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EXPLICIT CONSTRUCTIONS OF ASTHENO-KÄHLER MANIFOLDS

BY

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Abstract. We investigate the conditions under which astheno-Kähler structures can be identified on the product of two compact trans-Sasakian manifolds of dimensions greater than 2.

1. Introduction. Over the past three decades, astheno-Kähler manifolds have garnered considerable attention from geometers globally, demonstrating their potential significance in both differential and algebraic frameworks. The phrase astheno-Kähler manifold, derived from the Greek word for weak, was introduced by Jost and Yau in their influential papers [JY93, JY94]. A Hermitian metric g on a complex manifold (M, J) of complex dimension m is called *astheno-Kähler* if its associated fundamental 2-form Ω satisfies

$$\partial\bar{\partial}\Omega^{m-2} = 0,$$

where ∂ and $\bar{\partial}$ denote the Dolbeault operators and

$$\Omega^{m-2} = \underbrace{\Omega \wedge \cdots \wedge \Omega}_{m-2 \text{ times}}$$

is the $(m-2)$-fold wedge product of Ω with itself. A Hermitian manifold with an astheno-Kähler metric is known as an *astheno-Kähler manifold*.

For $m = 2$, every Hermitian metric is astheno-Kähler. For $m = 3$, the notion of an astheno-Kähler metric coincides with that of an SKT (strong Kähler with torsion) metric, also known as a pluriclosed metric. Astheno-Kähler metrics have attracted sustained interest since their introduction by Jost and Yau [JY93], due to their relevance in both differential and complex geometry. Subsequent important results were obtained by Matsuo and Takahashi [MT01] on compact balanced astheno-Kähler manifolds, as well as on the construction of such metrics on Calabi–Eckmann manifolds.

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Matsuo and Takahashi [MT01] demonstrated a pivotal finding, establishing that every compact balanced astheno-Kähler manifold is indeed Kähler. Notably, this implies that compact Hermitian-flat astheno-Kähler manifolds are also Kähler. Their research further revealed several important results:

- Any product manifold composed of curves and surfaces qualifies as astheno-Kähler.
- For two compact Sasakian manifolds (M_i, g_i) in three dimensions, the product manifold $M = M_1 \times M_2$ equipped with the specified Hermitian structure and Kähler form is astheno-Kähler.
- If (M_1, g_1) is a three-dimensional compact Sasakian manifold and (M_2, g_2) is a compact cosymplectic manifold of dimension at least 3, then the product manifold $M = M_1 \times M_2$ with the given Hermitian structure is also astheno-Kähler.

In a separate study, Matsuo [M09] identified the existence of astheno-Kähler metrics on Calabi–Eckmann manifolds, which are principal T^2-bundles over $\mathbb{C}P^n \times \mathbb{C}P^m$, thereby facilitating the construction of astheno-Kähler structures on torus bundles over Kähler manifolds.

Building on these findings, this work aims to comprehensively explore the conditions under which astheno-Kähler structures can be identified on various manifolds.

2. Preliminaries. An almost complex manifold (M, J) [B06] together with a compatible Riemannian metric g, that is, $g(JX, JY) = g(X, Y)$ for all vector fields X, Y on M, is called an *almost Hermitian manifold* and the metric g is called an *almost Hermitian metric*. A complex manifold M [B06] together with a compatible Riemannian metric g is called a *Hermitian manifold* and the metric g is called a *Hermitian metric*. The alternating 2-form

$$\Omega(X, Y) = g(JX, Y)$$

is called the *Kähler form*. We can retrieve g from Ω:

$$g(X, Y) = \Omega(X, JY).$$

If Ω is closed, then (M, g) is a *Kähler manifold* [B06] and g is a *Kähler metric*.

There are certain complex manifolds that do not support Kähler metrics, leading to the natural emergence of the question regarding a suitable generalisation. While a universal type of Hermitian non-Kähler metric has yet to be discovered, various classes associated with different geometric or physical applications have been proposed and examined.

In 1993, Jost and Yau first coined the term astheno-Kähler manifold in the remarkable papers [JY93, JY94].

DEFINITION 2.1 ([JY93, JY94]). A Hermitian manifold (M, J, g) of complex dimension m is called an *astheno-Kähler manifold* if it carries a fundamental 2-form (Kähler form) Ω satisfying

$$\partial\bar{\partial}\Omega^{m-2} = 0,$$

where ∂ and $\bar{\partial}$ are complex exterior differentials and $\Omega^{m-2} = \Omega \wedge \cdots \wedge \Omega$ ($m-2$ times); the metric g is then called an *astheno-Kähler metric*.

The above condition is automatically satisfied for $m = 2$. Thus, any Hermitian metric on a complex surface is astheno-Kähler.

DEFINITION 2.2 ([ST10]). Let (M, J, g) be a Hermitian manifold of complex dimension m. The Hermitian structure (J, g), or the metric g, is called *strong Kähler with torsion* (strong KT or SKT) if $\partial\bar{\partial}\Omega = 0$. An SKT metric is also called a *pluriclosed metric*.

DEFINITION 2.3 ([G84]). Let (M, J, g) be a Hermitian manifold of complex dimension m. The Hermitian structure (J, g) is called *standard* or *Gauduchon* and g is called a *Gauduchon metric* if $\partial\bar{\partial}\Omega^{m-1} = 0$, or equivalently if the Lee form of Ω^{m-1} is co-closed.

REMARK 2.4. Fino et al. [FT11] mentioned that for $m = 4$, a Hermitian metric that is simultaneously strong KT and astheno-Kähler metric, must also be Gauduchon. For $m = 3$, astheno-Kähler structure means a strong KT metric.

An odd-dimensional Riemannian manifold $\left(M^{2n+1}, g\right)$ is said to be an *almost contact metric manifold* if there exists on M a $(1,1)$ tensor field ϕ, a vector field ξ (called the *structure vector field*) and a 1-form η such that

$$\eta(\xi) = 1, \quad \phi^2(X) = -X + \eta(X)\xi, \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields X, Y on M.

In particular, in an almost contact metric manifold, we also have $\phi\xi = 0$ and $\eta \circ \phi = 0$. Such a manifold is said to be a *contact metric manifold* if $d\eta = \Phi$, where $\Phi(X, Y) = g(X, \phi Y)$ is called the *fundamental 2-form* of M. Such an almost contact metric structure is called a *contact metric structure*. Moreover, if a contact metric structure is normal, then it is called a *Sasakian structure*. On the other hand, if $d\Phi = 0$ and $d\eta = 0$, then M is said to have an *almost cosymplectic structure*. In addition, if an almost cosymplectic structure is normal, then it is called a *cosymplectic structure*. If N is a compact Kähler manifold, then $N \times S^1$ is the trivial example of a compact cosymplectic manifold. If a contact metric structure (ϕ, ξ, η) is normal and $d\eta = 0$, $d\Phi = 2\Phi \wedge \eta$, then M is said to be a *Kenmotsu manifold*.

These manifolds can be characterised through their Levi-Civita connection by requiring

$$\begin{cases} \text{Sasakian: } (\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \\ \text{Cosymplectic: } (\nabla_X \phi)Y = 0, \\ \text{Kenmotsu: } (\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X. \end{cases}$$

DEFINITION 2.5 ([JV81]). An *almost α-Sasakian manifold* M is an almost co-Hermitian manifold (or almost contact metric manifold) such that $\Phi = (1/\alpha)\, d\eta$, $\alpha \in \mathbb{R}_0 = \mathbb{R}\backslash\{0\}$; M is an *α-Sasakian manifold* if the structure is normal.

THEOREM 2.6 ([JV81]).

(i) *An almost co-Hermitian (or almost contact metric) manifold M is α-Sasakian if and only if*

$$(\nabla_X \phi)\, Y = \alpha\{g(X,Y)\xi - \eta(Y)X\}$$

for all $X, Y \in \mathfrak{X}(M)$.

(ii) *If M is α-Sasakian, then ξ is a Killing vector field and*

$$\nabla_X \xi = -\alpha\phi X$$

for all $X \in \mathfrak{X}(M)$.

