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COMPLETE SPACE-LIKE SELF-EXPANDERS IN THE MINKOWSKI SPACE

BY

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Abstract. We study complete space-like self-expanders in the Minkowski space. By using a maximum principle of Omori–Yau type, we obtain rigidity theorems on n-dimensional complete space-like self-expanders in the Minkowski space $\mathbb{R}_1^{n+1}$. For dimensions 2 and 3, we also classify them under the assumption of constant squared norm of the second fundamental form.

1. Introduction. An n-dimensional smooth immersed hypersurface $x : M^n \to \mathbb{R}_1^{n+1}$ is called a *self-expander* of the mean curvature flow if its mean curvature vector $\vec{H}$ satisfies the equation

$$\vec{H} = x^{\perp}, \tag{1.1}$$

where $\vec{H} = He_{n+1}$, while x and e_{n+1} denote the position vector and the inward unit normal field, respectively. If we denote by $\langle \cdot, \cdot \rangle$ the standard inner product on $\mathbb{R}_1^{n+1}$, then equation (1.1) is equivalent to

$$H = -\langle x, e_{n+1} \rangle, \quad \langle e_{n+1}, e_{n+1} \rangle = -1. \tag{1.2}$$

Self-expanders are self similar solutions of the mean curvature flow, that is, the family of hypersurfaces $x_t = \sqrt{2t}x$, $t > 0$, satisfying the equation of mean curvature flow. Self-expanders are important as they model the behavior of a mean curvature flow coming out of a conical singularity [3] and also model the long time behaviors of the flows starting from entire graphs [11]. There are many classification results on self-expanders. Ishimura [13] and Halldorsson [12] have classified self-expander curves in $\mathbb{R}^2$. In [12], Halldorsson states that each of the complete self-expander curves immersed in $\mathbb{R}^2$ is convex, properly embedded and asymptotic to the boundary of a cone with vertex at the origin. In 2018, Cheng and Zhou [7] studied some properties of complete properly immersed self-expanders. Recently, Ancari and Cheng [2] studied immersed self-expander hypersurfaces whose mean curvatures have

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some linear growth controls and obtained a rigidity property of hyperplanes as self-expanders in Euclidean space. Further results about self-expanders have been obtained (cf. [4, 8–10, 14–16]).

In this paper, by using a maximum principle of Omori–Yau type, we obtain a classification of complete space-like self-expanders in the Minkowski space $\mathbb{R}_1^{n+1}$.

THEOREM 1.1. *Let $x : M^n \to \mathbb{R}_1^{n+1}$ be a complete space-like self-expander with the constant squared norm S of the second fundamental form. If $\inf H^2 > 0$, then $S = 1$ and $x(M^n)$ is the hyperbolic space $\mathbb{H}^n(\sqrt{n})$ or the hyperbolic cylinder $\mathbb{H}^k(\sqrt{k}) \times \mathbb{R}^{n-k}$, $1 \le k \le n-1$.*

When $n = 2$, we completely classify complete self-expanders with constant squared norm of the second fundamental form in the Minkowski space $\mathbb{R}_1^3$.

THEOREM 1.2. *Let $x : M^2 \to \mathbb{R}_1^3$ be a 2-dimensional complete space-like self-expander in $\mathbb{R}_1^3$. If the squared norm S of the second fundamental form is constant, then $x : M^2 \to \mathbb{R}_1^3$ is either the space-like affine plane $\mathbb{R}^2$ through the origin, the hyperbolic cylinder $\mathbb{H}^1(1) \times \mathbb{R}^1$ or the hyperbolic space $\mathbb{H}^2(\sqrt{2})$.*

REMARK. In fact, using the proof method of Theorem 1.1, we can also classify complete self-expanders in the Euclidean space $\mathbb{R}^3$ under the condition of the constant squared norm of the second fundamental form, and the classification yields only the plane $\mathbb{R}^2$ through the origin, which shows that the conclusion of Ancari and Cheng [2] still holds without the assumption on the scalar curvature.

For higher dimensions n, it is not easy to classify space-like self-expanders in the Minkowski space with constant squared norm S. Next, under the assumption that f_3 is constant, by considering the supremum and the infimum of H, we give a complete classification of 3-dimensional complete space-like self-expanders in $\mathbb{R}_1^4$ with constant squared norm S. Specifically, we prove

THEOREM 1.3. *Let $x : M^3 \to \mathbb{R}_1^4$ be a 3-dimensional complete space-like self-expander with the constant squared norm S of the second fundamental form in $\mathbb{R}_1^4$. If f_3 is constant, then $x(M^3)$ is isometric to one of the following:*

(1) *the space-like hyperplane $\mathbb{R}^3$ through the origin,*
(2) *the hyperbolic cylinder $\mathbb{H}^1(1) \times \mathbb{R}^2$,*
(3) *the hyperbolic cylinder $\mathbb{H}^2(\sqrt{2}) \times \mathbb{R}^1$,*
(4) *the hyperbolic space $\mathbb{H}^3(\sqrt{3})$.*

In particular, S must be 0 or 1 and f_3 must be 0, 1, $\frac{\sqrt{2}}{2}$ or $\frac{\sqrt{3}}{3}$, where $S = \sum_{i,j} h_{ij}^2$ and $f_3 = \sum_{i,j,k} h_{ij} h_{jk} h_{ki}$.

REMARK. Obviously, if $S \equiv 0$, then $x(M^n)$ is isometric to the space-like hyperplane $\mathbb{R}^n$ through the origin. Therefore, from now on, we always assume that S is a non-zero constant.

2. Preliminaries. Let $x : M^n \to \mathbb{R}_1^{n+1}$ be an n-dimensional space-like hypersurface immersed in the $(n+1)$-dimensional Minkowski space $\mathbb{R}_1^{n+1}$. We choose a local orthonormal frame field $\{e_A\}_{A=1}^{n+1}$ in $\mathbb{R}_1^{n+1}$, with dual coframe field $\{\omega_A\}_{A=1}^{n+1}$, such that, restricted to M^n, $e_1, \ldots, e_n$ are tangent to M^n. In this paper, we make the following conventions on the ranges of indices:

$$1 \le i, j, k, l \le n.$$

Then we have the structure equations

$$dx = \sum_i \omega_i e_i, \qquad de_i = \sum_j \omega_{ij} e_j + \omega_{i\,n+1} e_{n+1},$$

$$de_{n+1} = \omega_{n+1\,i} e_i, \qquad \omega_{n+1\,i} = \omega_{i\,n+1},$$

where ω_{ij} is the Levi-Civita connection of the hypersurface.

