensional subtractive Euclidean algorithm to the pair (v_1, v_2) in order to zero v_1 out. Here we make the subtractions following the Euclidean algorithm for the pair $(p_{1,i}, p_{2,i})$ until we make $p_{1,i}$ zero. Once this is done, the plane under consideration is in the

plane $x_1 = 0$ in the current basis. At this point we exclude the first coordinate. This reduces the dimension by one.

(2) Assume that $p_{1,i} = 0$ for all i. This implies that $w_1 = 0$. So we exclude the first coordinate.

- *Go to Zeroing Step or terminate.* Once we eliminate one of the coordinates, we move to Zeroing Step with one dimension less. In case we have reduced the dimension to d, we terminate.

The corresponding basis transformation matrices of these stages are *added to the continued-fraction sequence.*

REMARK 3.3. For simplicity we work with $\mathrm{GL}(n, \mathbb{Z})$: we send the excluded coordinate to the last position. Further, the algorithm will work in dimensions $< n$, so all the lower-dimensional transformations will be supplemented with the identity transformation for the excluded coordinates.

3.3. Minimal Element Elimination for $G(d, n)$. The above algorithm does not work for general d due to Example 2.6. Hence we propose a different approach where we do not intend to make all the Plücker coordinates positive.

3.3.1. *Zeroing Step*

- *Stage I.* Select a Plücker coordinate $p_{i_1,\dots,i_d}$ (for $i_1 < \dots < i_d$) with minimal absolute value. In case of several such coordinates take the one with the lexicographically smallest index.
- *Stage II.* Select a Plücker coordinate $p_{j_1,\dots,j_d}$ (for $j_1 < \dots < j_d$) with $j_s = i_s$ except for $s = t$. The values of t and j_t are defined as follows:
  - t is the minimal index such that either $i_t - 1$ (considered only if $t > 1$) or $i_t + 1$ (considered only if $t < n$) is not in the set $\{i_1, \dots, i_k\}$;
  - we select $j_t = i_t - 1$ if this option is available and $j_t = i_t + 1$ otherwise.
- *Stage III.* Subtract
$$v_{i_t} \mapsto v_{i_t} - \left\lfloor \frac{p_{j_1,\dots,j_k}}{p_{i_1,\dots,i_k}} \right\rfloor v_{j_t}.$$
The corresponding basis transformation matrix is *added to the continued-fraction sequence.*
If all Plücker coordinates are non-zero, then we return to Stage I.
- *Go to Dimension Reduction Step.* Once we get some zero Plücker coordinate $p_{s,t} = 0$ for some $s < t$ we stop the iterations and move to the Dimension Reduction Step.

3.3.2. *Dimension Reduction Step.* This is the multidimensional analog of Dimension Reduction Step for $G(2, n)$ described in the Maximal Element Elimination algorithm.

- *Stage I.* Swap the coordinates of $\mathbb{R}^n$ so that in the new coordinates $p_{1,\dots,d} = 0$.
- *Stage II.* Find a non-zero Plücker coordinate $p_{1,\dots,s,i_{s+1},\dots,i_d}$ such that
  - all the coordinates $p_{1,\dots,s,s+1,j_{s+2},\dots,j_d}$ are zero;
  - $(i_{s+1},\dots,i_d)$ is the lexicographically first $(d-s)$-tuple that provides a non-zero Plücker coordinate.

  Note that $s < d$.
- *Stage III.* From the first condition of Stage II we see that the vectors $v_1,\dots,v_{s+1}$ are linearly dependent. Moreover, they satisfy the following linear equation with integer coefficients:
$$\Big(\sum_{j=1}^{s} p_{1,\dots,j-1,s+1,j+1,\dots,s,i_{s+1},\dots,i_d} v_j\Big) - p_{1,\dots,s,i_{s+1},\dots,i_d} v_{s+1} = 0.$$
  Denote these coefficients by $(a_1,\dots,a_{s+1})$.
- *Stage IV.* Here we again have two cases.
  (1) Let $(a_1,\dots,a_s) = (0,\dots,0)$. Then v_{s+1} is a zero vector. We swap v_1 and v_{s+1} to get $v_1 = 0$.
  (2) Let $a_i \neq 0$ for some $1 \leq i \leq s+1$. Then we apply the Jacobi–Perron algorithm to $(a_1,\dots,a_{s+1})$ that will bring the vector to $(a,0,\dots,0)$. As a result, in the new basis we have $av_1 = 0$, and hence $v_1 = 0$.