DEFINITION 2.7 ([JV81]). An *almost β-Kenmotsu manifold* is an almost co-Hermitian (or almost contact metric) manifold such that

$$d\eta = 0, \quad (d\Phi)(X,Y,Z) = \frac{2}{3}\beta \mathop{\mathfrak{S}}_{X,Y,Z}\{\eta(X)\Phi(Y,Z)\}$$

for $\beta \in \mathbb{R}_0$ and all $X, Y, Z \in \mathfrak{X}(M)$, where $\mathfrak{S}$ denotes the cyclic sum; and M is a *β-Kenmotsu manifold* if the structure is normal.

THEOREM 2.8 ([JV81]).

(i) *An almost co-Hermitian (or almost contact metric) manifold is a β-Kenmotsu manifold if and only if*

$$(\nabla_X \phi)\, Y = \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}$$

for all $X, Y \in \mathfrak{X}(M)$.

DEFINITION 2.9 ([O85]). A structure (ϕ, ξ, η, g) is a *trans-Sasakian structure* if it is normal and

$$d\eta = \alpha\Phi, \quad d\Phi = 2\beta\eta \wedge \Phi,$$

where $\alpha = \frac{1}{2n}\delta\phi(\xi)$, $\beta = \frac{1}{2n}$ div ξ and δ is the codifferential of g; the manifold is then said to be *trans-Sasakian of type* (α, β).

A trans-Sasakian structure of type (α, β) may be expressed as an almost contact metric structure satisfying

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X).$$

From this formula, one can easily obtain

$$\nabla_X \xi = -\alpha\phi X - \beta\phi^2 X,$$
$$(\nabla_X \eta)\, Y = \alpha g(X, \phi Y) + \beta g(\phi X, \phi Y).$$

It is clear that a trans-Sasakian manifold of type $(1,0)$ is a Sasakian manifold, a trans-Sasakian manifold of type $(0,1)$ is a Kenmotsu manifold, and a trans-Sasakian manifold of type $(0,0)$ is a cosymplectic manifold. An (α, β) trans-Sasakian manifold is α-Sasakian when $\alpha \in \mathbb{R}_0$ and $\beta = 0$ and is β-Kenmotsu when $\beta \in \mathbb{R}_0$ and $\alpha = 0$.

Now, consider the product manifold $M = M_1 \times M_2$, where M_i $(i = 1, 2)$ is a $(2m_i + 1)$-dimensional compact normal almost contact metric manifold with the structure tensor fields (ϕ_i, ξ_i, η_i). We consider an almost complex structure J (Morimoto's complex structure) [M63] defined by

$$J = \phi_1 - \eta_2 \otimes \xi_1 + \phi_2 + \eta_1 \otimes \xi_2. \tag{2.1}$$

Here M endowed with J is a compact complex manifold of complex dimension $m = m_1 + m_2 + 1$. Moreover, if g_i is the compatible Riemannian metric on M_i for each $i = 1, 2$, then the product Riemannian metric $g = g_1 + g_2$ on M is compatible with J, that is, g is a Hermitian metric on M. Then its Kähler form Ω is given by

$$\Omega = \Phi_1 + \Phi_2 - 2\eta_1 \wedge \eta_2, \tag{2.2}$$

where Φ_i denotes the fundamental 2-form on M_i for each $i = 1, 2$.

Marrero [M92] proved the following result for trans-Sasakian manifolds:

THEOREM 2.10. *An (α, β) trans-Sasakian manifold of dimension $m = 2n + 1 \geq 5$ is either α-Sasakian, β-Kenmotsu or cosymplectic.*

3. Construction of astheno-Kähler manifolds. Let $(M_i, \phi_i, \xi_i, \eta_i, g_i)$, $i = 1, 2$, be two compact trans-Sasakian manifolds of type and dimension (α_i, β_i) and $2m_i + 1$, respectively, where α_i, β_i are functions on M_i and $\alpha_i = \frac{1}{2n_i}\delta_i\phi_i(\xi_i)$, $\beta_i = \frac{1}{2n_i}\operatorname{div}\xi_i$, where δ_i is the codifferential of g_i.

Let $M = M_1 \times M_2$ be the product manifold with complex dimension $m = m_1 + m_2 + 1$.

Consider the Kähler form

$$\Omega = \Phi_1 + \Phi_2 - 2\eta_1 \wedge \eta_2. \tag{3.1}$$

By (3.1), we get

$$\begin{aligned} d\Omega &= d\Phi_1 + d\Phi_2 - 2\,(d\eta_1 \wedge \eta_2 + \eta_1 \wedge d\eta_2) \\ &= 2\beta_1\eta_1 \wedge \Phi_1 + 2\beta_2\eta_2 \wedge \Phi_2 - 2\,(\alpha_1\Phi_1 \wedge \eta_2 + \eta_1 \wedge \alpha_2\Phi_2) \end{aligned} \tag{3.2}$$

and

$$
\begin{aligned}
(3.3)\qquad d^c\Omega &= Jd\Omega \\
&= 2(J\beta_1\eta_2\wedge\Phi_1 - J\beta_2\eta_1\wedge\Phi_2 + J\alpha_1\Phi_1\wedge\eta_1 - \eta_2\wedge J\alpha_2\Phi_2), \\
&= 2(\beta_1\eta_2\wedge\Phi_1 - \beta_2\eta_1\wedge\Phi_2 + \alpha_1\Phi_1\wedge\eta_1 - \alpha_2\eta_2\wedge\Phi_2).
\end{aligned}
$$

Then

$$
\begin{aligned}
(3.4)\qquad dd^c\Omega &= 2\{\beta_1 d\eta_2\wedge\Phi_1 + \beta_1\eta_2\wedge d\Phi_1 - \beta_2 d\eta_1\wedge\Phi_2 - \beta_2\eta_1\wedge d\Phi_2 \\
&\quad + \alpha_1 d\Phi_1\wedge\eta_1 + \alpha_1\Phi_1\wedge d\eta_1 - \alpha_2 d\eta_1\wedge\Phi_2 - \alpha_2\eta_2\wedge d\Phi_2\} \\
&= 2\{\beta_1\alpha_2\Phi_2\wedge\Phi_1 + \beta_1\eta_2\wedge 2\beta_1\eta_1\wedge\Phi_1 - \beta_2\alpha_1\Phi_1\wedge\Phi_2 \\
&\quad - \beta_2\eta_1\wedge 2\beta_2\eta_2\wedge\Phi_2 + 2\alpha_1\beta_1\eta_1\wedge\Phi_1\wedge\eta_1 + \alpha_1\Phi_1\wedge\alpha_1\Phi_1 \\
&\quad - \alpha_2\alpha_2\Phi_2\wedge\Phi_2 - \alpha_2\eta_2\wedge 2\beta_2\eta_2\wedge\Phi_2\} \\
&= 2\{\beta_1\alpha_2\Phi_2\wedge\Phi_1 + 2\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1 - \beta_2\alpha_1\Phi_1\wedge\Phi_2 \\
&\quad - 2\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2 + \alpha_1^2\Phi_1^2 - \alpha_2^2\Phi_2^2\}.
\end{aligned}
$$