By restricting these forms to M, we get

$$\omega_{n+1} = 0. \tag{2.1}$$

Taking exterior derivatives of (2.1), we obtain

$$\omega_{i\,n+1} = \sum_j h_{ij}\omega_j, \qquad h_{ij} = h_{ji}. \tag{2.2}$$

Here

$$h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j, \qquad H = \sum_i h_{ii}$$

are called the *second fundamental form* and the *mean curvature* of $x : M \to \mathbb{R}_1^{n+1}$, respectively. Let $S = \sum_{i,j}(h_{ij})^2$ be the squared norm of the second fundamental form of $x : M \to \mathbb{R}_1^{n+1}$. The Gauss equations are given by

$$R_{ijkl} = -(h_{ik}h_{jl} - h_{il}h_{jk}). \tag{2.3}$$

Defining the covariant derivative of h_{ij} by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{ik}\omega_{kj} + \sum_k h_{kj}\omega_{ki}, \tag{2.4}$$

we obtain the Codazzi equations

$$h_{ijk} = h_{ikj}. \tag{2.5}$$

By taking the exterior derivative of (2.4), and setting

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_l h_{ljk}\omega_{li} + \sum_l h_{ilk}\omega_{lj} + \sum_l h_{ijl}\omega_{lk}, \tag{2.6}$$

we obtain the Ricci identities

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}. \tag{2.7}$$

Setting

$$\sum_m h_{ijklm}\omega_m = dh_{ijkl} + \sum_m h_{mjkl}\omega_{mi} + \sum_m h_{imkl}\omega_{mj} + \sum_m h_{ijml}\omega_{mk} + \sum_m h_{ijkm}\omega_{ml} \tag{2.8}$$

and taking the exterior derivative of (2.6), we get

$$h_{ijklq} - h_{ijkql} = \sum_m h_{mjk}R_{milq} + \sum_m h_{imk}R_{mjlq} + \sum_m h_{ijm}R_{mklq}. \tag{2.9}$$

Let V be a tangent C^1-vector field on M^n and denote by $Ric_V := Ric - \frac{1}{2}L_V g$ the Bakry–Émery Ricci tensor with L_V standing for the Lie derivative along the vector field V. Define a differential operator

$$\mathcal{L}f = \Delta f + \langle V, \nabla f\rangle,$$

where Δ and ∇ denote the Laplacian and the gradient operator, respectively. The following maximum principle of Omori–Yau type concerning the operator $\mathcal{L}$ will be used in this paper. It was proved by Chen and Qiu [5].

LEMMA 2.1. *Let (M^n, g) be a complete Riemannian manifold, and V a C^1 vector field on M^n. If the Bakry–Émery Ricci tensor Ric_V is bounded from below, then for any $f \in C^2(M^n)$ bounded from above, there exists a sequence $\{p_t\} \subset M^n$ such that*

$$\lim_{t\to\infty} f(p_t) = \sup f, \qquad \lim_{t\to\infty} |\nabla f|(p_t) = 0, \qquad \lim_{t\to\infty} \mathcal{L}f(p_t) \le 0.$$

We next suppose that $x : M \to \mathbb{R}^{n+1}_1$ is a self-expander, that is, $H = -\langle x, e_{n+1}\rangle$. By a simple calculation, we have the following basic formulas:

$$\begin{aligned} \nabla_i H &= -\sum_k h_{ik}\langle x, e_k\rangle, \\ \nabla_j\nabla_i H &= -\sum_k h_{ijk}\langle x, e_k\rangle - h_{ij} + H\sum_k h_{ik}h_{kj}. \end{aligned} \tag{2.10}$$

In this paper, we shall always take $V = x^\top$ in all formulas. Using (2.10) and the Ricci identities, we obtain the following lemmas.

LEMMA 2.2. *Let $x : M^n \to \mathbb{R}^{n+1}_1$ be an n-dimensional complete space-like self-expander in $\mathbb{R}^{n+1}_1$. We have*

$$\mathcal{L}H = H(S-1), \tag{2.11}$$

$$\frac{1}{2}\mathcal{L}S = \sum_{i,j,k} h^2_{ijk} + S(S-1), \tag{2.12}$$

$$\frac{1}{3}\mathcal{L}f_3 = 2\sum_{i,j,k,l} h_{ijl}h_{jkl}h_{ki} + f_3(S-1). \tag{2.13}$$

LEMMA 2.3. *Let $x : M^n \to \mathbb{R}^{n+1}_1$ be an n-dimensional complete space-like self-expander in $\mathbb{R}^{n+1}_1$. If S is constant, we have*

$$\sum_{i,j,k,l}(h_{ijkl})^2 = (2-S)\sum_{i,j,k}(h_{ijk})^2 + 6\sum_{i,j,k,l,p} h_{ijk}h_{il}h_{jp}h_{klp} - 3\sum_{i,j,k,l,p} h_{ijk}h_{ijl}h_{kp}h_{lp}. \tag{2.14}$$

3. Proof of the main result. For an arbitrary point p in M^n, we always choose a local frame field $\{e_i\}_{i=1}^n$ such that

$$h_{ij} = \lambda_i\delta_{ij}, \quad S = \sum_i \lambda_i^2, \quad H^2 \le nS,$$

and equality holds in the above inequality if and only if $\lambda_1 = \cdots = \lambda_n$. Taking the exterior derivative of $S = \sum_{i,j}(h_{ij})^2$, we find that

$$\sum_{i,j} h_{ij}h_{ijk} = 0, \quad \sum_{i,j} h_{ij}h_{ijkl} + \sum_{i,j} h_{ijk}h_{ijl} = 0, \quad k,l = 1,\dots,n. \tag{3.1}$$

On the other hand, from the Ricci identities (2.7), we obtain

$$h_{ijkl} - h_{ijlk} = -(\lambda_i\lambda_j\lambda_k\delta_{il}\delta_{jk} - \lambda_i\lambda_j\lambda_l\delta_{ik}\delta_{jl} + \lambda_i\lambda_j\lambda_k\delta_{ik}\delta_{jl} - \lambda_i\lambda_j\lambda_l\delta_{il}\delta_{jk}).$$

That is,

$$h_{iijj} - h_{jjii} = -\lambda_i\lambda_j(\lambda_i - \lambda_j), \quad h_{iikl} - h_{iilk} = 0, \quad k \ne l. \tag{3.2}$$

In 2015, Cheng and Peng [6] obtained some rigidity theorems on complete self-shrinkers without the assumption on polynomial volume growth. Motivated by the above work, we establish Theorem 1.1.

