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Some novel ways of generating Cantor and Julia type sets

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To Professor Józef Siciak on the occasion of his 80th birthday

Abstract. It is a survey article showing how an enhanced version of the Banach contraction principle can lead to generalizations of attractors of iterated function systems and to Julia type sets.

1. Introduction. Note that the standard proof of the Banach contraction principle deals with a sequence of images of a point under iterations of a contraction. One should therefore expect that this principle can be a useful tool in looking for special sets associated with iterations. Actually one can state the principle in the following way: If we have a complete metric space (Y, ϱ) and a contraction $f : Y \to Y$, then the sequence $(f^n)_{n=1}^{\infty}$ of iterates converges pointwise to a constant mapping, whose value is exactly the desired fixed point. By f^n we mean of course the composition of n copies of f.

Iterating means repeating the same thing again and again. However, since as in Szymborska's poem, "Nothing can ever happen twice" (cf. [S, p. 15]), one might want to be able to change the mapping at each step, and that is the reason for considering the following result [KK1, see Lemma 4.5 and its proof].

Theorem 1.1 (Enhanced version of the Banach contraction principle). *Let (Y, ϱ) be a complete metric space and let $(H_n)_{n=1}^{\infty}$ be a sequence of contractions of Y with contraction ratios not greater than $L < 1$. If*

$$\forall x \in Y : \quad M_x := \sup_{n\geq 1} \varrho(H_n(x), x) < \infty,$$

then the sequence $(H_1 \circ \cdots \circ H_n)_{n=1}^{\infty}$ converges pointwise to a constant mapping.

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Moreover, if $M_K := \sup_{x\in K} M_x < \infty$ *for some* $K \subset Y$, *then the convergence is uniform on* K.

Hutchinson uses the standard version of the Banach contraction principle (see [H, 3.3(1)]) to show (in one of two independent approaches) the existence, uniqueness and compactness of the so called *invariant set* associated with a system of contractions $\{f_1, \ldots, f_k\}$ on a complete metric space X. The invariant set appears namely to be the unique fixed point of the contraction

$$\Phi : \mathcal{CB}(X) \ni A \mapsto \bigcup_{j=1}^{k} f_j(A) \in \mathcal{CB}(X),$$

where $\mathcal{CB}(X)$ is the family of all non-empty closed bounded subsets of X furnished with the Hausdorff metric.

In a similar way Baribeau and Roy deal with a special type of countable (not necessarily finite) iterated function systems (see [BR, Lemma 1]), called for short IFSs. The corresponding invariant set, called here the *attractor* of the IFS, is not always closed in the infinite case. In order to apply the Banach contraction principle as before, one has to take the closure in the formula of the mapping, i.e.

$$\Phi : A \mapsto \overline{\bigcup_{j\in J} f_j(A)},$$

and the unique fixed point is the closure of the attractor.

It is natural to ask what would happen if we took a sequence (H_n) as in Theorem 1.1 instead of the sequence (Φ^n) of iterates. It turns out that this leads to a (bigger) family of (generalized) attractors, but we deal only with the closures.

Another area in which we can apply Theorem 1.1 is complex dynamics. The Julia set associated with a holomorphic mapping f is also defined by iterates $(f^n)_{n=1}^{\infty}$. Let us restrict our investigations to a polynomial mapping $P : \mathbb{C}^N \to \mathbb{C}^N$ (this choice will be explained later). The associated *filled-in Julia set* can be defined by

$$K_+[P] := \{z \in \mathbb{C}^N : (P^n(z))_{n=1}^{\infty} \text{ is bounded}\}.$$

Can the Banach contraction principle be used here? It can. Klimek [K2] defined a special complete metric space consisting of specific non-empty compact subsets of $\mathbb{C}^N$ such that if the *Łojasiewicz exponent* of P, i.e.

$$\operatorname{expt}(P) := \sup\left\{\delta : \liminf_{|z|\to\infty} \frac{|P(z)|}{|z|^\delta} > 0\right\}$$

($|\cdot|$ denotes the Euclidean norm in $\mathbb{C}^N$; for background on the Łojasiewicz exponent see [Pł]), is greater than 1, then the mapping

$$\Psi : K \mapsto P^{-1}(K)$$

is a contraction in this space and the filled-in Julia set $K_+[P]$ is the unique fixed point obtained from the Banach contraction principle (cf. [K2]).