Now

$$
\begin{aligned}
(3.5)\qquad d\Omega\wedge d^c\Omega &= 4\,(\beta_1\eta_1\wedge\Phi_1 + \beta_2\eta_2\wedge\Phi_2 - \alpha_1\Phi_1\wedge\eta_2 - \eta_1\alpha_2\wedge\Phi_2) \\
&\quad \wedge(\beta_1\eta_2\wedge\Phi_1 - \beta_2\eta_1\wedge\Phi_2 + \alpha_1\Phi_1\wedge\eta_1 - \alpha_2\eta_2\wedge\Phi_2) \\
&= 4[\beta_1^2\eta_1\wedge\Phi_1\wedge\eta_2\wedge\Phi_1 - \beta_1\beta_2\eta_1\wedge\Phi_1\wedge\eta_1\wedge\Phi_2 \\
&\quad + \beta_1\alpha_1\eta_1\wedge\Phi_1\wedge\Phi_1\wedge\eta_1 - \beta_1\alpha_2\eta_1\wedge\Phi_1\wedge\eta_2\Phi_2 \\
&\quad + \beta_2\beta_1\eta_2\wedge\eta_2\wedge\Phi_2\wedge\Phi_1 - \beta_2^2\eta_2\wedge\Phi_2\wedge\eta_1\wedge\Phi_2 \\
&\quad + \beta_2\alpha_1\eta_2\wedge\Phi_2\wedge\Phi_1\wedge\eta_1 - \beta_2\alpha_2\eta_2\wedge\Phi_2\wedge\eta_2\wedge\Phi_2 \\
&\quad - \alpha_1\beta_1\Phi_1\wedge\eta_2\wedge\eta_2\wedge\Phi_1 + \alpha_1\beta_2\Phi_1\wedge\eta_2\wedge\eta_1\wedge\Phi_2 \\
&\quad - \alpha_1^2\Phi_1\wedge\eta_2\wedge\Phi_1\wedge\eta_1 + \alpha_1\alpha_2\Phi_1\wedge\eta_2\wedge\eta_2\wedge\Phi_2 \\
&\quad - \beta_1\alpha_2\eta_1\wedge\Phi_2\wedge\eta_2\wedge\Phi_1 + \alpha_2\beta_2\eta_1\wedge\Phi_2\wedge\eta_1\wedge\Phi_2 \\
&\quad - \alpha_2\alpha_1\eta_1\wedge\Phi_2\wedge\Phi_1\wedge\eta_1 + \alpha_2^2\eta_1\wedge\Phi_2\wedge\eta_2\wedge\Phi_2] \\
&= 4[\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^2 - \beta_1\alpha_2\eta_1\wedge\eta_2\wedge\Phi_1\wedge\Phi_2 - \beta_2^2\eta_2\wedge\eta_1\wedge\Phi_2^2 \\
&\quad + \beta_2\alpha_1\eta_2\wedge\eta_1\wedge\Phi_2\wedge\Phi_1 + \alpha_1\beta_2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2 \\
&\quad - \alpha_1^2\eta_2\wedge\eta_1\wedge\Phi_1^2 - \beta_1\alpha_2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1 + \alpha_2^2\eta_1\wedge\eta_2\wedge\Phi_2^2].
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
(3.6)\qquad dd^c\Omega\wedge\Omega &= 2\{\beta_1\alpha_2\Phi_2\wedge\Phi_1 + 2\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1 - \beta_2\alpha_1\Phi_1\wedge\Phi_2 \\
&\quad - 2\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2 + \alpha_1^2\Phi_1^2 - \alpha_2^2\Phi_2^2\}\wedge\{\Phi_1 + \Phi_2 - 2\eta_1\wedge\eta_2\} \\
&= 2[\beta_1\alpha_2\Phi_2\wedge\Phi_1^2 + \beta_1\alpha_2\Phi_2^2\wedge\Phi_1 - 2\beta_1\alpha_2\Phi_2\wedge\Phi_1\wedge\eta_1\wedge\eta_2 \\
&\quad + 2\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1^2 + 2\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2 \\
&\quad - 4\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\eta_1\wedge\eta_2 - \beta_2\alpha_1\Phi_1^2\wedge\Phi_2 - \beta_2\alpha_1\Phi_1\wedge\Phi_2^2
\end{aligned}
$$

$$+2\beta_2\alpha_1\Phi_1\wedge\Phi_2\wedge\eta_1\wedge\eta_2-2\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1$$
$$-2\beta_2^2\wedge\eta_1\wedge\eta_2\wedge\Phi_2^2+4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\eta_1\wedge\eta_2+\alpha_1^2\Phi_1^3$$
$$+\alpha_1^2\Phi_1^2\wedge\Phi_2-2\alpha_1^2\Phi_1^2\wedge\eta_1\wedge\eta_2-\alpha_2^2\Phi_2^2\wedge\Phi_1-\alpha_2^2\Phi_2^3+2\alpha_2^2\Phi_2^2\wedge\eta_1\wedge\eta_2].$$

If the complex dimension of the manifold is m, then to get an astheno-Kähler manifold we need to show

$$dd^c\Omega^{m-2}=(m-2)[dd^c\Omega\wedge\Omega+(m-3)d\Omega\wedge d^c\Omega]\wedge\Omega^{m-4}=0. \tag{3.7}$$

The possibilities for the dimension m of the product manifold M (over $\mathbb{C}$) are the following:

Case 1. For $m=3$ (complex dimension), we take the product of two 3-trans-Sasakian manifolds: the dimension of M_1 is $2m_1+1=3$ and the dimension of M_2 is $2m_2+1=3$. To show that a manifold of dimension $m=3$ is astheno-Kähler, we need to prove $dd^c\Omega=0$. From (3.4),

$$dd^c\Omega=2\{\beta_1\alpha_2\Phi_2\wedge\Phi_1+2\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1-\beta_2\alpha_1\wedge\Phi_1\wedge\Phi_2$$
$$-2\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2+\alpha_1^2\Phi_1^2-\alpha_2^2\Phi_2^2\}.$$

Table 1

	M_1	M_2	α_1	α_2	β_1	β_2	$dd^c\Omega$
1	α_1-Sasakian	α_2-Sasakian	1	1	0	0	0
2	α_1-Sasakian	β_2-Kenmotsu	1	0	0	1	$-2\beta_2\alpha_1\Phi_1\wedge\Phi_2-4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2$ $+2\alpha_1^2\Phi_1^2$
3	α_1-Sasakian	Cosymplectic	1	0	0	0	0
4	β_1-Kenmotsu	β_2-Kenmotsu	0	0	1	1	$2\{2\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1-2\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\}$
5	β_1-Kenmotsu	α_2-Sasakian	0	1	1	0	$2\{\beta_1\alpha_2\Phi_2\wedge\Phi_1+2\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\}$
6	β_1-Kenmotsu	Cosymplectic	0	0	1	0	$2\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1$
7	Cosymplectic	α_2-Sasakian	0	1	0	0	0
8	Cosymplectic	β_2-Kenmotsu	0	0	0	1	$-2\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2$
9	Cosymplectic	Cosymplectic	0	0	0	0	0

Remark 3.1. (1) In Table 1, Cases 1, 3, 7 and 9 yield astheno-Kähler manifolds.

(2) Specific instances of Case 1 and Cases 3 and 7 can be observed in Theorems 4.1 and 4.2, respectively, in the paper by Matsuo and Takahashi [MT01].

Case 2. For $m=4$ (complex dimension), there are two possibilities.

(i) One possibility is $m_1=1$, $m_2=2$, that is, $\dim M_1=3$, $\dim M_2=5$. From (3.7), we have $dd^c\Omega^2=2\,[dd^c\Omega\wedge\Omega+d\Omega\wedge d^c\Omega]=0$. Using (3.5)