Proof of Theorem 1.1. For any unit vector $e \in TM^n$, we can choose a local tangent orthonormal frame field $\{e_i\}_{i=1}^n$ such that $e = e_i$. Then by the definition of self-expander, direct computation gives

$$\begin{aligned} Ric_{x^\top}(e_i,e_i) &= Ric(e_i,e_i) - \frac{1}{2}L_{x^\top}g(e_i,e_i) \\ &= -\Big(Hh_{ii} - \sum_j h_{ij}^2\Big) - (1 - Hh_{ii}) \\ &= \sum_j h_{ij}^2 - 1 \ge -1. \end{aligned}$$

Thus, the Bakry–Émery Ricci curvature $Ric_{x^\top}$ is bounded from below and the maximum principle of Omori–Yau type holds on $x(M^n)$. For the constant S, (2.12) yields

$$\sum_{i,j,k} h_{ijk}^2 + S(S-1) = 0. \tag{3.3}$$

Since $\inf H^2 > 0$, without loss of generality we can assume that $\inf H > 0$. By the maximum principle of Omori–Yau type for the operator $\mathcal{L}$ applied to the function $-H$, there exists a sequence $\{p_t\} \in M^2$ such that

$$0 \leq \lim_{t\to\infty} \mathcal{L}H(p_t) = \inf H(S-1).$$

It is obvious that $S = 1$ and $\sum_{i,j,k} h_{ijk}^2 = 0$ from (3.3). Namely, the second fundamental form of $x(M^n)$ is parallel and H is constant. It follows from (2.10) that $\lambda_i(H\lambda_i - 1) = 0$, $1 \leq i \leq n$. Then we infer that λ_i equals 0 or $1/H$ for any $1 \leq i \leq n$, which indicates that the number of distinct principal curvatures is at most two. If $x(M^n)$ has only one principal curvature, that is, $x(M^n)$ is totally umbilic, then $x(M^n)$ is the hyperbolic space $\mathbb{H}^n(\sqrt{n})$. If $x(M^n)$ has just two distinct constant principal curvatures, by the congruence theorem of Abe, Koike and Yamaguchi [1], $x(M^n)$ is the hyperbolic cylinder $\mathbb{H}^k(\sqrt{k}) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n-1$. ∎

In order to prove Theorem 1.2, we need the following proposition.

PROPOSITION 3.1. *Let $x : M^2 \to \mathbb{R}^3_1$ be a complete space-like self-expander with the non-zero constant squared norm S of the second fundamental form. Then* $\inf H^2 > 0$.

Proof. We argue by contradiction. If we had $\inf H^2 = 0$, there would exist a sequence $\{p_t\}$ in M^2 such that

$$\lim_{t\to\infty} H^2(p_t) = \inf H^2 = \bar{H}^2 = 0.$$

By (2.12), (2.14) and S being constant, we know that $\{h_{ij}(p_t)\}$, $\{h_{ijk}(p_t)\}$ and $\{h_{ijkl}(p_t)\}$ are bounded sequences, and so one can assume that

$$\lim_{t\to\infty} h_{ij}(p_t) = \bar{h}_{ij} = \bar{\lambda}_i\delta_{ij}, \quad \lim_{t\to\infty} h_{ijk}(p_t) = \bar{h}_{ijk},$$
$$\lim_{t\to\infty} h_{ijkl}(p_t) = \bar{h}_{ijkl}, \quad i,j,k,l = 1,2.$$

It follows from (2.10) that

$$\bar{h}_{11i} + \bar{h}_{22i} = -\bar{\lambda}_i \lim_{t\to\infty} \langle x, e_i\rangle(p_t), \quad i = 1,2, \tag{3.4}$$

and

$$\bar{h}_{11ij} + \bar{h}_{22ij} = -\sum_k \bar{h}_{ijk} \lim_{t\to\infty} \langle x, e_k\rangle(p_t) - \bar{\lambda}_i\delta_{ij}, \quad i,j = 1,2. \tag{3.5}$$

Using (3.1), we obtain

$$\bar{\lambda}_1\bar{h}_{11k} + \bar{\lambda}_2\bar{h}_{22k} = 0, \quad k = 1,2, \tag{3.6}$$

and

$$\bar{\lambda}_1\bar{h}_{11kl} + \bar{\lambda}_2\bar{h}_{22kl} = -\sum_{ij} \bar{h}_{ijk}\bar{h}_{ijl}, \quad k,l = 1,2. \tag{3.7}$$

Next, (3.2) yields

$$\bar{h}_{1112}-\bar{h}_{1121}=0,\qquad \bar{h}_{2221}-\bar{h}_{2212}=0. \tag{3.8}$$

Since $\bar{H}^2=0$ and S is a non-zero constant, we see that

$$\bar{\lambda}_1=-\bar{\lambda}_2\neq 0,\qquad S=2\bar{\lambda}_1^2.$$

Then (3.4) and (3.6) yield

$$\bar{h}_{11k}=\bar{h}_{22k},\qquad \lim_{t\to\infty}\langle x,e_k\rangle(p_t)=-\frac{2\bar{h}_{11k}}{\bar{\lambda}_k},\qquad k=1,2. \tag{3.9}$$

It follows from (3.5), (3.7) and (3.9) that

$$\begin{aligned}
&\bar{h}_{1111}+\bar{h}_{2211}=\frac{2(\bar{h}_{111}^2-\bar{h}_{112}^2)}{\bar{\lambda}_1}-\bar{\lambda}_1,\\
&\bar{h}_{1122}+\bar{h}_{2222}=\frac{2(\bar{h}_{111}^2-\bar{h}_{112}^2)}{\bar{\lambda}_1}+\bar{\lambda}_1,\\
&\bar{h}_{1112}+\bar{h}_{2212}=0
\end{aligned}$$

and

$$\begin{aligned}
&\bar{\lambda}_1(\bar{h}_{1111}-\bar{h}_{2211})=-2\bar{h}_{111}^2-2\bar{h}_{112}^2,\\
&\bar{\lambda}_1(\bar{h}_{1122}-\bar{h}_{2222})=-2\bar{h}_{111}^2-2\bar{h}_{112}^2,\\
&\bar{\lambda}_1(\bar{h}_{1112}-\bar{h}_{2212})=-4\bar{h}_{111}\bar{h}_{112}.
\end{aligned}$$

