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Spectral theory of SG pseudo-differential operators on $L^p(\mathbb{R}^n)$

by

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Abstract. To every elliptic SG pseudo-differential operator with positive orders, we associate the minimal and maximal operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and prove that they are equal. The domain of the minimal (= maximal) operator is explicitly computed in terms of a Sobolev space. We prove that an elliptic SG pseudo-differential operator is Fredholm. The essential spectra of elliptic SG pseudo-differential operators with positive orders and bounded SG pseudo-differential operators with orders $0, 0$ are computed.

1. SG pseudo-differential operators. We give in this section a precise introduction to the formal properties of SG pseudo-differential operators, also known as pseudo-differential operators with symbols of global type. In [7], they are also called pseudo-differential operators with exit behavior. SG pseudo-differential operators and related topics can be found in [3], [4], [6], [10], [12] and the references therein.

Let $m_1, m_2 \in (-\infty, \infty)$. Then we let S^{m_1,m_2} be the set of all functions in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$ for which

$$|(D_x^\alpha D_\xi^\beta \sigma)(x,\xi)| \le C_{\alpha,\beta} \langle x\rangle^{m_2-|\alpha|} \langle \xi\rangle^{m_1-|\beta|}, \qquad x, \xi \in \mathbb{R}^n,$$

where $\langle\,\rangle$ denotes the function on $\mathbb{R}^N$ given by

$$\langle z\rangle = (1+|z|^2)^{1/2}, \qquad z \in \mathbb{R}^N,$$

for every positive integer N. A function in S^{m_1,m_2} is said to be an *SG symbol of orders* m_1, m_2. It is clear that if $\sigma \in S^{m_1,m_2}$, then $\sigma \in S^{m_1}$, where S^{m_1} is the class of symbols of classical pseudo-differential operators studied extensively in the book [19] by Wong.

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Let $\sigma \in S^{m_1,m_2}$. Then we define the *pseudo-differential operator* T_σ *with symbol* σ by

$$(T_\sigma\varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\hat{\varphi}(\xi)\,d\xi, \qquad x \in \mathbb{R}^n, \tag{1.1}$$

for all functions φ in the Schwartz space $\mathcal{S}$, where

$$\hat{\varphi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)\,dx, \qquad \xi \in \mathbb{R}^n.$$

It can be proved easily that $T_\sigma : \mathcal{S} \to \mathcal{S}$ is a continuous linear mapping.

The following two results on the basic calculus of SG pseudo-differential operators can be found on page 251 of [7].

THEOREM 1.1. *Let* $\sigma \in S^{m_1,m_2}$ *and* $\tau \in S^{\mu_1,\mu_2}$. *Then* $T_\sigma T_\tau = T_\lambda$, *where* $\lambda \in S^{m_1+\mu_1,m_2+\mu_2}$ *and*

$$\lambda \sim \sum_\mu \frac{(-i)^{|\mu|}}{\mu!}(\partial_\xi^\mu\sigma)(\partial_x^\mu\tau).$$

Here, *the asymptotic expansion means that for every positive integer* M, *there exists a positive integer* N *such that*

$$\lambda - \sum_{|\mu|<N} \frac{(-i)^{|\mu|}}{\mu!}(\partial_\xi^\mu\sigma)(\partial_x^\mu\tau) \in S^{m_1+\mu_1-M,m_2+\mu_2-M}.$$

THEOREM 1.2. *Let* $\sigma \in S^{m_1,m_2}$. *Then the formal adjoint* T_σ^* *of* T_σ *is a pseudo-differential operator* T_τ, *where* $\tau \in S^{m_1,m_2}$ *and*

$$\tau \sim \sum_\mu \frac{(-i)^{|\mu|}}{\mu!}\partial_x^\mu\partial_\xi^\mu\bar{\sigma}.$$

Here, *the asymptotic expansion means that for every positive integer* M, *there exists a positive integer* N *such that*

$$\tau - \sum_{|\mu|<N} \frac{(-i)^{|\mu|}}{\mu!}\partial_x^\mu\partial_\xi^\mu\bar{\sigma} \in S^{m_1-M,m_2-M}.$$

Using the formal adjoint, we can extend the definition of a pseudo-differential operator from the Schwartz space $\mathcal{S}$ to the space $\mathcal{S}'$ of all tempered distributions. Indeed, let $\sigma \in S^{m_1,m_2}$. Then for all u in $\mathcal{S}'$, we define the linear functional $T_\sigma : \mathcal{S} \to \mathbb{C}$ by

$$(T_\sigma u)(\varphi) = u(\overline{T_\sigma^*\overline{\varphi}}), \qquad \varphi \in \mathcal{S}.$$

It is easy to check that T_σ maps $\mathcal{S}'$ into $\mathcal{S}'$ continuously. In fact, we have the following theorem.