and (3.6), we have

$$\begin{aligned}
dd^c\Omega^2 &= 2\big[2(\beta_1\alpha_2\Phi_2\wedge\Phi_1^2+\beta_1\alpha_2\Phi_2^2\wedge\Phi_1-2\beta_1\alpha_2\Phi_2\wedge\Phi_1\wedge\eta_1\wedge\eta_2\\
&\quad+2\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1^2+2\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2-\beta_2\alpha_1\wedge\Phi_1^2\wedge\Phi_2\\
&\quad-\beta_2\alpha_1\Phi_1\wedge\Phi_2^2+2\beta_2\alpha_1\wedge\Phi_1\wedge\Phi_2\wedge\eta_1\wedge\eta_2\\
&\quad-2\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1-2\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^2+\alpha_1^2\Phi_1^3+\alpha_1^2\Phi_1^2\wedge\Phi_2\\
&\quad-2\alpha_1^2\Phi_1^2\wedge\eta_1\wedge\eta_2-\alpha_2^2\Phi_2^2\wedge\Phi_1-\alpha_2^2\Phi_2^3+2\alpha_2^2\Phi_2^2\wedge\eta_1\wedge\eta_2)\\
&\quad+4(\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^2-\beta_1\alpha_2\eta_1\wedge\eta_2\wedge\Phi_1\wedge\Phi_2-\beta_2^2\eta_2\wedge\eta_1\wedge\Phi_2^2\\
&\quad+\beta_2\alpha_1\eta_2\wedge\eta_1\wedge\Phi_2\wedge\Phi_1+\alpha_1\beta_2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2\\
&\quad-\alpha_1^2\eta_2\wedge\eta_1\wedge\Phi_1^2-\beta_1\alpha_2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1+\alpha_2^2\eta_1\wedge\eta_2\wedge\Phi_2^2)\big]\\
&= 2\big[2\beta_1\alpha_2\Phi_2\wedge\Phi_1^2+2\beta_1\alpha_2\Phi_2^2\wedge\Phi_1-12\beta_1\alpha_2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1\\
&\quad+4\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1\wedge\Phi_2-2\beta_2\alpha_1\Phi_1^2\wedge\Phi_2-2\beta_2\alpha_1\Phi_1\wedge\Phi_2^2\\
&\quad+4\alpha_1\beta_2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2-4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1+2\alpha_1^2\Phi_1^3\\
&\quad+2\alpha_1^2\Phi_1^2\wedge\Phi_2-2\alpha_2^2\Phi_2^2\wedge\Phi_1-2\alpha_2^2\Phi_2^3+8\alpha_2^2\eta_1\wedge\eta_2\wedge\Phi_2^2\big].
\end{aligned}$$

Table 2

	M_1	M_2	α_1	α_2	β_1	β_2	$dd^c\Omega^2$
1	α_1-Sasakian	α_2-Sasakian	1	1	0	0	$16\alpha_2^2\eta_1\wedge\eta_2\wedge\Phi_2^2-4\alpha_2^2\Phi_2^2\wedge\Phi_1-4\alpha_2^2\Phi_2^2$
2	α_1-Sasakian	β_2-Kenmotsu	1	0	0	1	$8\beta_2\alpha_1\Phi_1\wedge\Phi_2\wedge\eta_2\wedge\eta_1-4\beta_2\alpha_1\Phi_1\wedge\Phi_2^2$ $-8\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1$
3	α_1-Sasakian	Cosymplectic	1	0	0	0	0
4	β_1-Kenmotsu	β_2-Kenmotsu	0	0	1	1	$8\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2-8\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1$
5	β_1-Kenmotsu	α_2-Sasakian	0	1	1	0	$-24\beta_1\alpha_2\Phi_1\wedge\Phi_2\wedge\eta_1\wedge\eta_2$ $+16\alpha_2^2\eta_1\wedge\eta_2\wedge\Phi_2^2+4\beta_1\alpha_2\Phi_2^2\wedge\Phi_1$ $+8\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2$ $-4\alpha_2^2\Phi_2^2\wedge\Phi_1+4\beta_1\alpha_2\Phi_2\wedge\Phi_1^2$
6	β_1-Kenmotsu	Cosymplectic	0	0	1	0	$8\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2$
7	Cosymplectic	α_2-Sasakian	0	1	0	0	$16\alpha_2^2\eta_1\wedge\eta_2\wedge\Phi_2^2-4\alpha_2^2\Phi_2^2\wedge\Phi_1$
8	Cosymplectic	β_2-Kenmotsu	0	0	0	1	$-8\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1$
9	Cosymplectic	Cosymplectic	0	0	0	0	0

REMARK 3.2. (1) In Table 2, Cases 3 and 9 yield astheno-Kähler manifolds.

(2) A specific instance of Case 3 can be observed in [MT01, Theorem 4.2].

(ii) Another possibility is $\dim M_1 = 5$, $\dim M_2 = 3$. We have

$$\begin{aligned}
dd^c\Omega^2 &= 2\big[2\beta_1\alpha_2\Phi_2\wedge\Phi_1^2+2\beta_1\alpha_2\Phi_2^2\wedge\Phi_1-12\beta_1\alpha_2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1\\
&\quad+4\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1\wedge\Phi_2-2\beta_2\alpha_1\Phi_1^2\wedge\Phi_2-2\beta_2\alpha_1\Phi_1\wedge\Phi_2^2
\end{aligned}$$

$$+ 4\alpha_1\beta_2\eta_2 \wedge \eta_1 \wedge \Phi_1 \wedge \Phi_2 - 4\beta_2^2\eta_1 \wedge \eta_2 \wedge \Phi_2 \wedge \Phi_1 + 2\alpha_1^2\Phi_1^3$$
$$+ 2\alpha_1^2\Phi_1^2 \wedge \Phi_2 - 2\alpha_2^2\Phi_2^2 \wedge \Phi_1 - 2\alpha_2^2\Phi_2^3 + 8\alpha_2^2\eta_1 \wedge \eta_2 \wedge \Phi_2^2\big].$$

Table 3

	M_1	M_2	α_1	α_2	β_1	β_2	$dd^c\Omega^2$
1	α_1- Sasakian	α_2-Sasakian	1	1	0	0	$4\alpha_1^2\Phi_1^2 \wedge \Phi_2$
2	α_1-Sasakian	β_2-Kenmotsu	1	0	0	1	$8\beta_2\alpha_1\Phi_1\wedge\Phi_2\wedge\eta_2\wedge\eta_1 - 4\beta_2\alpha_1\Phi_1\wedge\Phi_2^2$ $-8\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1 - 4\beta_2\alpha_1\Phi_1^2\wedge\Phi_2$ $+4\alpha_1^2\Phi_1^2\wedge\Phi_2$
3	α_1-Sasakian	Cosymplectic	1	0	0	0	$4\alpha_1^2\Phi_1^2 \wedge \Phi_2$
4	β_1-Kenmotsu	β_2-Kenmotsu	0	0	1	1	$-8\beta_2^2\eta_1 \wedge \eta_2 \wedge \Phi_2 \wedge \Phi_1$ $+ 8\beta_1^2\eta_1 \wedge \eta_2 \wedge \Phi_1 \wedge \Phi_2$
5	β_1-Kenmotsu	α_2-Sasakian	0	1	1	0	$4\beta_1\alpha_2\Phi_2\wedge\Phi_1^2 - 24\beta_1\alpha_2\Phi_1\wedge\Phi_2\wedge\eta_1\wedge\eta_2$ $+8\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1\wedge\Phi_2$
6	β_1-Kenmotsu	Cosymplectic	0	0	1	0	$8\beta_1^2\eta_1 \wedge \eta_2 \wedge \Phi_1 \wedge \Phi_2$
7	Cosymplectic	α_2-Sasakian	0	1	0	0	0
8	Cosymplectic	β_2-Kenmotsu	0	0	0	1	$-8\beta_2^2\eta_1 \wedge \eta_2 \wedge \Phi_2 \wedge \Phi_1$
9	Cosymplectic	Cosymplectic	0	0	0	0	0

REMARK 3.3. (1) In Table 3, Cases 7 and 9 are astheno-Kähler manifolds.
(2) A specific instance of Case 7 can be observed in [MT01, Theorem 4.2].

REMARK 3.4. In Case 4 of Tables 1, 2 and 3, if $\beta_1 = \pm\beta_2$, it will become an astheno-Kähler manifold.

CASE 3. For $\dim_{\mathbb{C}} M = m = 5$, that is, $\dim_{\mathbb{R}} M = 10$, we have the following possibilities:

(i) $\dim_{\mathbb{R}} M_1 = 3$ and $\dim_{\mathbb{R}} M_2 = 7$,
(ii) $\dim_{\mathbb{R}} M_1 = 5$ and $\dim_{\mathbb{R}} M_2 = 5$,
(iii) $\dim_{\mathbb{R}} M_1 = 7$ and $\dim_{\mathbb{R}} M_2 = 3$.