Thus,

$$\begin{cases}
\bar{h}_{1111}=-\frac{2\bar{h}_{112}^2}{\bar{\lambda}_1}-\frac{\bar{\lambda}_1}{2},\quad \bar{h}_{2211}=\frac{2\bar{h}_{111}^2}{\bar{\lambda}_1}-\frac{\bar{\lambda}_1}{2},\\
\bar{h}_{1122}=-\frac{2\bar{h}_{112}^2}{\bar{\lambda}_1}+\frac{\bar{\lambda}_1}{2},\quad \bar{h}_{2222}=\frac{2\bar{h}_{111}^2}{\bar{\lambda}_1}+\frac{\bar{\lambda}_1}{2},\\
\bar{h}_{1112}=-\frac{2\bar{h}_{111}\bar{h}_{112}}{\bar{\lambda}_1},\quad \bar{h}_{2212}=\frac{2\bar{h}_{111}\bar{h}_{112}}{\bar{\lambda}_1}.
\end{cases} \tag{3.10}$$

It follows from (2.12) that

$$\sum_{i,j,k}h_{ijk}^2=S(1-S),\qquad \sum_{i,j,k}\bar{h}_{ijk}\bar{h}_{ijkl}=0,\qquad l=1,2. \tag{3.11}$$

Specifically,

$$\begin{aligned}
&\bar{h}_{111}\bar{h}_{1111}+\bar{h}_{222}\bar{h}_{2221}+3\bar{h}_{112}\bar{h}_{1121}+3\bar{h}_{221}\bar{h}_{2211}=0,\\
&\bar{h}_{111}\bar{h}_{1112}+\bar{h}_{222}\bar{h}_{2222}+3\bar{h}_{112}\bar{h}_{1122}+3\bar{h}_{221}\bar{h}_{2212}=0.
\end{aligned}$$

Then, from (3.8) and (3.9), we conclude that

$$\begin{aligned}
&\bar{h}_{111}\bar{h}_{1111}+\bar{h}_{112}\bar{h}_{2212}+3\bar{h}_{112}\bar{h}_{1112}+3\bar{h}_{111}\bar{h}_{2211}=0,\\
&\bar{h}_{111}\bar{h}_{1112}+\bar{h}_{112}\bar{h}_{2222}+3\bar{h}_{112}\bar{h}_{1122}+3\bar{h}_{111}\bar{h}_{2212}=0.
\end{aligned} \tag{3.12}$$

Combining (3.10) with (3.12), we get

$$\bar{h}_{111}(3\bar{h}_{111}^2-3\bar{h}_{112}^2-\bar{\lambda}_1^2)=0,\qquad \bar{h}_{112}(3\bar{h}_{111}^2-3\bar{h}_{112}^2+\bar{\lambda}_1^2)=0. \tag{3.13}$$

If $\bar{h}_{111}\bar{h}_{112} \neq 0$, (3.13) implies that $\bar{\lambda}_1^2 = S = 0$. This is a contradiction. If $\bar{h}_{111}\bar{h}_{112} = 0$, we also obtain contradictions. In fact, we may assume that $\sum_{i,j,k} \bar{h}_{ijk}^2 \neq 0$. Otherwise, we know that $\bar{h}_{ijkl} = 0$ for $i, j, k, l = 1, 2$ from (2.14). Then, using (3.10), we infer that $\bar{\lambda}_1 = 0$, a contradiction. Assuming $\bar{h}_{111} \neq 0$, we get $\bar{h}_{112} = 0$. From the first equation of (3.11) and (3.13), it follows that

$$\bar{h}_{111}^2 = \frac{1}{4}S(1-S), \quad \bar{h}_{111}^2 = \frac{1}{3}\bar{\lambda}_1^2 = \frac{1}{6}S.$$

Then

$$S = \frac{1}{3}, \quad \bar{\lambda}_1^2 = \frac{1}{6}, \quad \bar{h}_{111}^2 = \frac{1}{18}. \tag{3.14}$$

Consequently, the relations

$$\begin{aligned} &\bar{h}_{1111} = -\frac{\bar{\lambda}_1}{2}, \quad \bar{h}_{2211} = \frac{\bar{\lambda}_1}{6}, \quad \bar{h}_{1122} = \frac{\bar{\lambda}_1}{2}, \\ &\bar{h}_{2222} = \frac{7\bar{\lambda}_1}{6}, \quad \bar{h}_{1112} = \bar{h}_{2212} = 0 \end{aligned} \tag{3.15}$$

can be derived by a simple calculation from (3.10).

It follows from (2.14), (3.14) and (3.15) that

$$\sum_{i,j,k,l} (\bar{h}_{ijkl})^2 = \bar{h}_{1111}^2 + \bar{h}_{2222}^2 + 3\bar{h}_{1122}^2 + 3\bar{h}_{2211}^2 + 4\bar{h}_{1112}^2 + 4\bar{h}_{2212}^2 = \frac{11}{27}$$

and

$$\begin{aligned} \sum_{i,j,k,l} (\bar{h}_{ijkl})^2 &= (2-S)\sum_{i,j,k} (\bar{h}_{ijk})^2 + 6\sum_{i,j,k} \bar{\lambda}_i\bar{\lambda}_j\bar{h}_{ijk}^2 - 3\sum_{i,j,k} \bar{\lambda}_k^2\bar{h}_{ijk}^2 \\ &= 4(2-S)\bar{h}_{111}^2 - 12\bar{\lambda}_1^2\bar{h}_{111}^2 = \frac{7}{27}. \end{aligned}$$

This is impossible. Assuming that $\bar{h}_{111} = 0$ and $\bar{h}_{112} \neq 0$ we similarly obtain contradictions. ∎

Proof of Theorem 1.2. If $S = 0$, we know that $x : M^2 \to \mathbb{R}_1^3$ is a space-like affine plane $\mathbb{R}^2$. If $S \neq 0$, it is obvious that $x(M^2)$ is either $\mathbb{H}^1(1) \times \mathbb{R}^1$ or $\mathbb{H}^2(\sqrt{2})$ from Theorem 1.1 and Proposition 3.1. ∎

For the constant S, if $S \geq 1$, we get $S = 1$ from (2.12). Then, from Theorem 1.1, we obtain the conclusion of Theorem 1.3. Thus, we only need to consider the case $0 < S < 1$.

