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NEW RESULTS ANALOGOUS TO LAGRANGE'S FOUR-SQUARE THEOREM

BY

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Abstract. We establish some new results analogous to Lagrange's four-square theorem. For example, we prove that each integer $n > 1$ can be written as $w(5w+1)/2 + x(5x+1)/2 + y(5y+1)/2 + z(5z+1)/2$ with $w, x, y, z \in \mathbb{Z}$.

Let a and b be integers with $a > 0$, $b > -a$ and $\gcd(a, b) = 1$. When $2 \nmid ab$, we show that the set $\{n(an+b)/2 : n = 0, 1, 2, \ldots\}$ is an asymptotic base of order four, i.e., any sufficiently large integer can be written as

$$\frac{w(aw+b)}{2} + \frac{x(ax+b)}{2} + \frac{y(ay+b)}{2} + \frac{z(az+b)}{2}$$

with w, x, y, z nonnegative integers. When $2 \mid a$ and $2 \nmid b$, we prove that the set $\{n(an+b) : n = 0, 1, 2, \ldots\}$ is an asymptotic base of order four.

1. Introduction. For any integer $m \geqslant 3$, the *polygonal numbers of order* m (or *m-gonal numbers*) are the nonnegative integers

$$p_m(n) = (m-2)\binom{n}{2} + n = \frac{n((m-2)n - (m-4))}{2} \quad (n = 0, 1, 2, \ldots).$$

The numbers

$$\bar{p}_m(n) = p_m(-n) = \frac{n((m-2)n + (m-4))}{2} \quad (n = 0, 1, 2, \ldots)$$

are called *second m-gonal numbers*, and those $p_m(x)$ with $x \in \mathbb{Z}$ are called *generalized m-gonal numbers*. Note that

$$p_3(x) = \frac{x(x+1)}{2}, \quad p_4(x) = x^2, \quad p_5(x) = \frac{x(3x-1)}{2},$$

$$p_6(x) = x(2x-1) = p_3(2x-1), \quad p_7(x) = \frac{x(5x-3)}{2}, \quad p_8(x) = x(3x-2).$$

In 1638 Fermat claimed that each $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ is a sum of m m-gonal numbers; this was confirmed by Lagrange, Gauss and Cauchy for

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the cases $m=4$, $m=3$, and $m\geqslant 5$, respectively (cf. [6, pp. 54–57] and [8, Chapter 1]). Lagrange's four-square theorem states that any $n\in\mathbb{N}$ can be written as $w^2+x^2+y^2+z^2$ with $w,x,y,z\in\mathbb{N}$. In 1830 Legendre refined Cauchy's polygonal number theorem by showing that for any integer $m\geqslant 5$ we can write each integer $n\geqslant 28(m-2)^3$ as

$$p_m(w)+p_m(x)+p_m(y)+p_m(z)+\delta_m(n),$$

where $w,x,y,z\in\mathbb{N}$, $\delta_m(n)=0$ if $2\nmid m$, and $\delta_m(n)\in\{0,1\}$ if $2\mid m$. M. B. Nathanson ([7] and [8, pp. 27–33]) simplified the proofs of Cauchy's and Legendre's theorems. As a supplement to Legendre's theorem, X.-Z. Meng and the author [5] proved that for any integer $m\geqslant 6$ with $m\equiv 2\pmod 4$ each integer $n\geqslant 28(m-2)^3$ can be expressed as $p_m(w)+p_m(x)+p_m(y)+p_m(z)$ with $w,x,y,z\in\mathbb{N}$, and that for any integer $m\geqslant 8$ with $4\mid m$ there are infinitely many positive integers not of the form $p_m(w)+p_m(x)+p_m(y)+p_m(z)$ with $w,x,y,z\in\mathbb{N}$.

In additive number theory, a subset A of $\mathbb{N}$ is called a *base of order* n if

$$nA=\{a_1+\cdots+a_n: a_1,\ldots,a_n\in A\}$$

coincides with $\mathbb{N}$. When $\mathbb{N}\setminus nA$ is finite, we call A an *asymptotic base of order* n. In this terminology, for each integer $m\geqslant 3$ the set $\{p_m(n): n\in\mathbb{N}\}$ is a base of order m, and $\{p_m(n): n\in\mathbb{N}\}$ is an asymptotic base of order four if and only if $m=4$ or $4\nmid m$.

In 1994 R. K. Guy [2] observed that each $n\in\mathbb{N}$ is a sum of three generalized pentagonal numbers (see also the comments around [9, (1.4)]), that is, the set $\{p_5(x): x\in\mathbb{Z}\}$ is a base of order three. In 2016 the author [10] proved that $\{p_8(x): x\in\mathbb{Z}\}$ is a base of order four; moreover, each $n\in\mathbb{Z}^+=\{1,2,3,\ldots\}$ can be written as a sum of four generalized octagonal numbers one of which is odd. This is quite similar to Lagrange's four-square theorem. For any integer $m\geqslant 5$, J. Li and H. Wang [4] used Jacobi forms to obtain a formula for

$$R_m(n)=|\{(w,x,y,z)\in\mathbb{Z}^4: p_m(w)+p_m(x)+p_m(y)+p_m(z)=n\}|$$

in terms of Hurwitz class numbers. Consequently, for any integer $n\geqslant (m-2)^3/8+m/2+1$, we have $R_m(n)>0$ if m or n is odd.

For any $a\in\mathbb{Z}^+$ and $b\in\mathbb{Z}$ with $b>-a$ and $a\equiv b\pmod 2$, we clearly have

$$\left\{\frac{x(ax+b)}{2}: x\in\mathbb{N}\right\}\subseteq\mathbb{N}.$$

In [11, 12] the author investigated those tuples $(a,b,c,d,e,f)\in\mathbb{Z}^6$ with $a,c,e\in\mathbb{Z}^+$ and $a+b,c+d,e+f\in 2\mathbb{Z}^+$ such that each $n\in\mathbb{N}$ can be written as

$$\frac{x(ax+b)}{2}+\frac{y(cy+d)}{2}+\frac{z(ez+f)}{2}$$

with $x, y, z \in \mathbb{N}$. For example, the author [12] conjectured that each $n \in \mathbb{N}$ can be written as $x(x+1)/2 + y(3y+1)/2 + z(5z+1)/2$ with $x, y, z \in \mathbb{N}$, and proved that

$$\left\{\frac{x(x+1)}{2} + \frac{y(3y+1)}{2} + \frac{z(5z+1)}{2} : x, y, z \in \mathbb{Z}\right\} = \mathbb{N}.$$

Motivated by the above work, we establish some new results analogous to Lagrange's four-square theorem.

THEOREM 1.1. *Let $a \in \mathbb{Z}^+$ and $b \in \mathbb{Z}$ with $b > -a$, $2 \nmid ab$ and $\gcd(a, b) = 1$. Then the set $\{x(ax+b)/2 : x \in \mathbb{N}\}$ is an asymptotic base of order four. Moreover, for any $n \in \mathbb{N}$ with*

$$n > 2a(7a(a-1)+1) + (7a-2)b + (2a(2a-1)+b)\sqrt{6(2a(a-1)+b)}, \tag{1.1}$$

there are $w, x, y, z \in \mathbb{N}$ such that

$$n = \frac{w(aw+b)}{2} + \frac{x(ax+b)}{2} + \frac{y(ay+b)}{2} + \frac{z(az+b)}{2}.$$

Via Theorem 1.1 and some easy numerical computations, we obtain the following corollary involving second pentagonal numbers and second heptagonal numbers.