THEOREM 1.3. *Let* $\sigma \in S^{0,0}$. *Then* $T_\sigma : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ *is a bounded linear operator for* $1 < p < \infty$.

Theorem 1.3 follows from Theorem 10.7 in [19] and the fact that every symbol in $S^{0,0}$ is in S^0.

Let $\sigma \in S^{m_1,m_2}$, $-\infty < m_1, m_2 < \infty$. Then σ is said to be *elliptic* if there exist positive constants C and R such that

$$|\sigma(x,\xi)| \geq C\langle x\rangle^{m_2}\langle\xi\rangle^{m_1}, \qquad |x|^2 + |\xi|^2 \geq R.$$

THEOREM 1.4. *Let $\sigma \in S^{m_1,m_2}$, $-\infty < m_1, m_2 < \infty$, be elliptic. Then there exists a symbol τ in $S^{-m_1,-m_2}$ such that*

$$T_\tau T_\sigma = I + R \quad \text{and} \quad T_\sigma T_\tau = I + S,$$

where R and S are infinitely smoothing in the sense that they are SG pseudo-differential operators with symbols in $\bigcap_{k_1,k_2\in\mathbb{R}} S^{k_1,k_2}$.

The SG pseudo-differential operator T_τ in Theorem 1.4 is known as a *parametrix* of T_σ.

The aim of this paper is to investigate the spectral theory of SG pseudo-differential operators with symbols in S^{m_1,m_2}, $m_1, m_2 > 0$, on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, in the context of minimal and maximal operators, the domains of elliptic SG pseudo-differential operators, Fredholm operators and essential spectra. Results on essential spectra of bounded SG pseudo-differential operators of orders $0, 0$ on $L^p(\mathbb{R}^n)$ are also given. An essential ingredient in the spectral theory is the family of L^p-Sobolev spaces of orders s_1, s_2, $1 < p < \infty$, $-\infty < s_1, s_2 < \infty$, which we introduce in Section 2. In Section 3, we present the minimal and maximal operators of SG pseudo-differential operators with symbols in S^{m_1,m_2}, $m_1, m_2 > 0$, and show that they are equal for elliptic SG pseudo-differential operators. The main tool is an analogue of the Agmon–Douglis–Nirenberg inequalities for elliptic SG pseudo-differential operators, which we also establish in Section 3. Section 4 is devoted to Fredholmness and essential spectra of elliptic SG pseudo-differential operators with positive orders and bounded SG pseudo-differential operators with orders $0, 0$.

The spectral theory for another class of elliptic pseudo-differential operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, can be found in [20].

2. Sobolev spaces. For $s_1, s_2 \in (-\infty, \infty)$, we let J_{s_1,s_2} be the Bessel potential of orders s_1, s_2 defined by

$$J_{s_1,s_2} = T_{\sigma_{s_1,s_2}},$$

where

$$\sigma_{s_1,s_2}(x,\xi) = \langle x\rangle^{-s_2}\langle\xi\rangle^{-s_1}, \qquad x,\xi \in \mathbb{R}^n.$$

Obviously, $\sigma_{s_1,s_2} \in S^{-s_1,-s_2}$. It can be shown easily that the mapping J_{s_1,s_2} : $\mathcal{S}' \to \mathcal{S}'$ is a bijection and

$$J_{s_1,s_2}^{-1} = J_{-s_1,0}J_{0,-s_2} \tag{2.1}$$

and hence, by Theorem 1.1, $J^{-1}_{s_1,s_2}$ is an SG pseudo-differential operator of orders $-s_1, -s_2$.

For $1 < p < \infty$ and $-\infty < s_1, s_2 < \infty$, we define the L^p-Sobolev space $H^{s_1,s_2,p}$ of orders s_1, s_2 by

$$H^{s_1,s_2,p} = \{u \in \mathcal{S}' : J_{-s_1,-s_2} u \in L^p(\mathbb{R}^n)\}.$$

Then $H^{s_1,s_2,p}$ is a Banach space in which the norm $\| \, \|_{s_1,s_2,p}$ is given by

$$\|u\|_{s_1,s_2,p} = \|J_{-s_1,-s_2} u\|_{L^p(\mathbb{R}^n)}, \qquad u \in H^{s_1,s_2,p},$$

where $\| \, \|_{L^p(\mathbb{R}^n)}$ is the norm in $L^p(\mathbb{R}^n)$. Obviously,

$$H^{0,0,p} = L^p(\mathbb{R}^n).$$

We also have the following simple proposition.