It was quite natural for Klimek to follow in a way the idea of Hutchinson, namely to take a finite family of polynomial mappings $\{P_1, \ldots, P_k\}$ with Łojasiewicz exponents greater than 1 and to define a mapping

$$\Psi : K \mapsto \widehat{\bigcup_{j=1}^{k} P_j^{-1}(K)}$$

($\widehat{A}$ denotes the polynomially convex hull of the set A; just like the closure in the case of infinite iterated function systems, here the polynomially convex hull is needed). Since one deals here with inverse images, Klimek called the family $\{P_1, \ldots, P_k\}$ the *inverse iteration system* and the unique fixed point the *composite Julia set* (see also [K3]). The family of all composite Julia sets defined in this way is a proper and dense subset in a special metric space defined by Klimek (see [K4, Theorem 3]).

Once again a natural question is what happens if one takes a sequence of mappings instead of the iterations of Ψ and if one applies Theorem 1.1. Well, one can obtain a (bigger) family of Julia type sets.

2. Generalized attractors in Banach spaces. Let $(E, \|\cdot\|)$ be a Banach space and $\mathcal{L}(E)$ be the space of bounded linear operators on E furnished with the operator norm. Denote by $\mathcal{A}(E)$ the space of continuous affine operators on E. One can write $\mathcal{A}(E) = \mathcal{L}(E) \oplus E$, since

$$\forall T \in \mathcal{A}(E): \quad \widetilde{T} := T - T(0) \in \mathcal{L}(E),$$

and consider it with the natural norm $\|T\| = \|\widetilde{T}\| + \|T(0)\|$. Note that $T \in \mathcal{A}(E)$ is a contraction if and only if $\|\widetilde{T}\| < 1$.

We have a first consequence of Theorem 1.1 (cf. [KK3, Lemma 2.2]):

Proposition 2.1. *If $(T_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{A}(E)$ such that*

$$\sup_{n\geq 1} \|\widetilde{T_n}\| < 1,$$

then $(T_1 \circ \cdots \circ T_n)_{n=1}^{\infty}$ converges uniformly on bounded sets to a constant mapping.

But we want to generalize iterated function systems. First of all take a bounded family $\mathcal{T} \subset \mathcal{A}(E)$. Then the mapping

$$\Phi_{\mathcal{T}} : \mathcal{CB}(E) \ni K \mapsto \overline{\bigcup_{T\in\mathcal{T}} T(K)} \in \mathcal{CB}(E)$$

is well defined, and if the closure of $\mathcal{T}$ in $\mathcal{A}(E)$ is compact, then $\Phi_{\mathcal{T}}$ maps compact sets to compact sets (see [KK3, Proposition 3.1]). This is the case in particular if E is of finite dimension or the family $\mathcal{T}$ is finite. Furthermore, if

$$\sup_{T\in\mathcal{T}} \|\widetilde{T}\| < 1,$$

then $\Phi_{\mathcal{T}}$ is a contraction on $\mathcal{CB}(E)$ furnished with the Hausdorff metric.

Take now a matrix $M := [T_{n,j}]_{n,j=1}^{\infty}$ with entries in $\mathcal{A}(E)$ such that

$$Q_M := \sup_{n,j\geq 1} \|T_{n,j}\| < \infty \quad \text{and} \quad L_M := \sup_{n,j\geq 1} \|\widetilde{T_{n,j}}\| < 1.$$

Put

$$\mathcal{T}_n := \{T_{n,j} : j \geq 1\}, \quad n \geq 1.$$

Note that $\mathcal{T}_n$ is bounded for each $n \geq 1$. One can apply the first part of Theorem 1.1 to the sequence $(\Phi_{\mathcal{T}_n})_{n=1}^{\infty}$ (see [KK3, Proposition 4.1]) and obtain a constant limit mapping. We call the value of this mapping the *attractor* of the matrix M. Note that actually the order in any row of the matrix does not matter for the attractor. It suffices to take a sequence $(\mathcal{T}_n)_{n=1}^{\infty}$ of countable (finite or not) families of continuous affine operators such that

$$Q_M = \sup_{T\in\bigcup_{n\geq 1}\mathcal{T}_n} \|T\| < \infty \quad \text{and} \quad L_M = \sup_{T\in\bigcup_{n\geq 1}\mathcal{T}_n} \|\widetilde{T}\| < 1$$

and one can define the attractor of the sequence $(\mathcal{T}_n)_{n=1}^{\infty}$. The setting with matrices was however useful to show analytic dependence of the attractor on the initial object. It would be more difficult to define any notion of analyticity for mappings with arguments being sequences of sets. We will not speak here about the analyticity, referring the reader to [KK3].