COROLLARY 1.2. *We have*

$$\{\bar{p}_5(w) + \bar{p}_5(x) + \bar{p}_5(y) + \bar{p}_5(z) : w, x, y, z \in \mathbb{N}\} = \mathbb{N} \setminus \{1, 3, 5, 10, 12, 20, 25, 27, 38, 53, 65, 153, 165\}.$$

Also, every integer $n \geqslant 786$ can be written as $\bar{p}_7(w) + \bar{p}_7(x) + \bar{p}_7(y) + \bar{p}_7(z)$ with $w, x, y, z \in \mathbb{N}$.

REMARK. By Legendre's refinement of Cauchy's polygonal number theorem, any integer $n \geqslant 28 \times (5-2)^3 = 756$ is a sum of four pentagonal numbers, and integer $n \geqslant 28 \times (7-2)^3 = 3500$ is a sum of four heptagonal numbers. Combining this with direct checks for small numbers, we find that

$$\{p_5(w) + p_5(x) + p_5(y) + p_5(z) : w, x, y, z \in \mathbb{N}\} = \mathbb{N} \setminus \{9, 21, 31, 43, 55, 89\}$$

and

$$\{p_7(w) + p_7(x) + p_7(y) + p_7(z) : w, x, y, z \in \mathbb{N}\} \supseteq \{n \in \mathbb{N} : n \geqslant 1004\}.$$

Theorem 1.1 with $(a, b) = (5, \pm 1)$ has the following consequence.

COROLLARY 1.3. *Each integer $n > 1$ can be written as*

$$\frac{w(5w+1)}{2} + \frac{x(5x+1)}{2} + \frac{y(5y+1)}{2} + \frac{z(5z+1)}{2}$$

with $w, x \in \mathbb{N}$ and $y, z \in \mathbb{Z}$. Moreover, every integer $n \geqslant 776$ can be written as

$$\frac{w(5w+1)}{2} + \frac{x(5x+1)}{2} + \frac{y(5y+1)}{2} + \frac{z(5z+1)}{2}$$

with $w,x,y,z\in\mathbb{N}$, and every integer $n\geqslant 884$ can be written as

$$\frac{w(5w-1)}{2}+\frac{x(5x-1)}{2}+\frac{y(5y-1)}{2}+\frac{z(5z-1)}{2}$$

with $w,x,y,z\in\mathbb{N}$.

REMARK. Via a computer we can easily verify that the set

$$\left\{\frac{w(5w+1)}{2}+\frac{x(5x+1)}{2}+\frac{y(5y+1)}{2}+\frac{z(5z+1)}{2}: w,x\in\mathbb{N}\text{ and }y,z\in\mathbb{Z}\right\}$$

contains $2,3,\ldots,775$. So the first assertion in Corollary 1.3 follows from the second assertion.

For convenience, for $a\in\mathbb{Z}^+$ and $b\in\mathbb{Z}$ with $a\equiv b\pmod 2$, we set

$$S_{a,b}=\left\{\frac{w(aw+b)}{2}+\frac{x(ax+b)}{2}+\frac{y(ay+b)}{2}+\frac{z(az+b)}{2}: w,x,y,z\in\mathbb{Z}\right\}.$$

Via Theorem 1.1 and some easy computations via a computer, we have the following corollary.

COROLLARY 1.4. *We have*

$$S_{7,1}=\mathbb{N}\setminus\{1,2,5\},\quad S_{7,3}=\mathbb{N}\setminus\{1,3,25\},\quad S_{7,5}=\mathbb{N}\setminus\{5,23\}.$$

Also,

$$\begin{aligned}S_{9,1}&=\mathbb{N}\setminus\{1,2,3,6,7,11,35,37\},\\ S_{9,5}&=\mathbb{N}\setminus\{1,3,5,10,12,31,67\},\\ S_{9,7}&=\mathbb{N}\setminus\{5,6,7,15,29,65\}.\end{aligned}$$

Now we state our second theorem.

THEOREM 1.5. *Let $a\in\mathbb{Z}^+$ and $b\in\mathbb{Z}$ with $b>-a$, $2\mid a$ and $\gcd(a,b)=1$. Then the set $\{x(ax+b): x\in\mathbb{N}\}$ is an asymptotic base of order four. Moreover, if n is an odd integer greater than*

$$4a(7a(a-1)+1)+2(7a-2)b+2(2a(2a-1)+b)\sqrt{6(2a(a-1)+b)},$$

or n is an even integer with $n/4$ greater than

$$28a^3-14a^2+a+(7a-1)b+(2a(4a-1)+b)\sqrt{3(2a(2a-1)+b)},$$

then we can write n as

$$w(aw+b)+x(ax+b)+y(ay+b)+z(az+b)$$

with $w,x,y,z\in\mathbb{N}$.

For $a\in\mathbb{Z}^+$ and $b\in\mathbb{Z}$, we define

$$T_{a,b}=S_{2a,2b}=\{w(aw+b)+x(ax+b)+y(ay+b)+z(az+b): w,x,y,z\in\mathbb{Z}\}.$$

COROLLARY 1.6. *We have*

$$T_{4,1}=\mathbb{N}\setminus\{1,2,4,7,30\},\quad \max(\mathbb{N}\setminus T_{6,1})=168,\quad \max(\mathbb{N}\setminus T_{6,5})=182.$$

Moreover, every integer $n \geqslant 2283$ *can be written as* $w(4w+1)+x(4x+1)+y(4y+1)+z(4z+1)$ *with* $w,x,y,z\in\mathbb{N}$, *and each integer* $n>4500$ *can be written as* $w(4w-1)+x(4x-1)+y(4y-1)+z(4z-1)$ *with* $w,x,y,z\in\mathbb{N}$.

REMARK. Meng and Sun [5, Corollary 1.1] showed that $T_{4,3}=\{p_{10}(w)+p_{10}(x)+p_{10}(y)+p_{10}(z): w,x,y,z\in\mathbb{Z}\}$ coincides with $\mathbb{N}\setminus\{5,6,26\}$.

Now we present our third theorem.

THEOREM 1.7. *Let* $a\in\mathbb{Z}^+$ *and* $b\in\mathbb{Z}$ *with* $b>-a$, $2\mid b$ *and* $\gcd(a,b)=1$. *Then, for any integer* $n\not\equiv 0 \pmod 4$ *greater than*

$$4a(7a(a-1)+1)+2(7a-2)b+2(2a(2a-1)+b)\sqrt{6(2a(a-1)+b)},$$

there are $w,x,y,z\in\mathbb{N}$ *such that*

$$n=w(aw+b)+x(ax+b)+y(ay+b)+z(az+b).$$

REMARK. Let $a\in\mathbb{Z}^+$ and $b\in\mathbb{Z}$ with $b>-a$, $2\mid b$ and $\gcd(a,b)=1$. As $a+b$ is odd, for each $n\in\mathbb{N}$ either n or $n-(a+b)$ is not divisible by 4. So Theorem 1.7 implies that the set $\{x(ax+b): x\in\mathbb{N}\}$ is an asymptotic base of order five.