PROPOSITION 3.2. *For a 3-dimensional complete space-like self-expander $x : M^3 \to \mathbb{R}_1^4$ with constant squared norm S of the second fundamental form and constant f_3, if $0 < S < 1$, then either $\sup H = \frac{3f_3}{2S}$ or $\sup H = \frac{1-S^2}{Sf_3}$ with $f_3^2 = (1-S^2)^2$ and $3S^3 + S^2 - S - 1 = 0$.*

Proof. For an arbitrary fixed point $p \in M^3$, we choose a local frame field $\{e_i\}_{i=1}^3$ such that

$$h_{ij} = \lambda_i \delta_{ij}, \qquad S = \sum_i \lambda_i^2, \qquad H^2 \leq 3S.$$

The proof of Theorem 1.1 states that the Bakry–Émery Ricci curvature $Ric_{x^\top}$ is bounded from below. By the maximum principle of Omori–Yau type for the operator $\mathcal{L}$ applied to the function H, there exists a sequence $\{p_t\}$ in M^3 such that

$$\lim_{t\to\infty} H(p_t) = \sup H, \qquad \lim_{t\to\infty} |\nabla H|(p_t) = 0, \qquad \lim_{t\to\infty} \mathcal{L}H(p_t) \leq 0. \tag{3.16}$$

For the constant S, by (2.12) and (2.14), we know that $\{h_{ij}(p_t)\}$, $\{h_{ijk}(p_t)\}$ and $\{h_{ijkl}(p_t)\}$ are bounded sequences for $i,j,k,l = 1,2,3$. Hence, one can assume that

$$\begin{aligned}
&\lim_{t\to\infty} H(p_t) = \bar{H}, \qquad \lim_{t\to\infty} h_{ij}(p_t) = \bar{h}_{ij} = \bar{\lambda}_i \delta_{ij},\\
&\lim_{t\to\infty} h_{ijk}(p_t) = \bar{h}_{ijk}, \qquad \lim_{t\to\infty} h_{ijkl}(p_t) = \bar{h}_{ijkl}, \qquad i,j,k,l = 1,2,3.
\end{aligned}$$

Taking the exterior derivative of H, we get

$$\bar{H}_{,i} = \bar{h}_{11i} + \bar{h}_{22i} + \bar{h}_{33i} = 0, \qquad i = 1,2,3. \tag{3.17}$$

Also, (3.1) yields

$$\bar{H}_{,i} = \bar{h}_{11i} + \bar{h}_{22i} + \bar{h}_{33i} = -\bar{\lambda}_i \lim_{t\to\infty} \langle x, e_i\rangle(p_t), \qquad i = 1,2,3, \tag{3.18}$$

and

$$\bar{H}_{,ij} = -\sum_k \bar{h}_{ijk} \lim_{t\to\infty} \langle x, e_k\rangle(p_t) - \bar{\lambda}_i \delta_{ij} + \bar{H}\bar{\lambda}_i^2 \delta_{ij}, \qquad i,j = 1,2,3. \tag{3.19}$$

Since S is constant, we obtain

$$\bar{\lambda}_1 \bar{h}_{11k} + \bar{\lambda}_2 \bar{h}_{22k} + \bar{\lambda}_3 \bar{h}_{33k} = 0, \qquad k = 1,2,3, \tag{3.20}$$

and

$$\sum_i \bar{\lambda}_i \bar{h}_{iikl} = -\sum_{i,j} \bar{h}_{ijk} \bar{h}_{ijl}, \qquad k,l = 1,2,3. \tag{3.21}$$

Since f_3 is constant, Lemma 2.2 gives

$$\bar{\lambda}_1^2 \bar{h}_{11k} + \bar{\lambda}_2^2 \bar{h}_{22k} + \bar{\lambda}_3^2 \bar{h}_{33k} = 0, \qquad k = 1,2,3 \tag{3.22}$$

and

$$\sum_i \bar{\lambda}_i^2 \bar{h}_{iikl} = -2\sum_{i,j} \bar{\lambda}_i \bar{h}_{ijk} \bar{h}_{ijl}, \qquad k,l = 1,2,3. \tag{3.23}$$

Using the above equations, we now prove that if $0 < S < 1$, then either

$$\sup H = \bar{H} = \frac{3f_3}{2S}$$

or

$$\sup H = \bar{H} = \frac{1-S^2}{Sf_3}, \qquad f_3^2 = (1-S^2)^2, \qquad 3S^3 + S^2 - S - 1 = 0.$$

In fact, if $\bar{\lambda}_1 = \bar{\lambda}_2 = \bar{\lambda}_3$, from (3.21) and (3.23) we have

$$\begin{cases} \bar{\lambda}_1 \sum_i \bar{h}_{ii11} = -(\bar{h}_{111}^2 + \bar{h}_{221}^2 + \bar{h}_{331}^2) - 2(\bar{h}_{112}^2 + \bar{h}_{113}^2 + \bar{h}_{123}^2), \\ \bar{\lambda}_1 \sum_i \bar{h}_{ii22} = -(\bar{h}_{112}^2 + \bar{h}_{222}^2 + \bar{h}_{332}^2) - 2(\bar{h}_{221}^2 + \bar{h}_{223}^2 + \bar{h}_{123}^2), \\ \bar{\lambda}_1 \sum_i \bar{h}_{ii33} = -(\bar{h}_{113}^2 + \bar{h}_{223}^2 + \bar{h}_{333}^2) - 2(\bar{h}_{331}^2 + \bar{h}_{332}^2 + \bar{h}_{123}^2) \end{cases}$$

and

$$\begin{cases} \bar{\lambda}_1^2 \sum_i \bar{h}_{ii11} = -2\bar{\lambda}_1(\bar{h}_{111}^2 + \bar{h}_{221}^2 + \bar{h}_{331}^2) - 4\bar{\lambda}_1(\bar{h}_{112}^2 + \bar{h}_{113}^2 + \bar{h}_{123}^2), \\ \bar{\lambda}_1^2 \sum_i \bar{h}_{ii22} = -2\bar{\lambda}_1(\bar{h}_{112}^2 + \bar{h}_{222}^2 + \bar{h}_{332}^2) - 4\bar{\lambda}_1(\bar{h}_{221}^2 + \bar{h}_{223}^2 + \bar{h}_{123}^2), \\ \bar{\lambda}_1^2 \sum_i \bar{h}_{ii33} = -2\bar{\lambda}_1(\bar{h}_{113}^2 + \bar{h}_{223}^2 + \bar{h}_{333}^2) - 4\bar{\lambda}_1(\bar{h}_{331}^2 + \bar{h}_{332}^2 + \bar{h}_{123}^2). \end{cases}$$

Thus, we infer that

$$\bar{h}_{ijk} = 0, \qquad i,j,k = 1,2,3.$$

Then it follows that $S = 1$ from (2.12). This contradicts the hypothesis.