  In both cases we exclude the first coordinate.
- *Go to Zeroing Step or terminate.* Once we eliminated one of the coordinates, we move to Zeroing Step with one dimension less. In case we have reduced the dimension to d, we terminate.

The corresponding basis transformation matrices of these stages are *added to the continued-fraction sequence.*

Remark 3.4. Similar to the case of Maximal Element Elimination (see Remark 3.3) we intend to work with $\mathrm{GL}(n,\mathbb{Z})$ basis transformations. For that reason we send the excluded coordinate to the last position. Further, the algorithm will work in dimensions $< n$, so all the lower-dimensional transformations will be supplemented with the identity transformation for the excluded coordinates.

Remark 3.5. The Jacobi–Perron algorithm can be replaced by any other subtractive algorithm (for subtractive algorithms we refer to [18] and [7, Section 27.4]).

3.4. Output of the algorithms. Termination of the algorithm. The algorithm performs $n-d$ dimension reductions ($d = 2$ in the case of Maximal Element Elimination). Once we arrive at $G(d,d)$ we terminate. At this stage we have

- $\hat{p} = \hat{p}_{1,\dots,d}$, the only Plücker coordinate of $G(d,d)$;
- the continued-fraction sequence $(A_i)_{i=1}^N$ of $\mathrm{GL}(n,\mathbb{Z})$ transformations.

Finally, to complete the output we should produce vectors whose Plücker coordinates are as in the input.

- The basis $(\hat{v}_1,\dots,\hat{v}_d)$ of the sublattice with Plücker coordinates P is obtained from the basis vectors $\hat{p}e_1, e_2, \dots, e_d$ by multiplying by the inverses of all matrices of the continued-fraction sequence:

$$\hat{v}_1 = \Big(\prod_{i=1}^{n}(A_i)^{-1}\Big)\hat{p}e_1, \qquad \hat{v}_i = \Big(\prod_{i=1}^{n}(A_i)^{-1}\Big)e_i \quad \text{for } i = 2,\dots,d. \tag{3.1}$$

  Here, when we increase the dimension, we simply add zero coordinates to all vectors.

Output: $((\hat{v}_1,\dots,\hat{v}_d),(A_i)_{i=1}^N)$.
Finite termination time. Each Dimension Reduction Step and each Zeroing Step of the algorithm take finitely many operations. It is sufficient to make $n-d$ iterations of pairs of these steps in order to arrive at the termination state.

REMARK 3.6. If the input is irrational then the running time of the algorithm can be infinite; in this case we will have an infinite continued fraction as an output. It is of interest to study periodic algorithms (Lagrange's theorem), distribution of partial quotients (Gauss–Kuz'min statistics), and further properties of the corresponding generalized Gauss map.

3.5. Basic properties of the algorithms. Finally, we have the following property.

THEOREM 3.7. *The Plücker coordinate $\hat{p}$ is the volume of the sublattice generated by the vectors $(\hat{v}_1,\dots,\hat{v}_d)$.*

Proof. On the one hand, the volume of the sublattice is invariant under all the basis transformations of the steps of the algorithm. On the other hand, the index of the d-dimensional sublattice in $\mathbb{Z}^d$ coincides with the absolute value of its Plücker coordinate (i.e., the determinant of the basis for the sublattice). ∎

THEOREM 3.8. *For a given set P of integer Plücker coordinates both algorithms generate a d-tuple of lattice vectors that coincides with P.*

Proof. This follows from the construction. Here we have a sequence of integer linear transformations (see (3.1)) bringing $\hat{p}e_1, e_2, \dots, e_d$ to the required basis. ∎

Note that Maximal Element Elimination works only for the case of $G(2,n)$ due to the discussion in Section 2.2 and Example 2.6 above. So we arrive at the following reformulation of Problem 2.8.