COROLLARY 1.8. *We have*

$$\{n\in\mathbb{N}: 4\nmid n \text{ and } n\notin T_{5,2}\}=\{1,2,5,11,15,18\},$$
$$\{n\in\mathbb{N}: 4\nmid n \text{ and } n\notin T_{5,4}\}=\{5,6,7,17\}.$$

REMARK. Sun [10] proved that $T_{3,2}=\mathbb{N}$. It seems that both $\mathbb{N}\setminus T_{5,2}$ and $\mathbb{N}\setminus T_{5,4}$ contain infinitely many multiples of 4.

In 2020 Krachun and Sun [3] proved that

$$\{2p_5(w)+p_5(x)+p_5(y)+p_5(z): w,x,y,z\in\mathbb{N}\}=\mathbb{N},$$

which was previously conjectured by Sun [10]. Motivated by this, we establish the following theorem.

THEOREM 1.9. *Let* $a\in\mathbb{Z}^+$ *and* $b\in\mathbb{Z}$ *with* $b>-a$ *and* $\gcd(a,5b)=1$. *Then, for any integer*

$$n\geqslant 550a(60a^2+b)+25(120a^2+b)\sqrt{2(60a^2+b)} \tag{1.2}$$

with $2\mid n$ *if* $2\nmid ab$, *there are* $w,x,y,z\in\mathbb{N}$ *such that*

$$2w(aw+b)+x(ax+b)+y(ay+b)+z(az+b)=n. \tag{1.3}$$

REMARK. We guess that $\gcd(a,5b)$ in Theorem 1.9 can be replaced by $\gcd(a,b)$, but we have been unable to prove this.

Theorem 1.9 in the case $2\nmid ab$ yields the following result.

COROLLARY 1.10. *Let $a \in \mathbb{Z}^+$ and $b \in \mathbb{Z}$ with $b > -a$, $2 \nmid ab$ and $\gcd(a, 5b) = 1$. Then every integer*

$$\geqslant 275a(60a^2+b)+\frac{25}{2}(120a^2+b)\sqrt{2(60a^2+b)}.$$

can be written as

$$w(aw+b)+\frac{x(ax+b)}{2}+\frac{y(ay+b)}{2}+\frac{z(az+b)}{2} \tag{1.4}$$

with $w, x, y, z \in \mathbb{N}$.

In the case $(a, b) = (3, 1)$, Corollary 1.10 yields the following concrete result.

COROLLARY 1.11. *We have*

$$\{2\bar{p}_5(w)+\bar{p}_5(x)+\bar{p}_5(y)+\bar{p}_5(z) : w, x, y, z \in \mathbb{N}\} = \mathbb{N} \setminus \{1, 3, 5, 12, 27, 50\},$$

Theorem 1.9 with $(a, b) = (2, \pm 1)$ yields the following corollary.

COROLLARY 1.12. *We have*

$$\{2w(2w-1)+x(2x-1)+y(2y-1)+z(2z-1) : w, x, y, z \in \mathbb{N}\} = \mathbb{N} \setminus \{11, 26, 50\}.$$

Also, every integer $n \geqslant 90$ can be written as $2w(2w+1)+x(2x+1)+y(2y+1)+z(2z+1)$ with $w, x, y, z \in \mathbb{N}$.

Meng and Sun [5, Theorem 1.3] proved that for each integer $m \geqslant 5$, every integer $n \geqslant 924(m-2)^3$ can be written as $3p_m(w)+p_m(x)+p_m(y)+p_m(z)$ with $w, x, y, z \in \mathbb{N}$. We extend this as follows.

THEOREM 1.13. *Let $a \in \mathbb{Z}^+$ and $b \in \mathbb{Z}$ with $b > -a$ and $\gcd(a, b) = 1$. Let $n \in \mathbb{N}$ with ab or n even. Suppose that*

$$\frac{n}{6} > 66a^2(9a-1)+a+(33a-1)b+(2a(18a-1)+b)\sqrt{15(2a(9a-1)+b)}. \tag{1.5}$$

Then there are $w, x, y, z \in \mathbb{N}$ such that

$$3w(aw+b)+x(ax+b)+y(ay+b)+z(az+b) = n.$$

Theorem 1.13 in the case $2 \nmid ab$ yields the following result.

COROLLARY 1.14. *Let a and b be odd integers with $a > 0$, $b > -a$ and $\gcd(a, b) = 1$. Then every integer n with*

$$\frac{n}{3} > 66a^2(9a-1)+(33a-1)b+(2a(18a-1)+b)\sqrt{15(2a(9a-1)+b)} \tag{1.6}$$

can be written as

$$3\frac{w(aw+b)}{2}+\frac{x(ax+b)}{2}+\frac{y(ay+b)}{2}+\frac{z(az+b)}{2}$$

with $w, x, y, z \in \mathbb{N}$.

Applying Corollary 1.14 with some concrete odd integers a and b, we obtain the following corollary.

COROLLARY 1.15. (i) *We have*

$$\{3\bar{p}_5(w)+\bar{p}_5(x)+\bar{p}_5(y)+\bar{p}_5(z): w,x,y,z\in\mathbb{N}\}=\mathbb{N}\setminus\{1,3,5,18,31\}.$$

(ii) *Every integer $n>1$ can be written as*

$$3\frac{w(5w+1)}{2}+\frac{x(5x+1)}{2}+\frac{y(5y+1)}{2}+\frac{z(5z+1)}{2}$$

with $w,x,y,z\in\mathbb{Z}$. Moreover, each integer $n>151$ can be written as

$$3\frac{w(5w-1)}{2}+\frac{x(5x-1)}{2}+\frac{y(5y-1)}{2}+\frac{z(5z-1)}{2}$$

with $w,x,y,z\in\mathbb{N}$, and every integer $n>682$ can be written as

$$3\frac{w(5w+1)}{2}+\frac{x(5x+1)}{2}+\frac{y(5y+1)}{2}+\frac{z(5z+1)}{2}$$

with $w,x,y,z\in\mathbb{N}$.

(iii) $1,2,5,14$ *are the only natural numbers not of the form*

$$3\frac{w(7w+1)}{2}+\frac{x(7x+1)}{2}+\frac{y(7y+1)}{2}+\frac{z(7z+1)}{2}\quad(w,x,y,z\in\mathbb{Z}),$$

and $1,3,14$ are the only natural numbers not of the form

$$3\frac{w(7w+3)}{2}+\frac{x(7x+3)}{2}+\frac{y(7y+3)}{2}+\frac{z(7z+3)}{2}\quad(w,x,y,z\in\mathbb{Z}).$$

REMARK. Meng and Sun [5, Corollary 1.3] proved that

$$\{3p_7(w)+p_7(x)+p_7(y)+p_7(z): w,x,y,z\in\mathbb{N}\}$$
$$=\mathbb{N}\setminus\{13,16,27,31,33,49,50,67,87,178,181,259\}$$

and

$$\{3p_9(w)+p_9(x)+p_9(y)+p_9(z): w,x,y,z\in\mathbb{Z}\}=\mathbb{N}\setminus\{17\}.$$

Theorem 1.13 with $(a,b)=(4,\pm1)$ yields the following corollary.