If two of $\bar{\lambda}_1$, $\bar{\lambda}_2$ and $\bar{\lambda}_3$ are equal, without loss of generality we assume that $\bar{\lambda}_1 \neq \bar{\lambda}_2 = \bar{\lambda}_3$. Then we deduce from (3.20) and (3.22) that

$$\bar{h}_{11k} = 0, \qquad \bar{h}_{22k} + \bar{h}_{33k} = 0, \qquad k = 1,2,3. \tag{3.24}$$

When $\bar{\lambda}_1 = 0$, we know that $\bar{\lambda}_2 = \bar{\lambda}_3 \neq 0$ since $S \neq 0$. By use of (3.21), (3.23) and (3.24), we obtain

$$\begin{cases} \bar{\lambda}_2(\bar{h}_{2211} + \bar{h}_{3311}) = -2(\bar{h}_{221}^2 + \bar{h}_{123}^2), \\ \bar{\lambda}_2(\bar{h}_{2222} + \bar{h}_{3322}) = -2(\bar{h}_{222}^2 + \bar{h}_{223}^2) - 2(\bar{h}_{221}^2 + \bar{h}_{123}^2) \end{cases}$$

and

$$\begin{cases} \bar{\lambda}_2^2(\bar{h}_{2211} + \bar{h}_{3311}) = -4\bar{\lambda}_2(\bar{h}_{221}^2 + \bar{h}_{123}^2), \\ \bar{\lambda}_2^2(\bar{h}_{2222} + \bar{h}_{3322}) = -4\bar{\lambda}_2(\bar{h}_{222}^2 + \bar{h}_{223}^2) - 2\bar{\lambda}_2(\bar{h}_{221}^2 + \bar{h}_{123}^2). \end{cases}$$

Thus,

$$\bar{h}_{221} = \bar{h}_{123} = 0, \qquad \bar{h}_{222} = \bar{h}_{223} = 0.$$

Consequently,

$$\bar{h}_{ijk} = 0, \qquad i,j,k = 1,2,3.$$

Then $S = 1$ from (2.12). This contradicts the hypothesis.

When $\bar{\lambda}_1 \neq 0$, $\bar{\lambda}_2 = \bar{\lambda}_3 = 0$, it follows from (3.21), (3.23) and (3.24) that

$$\begin{cases} \bar{\lambda}_1 \bar{h}_{1111} = -2(\bar{h}_{221}^2 + \bar{h}_{123}^2), \\ \bar{\lambda}_1 \bar{h}_{1122} = -2(\bar{h}_{222}^2 + \bar{h}_{223}^2) - 2(\bar{h}_{221}^2 + \bar{h}_{123}^2) \end{cases}$$

and

$$\begin{cases} \bar{\lambda}_1^2 \bar{h}_{1111} = 0, \\ \bar{\lambda}_1^2 \bar{h}_{1122} = -2\bar{\lambda}_1(\bar{h}_{221}^2 + \bar{h}_{123}^2). \end{cases}$$

Thus,

$$\bar{h}_{221} = \bar{h}_{123} = 0, \quad \bar{h}_{222} = \bar{h}_{223} = 0.$$

Consequently,

$$\bar{h}_{ijk} = 0, \quad i,j,k = 1,2,3.$$

Then $S = 1$ from (2.12). This contradicts the hypothesis.

When $\bar{\lambda}_1 \neq 0$, $\bar{\lambda}_2 = \bar{\lambda}_3 \neq 0$, (3.17) and (3.18) yield

$$\lim_{t\to\infty} \langle x, e_k \rangle(p_t) = 0, \quad k = 1,2,3. \tag{3.25}$$

Using (3.19) and (3.25), we can rewrite (3.21), (3.23) and (3.24) as

$$\begin{cases} \bar{h}_{1111} + \bar{h}_{2211} + \bar{h}_{3311} = -\bar{\lambda}_1 + \bar{H}\bar{\lambda}_1^2, \\ \bar{h}_{1122} + \bar{h}_{2222} + \bar{h}_{3322} = -\bar{\lambda}_2 + \bar{H}\bar{\lambda}_2^2, \\ \bar{h}_{1112} + \bar{h}_{2212} + \bar{h}_{3312} = 0, \\ \bar{h}_{1113} + \bar{h}_{2213} + \bar{h}_{3313} = 0, \end{cases} \tag{3.26}$$

$$\begin{cases} \bar{\lambda}_1\bar{h}_{1111} + \bar{\lambda}_2(\bar{h}_{2211} + \bar{h}_{3311}) = -2(\bar{h}_{221}^2 + \bar{h}_{123}^2), \\ \bar{\lambda}_1\bar{h}_{1122} + \bar{\lambda}_2(\bar{h}_{2222} + \bar{h}_{3322}) = -2(\bar{h}_{221}^2 + \bar{h}_{123}^2) - 2(\bar{h}_{222}^2 + \bar{h}_{223}^2), \\ \bar{\lambda}_1\bar{h}_{1112} + \bar{\lambda}_2(\bar{h}_{2212} + \bar{h}_{3312}) = -2(\bar{h}_{221}\bar{h}_{222} + \bar{h}_{223}\bar{h}_{123}), \\ \bar{\lambda}_1\bar{h}_{1113} + \bar{\lambda}_2(\bar{h}_{2213} + \bar{h}_{3313}) = -2(\bar{h}_{221}\bar{h}_{223} - \bar{h}_{222}\bar{h}_{123}) \end{cases} \tag{3.27}$$