To get an astheno-Kähler manifold with $m = 5$, we need to show

$$dd^c\Omega^3 = 3(dd^c\Omega \wedge \Omega + 2d\Omega \wedge d^c\Omega) \wedge \Omega = 0.$$

By (3.5) and (3.6), we have

$$\begin{aligned} dd^c\Omega^3 = 3\big[&2(\beta_1\alpha_2\Phi_2 \wedge \Phi_1^2 + \beta_1\alpha_2\Phi_2^2 \wedge \Phi_1 - 2\beta_1\alpha_2\Phi_2 \wedge \Phi_1 \wedge \eta_1 \wedge \eta_2 \\ &+ 2\beta_1^2\eta_2 \wedge \eta_1 \wedge \Phi_1^2 + 2\beta_1^2\eta_2 \wedge \eta_1 \wedge \Phi_1 \wedge \Phi_2 - \beta_2\alpha_1\Phi_1^2 \wedge \Phi_2 \\ &- \beta_2\alpha_1\Phi_1 \wedge \Phi_2^2 + 2\beta_2\alpha_1\Phi_1 \wedge \Phi_2 \wedge \eta_1 \wedge \eta_2 - 2\beta_2^2\eta_1 \wedge \eta_2 \wedge \Phi_2 \wedge \Phi_1 \\ &- 2\beta_2^2 \wedge \eta_1 \wedge \eta_2 \wedge \Phi_2^2 + \alpha_1^2\Phi_1^3 + \alpha_1^2\Phi_1^2 \wedge \Phi_2 - 2\alpha_1^2\Phi_1^2 \wedge \eta_1 \wedge \eta_2 \\ &- \alpha_2^2\Phi_2^2 \wedge \Phi_1 - \alpha_2^2\Phi_2^3 + 2\alpha_2^2\Phi_2^2 \wedge \eta_1 \wedge \eta_2) \end{aligned}$$

$$
\begin{aligned}
&+8(\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^2-\beta_1\alpha_2\eta_1\wedge\eta_2\wedge\Phi_1\wedge\Phi_2-\beta_2^2\eta_2\wedge\eta_1\wedge\Phi_2^2\\
&+\beta_2\alpha_1\eta_2\wedge\eta_1\wedge\Phi_2\wedge\Phi_1+\alpha_1\beta_2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2\\
&-\alpha_1^2\eta_2\wedge\eta_1\wedge\Phi_1^2-\beta_1\alpha_2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1+\alpha_2^2\eta_1\wedge\eta_2\wedge\Phi_2^2)\big]\wedge\Omega\\
&=3\big[2\beta_1\alpha_2\Phi_2\wedge\Phi_1^3+4\beta_1\alpha_2\Phi_2^2\wedge\Phi_1^2-24\beta_1\alpha_2\Phi_2\wedge\Phi_1^2\wedge\eta_1\wedge\eta_2\\
&+4\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3-2\beta_2\alpha_1\Phi_1^3\wedge\Phi_2-4\beta_2\alpha_1\Phi_1^2\Phi_2^2\\
&-4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1^2+2\alpha_1^2\Phi_1^4+4\alpha_1^2\Phi_1^4+4\alpha_1^2\Phi_1^3\wedge\Phi_2\\
&-4\alpha_2^2\Phi_2^3\wedge\Phi_1-2\alpha_2^2\Phi_2^2\wedge\Phi_1^2+16\alpha_2^2\eta_1\wedge\eta_2\wedge\Phi_2^2\wedge\Phi_1+2\beta_1\alpha_2\Phi_2^3\wedge\Phi_1\\
&-24\beta_1\alpha_2\Phi_2^2\wedge\Phi_1\wedge\eta_1\wedge\eta_2+4\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2^2-2\beta_2\alpha_1\Phi_1\wedge\Phi_2^3\\
&+8\beta_2\alpha_1\Phi_1\wedge\Phi_2^2\wedge\eta_2\wedge\eta_1+4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3+2\alpha_1^2\Phi_1^2\wedge\Phi_2^2-2\alpha_2^2\Phi_2^4\\
&+16\alpha_2^2\Phi_2^3\wedge\eta_1\wedge\eta_2\big].
\end{aligned}
$$

(i) $\dim_{\mathbb{R}} M_1 = 3$ and $\dim_{\mathbb{R}} M_2 = 7$

Table 4

	M_1	M_2	α_1	α_2	β_1	β_2	$dd^c\Omega^3$
1	α_1-Sasakian	α_2-Sasakian	1	1	0	0	$3(-4\alpha_2^2\Phi_2^3\wedge\Phi_1+16\alpha_2^2\eta_1\wedge\eta_2\wedge\Phi_2^2\wedge\Phi_1$ $+16\alpha_2^2\Phi_2^3\wedge\eta_1\wedge\eta_2)$
2	α_1-Sasakian	β_2-Kenmotsu	1	0	0	1	$3(-2\beta_2\alpha_1\Phi_1\wedge\Phi_2^3+8\beta_2\alpha_1\Phi_1\wedge\Phi_2^2\wedge\eta_2\wedge\eta_1$ $+4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3)$
3	α_1-Sasakian	Cosymplectic	1	0	0	0	0
4	β_1-Kenmotsu	β_2-Kenmotsu	0	0	1	1	$3(4\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3-4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1^2$ $+4\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2^2+4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3)$
5	β_1-Kenmotsu	α_2-Sasakian	0	1	1	0	$3(2\beta_1\alpha_2\Phi_2\wedge\Phi_1^3+4\beta_1\alpha_2\Phi_2^2\wedge\Phi_1^2$ $-24\beta_1\alpha_2\Phi_2\wedge\Phi_1^2\wedge\eta_1\wedge\eta_2$ $+4\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3$ $-4\alpha_2^2\Phi_2^3\wedge\Phi_1-2\alpha_2^2\Phi_2^2\wedge\Phi_1^2$ $+16\alpha_2^2\eta_1\wedge\eta_2\wedge\Phi_2^2\wedge\Phi_1+2\beta_1\alpha_2\Phi_2^3\wedge\Phi_1$ $-24\beta_1\alpha_2\Phi_2^2\wedge\Phi_1\wedge\eta_1\wedge\eta_2$ $+4\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2^2)$
6	β_1-Kenmotsu	Cosymplectic	0	0	1	0	$3(4\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3+4\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2^2)$
7	Cosymplectic	α_2-Sasakian	0	1	0	0	$3(-4\alpha_2^2\Phi_2^3\wedge\Phi_1+16\alpha_2^2\eta_1\wedge\eta_2\wedge\Phi_2^2\wedge\Phi_1$ $+16\alpha_2^2\Phi_2^3\wedge\eta_1\wedge\eta_2)$
8	Cosymplectic	β_2-Kenmotsu	0	0	0	1	$12\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3$
9	Cosymplectic	Cosymplectic	0	0	0	0	0

REMARK 3.5. (1) In Table 4, Cases 3 and 9 yield astheno-Kähler manifolds.

(2) A specific instance of Case 3 can be observed in [MT01, Theorem 4.2].