COROLLARY 1.16. *We have*

$$\{3w(4w+1)+x(4x+1)+y(4y+1)+z(4z+1): w,x,y,z\in\mathbb{Z}\}$$
$$=\mathbb{N}\setminus\{1,2,4,7,16\}.$$

Moreover, every integer $n>1204$ can be written as $3w(4w-1)+x(4x-1)+y(4y-1)+z(4z-1)$ with $w,x,y,z\in\mathbb{N}$, and every integer $n>2104$ can be written as $3w(4w+1)+x(4x+1)+y(4y+1)+z(4z+1)$ with $w,x,y,z\in\mathbb{N}$.

REMARK. In contrast with Corollary 1.16, Meng and Sun [5, Corollary 1.3] proved that

$$\{3p_{10}(w)+p_{10}(x)+p_{10}(y)+p_{10}(z): w,x,y,z\in\mathbb{Z}\}=\mathbb{N}\setminus\{16,19\}.$$

We are going to prove Theorems 1.1, 1.5, 1.7, Theorem 1.9, and Theorem 1.13 in Sections 2, 3, and 4, respectively. In Section 5, we give some additional remarks and pose some open conjectures.

2. Proofs of Theorems 1.1, 1.5 and 1.7. The following lemma is essentially [5, Lemma 2.4].

LEMMA 2.1. *Let $c, d \in \mathbb{N}$ with $c \equiv d \pmod 2$ and $4 \nmid c$. If*

$$4c > d^2 \quad \text{and} \quad 3c < d^2 + 2d + 4, \tag{2.1}$$

then there are $w, x, y, z \in \mathbb{N}$ such that

$$w^2 + x^2 + y^2 + z^2 = c \quad \text{and} \quad w + x + y + z = d. \tag{2.2}$$

REMARK. Lemma 2.1 in the case $2 \nmid cd$ is Cauchy's Lemma (see [8, p. 31]) used to prove Cauchy's polygonal number theorem.

LEMMA 2.2. *Suppose that $ac + bd = n > \max\{2a, b\}$ with $a \in \mathbb{Z}^+$, $b, c, d \in \mathbb{Z}$ and $b > -a$. Then*

$$d > 0 \quad \text{and} \quad \frac{d^2}{4} < c < \frac{d^2 + 2d + 4}{3} \tag{2.3}$$

if and only if

$$\frac{\sqrt{3a(n - a + b) + \frac{9}{4}b^2} - a - 3b/2}{a} < d < \frac{2}{a}(\sqrt{an + b^2} - b). \tag{2.4}$$

Proof. Clearly,

$$3a(n - a + b) + \frac{9}{4}b^2 \geqslant 3a^2 + 3ab + \frac{9}{4}b^2 = 2a^2 + \left(a + \frac{3b}{2}\right)^2.$$

Thus $d > 0$ if (2.4) holds.

Now assume $d > 0$. In the case $d = 1$, we have $c > 0$ since $ac + b = n > b$, and (2.4) holds if and only if

$$ac = n - b < \frac{7}{3}a \quad \text{and} \quad (n - b > a/4 \text{ or } a + 2b \leqslant 0),$$

so (2.3) and (2.4) are equivalent. When $d \geqslant 2$, we have $ad + 2b > a(d-2) \geqslant 0$ and

$$\begin{aligned} n > a\frac{d^2}{4} + bd &\iff 4an > (ad + 2b)^2 - 4b^2 \\ &\iff d < \frac{2}{a}\big(\sqrt{an + b^2} - b\big). \end{aligned}$$

As

$$ad + a + \frac{3b}{2} > \frac{3}{2}(a + b) > 0,$$

we have

$$n<\frac{a}{3}(d^2+2d+4)+bd$$

$$\iff 3an<a^2d^2+ad(2a+3b)+4a^2=\left(ad+a+\frac{3b}{2}\right)^2+3a(a-b)-\frac{9}{4}b^2$$

$$\iff \sqrt{3a(n-a+b)+\frac{9}{4}b^2}<ad+a+\frac{3b}{2}$$

$$\iff d>\frac{\sqrt{3a(n-a+b)+9b^2/4}-a-3b/2}{a}.$$

As $n=ac+bd$, by the above, $d^2/4<c<(d^2+2d+4)/3$ if and only if (2.4) holds.

In view of the above, we have completed the proof. ∎

Clearly, Theorems 1.1 and 1.7 follow from the following theorem.

THEOREM 2.3. *Let $a\in\mathbb{Z}^+$ and $b\in\mathbb{Z}$ with $b>-a$ and $\gcd(a,b)=1$. Let $n\in\mathbb{N}$ with*

$$n>4a(7a(a-1)+1)+2(7a-2)b+2(2a(2a-1)+b)\sqrt{6(2a(a-1)+b)}. \tag{2.5}$$

Suppose that $2\nmid ab$ and $2\mid n$, or $2\mid a$ and $2\nmid n$, or $2\mid b$ and $4\nmid n$. Then we can write n as

$$w(aw+b)+x(ax+b)+y(ay+b)+z(az+b)$$

with $w,x,y,z\in\mathbb{N}$.

Proof. Note that

$$2a(2a-1)+b>2a(a-1)+b\geqslant 2a(a-1)-a+1=a(2a-3)+1\geqslant 0.$$

If $a>1$, then by (2.5) we have

$$\frac{n}{2}>2(7a^2-7a+1)+(7a-2)b>2(7a^2-7a+1)-a(7a-2)=a(7a-12)+2>a.$$

When $a=1$, we have $b\geqslant 0$ and hence

$$\frac{n}{2}>2(7a^2-7a+1)+(7a-2)b\geqslant 14a^2-14a+2=2>a=1.$$

Thus we always have $n>\max\{2a,b\}$.

In view of Lemma 2.1, we want to find positive integers c and d with $c\equiv d\pmod 2$ and $4\nmid c$ such that

$$\frac{d^2}{4}<c<\frac{d^2+2d+4}{3}\quad\text{and}\quad n=ac+bd.$$

Motivated by Lemma 2.2, we consider the interval

$$I=\left(\frac{\sqrt{3a(n-a+b)+9b^2/4}-a-3b/2}{a},\ \frac{2}{a}\left(\sqrt{an+b^2}-b\right)\right). \tag{2.6}$$

Clearly the length of I is greater than $2a$ if and only if

$$2\sqrt{an+b^2}-a(2a-1)-\frac{b}{2}>\sqrt{3a(n-a+b)+\frac{9}{4}b^2}. \tag{2.7}$$

As $2a(2a-1)+b>0$, we have

$$2\sqrt{an+b^2}\geqslant a(2a-1)+\frac{b}{2}$$
$$\iff 4(an+b^2)\geqslant a^2(2a-1)^2+(2a-1)ab+\frac{b^2}{4}.$$