and

$$\begin{cases} \bar{\lambda}_1^2\bar{h}_{1111} + \bar{\lambda}_2^2(\bar{h}_{2211} + \bar{h}_{3311}) = -4\bar{\lambda}_2(\bar{h}_{221}^2 + \bar{h}_{123}^2), \\ \bar{\lambda}_1^2\bar{h}_{1122} + \bar{\lambda}_2^2(\bar{h}_{2222} + \bar{h}_{3322}) = -2(\bar{\lambda}_1 + \bar{\lambda}_2)(\bar{h}_{221}^2 + \bar{h}_{123}^2) \\ \qquad\qquad\qquad\qquad\qquad\qquad -4\bar{\lambda}_2(\bar{h}_{222}^2 + \bar{h}_{223}^2), \\ \bar{\lambda}_1^2\bar{h}_{1112} + \bar{\lambda}_2^2(\bar{h}_{2212} + \bar{h}_{3312}) = -4\bar{\lambda}_2(\bar{h}_{221}\bar{h}_{222} + \bar{h}_{223}\bar{h}_{123}), \\ \bar{\lambda}_1^2\bar{h}_{1113} + \bar{\lambda}_2^2(\bar{h}_{2213} + \bar{h}_{3313}) = -4\bar{\lambda}_2(\bar{h}_{221}\bar{h}_{223} - \bar{h}_{222}\bar{h}_{123}). \end{cases} \tag{3.28}$$

Combining (3.27) and (3.28), we obtain

$$\begin{cases} \bar{\lambda}_1(\bar{\lambda}_2 - \bar{\lambda}_1)\bar{h}_{1111} = 2\bar{\lambda}_2(\bar{h}_{221}^2 + \bar{h}_{123}^2), \\ \bar{\lambda}_1(\bar{\lambda}_2 - \bar{\lambda}_1)\bar{h}_{1122} = 2\bar{\lambda}_2(\bar{h}_{222}^2 + \bar{h}_{223}^2) + 2\bar{\lambda}_1(\bar{h}_{221}^2 + \bar{h}_{123}^2). \end{cases} \tag{3.29}$$

Next, (3.26) gives

$$\bar{h}_{2212} + \bar{h}_{3312} = -\bar{h}_{1112}, \quad \bar{h}_{2213} + \bar{h}_{3313} = -\bar{h}_{1113}. \tag{3.30}$$

Inserting (3.30) into (3.27) and (3.28), we conclude that

$$\bar{h}_{221}\bar{h}_{222} + \bar{h}_{223}\bar{h}_{123} = 0, \quad \bar{h}_{221}\bar{h}_{223} - \bar{h}_{222}\bar{h}_{123} = 0,$$

which yields $\bar{h}_{221}^2 + \bar{h}_{123}^2 = 0$ or $\bar{h}_{222}^2 + \bar{h}_{223}^2 = 0$.

When $\bar{h}^2_{221}+\bar{h}^2_{123}=0$ and $\bar{h}^2_{222}+\bar{h}^2_{223}=0$, we get $\bar{h}_{ijk}=0$, $i,j,k=1,2,3$. Thus, $S=1$ from (2.12). This contradicts the hypothesis.

When $\bar{h}^2_{221}+\bar{h}^2_{123}\neq 0$ and $\bar{h}^2_{222}+\bar{h}^2_{223}=0$, (3.29) implies

$$\bar{\lambda}_1(\bar{\lambda}_2-\bar{\lambda}_1)\bar{h}_{1111}=2\bar{\lambda}_2(\bar{h}^2_{221}+\bar{h}^2_{123}),$$
$$\bar{\lambda}_1(\bar{\lambda}_2-\bar{\lambda}_1)\bar{h}_{1122}=2\bar{\lambda}_1(\bar{h}^2_{221}+\bar{h}^2_{123}).$$

Then

$$\bar{h}_{1111}=\frac{2\bar{\lambda}_2}{\bar{\lambda}_1(\bar{\lambda}_2-\bar{\lambda}_1)}(\bar{h}^2_{221}+\bar{h}^2_{123}),\qquad \bar{h}_{1122}=\frac{2}{\bar{\lambda}_2-\bar{\lambda}_1}(\bar{h}^2_{221}+\bar{h}^2_{123}). \tag{3.31}$$

It follows from $(3.26)_2$, (3.27) and (3.31) that

$$\begin{aligned}-2(\bar{h}^2_{221}+\bar{h}^2_{123})&=\bar{\lambda}_1\bar{h}_{1122}+\bar{\lambda}_2(-\bar{\lambda}_2+\bar{H}\bar{\lambda}_2^2-\bar{h}_{1122})\\&=\bar{\lambda}_2(-\bar{\lambda}_2+\bar{H}\bar{\lambda}_2^2)+(\bar{\lambda}_1-\bar{\lambda}_2)\bar{h}_{1122}\\&=-\bar{\lambda}_2^2(1-\bar{H}\bar{\lambda}_2)-2(\bar{h}^2_{221}+\bar{h}^2_{123}).\end{aligned}$$

Then

$$\bar{\lambda}_2=\frac{1}{\bar{H}}. \tag{3.32}$$

By use of $(3.26)_1$, (3.27), (3.31) and (3.32), we get

$$\begin{aligned}-2(\bar{h}^2_{221}+\bar{h}^2_{123})&=\bar{\lambda}_1\bar{h}_{1111}+\bar{\lambda}_2(-\bar{\lambda}_1+\bar{H}\bar{\lambda}_1^2-\bar{h}_{1111})\\&=\bar{\lambda}_2(-\bar{\lambda}_1+\bar{H}\bar{\lambda}_1^2)+(\bar{\lambda}_1-\bar{\lambda}_2)\bar{h}_{1111}\\&=-\bar{\lambda}_1\bar{\lambda}_2\left(1-\frac{\bar{\lambda}_1}{\bar{\lambda}_2}\right)-\frac{2\bar{\lambda}_2}{\bar{\lambda}_1}(\bar{h}^2_{221}+\bar{h}^2_{123}),\end{aligned}$$

which implies

$$\bar{h}^2_{221}+\bar{h}^2_{123}=-\bar{\lambda}_1^2/2.$$

This is impossible.