(ii) $\dim_{\mathbb{R}} M_1 = 5$ and $\dim_{\mathbb{R}} M_2 = 5$

Table 5

	M_1	M_2	α_1	α_2	β_1	β_2	$dd^c\Omega^3$
1	α_1-Sasakian	α_2-Sasakian	1	1	0	0	$3(-2\alpha_2^2\Phi_2^2\wedge\Phi_1^2+16\alpha_2^2\eta_1\wedge\eta_2\wedge\Phi_2^2\wedge\Phi_1$ $+2\alpha_1^2\Phi_1^2\wedge\Phi_2^2)$
2	α_1-Sasakian	β_2-Kenmotsu	1	0	0	1	$3(-4\beta_2\alpha_1\Phi_1^2\Phi_2^2+8\beta_2\alpha_1\Phi_1^2\wedge\Phi_2\wedge\eta_2\wedge\eta_1$ $-4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1^2-2\beta_2\alpha_1\Phi_1\wedge\Phi_2^3$ $+8\beta_2\alpha_1\Phi_1\wedge\Phi_2^2\wedge\eta_2\wedge\eta_1$ $+4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3+2\alpha_1^2\Phi_1^2\wedge\Phi_2^2)$
3	α_1-Sasakian	Cosymplectic	1	0	0	0	$6\alpha_1^2\Phi_1^2\wedge\Phi_2^2$
4	β_1-Kenmotsu	β_2-Kenmotsu	0	0	1	1	$3(4\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3-4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1^2$ $+4\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2^2+4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3)$
5	β_1-Kenmotsu	α_2-Sasakian	0	1	1	0	$3(2\beta_1\alpha_2\Phi_2\wedge\Phi_1^3+4\beta_1\alpha_2\Phi_2^2\wedge\Phi_1^2$ $-24\beta_1\alpha_2\Phi_2\wedge\Phi_1^2\wedge\eta_1\wedge\eta_2$ $+4\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3-2\alpha_2^2\Phi_2^2\wedge\Phi_1^2$ $+16\alpha_2^2\eta_1\wedge\eta_2\wedge\Phi_2^2\wedge\Phi_1$ $-24\beta_1\alpha_2\Phi_2^2\wedge\Phi_1\wedge\eta_1\wedge\eta_2$ $-4\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2^2)$
6	β_1-Kenmotsu	Cosymplectic	0	0	1	0	$3(4\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3+4\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2^2)$
7	Cosymplectic	α_2-Sasakian	0	1	0	0	$3(-2\alpha_2^2\Phi_2^2\wedge\Phi_1^2+16\alpha_2^2\eta_1\wedge\eta_2\wedge\Phi_2^2\wedge\Phi_1)$
8	Cosymplectic	β_2-Kenmotsu	0	0	0	1	$3(-4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1^2+4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3)$
9	Cosymplectic	Cosymplectic	0	0	0	0	0

REMARK 3.6. In Table 5, Case 9 yields an astheno-Kähler manifold.

(iii) $\dim_{\mathbb{R}} M_1 = 7$ and $\dim_{\mathbb{R}} M_2 = 3$

Table 6

	M_1	M_2	α_1	α_2	β_1	β_2	$dd^c\Omega^3$
1	α_1-Sasakian	α_2-Sasakian	1	1	0	0	$12\alpha_1^2\Phi_1^3\wedge\Phi_2$
2	α_1-Sasakian	β_2-Kenmotsu	1	0	0	1	$3(-2\beta_2\alpha_1\Phi_1^3\wedge\Phi_2-4\beta_2\alpha_1\Phi_1^2\wedge\Phi_2^2$ $+8\beta_2\alpha_1\Phi_1^2\wedge\Phi_2\wedge\eta_2\wedge\eta_1$ $-4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1^2+4\alpha_1^2\Phi_1^3\wedge\Phi_2$ $-2\beta_2\alpha_1\Phi_1\wedge\Phi_2^3+8\beta_2\alpha_1\Phi_1\wedge\Phi_2^2\wedge\eta_2\wedge\eta_1$ $+4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3+2\alpha_1^2\Phi_1^2\Phi_1^2\wedge\Phi_2^2)$
3	α_1-Sasakian	Cosymplectic	1	0	0	0	$12\alpha_1^2\Phi_1^3\wedge\Phi_2$
4	β_1-Kenmotsu	β_2-Kenmotsu	0	0	1	1	$3(4\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3-4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1^2$ $+4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3+4\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2^2)$
5	β_1-Kenmotsu	α_2-Sasakian	0	1	1	0	$6\beta_1\alpha_2\Phi_2\wedge\Phi_1^3$

Table 6 [cont.]

6	β_1-Kenmotsu	Cosymplectic	0	0	1	0	$12\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3$
7	Cosymplectic	α_2-Sasakian	0	1	0	0	0
8	Cosymplectic	β_2-Kenmotsu	0	0	0	1	$3(-4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1^2+4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3)$
9	Cosymplectic	Cosymplectic	0	0	0	0	0

REMARK 3.7. (1) In Table 6, Cases 7 and 9 yield astheno-Kähler manifolds.

(2) A specific instance of Case 7 can be observed in [MT01, Theorem 4.2].

CASE 4: For $\dim_{\mathbb{C}} M = m = 6$, that is, $\dim_{\mathbb{R}} M = 12$, we have the following possibilities :

(i) $\dim_{\mathbb{R}} M_1 = 3$ and $\dim_{\mathbb{R}} M_2 = 9$,
(ii) $\dim_{\mathbb{R}} M_1 = 5$ and $\dim_{\mathbb{R}} M_2 = 7$,
(iii) $\dim_{\mathbb{R}} M_1 = 7$ and $\dim_{\mathbb{R}} M_2 = 5$,
(iv) $\dim_{\mathbb{R}} M_1 = 9$ and $\dim_{\mathbb{R}} M_2 = 3$.

To get an astheno-Kähler manifold with $m = 6$, we need to show

$$dd^c\Omega^4 = 4(dd^c\Omega\wedge\Omega + 3d\Omega\wedge d^c\Omega)\wedge\Omega^2 = 0.$$

By (3.5) and (3.6), we have

$$\begin{aligned}
dd^c\Omega^4 &= 4\big[2(\beta_1\alpha_2\Phi_2\wedge\Phi_1^2+\beta_1\alpha_2\Phi_2^2\wedge\Phi_1-2\beta_1\alpha_2\Phi_2\wedge\Phi_1\wedge\eta_1\wedge\eta_2\\
&\quad+2\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1^2+2\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2\\
&\quad-4\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\eta_1\wedge\eta_2-\beta_2\alpha_1\Phi_1^2\wedge\Phi_2-\beta_2\alpha_1\Phi_1\wedge\Phi_2^2\\
&\quad+2\beta_2\alpha_1\Phi_1\wedge\Phi_2\wedge\eta_1\wedge\eta_2-2\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1\\
&\quad-2\beta_2^2\wedge\eta_1\wedge\eta_2\wedge\Phi_2^2+4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\eta_1\wedge\eta_2+\alpha_1^2\Phi_1^3\\
&\quad+\alpha_1^2\Phi_1^2\wedge\Phi_2-2\alpha_1^2\Phi_1^2\wedge\eta_1\wedge\eta_2-\alpha_2^2\Phi_2^2\wedge\Phi_1-\alpha_2^2\Phi_2^3+2\alpha_2^2\Phi_2^2\wedge\eta_1\wedge\eta_2)\\
&\quad+12(\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^2-\beta_1\alpha_2\eta_1\wedge\eta_2\wedge\Phi_1\wedge\Phi_2-\beta_2^2\eta_2\wedge\eta_1\wedge\Phi_2^2\\
&\quad+\beta_2\alpha_1\eta_2\wedge\eta_1\wedge\Phi_2\wedge\Phi_1+\alpha_1\beta_2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2\\
&\quad-\alpha_1^2\eta_2\wedge\eta_1\wedge\Phi_1^2-\beta_1\alpha_2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1+\alpha_2^2\eta_1\wedge\eta_2\wedge\Phi_2^2)\big]\\
&\quad\wedge[\Phi_1+\Phi_2-2\eta_1\wedge\eta_2]^2\\
&= 4\big[2\beta_1\alpha_2\Phi_2\wedge\Phi_1^4+6\beta_1\alpha_2\Phi_2^2\wedge\Phi_1^3-36\beta_1\alpha_2\Phi_2\wedge\Phi_1^3\wedge\eta_1\wedge\eta_2\\
&\quad+8\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^4+12\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3\wedge\Phi_2-2\beta_2\alpha_1\Phi_1^4\wedge\Phi_2\\
&\quad-6\beta_2\alpha_1\Phi_1^3\wedge\Phi_2^2-4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1^3+2\alpha_1^2\Phi_1^5+6\alpha_1^2\Phi_1^4\wedge\Phi_2\\
&\quad-2\alpha_2^2\Phi_2^2\wedge\Phi_1^3-6\alpha_2^2\Phi_2^3\wedge\Phi_1^2+24\alpha_2^2\Phi_2^2\wedge\Phi_1^2\wedge\eta_1\wedge\eta_2\\
&\quad+6\beta_1\alpha_2\Phi_2^3\wedge\Phi_1^2+2\beta_1\alpha_2\Phi_2^4\wedge\Phi_1-36\beta_1\alpha_2\Phi_2^3\wedge\Phi_1\wedge\eta_1\wedge\eta_2\\
&\quad+4\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2^3-6\beta_2\alpha_1\Phi_1^2\wedge\Phi_2^3-2\beta_2\alpha_1\Phi_1\wedge\Phi_2^4
\end{aligned}$$