But the idea of starting with a sequence of sets may make it easier to realize that this is a generalization of IFSs. Namely if $\mathcal{T}_n = \mathcal{T}_1$ for all $n \geq 1$, then we have exactly the iterated function system $\mathcal{T}_1$ and "our" attractor is the closure of the attractor in the sense of IFSs. However, our construction leads to new attractors, hence is a real generalization. Namely, if $(l_n)_{n=0}^{\infty}$ is a given sequence of positive numbers such that $l_0 = 1$ and $2l_n < l_{n-1}$, $n \geq 2$, and we put

$$\mathcal{T}_n := \left\{\mathbb{C} \ni z \mapsto \frac{l_n}{l_{n-1}} z \in \mathbb{C},\, \mathbb{C} \ni z \mapsto \frac{l_n}{l_{n-1}} z + 1 - \frac{l_n}{l_{n-1}} \in \mathbb{C}\right\}, \quad n \geq 1,$$

then the sequence $(\mathcal{T}_n)_{n=1}^{\infty}$ satisfies the above assumptions (cf. [KK3, Section 7]). We can choose $(l_n)_{n\geq 1}$ in such a way that the attractor cannot be obtained by any IFS (which follows from a result in [CR]). Let us note that the Cantor type sets we obtained here are important examples in the constructive theory of functions (cf. [P]).

3. Generalized filled-in Julia sets. If K is a compact set in $\mathbb{C}^N$, we denote by V_K the pluricomplex Green function of K with a pole at infinity (see [K1, Chapter 5]). The set K is *pluriregular* if V_K is continuous. Let us recall a beautiful transformation formula for the pluricomplex Green function of inverse images ([K1, Theorem 5.3.1]):

THEOREM 3.1. *Let α, β be positive numbers and let $f : \mathbb{C}^N \to \mathbb{C}^N$ be a holomorphic mapping. The following conditions are equivalent:*

- *f is a polynomial mapping of degree not greater than β and of Łojasiewicz exponent not smaller than α;*
- *f is proper and for every compact set $K \subset \mathbb{C}^N$,*

$$\frac{1}{\beta} V_K \circ f \le V_{f^{-1}(K)} \le \frac{1}{\alpha} V_K \circ f.$$

Let $\mathcal{R}$ denote the family of all compact pluriregular polynomially convex subsets of $\mathbb{C}^N$ and put

$$\Gamma(K_1, K_2) := \|V_{K_1} - V_{K_2}\|_{\mathbb{C}^N} = \max\{\|V_{K_1}\|_{K_2}, \|V_{K_2}\|_{K_1}\}, \qquad K_1, K_2 \in \mathcal{R}.$$

[K2, Theorem 1] says that $(\mathcal{R}, \Gamma)$ is a complete metric space. Hence in view of Theorem 3.1 the mapping

$$\Psi_P : \mathcal{R} \ni K \mapsto P^{-1}(K) \in \mathcal{R}$$

is a contraction if P is a polynomial mapping of Łojasiewicz exponent greater than 1. Moreover, because of the equivalence in Theorem 3.1 it seems that we cannot use in this construction a holomorphic mapping which is not polynomial.

For our generalization we will consider a simpler situation. If $P : \mathbb{C}^N \to \mathbb{C}^N$ is a polynomial mapping of degree d and $\widetilde{P}$ is the homogeneous part of P of degree d, we say that P is *regular* if $\widetilde{P}^{-1}(0) = \{0\}$. If P is regular of degree $d > 2$, then $\operatorname{expt}(P) = d$ and the contraction ratio of Ψ_P equals $1/d$. The regularity of P is equivalent to the condition

$$\lfloor P \rfloor := \inf_{|z|=1} |\widetilde{P}(z)| > 0.$$

Take now any complex norm $\|\cdot\|$ in the space $\mathcal{P}_d$ of all polynomial mappings $P : \mathbb{C}^N \to \mathbb{C}^N$ of degree not greater than d. We have (cf. [KK1, Proof of Theorem 4.6])

PROPOSITION 3.2. *If $(P_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{P}_d$ such that*

$$\sup_{n\ge 1} \|P_n\| < \infty \quad \text{and} \quad \inf_{n\ge 1} \lfloor P_n \rfloor > 0,$$

then

$$\forall K \in \mathcal{R} : \quad \sup_{n\ge 1} \Gamma(K, \Psi_{P_n}(K)) < \infty.$$