If $b\geqslant a(2a-1)$, then

$$4b^2\geqslant a^2(2a-1)^2+a(2a-1)b+2b^2>a^2(2a-1)^2+(2a-1)ab+\frac{b^2}{4}.$$

When $b<a(2a-1)$, if $a>1$ then

$$\begin{aligned}\frac{n}{2}&>2a(7a^2-7a+1)+(7a-2)b\\&>2a(7a^2-7a+1)-a(7a-2)=a(14a^2-21a+4)\\&>a\frac{(2a-1)^2}{4},\end{aligned}$$

hence we always have (whether $a=1$ or not)

$$4an>2a^2(2a-1)^2>a^2(2a-1)^2+a(2a-1)b.$$

Thus the left-hand side of (2.7) is nonnegative. Let $t=\sqrt{an+b^2}$. Then (2.7) holds

$$\iff \left(2t-a(2a-1)-\frac{b}{2}\right)^2>3a(n-a+b)+\frac{9}{4}b^2$$
$$\iff 4t^2-2t(2a(2a-1)+b)+\left(a(2a-1)+\frac{b}{2}\right)^2>3a(n-a+b)+\frac{9}{4}b^2$$
$$\iff (t-(2a(2a-1)+b))^2>6a^2(2a(a-1)+b),$$

and

$$\begin{aligned}&\left(2a(2a-1)+b+\sqrt{6a^2(2a(a-1)+b)}\right)^2\\&\qquad-2(2a(2a-1)+b)a\sqrt{6(2a(a-1)+b)}\\&=(2a(2a-1)+b)^2+6a^2(2a(a-1)+b)\\&=4a^2(7a^2-7a+1)+14a^2b-4ab+b^2\\&<b^2+a(n-2(2a(2a-1)+b)\sqrt{6(2a(a-1)+b)})\end{aligned}$$

by (2.5). Thus

$$t=\sqrt{an+b^2}>2a(2a-1)+b+\sqrt{6a^2(2a(a-1)+b)}$$

and hence (2.7) does hold.

We claim that there are $c, d \in \mathbb{Z}$ with $c \equiv d \pmod 2$ and $4 \nmid c$ such that $d \in I$ and $ac + bd = n$. To prove this, we distinguish two cases.

Case 1: $2 \nmid ab$ *and* $2 \mid n$. In this case, $\gcd(b, 2a) = 1$. As the length of the interval I is greater than $2a$, there is an integer $d \in I$ with $n - bd \equiv a \pmod{2a}$. As $2 \nmid a$ and $2 \mid n$, we must have $2 \nmid d$. Write $n - bd = ac$ with c an odd integer.

Case 2: $2 \mid a$ *and* $2 \nmid n$, *or* $2 \mid b$ *and* $4 \nmid n$. In this case, $a - b$ is odd since $\gcd(a, b) = 1$. Since $\gcd(a - b, 2a) = 1$ and the length of the interval I is greater than $2a$, there is an integer $d \in I$ with $(b-a)d \equiv n \pmod{2a}$. Write $n = (b - a)d + 2ac_0$ with $c_0 \in \mathbb{Z}$. Then $n = ac + bd$ with $c = 2c_0 - d \equiv d \equiv n \pmod 2$. If $2 \nmid n$, then we have $2 \nmid cd$. If $2 \mid b$ and $n \equiv 2 \pmod 4$, then $ac \equiv ac + bd = n \equiv 2 \pmod 4$ and hence $c \equiv 2 \pmod 4$.

Since $d \in I$ in either case, we have (2.3) by Lemma 2.2. In light of Lemma 2.1, there are $w, x, y, z \in \mathbb{N}$ satisfying (2.2). Therefore

$$\begin{aligned} n = ac + bd &= a(w^2 + x^2 + y^2 + z^2) + b(w + x + y + z) \\ &= w(aw + b) + x(ax + b) + y(ay + b) + z(az + b). \end{aligned}$$

This concludes our proof. ∎

Theorem 2.4. *Let $a \in \mathbb{Z}^+$ and $b \in \mathbb{Z}$ with $2 \mid a$, $\gcd(a, b) = 1$ and $b > -a$. Let $n \in \mathbb{N}$ with $2 \mid n$ and*

$$\frac{n}{4} > 28a^3 - 14a^2 + a + (7a - 1)b + (2a(4a - 1) + b)\sqrt{3(2a(2a - 1) + b)}. \tag{2.8}$$

Then there are $w, x, y, z \in \mathbb{N}$ such that

$$n = w(aw + b) + x(ax + b) + y(ay + b) + z(az + b).$$

Proof. As $b > -a$, we have

$$\frac{n}{4} > 28a^3 - 14a^2 + a - a(7a - 1) = 8a^3 + a(20a^2 - 21a + 2) > 8a^3.$$

Note also that $n > b$ by (2.8). For the interval I given by (2.6), its length is greater than $4a$ if and only if

$$2\sqrt{an + b^2} - a(4a - 1) - \frac{b}{2} > \sqrt{3a(n - a + b) + \frac{9}{4}b^2}. \tag{2.9}$$

As $2a(4a - 1) + b > 2a(4a - 1) - a > 0$, we have

$$\begin{aligned} 2\sqrt{an + b^2} \geqslant a(4a - 1) + \frac{b}{2} \\ \iff 4(an + b^2) \geqslant a^2(4a - 1)^2 + (4a - 1)ab + \frac{b^2}{4}. \end{aligned}$$

If $b \geqslant a(4a - 1)$, then

$$4b^2 \geqslant a^2(4a - 1)^2 + a(4a - 1)b + 2b^2 > a^2(4a - 1)^2 + (4a - 1)ab + \frac{b^2}{4}.$$

When $b<a(4a-1)$, we have

$$4an>4a\times 8a^3>2a^2(4a-1)^2>a^2(4a-1)^2+a(4a-1)b.$$

Thus the left-hand side of (2.9) is nonnegative. Let $t=\sqrt{an+b^2}$. Then (2.9) holds

$$\begin{aligned}&\iff \left(2t-a(4a-1)-\frac{b}{2}\right)^2>3a(n-a+b)+\frac{9}{4}b^2\\&\iff 4t^2-2t(2a(4a-1)+b)+\left(a(4a-1)+\frac{b}{2}\right)^2>3a(n-a+b)+\frac{9}{4}b^2\\&\iff (t-(2a(4a-1)+b))^2>12a^2(2a(2a-1)+b),\end{aligned}$$

and

$$\begin{aligned}&\big(2a(4a-1)+b+\sqrt{12a^2(2a(2a-1)+b)}\,\big)^2\\&\qquad -4a(2a(4a-1)+b)\sqrt{3(2a(2a-1)+b)}\\&=(2a(4a-1)+b)^2+12a^2(2a(2a-1)+b)\\&=b^2+28a^2(4a^2-2a+b)+4a^2-4ab\\&<b^2+an-4a(2a(4a-1)+b)\sqrt{3(2a(2a-1)+b)}\end{aligned}$$

by (2.8). Thus

$$t=\sqrt{an+b^2}>2a(4a-1)+b+\sqrt{12a^2(2a(2a-1)+b)}$$

and hence (2.9) holds. So the length of the interval I is greater than $4a$.