$$+12\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3\wedge\Phi_1+8\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^4+6\alpha_1^2\Phi_1^3\wedge\Phi_2^2$$
$$+2\alpha_1^2\Phi_1^2\wedge\Phi_2^3-6\alpha_2^2\Phi_2^4\wedge\Phi_1-2\alpha_2^2\Phi_2^5+24\alpha_2^2\Phi_2^4\wedge\eta_1\wedge\eta_2$$
$$-60\beta_1\alpha_2\Phi_2^2\wedge\Phi_1^2\wedge\eta_1\wedge\eta_2+8\beta_2\alpha_1\eta_2\wedge\eta_1\wedge\Phi_2^2\wedge\Phi_1^2$$
$$-16\alpha_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3\wedge\Phi_2+48\alpha_2^2\Phi_2^3\wedge\Phi_1\wedge\eta_1\wedge\eta_2\big].$$

(i) $\dim_{\mathbb{R}} M_1 = 3$ and $\dim_{\mathbb{R}} M_2 = 9$

Table 7

	M_1	M_2	α_1	α_2	β_1	β_2	$dd^c\Omega^4$
1	α_1-Sasakian	α_2-Sasakian	1	1	0	0	$4(-6\alpha_2^2\Phi_2^4\wedge\Phi_1+24\alpha_2^2\Phi_2^4\wedge\eta_1\wedge\eta_2$ $+48\alpha_2^2\Phi_2^3\wedge\Phi_1\wedge\eta_1\wedge\eta_2)$
2	α_1-Sasakian	β_2-Kenmotsu	1	0	0	1	$4(-2\beta_2\alpha_1\Phi_1\wedge\Phi_2^4+12\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3\wedge\Phi_1$ $+8\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^4)$
3	α_1-Sasakian	Cosymplectic	1	0	0	0	0
4	β_1-Kenmotsu	β_2-Kenmotsu	0	0	1	1	$4(8\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^4+12\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3\wedge\Phi_2$ $-4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1^3$ $+4\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2^3$ $+12\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3\wedge\Phi_1$ $+8\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^4)$
5	β_1-Kenmotsu	α_2-Sasakian	0	1	1	0	$4(2\beta_1\alpha_2\Phi_2\wedge\Phi_1^4+6\beta_1\alpha_2\Phi_2^2\wedge\Phi_1^3$ $-36\beta_1\alpha_2\Phi_2\wedge\Phi_1^3\wedge\eta_1\wedge\eta_2$ $+8\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^4+12\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^4$ $-2\alpha_2^2\Phi_2^2\wedge\Phi_1^3$ $-6\alpha_2^2\Phi_2^3\wedge\Phi_1^2+24\alpha_2^2\Phi_2^2\wedge\Phi_1^2\wedge\eta_1\wedge\eta_2$ $+6\beta_1\alpha_2\Phi_2^3\wedge\Phi_1^2+2\beta_1\alpha_2\Phi_2^4\wedge\Phi_1$ $-36\beta_1\alpha_2\Phi_2^3\wedge\Phi_1\wedge\eta_1\wedge\eta_2$ $+4\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2^3-6\alpha_2^2\Phi_2^4\wedge\Phi_1$ $+24\alpha_2^2\Phi_2^4\wedge\eta_1\wedge\eta_2$ $-60\beta_1\alpha_2\Phi_2^2\wedge\Phi_1^2\wedge\eta_1\wedge\eta_2$ $+48\alpha_2^2\Phi_2^3\wedge\Phi_1\wedge\eta_1\wedge\eta_2)$
6	β_1-Kenmotsu	Cosymplectic	0	0	1	0	$4(8\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^4+12\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3\wedge\Phi_2$ $+4\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2^3)$
7	Cosymplectic	α_2-Sasakian	0	1	0	0	$4(-6\alpha_2^2\Phi_2^4\wedge\Phi_1+24\alpha_2^2\Phi_2^4\wedge\eta_1\wedge\eta_2$ $+48\alpha_2^2\Phi_2^3\wedge\Phi_1\wedge\eta_1\wedge\eta_2)$
8	Cosymplectic	β_2-Kenmotsu	0	0	0	1	$4(12\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3\wedge\Phi_1$ $+8\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^4)$
9	Cosymplectic	Cosymplectic	0	0	0	0	0

REMARK 3.8. (1) In Table 7, Cases 3 and 9 yield astheno-Kähler manifolds.

(2) A specific instance of Case 3 can be observed in [MT01, Theorem 4.2].

(ii) $\dim_{\mathbb{R}} M_1 = 5$ and $\dim_{\mathbb{R}} M_2 = 7$

Table 8

	M_1	M_2	α_1	α_2	β_1	β_2	$dd^c\Omega^4$
1	α_1-Sasakian	α_2-Sasakian	1	1	0	0	$4(2\alpha_1^2\Phi_1^2\wedge\Phi_2^3-6\alpha_2^2\Phi_2^3\wedge\Phi_1^2$ $+24\alpha_2^2\Phi_2^2\wedge\Phi_1^2\wedge\eta_1\wedge\eta_2$ $+48\alpha_2^2\Phi_2^3\wedge\Phi_1\wedge\eta_1\wedge\eta_2)$
2	α_1-Sasakian	β_2-Kenmotsu	1	0	0	1	$4(-6\beta_2\alpha_1\Phi_1^2\wedge\Phi_2^3-2\beta_2\alpha_1\Phi_1\wedge\Phi_2^4$ $+12\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3\wedge\Phi_1+8\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^4$ $+2\alpha_1^2\Phi_1^2\wedge\Phi_2^3+8\beta_2\alpha_1\eta_2\wedge\eta_1\wedge\Phi_2^2\wedge\Phi_1^2)$
3	α_1-Sasakian	Cosymplectic	1	0	0	0	$8\alpha_1^2\Phi_1^2\wedge\Phi_2^3$
4	β_1-Kenmotsu	β_2-Kenmotsu	0	0	1	1	$4(8\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^4+12\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3\wedge\Phi_2$ $-4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1^3$ $+4\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2^3$ $+12\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3\wedge\Phi_1+8\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^4)$
5	β_1-Kenmotsu	α_2-Sasakian	0	1	1	0	$4(2\beta_1\alpha_2\Phi_2\wedge\Phi_1^4+6\beta_1\alpha_2\Phi_2^2\wedge\Phi_1^3$ $-36\beta_1\alpha_2\Phi_2\wedge\Phi_1^3\wedge\eta_1\wedge\eta_2$ $+8\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^4+12\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^4$ $-2\alpha_2^2\Phi_2^2\wedge\Phi_1^3-6\alpha_2^2\Phi_2^3\wedge\Phi_1^2$ $+24\alpha_2^2\Phi_2^2\wedge\Phi_1^2\wedge\eta_1\wedge\eta_2+6\beta_1\alpha_2\Phi_2^3\wedge\Phi_1^2$ $-36\beta_1\alpha_2\Phi_2^3\wedge\Phi_1\wedge\eta_1\wedge\eta_2$ $+4\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2^3$ $-60\beta_1\alpha_2\Phi_2^2\wedge\Phi_1^2\wedge\eta_1\wedge\eta_2$ $+48\alpha_2^2\Phi_2^3\wedge\Phi_1\wedge\eta_1\wedge\eta_2)$
6	β_1-Kenmotsu	Cosymplectic	0	0	1	0	$4(8\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^4+12\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3\wedge\Phi_2$ $+4\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2^3)$
7	Cosymplectic	α_2-Sasakian	0	1	0	0	$4(-6\alpha_2^2\Phi_2^3\wedge\Phi_1^2+24\alpha_2^2\Phi_2^2\wedge\Phi_1^2\wedge\eta_1\wedge\eta_2$ $+48\alpha_2^2\Phi_2^3\wedge\Phi_1\wedge\eta_1\wedge\eta_2)$
8	Cosymplectic	β_2-Kenmotsu	0	0	0	1	$4(12\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3\wedge\Phi_1+8\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^4)$
9	Cosymplectic	Cosymplectic	0	0	0	0	0

REMARK 3.9. In Table 8, Case 9 yields an astheno-Kähler manifold.