PROBLEM 3.9. *Generalize the Maximal Element Elimination algorithm to the case of $G(d,n)$ for $d \geq 3$.*

3.6. Admissible d-tuple of vectors. *Which integer vectors have the same Plücker coordinates?* The answer is rather simple. Let $\hat{p}$ be the last Plücker coordinate obtained as the first output of the algorithm. These are the d-tuples of integer linearly independent vectors that generate a sublattice on the integer lattice in the d-plane of index $\hat{p}$. The Plücker coordinates of these d-tuples will be as desired (up to a simultaneous choice of the sign).

Let us indicate how to construct them. In the output of the algorithms we have chosen the vectors $(\hat{p}e_1, e_2, \ldots, e_d)$ and applied to them the elements of the constructed continued-fraction sequence. Instead of taking $(\hat{p}e_1, e_2, \ldots, e_d)$ one can choose any other integer basis of $\mathbb{R}^d$ with determinant $\hat{p}$. Equivalently, one might say that we start with an integer matrix of determinant $\hat{p}$. Then we again apply the elements of the constructed continued-fraction sequence (as in (3.1)). This will generate the set of integer d-tuples in $\mathbb{R}^n$ with the given Plücker coordinates.

3.7. Working example. Recall that there is a natural bijection between the set of all rank-2 skew-symmetric matrices in $\mathrm{PSL}(n,\mathbb{R})$ and the Grassmannian $G(2,n)$. For instance, thanks to the Plücker relations

$$p_{i,k}p_{j,l} = p_{i,j}p_{k,l} + p_{jk}p_{ki} \qquad \text{for } i<j<k<l,$$

one can consider any $G(2,n)$ as a decorated frieze pattern. Thus, for example if $p_{j,j+1}=1$ for all j, one recovers the usual frieze (see [13] for more on friezes and [3] for decorated friezes). In this regard, our algorithms correspond to the natural operations on the corresponding cluster variety. Here we illustrate our Maximal Element Elimination algorithm for the first non-trivial case of $G(2,4)$.

EXAMPLE 3.10. Consider the following set of Plücker coordinates:

$$(p_{1,2}, p_{1,3}, p_{1,4}, p_{2,3}, p_{2,4}, p_{3,4}) = (10, 10, 12, -15, 3, 21).$$

Then after the reflections and reordering step, we get $(10, 10, 12, 15, 21, 3)$. Maximal Element Elimination then proceeds as follows:

$$\begin{pmatrix} 0 & 10 & 10 & 12 \\ -10 & 0 & 15 & 21 \\ -10 & -15 & 0 & 3 \\ -12 & -21 & -3 & 0 \end{pmatrix} \overset{T_{4,3}}{\mapsto} \begin{pmatrix} 0 & 10 & 10 & 2 \\ -10 & 0 & 15 & 6 \\ -10 & -15 & 0 & 3 \\ -2 & -6 & -3 & 0 \end{pmatrix}.$$

The maximal element is now at position $(2,3)$. After the cyclic rotation $(v_1,v_2,v_3,v_4)\mapsto(-v_3,-v_4,v_1,v_2)$ the maximal element is at position $(1,4)$ and so the subtraction $T_{4,3}$ reduces it:

$$\begin{pmatrix} 0 & 3 & 10 & 15\\ -3 & 0 & 2 & 6\\ -10 & -2 & 0 & 10\\ -15 & -6 & -10 & 0 \end{pmatrix} \overset{T_{4,3}}{\mapsto} \begin{pmatrix} 0 & 3 & 10 & 5\\ -3 & 0 & 2 & 4\\ -10 & -2 & 0 & 10\\ -5 & -4 & -10 & 0 \end{pmatrix}.$$