As $\gcd(b,4a)=1$, there is an integer $d\in I$ with $n-bd\equiv 2a\pmod{4a}$. Write $n-bd=ac$ with $c\equiv 2\pmod 4$. As both a and n are even but b is odd, we must have $2\mid d$. As $d\in I$, we have (2.3) by Lemma 2.2. Now, by Lemma 2.1 there are $w,x,y,z\in\mathbb{N}$ satisfying (2.2). Hence

$$n=ac+bd=w(aw+b)+x(ax+b)+y(ay+b)+z(az+b).$$

This concludes our proof. ∎

Proof of Theorem 1.5. As $2\mid a$ and $2\nmid b$, combining Theorems 2.3 and 2.4 we immediately obtain the desired result. ∎

3. Proof of Theorem 1.9. We need the following lemma, which is [3, Lemma 2.1].

LEMMA 3.1. *Every even $n\in\mathbb{N}$ not in the set*

$$E=\{5^{2k+1}m: k,m\in\mathbb{N}\ \textit{and}\ m\equiv\pm 2\pmod 5\}\tag{3.1}$$

can be written as $x^2+y^2+z^2+(x+y+z)^2/2$ with $x,y,z\in\mathbb{Z}$.

Our proof of Theorem 1.9 extends the method in [3].

Proof of Theorem 1.9. Note that $\gcd(a+b,a)=\gcd(b,a)=1$ and $\gcd(5(a+b),2a)=\gcd(a+b,2)$ divides n. So the congruence $5(a+b)x\equiv n \pmod{2a}$ is solvable (over $\mathbb{Z}$). When $n=5q$ with $q\in\mathbb{Z}$, we choose $\delta\in\{1,-1\}$ such that $4aq+b^2+\delta\not\equiv\pm2 \pmod 5$, and hence the congruence $(2ax+b)^2\equiv 4aq+b^2+\delta \pmod 5$ is solvable. Thus, by the Chinese Remainder Theorem, among any $10a$ consecutive integers, there is an integer B such that $5(a+b)B\equiv n \pmod{2a}$, and

$$a(q-bB-aB^2)\equiv 4a(aB^2+bB-q)=(2aB+b)^2-(4aq+b^2)\equiv\delta \pmod 5$$

if $n=5q$ with $q\in\mathbb{N}$. Note that $(n-5bB)/a-5B^2$ is an even integer. If $n=5q$ with $q\in\mathbb{N}$, then

$$\frac{1}{5}\left(\frac{n-5bB}{a}-5B^2\right)=\frac{a(q-bB-aB^2)}{a^2}\equiv\frac{\delta}{a^2}\not\equiv 0,\pm2 \pmod 5.$$

So $(n-5bB)/a-5B^2$ is an even number not in the set E given by (3.1).

Suppose $B\in\mathbb{Z}^+$. Then $2aB+b\geqslant a+b>0$, $12aB+5b\geqslant 5(a+b)>0$, and

$$\begin{aligned}
0\leqslant\frac{n-5bB}{a}-5B^2\leqslant B^2
&\iff 5bB+5aB^2\leqslant n\leqslant 5bB+6aB^2\\
&\iff 4an\geqslant 5(2aB+b)^2-5b^2 \text{ and } 24an+25b^2\leqslant(12aB+5b)^2\\
&\iff \sqrt{\frac{4}{5}an+b^2}\geqslant 2aB+b \text{ and } \sqrt{24an+25b^2}\leqslant 12aB+5b\\
&\iff \frac{\sqrt{24an+25b^2}-5b}{12a}\leqslant B\leqslant\frac{\sqrt{4an/5+b^2}-b}{2a}.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\frac{\sqrt{4an/5+b^2}-b}{2a}-\frac{\sqrt{24an+25b^2}-5b}{12a}\geqslant 10a\\
&\iff 6\sqrt{\frac{4}{5}an+b^2}\geqslant 120a^2+b+\sqrt{24an+25b^2}\\
&\iff \frac{144}{5}an+36b^2\geqslant(120a^2+b)^2+24an+25b^2\\
&\qquad+2(120a^2+b)\sqrt{24an+25b^2}\\
&\iff \left(\sqrt{24an+25b^2}-5(120a^2+b)\right)^2\\
&\qquad\geqslant 25(2880a^4+48a^2b-b^2+(120a^2+b)^2)=2(60a^2+b)(60a)^2
\end{aligned}$$

and

$$\begin{aligned}\sqrt{24an+25b^2}&\geqslant 5(120a^2+b)+60a\sqrt{2(60a^2+b)}\\&\iff 24an+25b^2\geqslant 25(120a^2+b+12a\sqrt{2(60a^2+b)})^2\\&\iff \frac{n}{25}\geqslant 22a(60a^2+b)+(120a^2+b)\sqrt{2(60a^2+b)}.\end{aligned}$$

As (1.2) holds, by the above the length of the interval

$$J=\left[\frac{\sqrt{24an+25b^2}-5b}{12a},\ \frac{\sqrt{4an/5+b^2}-b}{2a}\right]$$

is at least $10a$. So, in view of the first paragraph of this proof, there is a positive integer $B\in J$ such that $(n-5bB)/a-5B^2$ is an even integer not in the set E given by (3.1). Since $B\in J$, we also have

$$0\leqslant \frac{n-5bB}{a}-5B^2\leqslant B^2.$$

By Lemma 3.1, there are $x,y,z\in\mathbb{Z}$ such that

$$\frac{n-5bB}{a}-5B^2=x^2+y^2+z^2+\frac{(x+y+z)^2}{2}=x^2+y^2+z^2+2w^2,$$

where $w=-(x+y+z)/2\in\mathbb{Z}$. As

$$x^2+y^2+z^2+2w^2=\frac{n-5bB}{a}-5B^2\leqslant B^2,$$

all the four integers

$$x_0=x+B,\quad y_0=y+B,\quad z_0=z+B,\quad w_0=w+B$$

are nonnegative. Note that

$$\begin{aligned}&2w_0(aw_0+b)+x_0(ax_0+b)+y_0(ay_0+b)+z_0(az_0+b)\\&\quad=a(2(w+B)^2+(x+B)^2+(y+B)^2+(z+B)^2)\\&\qquad+b(2(w+B)+(x+B)+(y+B)+(z+B))\\&\quad=a(2w^2+x^2+y^2+z^2)+(2aB+b)(2w+x+y+z)+5aB^2+5bB\\&\quad=(n-5bB-5aB^2)+(2aB+b)\times 0+5aB^2+5bB=n.\end{aligned}$$

This concludes our proof. ∎

4. Proof of Theorem 1.13. We need the following lemma, which is essentially [5, Lemma 4.1].

LEMMA 4.1. *Let $c,d\in\mathbb{Z}^+$ with $c\equiv d\pmod 2$, and $c\equiv 3\pmod 9$ or $3\nmid d$. If $6c>d^2$ and $5c<d^2+2d+6$, then there are $w,x,y,z\in\mathbb{N}$ such that*

$$3w^2+x^2+y^2+z^2=c\quad\text{and}\quad 3w+x+y+z=d. \tag{4.1}$$

Now we give another lemma.