(iii) $\dim_{\mathbb{R}} M_1 = 7$ and $\dim_{\mathbb{R}} M_2 = 5$

Table 9

	M_1	M_2	α_1	α_2	β_1	β_2	$dd^c\Omega^4$
1	α_1-Sasakian	α_2-Sasakian	1	1	0	0	$4(-2\alpha_2^2\Phi_2^2\wedge\Phi_1^3+24\alpha_2^2\Phi_2^2\wedge\Phi_1^2\wedge\eta_1\wedge\eta_2$ $+6\alpha_1^2\Phi_1^3\wedge\Phi_2^2-16\alpha_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3\wedge\Phi_2)$

Table 9 [cont.]

2	α_1-Sasakian	β_2-Kenmotsu	1	0	0	1	$4(-6\beta_2\alpha_1\Phi_1^3\wedge\Phi_2^2-4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1^3$ $-6\beta_2\alpha_1\Phi_1^2\wedge\Phi_2^3-2\beta_2\alpha_1\Phi_1\wedge\Phi_2^4$ $+12\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3\wedge\Phi_1+8\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^4$ $+6\alpha_1^2\Phi_1^3\wedge\Phi_2^2+2\alpha_1^2\Phi_1^2\wedge\Phi_2^3$ $+8\beta_2\alpha_1\eta_2\wedge\eta_1\wedge\Phi_2^2\wedge\Phi_1^2$ $-16\alpha_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3\wedge\Phi_2)$
3	α_1-Sasakian	Cosymplectic	1	0	0	0	$4(6\alpha_1^2\Phi_1^3\wedge\Phi_2^2-16\alpha_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3\wedge\Phi_2)$
4	β_1-Kenmotsu	β_2-Kenmotsu	0	0	1	1	$4(8\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^4+12\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3\wedge\Phi_2$ $-4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1^3+4\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2^3$ $+12\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3\wedge\Phi_1+8\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^4)$
5	β_1-Kenmotsu	α_2-Sasakian	0	1	1	0	$4(2\beta_1\alpha_2\Phi_2\wedge\Phi_1^4+6\beta_1\alpha_2\Phi_2^2\wedge\Phi_1^3$ $-36\beta_1\alpha_2\Phi_2\wedge\Phi_1^3\wedge\eta_1\wedge\eta_2$ $+8\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^4+12\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3\wedge\Phi_2$ $-2\alpha_2^2\Phi_2^2\wedge\Phi_1^3+24\alpha_2^2\Phi_2^2\wedge\Phi_1^2\wedge\eta_1\wedge\eta_2$ $-60\beta_1\alpha_2\Phi_2^2\wedge\Phi_1^2\wedge\eta_1\wedge\eta_2)$
6	β_1-Kenmotsu	Cosymplectic	0	0	1	0	$4(8\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^4+12\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3\wedge\Phi_2)$
7	Cosymplectic	α_2-Sasakian	0	1	0	0	$4(-2\alpha_2^2\Phi_2^2\wedge\Phi_1^3+24\alpha_2^2\Phi_2^2\wedge\Phi_1^2\wedge\eta_1\wedge\eta_2)$
8	Cosymplectic	β_2-Kenmotsu	0	0	0	1	$4(-4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1^3$ $+12\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3\wedge\Phi_1+8\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^4)$
9	Cosymplectic	Cosymplectic	0	0	0	0	0

REMARK 3.10. In Table 9, Case 9 is an astheno-Kähler manifold.

(iv) $\dim_{\mathbb{R}} M_1 = 9$ and $\dim_{\mathbb{R}} M_2 = 3$

Table 10

	M_1	M_2	α_1	α_2	β_1	β_2	$dd^c\Omega^4$
1	α_1-Sasakian	α_2-Sasakian	1	1	0	0	$4(6\alpha_1^2\Phi_1^4\wedge\Phi_2-16\alpha_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3\wedge\Phi_2)$
2	α_1-Sasakian	β_2-Kenmotsu	1	0	0	1	$4(-2\beta_2\alpha_1\Phi_1^4\wedge\Phi_2-6\beta_2\alpha_1\Phi_1^3\wedge\Phi_2^2$ $-4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1^3+6\alpha_1^2\Phi_1^4\wedge\Phi_2$ $-6\beta_2\alpha_1\Phi_1^2\wedge\Phi_2^3-2\beta_2\alpha_1\Phi_1\wedge\Phi_2^4$ $+12\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3\wedge\Phi_1+8\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^4$ $+6\alpha_1^2\Phi_1^3\wedge\Phi_2^2+2\alpha_1^2\Phi_1^2\wedge\Phi_2^3$ $+8\beta_2\alpha_1\eta_2\wedge\eta_1\wedge\Phi_2^2\wedge\Phi_1^2$ $-16\alpha_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3\wedge\Phi_2)$
3	α_1-Sasakian	Cosymplectic	1	0	0	0	$4(6\alpha_1^2\Phi_1^4\wedge\Phi_2-16\alpha_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3\wedge\Phi_2)$
4	β_1-Kenmotsu	β_2-Kenmotsu	0	0	1	1	$4(8\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^4+12\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3\wedge\Phi_2$ $-4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1^3+4\beta_1^2\eta_2\wedge\eta_1\wedge\Phi_1\wedge\Phi_2^3$ $+12\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3\wedge\Phi_1+8\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^4)$

Table 10 [cont.]

5	β_1-Kenmotsu	α_2-Sasakian	0	1	1	0	$4(2\beta_1\alpha_2\Phi_2\wedge\Phi_1^4 - 36\beta_1\alpha_2\Phi_2\wedge\Phi_1^3\wedge\eta_1\wedge\eta_2$ $+ 8\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^4 + 12\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3\wedge\Phi_2)$
6	β_1-Kenmotsu	Cosymplectic	0	0	1	0	$(8\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^4 + 12\beta_1^2\eta_1\wedge\eta_2\wedge\Phi_1^3\wedge\Phi_2)$
7	Cosymplectic	α_2-Sasakian	0	1	0	0	0
8	Cosymplectic	β_2-Kenmotsu	0	0	0	1	$4(-4\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2\wedge\Phi_1^3$ $+ 12\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^3\wedge\Phi_1 + 8\beta_2^2\eta_1\wedge\eta_2\wedge\Phi_2^4)$
9	Cosymplectic	Cosymplectic	0	0	0	0	0

REMARK 3.11. (1) In Table 10, Cases 7 and 9 yield astheno-Kähler manifolds.

(2) A specific instance of Case 7 can be observed in [MT01, Theorem 4.2].

REMARK 3.12. From [C84], when the manifold is a product of two cosymplectic manifolds, it will always be an astheno-Kähler manifold.

REMARK 3.13. By the above study and using the results of [MT01, Theorem 4.2] and [C84], we can say that if $\dim_{\mathbb{R}} M_1 \geq 5$ and $\dim_{\mathbb{R}} M_2 \geq 5$ then the only possibility for the product manifold M to be astheno-Kähler is that both M_1, M_2 should be cosymplectic. Also, we can observe that for $\dim_{\mathbb{C}} M = m \geq 3$, a product of an α-Sasakian manifold and a cosymplectic manifold is astheno-Kähler only when the real dimension of the α-Sasakian manifold is 3, which gives a more general example of an astheno-Kähler manifold and as mentioned in several remarks, in some cases it can be reduced to previously known examples.

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