Now the maximal elements are at positions $(1,3)$ and $(3,4)$. We use the *lexicographic order for the indices modulo n*, i.e. $(i_1,j_1)\prec(i_2,j_2)$ if $(i_1 \bmod n < i_2 \bmod n)$ or $i_1=i_2$ and $j_1 \bmod n < j_2 \bmod n$, hence the first maximum is at position $(1,3)$. Thus applying $T_{2,3}$ we get

$$\begin{pmatrix} 0 & 3 & 7 & 5\\ -3 & 0 & 2 & 4\\ -7 & -2 & 0 & 6\\ -5 & -4 & -6 & 0 \end{pmatrix} \overset{T_{3,2}T_{4,3}T_{3,2}}{\mapsto} \begin{pmatrix} 0 & 3 & 1 & 1\\ -3 & 0 & 2 & 2\\ -1 & -2 & 0 & 0\\ -1 & -2 & 0 & 0 \end{pmatrix}.$$

This is the end of the first stage of the algorithm, as $v_4=v_3$. Hence, we can project on first three coordinates and continue:

$$\begin{pmatrix} 0 & 3 & 1\\ -3 & 0 & 2\\ -1 & -2 & 0 \end{pmatrix} \overset{(1,2,3)\to(-2,-3,1)}{\mapsto} \begin{pmatrix} 0 & 2 & 3\\ -2 & 0 & 1\\ -3 & -1 & 0 \end{pmatrix} \overset{T_{3,2}}{\mapsto} \begin{pmatrix} 0 & 2 & 1\\ -2 & 0 & 1\\ -1 & -1 & 0 \end{pmatrix}$$

$$\overset{(1,2,3)\to(-2,-3,1)}{\mapsto} \begin{pmatrix} 0 & 1 & 2\\ -1 & 0 & 1\\ -2 & -1 & 0 \end{pmatrix} \overset{T_{3,2}T_{3,2}}{\mapsto} \begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 1\\ 0 & -1 & 0 \end{pmatrix}$$

$$\overset{(1,2,3)\to(-2,-3,1)}{\mapsto} \begin{pmatrix} 0 & 1 & 1\\ -1 & 0 & 0\\ -1 & 0 & 0 \end{pmatrix}.$$

The first, third and last basis transformations come from Stage III of the Zeroing Step (see Section 3.2.1).

We have arrived at a single coordinate $(p)=(1)$ in $G(2,2)$. Now applying the inverse transformations to the pair of vectors $(p\cdot e_1,e_2)=(e_1,e_2)$ we arrive at a basis (v_1,v_2) of the integer plane in $G(2,4)$:

$$\begin{pmatrix} v_1\\ v_2 \end{pmatrix} = \begin{pmatrix} 4 & 7 & 1 & 0\\ -6 & -8 & 1 & 3 \end{pmatrix},$$

which provides the given set of positive Plücker coordinates. Thus, performing the inverse reordering we obtain the final result

$$\begin{pmatrix} 4 & 1 & 7 & 0 \\ -6 & 1 & -8 & 3 \end{pmatrix}.$$

REMARK 3.11. The transformations (2.1) have a remarkable geometric interpretation due to the natural embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ as an affine plane $x_{n+1} = 1$, providing the bijection between the Grassmannian $G(d+1, n+1)$ and the affine Grassmannian $Gr(d, n)$.

For instance, for the transformation $T_{3,4}$, consider the above embedding of the element $\left[\begin{smallmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{smallmatrix}\right] \in G(2,4)$ as the line in $\mathbb{R}^3$.