LEMMA 4.2. *Let $a \in \mathbb{Z}^+$ and $b, c, d \in \mathbb{Z}$ with $b > -a$ and $ac + bd = n \geqslant 14a/5 + 2b$. Then*

$$d \geqslant 3 \quad \text{and} \quad \frac{d^2}{6} < c < \frac{d^2 + 2d + 6}{5}$$

if and only if

$$\frac{\sqrt{5a(n-a+b) + 25b^2/4} - a - 5b/2}{a} < d < \frac{\sqrt{6an + 9b^2} - 3b}{a}.$$

Proof. Observe that

$$3a + \frac{5}{2}b > 3a - \frac{5}{2}a = \frac{a}{2} > 0$$

and hence

$$\begin{aligned} &\frac{\sqrt{5a(n-a+b) + 25b^2/4} - a - 5b/2}{a} \geqslant 2 \\ &\qquad \iff 5a(n-a+b) + \frac{25}{4}b^2 \geqslant \left(3a + \frac{5}{2}b\right)^2 \\ &\qquad \iff n \geqslant \frac{14}{5}a + 2b. \end{aligned}$$

Since $n \geqslant 14a/5 + 2b$, by the above, if

$$d > \frac{\sqrt{5a(n-a+b) + 25b^2/4} - a - 5b/2}{a},$$

then $d > 2$ and hence $d \geqslant 3$.

Below we assume $d \geqslant 3$. Then

$$ad + 3b > a(d-3) \geqslant 0 \quad \text{and} \quad ad + a + \frac{5}{2}b > \frac{5}{2}(a+b) > 0.$$

Observe that

$$\begin{aligned} &\frac{d^2}{6} < c < \frac{d^2 + 2d + 6}{5} \\ &\iff \frac{a}{6}d^2 < ac = n - bd < \frac{a}{5}(d^2 + 2d + 6) \\ &\iff \sqrt{6an + 9b^2} > ad + 3b \text{ and } \sqrt{5a(n-a+b) + \frac{25}{4}b^2} < ad + a + \frac{5}{2}b. \end{aligned}$$

So the desired result follows. ∎

Proof of Theorem 1.13. By (1.5), we have

$$\begin{aligned} \frac{n}{6} &> 66a^2(9a-1) + (33a-1)b = 324a^3 + (270a^3 - 66a^2) + (33a-1)b \\ &> a(18a-1)^2 + (18a-1)b + (270a^3 - 66a^2 + 15ab) \\ &> a(18a-1)^2 + (18a-1)b + 66a^2(a-1) + 15a(a+b) \\ &> a(18a-1)^2 + (18a-1)b \end{aligned}$$

and hence

$$\frac{an}{6}+\frac{b^2}{4}>\left(a(18a-1)+\frac{b}{2}\right)^2.$$

Let L be the length of the interval

$$K=\left(\frac{\sqrt{5a(n-a+b)+25b^2/4}-a-5b/2}{a},\ \frac{\sqrt{6an+9b^2}-3b}{a}\right).$$

It is easy to verify that

$$\begin{aligned}&L>18a\\ &\iff \sqrt{6an+9b^2}-\left(a(18a-1)+\frac{b}{2}\right)>\sqrt{5a(n-a+b)+\frac{25}{4}b^2}\\ &\iff \left(\sqrt{6an+9b^2}-\left(a(18a-1)+\frac{b}{2}\right)\right)^2>5a(n-a+b)+\frac{25}{4}b^2\\ &\iff \left(\sqrt{6an+9b^2}-3(2a(18a-1)+b)\right)^2>36a^2\times 15(18a^2-2a+b).\end{aligned}$$

By (1.5), we have

$$6an+9b^2>\left(3(2a(18a-1)+b)+6a\sqrt{15(2a(9a-1)+b)}\right)^2$$

and hence

$$\left(\sqrt{6an+9b^2}-3(2a(18a-1)+b)\right)^2>36a^2\times 15(18a^2-2a+b).$$

So $L>18a$ by the above.

Now we claim that there are $c,d\in\mathbb{Z}$ with $c\equiv d\pmod 2$ such that $d\in K$, $ac+bd=n$, and $c\equiv 3\pmod 9$ if $3\mid d$. To prove this, we distinguish two cases.

Case 1: $3\nmid a$ *or* $3\nmid n$. As $\gcd(a-b,2a)=(a-b,2)$ divides n, we may choose the least integer $d_0\in K$ such that $(b-a)d_0\equiv n\pmod{2a}$. If $3\nmid a$, then we choose $d\in\{d_0,d_0+2a\}$ such that $3\nmid d$. If $3\mid a$ and $3\nmid n$, then $d=d_0\not\equiv 0\pmod 3$. As $L>4a$, we have $d\in K$. Write $n=(b-a)d+2ac_0$ with $c_0\in\mathbb{Z}$. Then $n=ac+bd$ with $c=2c_0-d\equiv d\pmod 2$.

Case 2: $3\mid a$ *and* $3\mid n$. As $3\mid a$, we have $3\nmid b$ in this case. Note that $\gcd(a-b,6a)=(a-b,2)$ divides n. As $L>18a$, we may choose the least integer $d_0\in K$ such that $(b-a)d_0\equiv n\pmod{6a}$. Since $3\mid n$ and $3\nmid b-a$, we must have $3\mid d_0$. Note that $(n-bd_0)/a=-d_0+6q$ for some $q\in\mathbb{Z}$. Choose $r\in\{0,1,2\}$ such that $br\equiv q+1+d_0/3\pmod 3$. Then $d=d_0+6ar\in K$ since $L>18a$. Note that

$$c:=\frac{n-bd}{a}=\frac{n-b(d_0+6ar)}{a}=-d_0+6q-6br\equiv -d_0-3q+3br\equiv 3\pmod 9$$

and also $c\equiv -d_0\equiv d\pmod 2$.

By the proved claim, we can write n as $ac+bd$ with $c,d\in\mathbb{Z}$ and $d\in K$ such that $c\equiv d\pmod 2$, and $c\equiv 3\pmod 9$ if $3\mid d$. Note that

$$\frac{n}{6} > 66a^2(9a-1)+a+(33a-1)(-a) = 495a^3+99a^2(a-1)+2a > 3a.$$

Whether $b<0$ or $b\geqslant 0$, we always have $n>3a+2b$. As $d\in K$, by Lemma 4.2 we have $d\geqslant 3$, $6c>d^2$ and $5c<d^2+2d+6$. Applying Lemma 4.1, we see that there are $w,x,y,z\in\mathbb{N}$ satisfying (4.1). Therefore

$$\begin{aligned} n = ac+bd &= a(3w^2+x^2+y^2+z^2)+b(3w+x+y+z)\\ &= 3w(aw+b)+x(ax+b)+y(ay+b)+z(az+b).\end{aligned}$$

This concludes our proof of Theorem 1.13. ∎

5. Some remarks and conjectures. It is easy to see that

$$\left\{\frac{n(n+1)}{2}: n\in\mathbb{N}\right\} = \{x(2x+1): x\in\mathbb{Z}\}.$$

For integers $a>b\geqslant 0$ and $a\equiv b\pmod 2$, clearly

$$\left\{\frac{x(ax+b)}{2}: x\in\mathbb{Z}\right\}\subseteq\mathbb{N}.$$

Sun [11, 12] investigated those tuples $(a,b,c,d,e,f)\in\mathbb{Z}^6$ with $a>b\geqslant 0$, $c>d\geqslant 0$, $e>f\geqslant 0$ and $a-b,c-d,e-f\in 2\mathbb{Z}^+$ such that each $n\in\mathbb{N}$ can be written as

$$\frac{x(ax+b)}{2}+\frac{y(cy+d)}{2}+\frac{z(ez+f)}{2}$$

with $x,y,z\in\mathbb{Z}$. Some further results along this direction were proved in [1, 15].