Hence we may apply Theorem 1.1 once again. We obtain a constant mapping, denote its value by $K_+[(P_n)_{n=1}^\infty]$ and call it the *filled-in Julia set* of the sequence $(P_n)_{n=1}^\infty$. This set is also given by the formula

$$K_+[(P_n)_{n=1}^\infty] = \{z \in \mathbb{C}^N : ((P_n \circ \cdots \circ P_1)(z))_{n=1}^\infty \text{ is bounded}\}.$$

4. Generalization of a composite Julia set. The following construction is similar to the one from Section 2 but we take inverse images under polynomial mappings instead of images under affine contractions. In order to obtain analytic dependence of the limit set on the mappings one has to use more complicated arguments (see [KK1] and [KK2]) but we will only sketch the construction leading to Julia type sets here. We will use the same spaces as in Section 3.

Take now a sequence $(\varUpsilon_n)_{n=1}^\infty$ of families $\varUpsilon_n \subset \mathcal{P}_d$, define $\varUpsilon := \bigcup_{n\geq 1} \varUpsilon_n$ and assume that

$$\sup_{P\in\varUpsilon} \|P\| < \infty \quad \text{and} \quad \inf_{P\in\varUpsilon} \lfloor P \rfloor > 0.$$

Under these assumptions the mapping

$$\varPsi_{\varUpsilon_n} : \mathcal{R} \ni K \mapsto \widehat{\bigcup_{P\in\varUpsilon_n} P^{-1}(K)} \in \mathcal{R}$$

is a well defined contraction with contraction ratio $1/d$ (cf. [KK1]). Note however that the relevant union is not always closed.

Now a straightforward generalization of Proposition 3.2 which follows from the behaviour of Γ on unions of sets allows an application of Theorem 1.1 to the sequence $(\varPsi_{\varUpsilon_n})$ and leads to a Julia type set $K_+[(\varUpsilon_n)_{n=1}^\infty]$. This set can also be defined in terms of bounded orbits, namely $K_+[(\varUpsilon_n)_{n=1}^\infty] = \widehat{k_+[(\varUpsilon_n)_{n=1}^\infty]}$, where $k_+[(\varUpsilon_n)_{n=1}^\infty]$ is defined to be the set of all $z \in \mathbb{C}^N$ for which the orbit $((P_n \circ \cdots \circ P_1)(z))_{n=1}^\infty$ is bounded for some sequence $(P_n)_n$ with $P_n \in \varUpsilon_n, n \in \mathbb{N}$ (see [KK2]). Note that this last set is the union of all generalized filled-in Julia sets associated with sequences (P_n) with $P_n \in \varUpsilon_n$, $n \geq 1$.

It may be worth noting that it is really necessary to take the polynomially convex hull in the relationship between $K_+[(\varUpsilon_n)_{n=1}^\infty]$ and $k_+[(\varUpsilon_n)_{n=1}^\infty]$ which can be shown by the following example in the one-dimensional case.

EXAMPLE 4.1. For $j \in \{1,2,3,4\}$, put $a_j := 2i^{j-1}$ and take the finite family $\varUpsilon_n = \{P_1, P_2, P_3, P_4\}$ for $n \geq 1$, where

$$P_j : \mathbb{C} \ni z \mapsto P_j(z) := -\frac{2i}{a_j}(z - a_j)^2 - ia_j + a_j \in \mathbb{C}, \quad j = 1,2,3,4.$$

Then $k_+[(\varUpsilon_n)_{n=1}^\infty]$ is not polynomially convex.

Proof. P_j is conjugate to $z \mapsto z^2 - 2$ and therefore its filled-in Julia set is $K_+[P_j] = [a_j - ia_j, a_j + ia_j]$. Since

$$\bigcup_{j\in\{1,2,3,4\}} K_+[P_j] \subset k_+[(\Upsilon_n)],$$

the origin lies in the polynomially convex hull of this set. However, $|P_j(0)| = 2\sqrt{10}$ and $|P_j(z) - a_j + ia_j| = |z - a_j|^2$, therefore $0 \notin k_+[(\Upsilon_n)]$. ∎

We end this short survey with open questions. It was shown in Section 2 that we deal there really with a generalization of attractors of iterated function systems. We have no such proof for the last two sections, though we are almost sure that we obtain here much bigger families of sets than only the class of all filled-in Julia sets (each associated with only one polynomial mapping). Can one exhibit a filled-in Julia set of a sequence of polynomial mappings which is not a filled-in Julia set of one polynomial mapping? Can one find such a generalized composite (filled-in) Julia set?

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