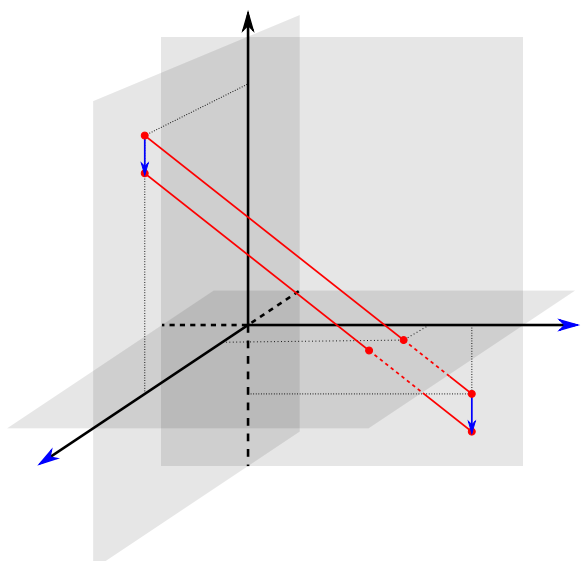


Fig. 1. Affine Grassmannian

This line intersects three coordinate planes at three points:

$A_1 = p_{2,3}^{-1}(0, -p_{3,4}, p_{2,4})$, $A_2 = p_{1,3}^{-1}(-p_{3,4}, 0, p_{1,4})$ and $A_3 = p_{1,2}^{-1}(-p_{2,4}, p_{1,4}, 0)$.

The transformation $T_{3,4}$ acts on these points as $A_1 \mapsto A_1 - (0,0,1)$, $A_2 \mapsto A_2 - (0,0,1)$, $A_3 \mapsto A_3 - (-p_{23}/p_{12}, p_{13}/p_{12}, 0)$ (see Figure 1) In other words, the transformations $T_{s,t}$ in the embedding $x_t = 1$ shift the affine lines by 1 in the direction of e_s.

4. Directions for further studies. In this paper we make the first steps in the study of Euclidean subtractive algorithms for Grassmannians. There are numerous natural questions coming from classical Euclidean settings that are awaiting investigation:

- *Quality of approximation.* Classical Euclidean algorithms provide a theory of best approximation for vectors in $\mathbb{R}^n$. What is the situation for the algorithms in Grassmannians? Is there a natural generalization of Markov numbers here? It would be interesting to find an analog of the Littlewood conjecture.
- *Algebraic periodicity.* Which planes in $G(d, n)$ have periodic continued-fraction sequences? Is there some analog of the Lagrange theorem here?
- *Dynamic properties.* Study the analogs of Gauss maps and corresponding Gauss–Kuz'min distribution. Are there links to lattice trigonometry?
- *Generic skew-symmetric matrices.* One can apply our algorithms to generic skew-symmetric matrices, not necessarily coming from the Plücker embedding of an oriented Grassmannian.

It seems that our algorithm will therefore be related to the reduction to the Hermite normal form of skew-symmetric matrices. There are numerous algorithms there, including some polynomial ones found in the 1970-ies; see, e.g., [4].

The case of Grassmannians corresponds to rank-2 matrices. It would be interesting to investigate the algorithms for the next non-trivial case of rank-4 matrices, coming from the first secant variety of $G(2, n)$.

Furthermore, it would be nice to have some version of the following algorithms:

- An analog of Maximal Element Elimination algorithms for $d \geq 3$.
- Lattice geometric algorithms generalizing Klein polyhedra and Minkowski–Voronoi continued fractions.
- Algorithms coming from algebraic number theory.

We believe that progress in the above directions will lead to new discoveries in the geometry of Grassmannians.

Acknowledgements. The authors are grateful to K. Khanin for useful remarks and discussions, and to the anonymous referee for greatly improving the exposition.

Funding. M. A. was partially supported by Simons Foundation grant MPS-TSM-00013259.

REFERENCES

[1] V. I. Arnold, *Continued Fractions*, Moscow Center of Continuous Mathematical Education, Moscow, 2002 (in Russian).