When $\bar{h}^2_{221}+\bar{h}^2_{123}=0$ and $\bar{h}^2_{222}+\bar{h}^2_{223}\neq 0$, (3.29) yields

$$\bar{h}_{1111}=0,\qquad \bar{h}_{1122}=\frac{2\bar{\lambda}_2}{\bar{\lambda}_1(\bar{\lambda}_2-\bar{\lambda}_1)}(\bar{h}^2_{222}+\bar{h}^2_{223}). \tag{3.33}$$

Noting that $\bar{h}_{1111}=0$ and using $(3.27)_1$, we get

$$\bar{h}_{2211}+\bar{h}_{3311}=0,\qquad \bar{H}_{,11}=0.$$

Hence, in view of $(3.26)_1$,

$$\bar{\lambda}_1=\frac{1}{\bar{H}}. \tag{3.34}$$

From $(3.26)_2$, (3.27), (3.33) and (3.34), we obtain

$$\begin{aligned}-2(\bar{h}^2_{222}+\bar{h}^2_{223}) &= \bar{\lambda}_1\bar{h}_{1122}+\bar{\lambda}_2(-\bar{\lambda}_2+\bar{H}\bar{\lambda}_2^2-\bar{h}_{1122})\\ &= \bar{\lambda}_2(-\bar{\lambda}_2+\bar{H}\bar{\lambda}_2^2)+(\bar{\lambda}_1-\bar{\lambda}_2)\cdot\frac{2\bar{\lambda}_2}{\bar{\lambda}_1(\bar{\lambda}_2-\bar{\lambda}_1)}(\bar{h}^2_{222}+\bar{h}^2_{223})\\ &= -\bar{\lambda}_2^2\left(1-\frac{\bar{\lambda}_2}{\bar{\lambda}_1}\right)-\frac{2\bar{\lambda}_2}{\bar{\lambda}_1}(\bar{h}^2_{222}+\bar{h}^2_{223}),\end{aligned}$$

which implies

$$\bar{h}^2_{222}+\bar{h}^2_{223}=\frac{\bar{\lambda}_2^2}{2}. \tag{3.35}$$

Since $\bar{h}^2_{221}+\bar{h}^2_{123}=0$, using (3.24) we get

$$\sum_{i,j,k}\bar{h}^2_{ijk}=4(\bar{h}^2_{222}+\bar{h}^2_{223}),\qquad 2\sum_{i,j,k,l}\bar{h}_{ijl}\bar{h}_{jkl}\bar{h}_{ki}=8\bar{\lambda}_2(\bar{h}^2_{222}+\bar{h}^2_{223}).$$

It follows from (2.12) and (2.13) that

$$\sum_{i,j,k}\bar{h}^2_{ijk}=S(1-S),\qquad 2\sum_{i,j,k,l}\bar{h}_{ijl}\bar{h}_{jkl}\bar{h}_{ki}=f_3(1-S).$$

Then

$$\bar{h}^2_{222}+\bar{h}^2_{223}=\frac{S(1-S)}{4},\qquad \bar{\lambda}_2=\frac{f_3}{2S}. \tag{3.36}$$

Combining (3.35) with (3.36), we infer that

$$\bar{\lambda}_2^2=\frac{S(1-S)}{2},\qquad f_3^2=2S^3(1-S). \tag{3.37}$$

From (3.34), (3.36) and (3.37) we obtain

$$\begin{aligned}&S=\bar{\lambda}_1^2+2\bar{\lambda}_2^2=\frac{1}{\bar{H}^2}+S(1-S),\qquad \bar{H}^2=\frac{1}{S^2},\\ &\bar{H}=\bar{\lambda}_1+2\bar{\lambda}_2=\frac{1}{\bar{H}}+\frac{f_3}{S},\qquad \bar{H}=\frac{1-S^2}{Sf_3},\qquad f_3^2=(1-S^2)^2.\end{aligned}$$

Consequently,

$$\bar{H}=\frac{1-S^2}{Sf_3},\qquad f_3^2=(1-S^2)^2,\qquad 3S^3+S^2-S-1=0,$$

where the last equation comes from the second equation of (3.37).

If $\bar{\lambda}_1$, $\bar{\lambda}_2$ and $\bar{\lambda}_3$ are distinct, it follows from (3.17), (3.20) and (3.22) that

$$\bar{h}_{11k}=\bar{h}_{22k}=\bar{h}_{33k}=0,\qquad k=1,2,3,$$

and

$$\sum_{i,j,k}\bar{h}^2_{ijk}=6\bar{h}^2_{123},\qquad 2\sum_{i,j,k,l}\bar{h}_{ijl}\bar{h}_{jkl}\bar{h}_{ki}=4\bar{H}\bar{h}^2_{123}.$$

Then, combining (2.12) and (2.13), we get

$$6\bar{h}^2_{123} = S(1-S), \quad 4\bar{H}\bar{h}^2_{123} = f_3(1-S),$$

Hence,

$$\bar{H} = \frac{3f_3}{2S}. \blacksquare$$

By the maximum principle of Omori–Yau type for the operator $\mathcal{L}$ applied to the function $-H$, we obtain

PROPOSITION 3.3. *For a 3-dimensional complete space-like self-expander $x : M^3 \to \mathbb{R}^4_1$ with non-zero constant squared norm S of the second fundamental form, if f_3 is constant, then either* $\inf H = \frac{3f_3}{2S}$ *or* $\inf H = \frac{1-S^2}{Sf_3}$, *where* $f_3^2 = (1-S^2)^2$, $3S^3 + S^2 - S - 1 = 0$.

Proof of Theorem 1.3. If $S = 0$, we know that $x : M^3 \to \mathbb{R}^4_1$ is the space-like hyperplane $\mathbb{R}^3$ through the origin. If $0 < S < 1$, we know that $\frac{3f_3}{2S}$ and $\frac{1-S^2}{Sf_3}$ cannot be equal since $f_3^2 = (1-S^2)^2$ and $3S^3 + S^2 - S - 1 = 0$. In fact, if $\frac{3f_3}{2S} = \frac{1-S^2}{Sf_3}$, then $3f_3^2 = 2(1-S^2)$. Then from $f_3^2 = (1-S^2)^2$, we infer that $S = \sqrt{3}/3$. This is a contradiction since $3S^3 + S^2 - S - 1 = 0$. Thus, from Propositions 3.2 and 3.3, we obtain $\sup H = \inf H = \frac{3f_3}{2S}$ or $\sup H = \inf H = \frac{1-S^2}{Sf_3}$, which shows that the mean curvature H is constant. Then, by (2.11), we know that $H = 0$ and $x(M^3)$ is isometric to the space-like hyperplane $\mathbb{R}^3$, which contradicts the relation $0 < S < 1$. This completes the proof of the main theorem. $\blacksquare$

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