The following lemma can be easily obtained based on the proof of Lemma 2.1 given in [5].

LEMMA 5.1. *Let $c,d\in\mathbb{N}$ with $4\nmid c$, $c\equiv d\pmod 2$ and $4c>d^2$. Then there are $s,t,u,v\in\mathbb{Z}$ such that*

$$s^2+t^2+u^2+v^2=c \quad \text{and} \quad s+t+u+v=d. \tag{5.1}$$

Using this lemma and some arguments similar to but easier than those in Section 2, we can deduce the following two theorems.

THEOREM 5.2. *Let a and b be positive odd integers with $a>b$ and $\gcd(a,b)=1$. Then every integer $n>a^3/2+ab$ can be written as*

$$\frac{w(aw+b)}{2}+\frac{x(ax+b)}{2}+\frac{y(ay+b)}{2}+\frac{z(az+b)}{2}$$

with $w,x,y,z\in\mathbb{Z}$.

THEOREM 5.3. *Let $a, b \in \mathbb{Z}^+$ with $a > b$, $a \not\equiv b \pmod 2$ and $\gcd(a,b) = 1$.*

(i) *If $2 \mid a$, $n \in \mathbb{N}$ and $n \geqslant 2^{1+(-1)^n}a^3 + (3+(-1)^n)ab$, then there are $w, x, y, z \in \mathbb{Z}$ such that*

$$n = w(aw+b) + x(ax+b) + y(ay+b) + z(az+b).$$

(ii) *If $2 \mid b$, then every integer $n \geqslant a^3 + 2ab$ with $4 \nmid n$ can be written as*

$$w(aw+b) + x(ax+b) + y(ay+b) + z(az+b)$$

with $w, x, y, z \in \mathbb{Z}$.

The following conjecture can be viewed as a supplement to Theorem 1.7 and Theorem 5.3(ii).

CONJECTURE 5.4. *For integers $a > b > 0$ with $2 \mid b$ and $\gcd(a,b) = 1$, if $(a,b) \neq (3,2)$ then there are infinitely many positive multiples of four not representable as $w(aw+b)+x(ax+b)+y(ay+b)+z(az+b)$ with $w, x, y, z \in \mathbb{Z}$.*

In contrast with Corollary 1.2 and the remark following it, we make the following conjecture.

CONJECTURE 5.5. *We have*

$$\{p_5(x)+p_5(y)+p_5(z) : x, y, z \in \mathbb{N}\} \supseteq \{n \in \mathbb{N} : n > 33066\},$$
$$\{\bar{p}_5(x)+\bar{p}_5(y)+\bar{p}_5(z) : x, y, z \in \mathbb{N}\} \supseteq \{n \in \mathbb{N} : n > 24036\},$$
$$\{p_6(x)+p_6(y)+p_6(z) : x, y, z \in \mathbb{N}\} \supseteq \{n \in \mathbb{N} : n > 146858\},$$
$$\{\bar{p}_6(x)+\bar{p}_6(y)+\bar{p}_6(z) : x, y, z \in \mathbb{N}\} \supseteq \{n \in \mathbb{N} : n > 138158\}.$$

Motivated by Corollary 1.3, we pose the following four conjectures.

CONJECTURE 5.6.

(i) *Every integer $n > 1$ can be written as $x(5x+1)+y(5y+1)/2+z(5z+1)/2$ with $x, y, z \in \mathbb{Z}$.*

(ii) *If $n \in \mathbb{N}$ can be written uniquely as $x(5x+1)+y(5y+1)/2+z(5z+1)/2$ with $x, y, z \in \mathbb{Z}$ and $y \leqslant z$, then it belongs to the set*

$$\{0,\ 2,\ 3,\ 5,\ 7,\ 14,\ 16,\ 19,\ 37,\ 43,\ 58,\ 61,\ 79\}.$$

REMARK. For the number of representations of $n \in \mathbb{N}$ as $x(5x+1) + y(5y+1)/2 + z(5z+1)/2$ with $x, y, z \in \mathbb{Z}$ and $y \leqslant z$, the reader may consult [13].

CONJECTURE 5.7. *We have*

$$\left\{\frac{x(5x+1)}{2} + \frac{y(5y+1)}{2} + \frac{z(5z+1)}{2} : x, y, z \in \mathbb{Z}\right\} = \mathbb{N} \setminus \{1, 10, 19, 94\},$$
$$\left\{x(5x+1) + \frac{y(5y+1)}{2} + \frac{3z(5z+1)}{2} : x, y, z \in \mathbb{Z}\right\} = \mathbb{N} \setminus \{1, 5, 32\},$$

$$\left\{x(5x+1)+2y(5y+1)+\frac{z(5z+1)}{2} : x,y,z\in\mathbb{Z}\right\}=\mathbb{N}\setminus\{1,5,70\},$$
$$\left\{x(5x+1)+\frac{y(5y+1)}{2}+\frac{5z(5z+1)}{2} : x,y,z\in\mathbb{Z}\right\}=\mathbb{N}\setminus\{1,5\}.$$

CONJECTURE 5.8. *Every integer $n>1$ with $n\neq 19$ can be written as*

$$\frac{w(5w-1)}{2}+\frac{x(5x-1)}{2}+\frac{y(5y+1)}{2}+\frac{z(5z+1)}{2}$$

with $w,x,y,z\in\mathbb{N}$.

REMARK. For more general conjectures of this type, the reader may consult [14].

CONJECTURE 5.9. *Let*

$$N(1)=114862,\quad N(-1)=166897,\quad N(3)=196987,\quad N(-3)=273118.$$

Then, for each $r\in\{\pm1,\pm3\}$, we can write every integer $n>N(r)$ as $x(5x+r)/2+y(5y+r)/2+z(5z+r)/2$ with $x,y,z\in\mathbb{N}$.

CONJECTURE 5.10. *We have*

$$\{p_7(x)+2p_7(y)+3p_7(z) : x,y,z\in\mathbb{Z}\}=\mathbb{N}\setminus\{31,\,77\},$$
$$\{p_7(x)+2p_7(y)+5p_7(z) : x,y,z\in\mathbb{Z}\}=\mathbb{N}\setminus\{10,\,16\}.$$

REMARK. In 2016, Sun [10, Conjecture 5.4(i)] conjectured that

$$\{p_7(x)+p_7(y)+2p_7(z) : x,y,z\in\mathbb{Z}\}=\mathbb{N}\setminus\{23\}.$$

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