[2] V. Brun, *Algorithmes euclidiens pour trois et quatre nombres*, in: Treizième congrès des mathématiciens scandinaves, tenu à Helsinki 18-23 août 1957, Mercators Tryckeri, Helsinki, 1958, 45–64.

[3] M. Cuntz, T. Holm, and P. Jørgensen, *Frieze patterns with coefficients*, Forum Math. Sigma 8 (2020), art. e17, 36 pp.
[4] M. Grötschel, L. Lovász, and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization*, Springer, 1988.
[5] C. Hermite, *Letter to C. D. J. Jacobi*, J. Reine Angew. Math. 40 (1839), 286.
[6] C. G. J. Jacobi, *Allgemeine Theorie der kettenbruchähnlichen Algorithmen, in welchen jede Zahl aus drei vorhergehenden gebildet wird* (Aus den hinterlassenen Papieren von C. G. J. Jacobi mitgetheilt durch Herrn E. Heine), J. Reine Angew. Math. 69 (1868), 29–64.
[7] O. Karpenkov, *Geometry of Continued Fractions*, 2nd ed., Algorithms Computation Math. 26, Springer, Berlin, 2022.
[8] O. Karpenkov, *On Hermite's problem, Jacobi–Perron type algorithms, and Dirichlet groups*, Acta Arith. 203 (2022), 27–48.
[9] O. Karpenkov, *On a periodic Jacobi–Perron type algorithm*, Monatsh. Math. 205 (2024), 531–601.
[10] F. Klein, *Ueber eine geometrische Auffassung der gewöhnliche Kettenbruchentwickelung*, Nachr. Ges. Wiss. Göttingen Math-Phys. Kl. 3 (1895), 352–357.
[11] F. Klein, *Sur une représentation géométrique de développement en fraction continue ordinaire*, Nouv. Ann. Math. 15 (1896), 327–331.
[12] H. Minkowski, *Généralisation du la théorie des fractions continues*, Ann. Sci. École Norm. Sup. (3) 13 (1896), 41–60.
[13] S. Morier-Genoud, *Coxeter's frieze patterns at the crossroads of algebra, geometry and combinatorics*, Bull. London Math. Soc. 47 (2015), 895–938.
[14] O. Perron, *Grundlagen für eine Theorie des Jacobischen Kettenbruchalgorithmus*, Math. Ann. 64 (1907), 1–76.
[15] F. Schweiger, *Ergodic properties of multi-dimensional subtractive algorithms*, in: New Trends in Probability and Statistics, Vol. 2 (Palanga, 1991), VSP, Utrecht, 1992, 91–100.
[16] F. Schweiger, *Invariant measures for fully subtractive algorithms*, Anz. Österreich. Akad. Wiss. Math.-Natur. Kl. 131 (1994), 25–30.
[17] F. Schweiger, *Fully subtractive algorithms*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II 204 (1995), 23–32.
[18] F. Schweiger, *Multidimensional Continued Fractions*, Oxford Sci. Publ., Oxford Univ. Press, Oxford, 2000.
[19] E. S. Selmer, *Continued fractions in several dimensions*, Nordisk Nat. Tidskr. 9 (1961), 37–43.
[20] V. N. Sorokin and J. Van Iseghem, *Matrix continued fractions*, J. Approx. Theory 96 (1999), 237–257.
[21] J. Tamura and S. Yasutomi, *A new multidimensional continued fraction algorithm*, Math. Comp. 78 (2009), 2209–2222.
[22] G. F. Voronoĭ, *On a generalization of the algorithm of continued fractions*, doctoral dissertation, Warsaw, 1896 (in Russian).

Maxim Arnold
Department of Mathematical Sciences
University of Texas at Dallas
Richardson, TX 75035, USA
E-mail: maxim.arnold@utdallas.edu

Oleg Karpenkov
Department of Mathematical Sciences
University of Liverpool
Liverpool L69 7ZL, UK
E-mail: karpenkov@liv.ac.uk