Proposition 2.1. *For $1 < p < \infty$ and $-\infty < s_1, s_2 < \infty$, $J_{-s_1,-s_2} : H^{s_1,s_2,p} \to L^p(\mathbb{R}^n)$ is a surjective isometry.*

Proof. Since

$$\|J_{-s_1,-s_2} u\|_{L^p(\mathbb{R}^n)} = \|u\|_{s_1,s_2,p}, \qquad u \in H^{s_1,s_2,p},$$

it follows that $J_{-s_1,-s_2} : H^{s_1,s_2,p} \to L^p(\mathbb{R}^n)$ is an isometry. For every v in $L^p(\mathbb{R}^n)$, let $u = J^{-1}_{-s_1,-s_2} v$. Then $J_{-s_1,-s_2} u = v \in L^p(\mathbb{R}^n)$, and hence $u \in H^{s_1,s_2,p}$ and the surjectivity is established. ∎

We can now extend the L^p-boundedness result in Theorem 1.3 from symbols in $S^{0,0}$ to symbols in S^{m_1,m_2}, $-\infty < m_1, m_2 < \infty$.

Theorem 2.2. *Let $\sigma \in S^{m_1,m_2}$, $-\infty < m_1, m_2 < \infty$. Then for $1 < p < \infty$ and $-\infty < s_1, s_2 < \infty$, $T_\sigma : H^{s_1,s_2,p} \to H^{s_1-m_1,s_2-m_2,p}$ is a bounded linear operator.*

$$\begin{tikzcd}
H^{s,p} \arrow[r, "T_\sigma"] \arrow[d, "J_{-s}"'] & H^{s-m,p} \arrow[d, "J_{m-s}"] \\
H^{0,p} \arrow[r, "T_\tau"'] & H^{0,p}
\end{tikzcd}$$

Fig. 1. The vector notation for subscripts and superscripts is used. Precisely, $s = (s_1, s_2)$, $m = (m_1, m_2)$ and $0 = (0, 0)$.

Proof. We factorize the pseudo-differential operator T_σ as in Figure 1 and get

$$T_\sigma = J^{-1}_{m_1-s_1,m_2-s_2} T_\tau J_{-s_1,-s_2}, \quad \text{where} \quad T_\tau = J_{m_1-s_1,m_2-s_2} T_\sigma J^{-1}_{-s_1,-s_2}.$$

By Theorem 1.1 and (2.1), we see that T_τ is an SG pseudo-differential operator with symbol in $S^{0,0}$. Hence, by Theorem 1.3, $T_\tau : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded linear operator, and this completes the proof of the theorem. ∎

The following Sobolev embedding theorem is the L^p-analogue of the one in [5].

THEOREM 2.3. *Let* $s_1, s_2, t_1, t_2 \in (-\infty, \infty)$ *be such that* $s_1 \le t_1$ *and* $s_2 \le t_2$. *Then* $H^{t_1,t_2,p} \subseteq H^{s_1,s_2,p}$ *and the inclusion* $i : H^{t_1,t_2,p} \hookrightarrow H^{s_1,s_2,p}$ *is a bounded linear operator.*

We need a lemma.

LEMMA 2.4. *Let* $s_1, s_2 \ge 0$. *Then* $H^{s_1,s_2,p} \subseteq L^p(\mathbb{R}^n)$ *and*

$$\|u\|_{L^p(\mathbb{R}^n)} \le \|u\|_{s_1,s_2,p}, \quad u \in H^{s_1,s_2,p}.$$

Proof. Let $u \in H^{s_1,s_2,p}$. Then $J_{-s_1,-s_2}u \in L^p(\mathbb{R}^n)$. So,

$$\langle\,\rangle^{s_1} J_{-s_2}u \in L^p(\mathbb{R}^n),$$

where J_{-s_2} is the classical pseudo-differential operator with symbol σ_{-s_2} given by

$$\sigma_{-s_2}(\xi) = \langle\xi\rangle^{s_2}, \quad \xi \in \mathbb{R}^n.$$

Therefore $J_{-s_2}u \in L^p(\mathbb{R}^n)$, which is the same as saying that u is the classical L^p-Sobolev space $H^{s_2,p}$ studied in Chapter 11 of [19]. By Theorem 11.5 in [19], $u \in L^p(\mathbb{R}^n)$ and

$$\|u\|_{L^p(\mathbb{R}^n)} \le \|u\|_{s_2,p} \le \|u\|_{s_1,s_2,p},$$

where $\|\;\|_{s_2,p}$ is the norm in $H^{s_2,p}$. ∎

Proof of Theorem 2.3. We first suppose that $s_1 \le t_1$ and $s_2 \le t_2$. By Lemma 2.4,

$$\|u\|_{s_1,s_2,p} = \|J_{-s_1,-s_2}u\|_{L^p(\mathbb{R}^n)} \le \|J_{-s_1,-s_2}u\|_{t_1-s_1,t_2-s_2,p}, \quad u \in H^{t_1,t_2,p}.$$

By Theorem 2.2, there exists a positive constant C such that

$$\|J_{-s_1,-s_2}u\|_{t_1-s_1,t_2-s_2,p} \le C\|u\|_{t_1,t_2,p}, \quad u \in H^{t_1,t_2,p},$$

and this completes the proof of the theorem. ∎

THEOREM 2.5. *Let* $s_1, s_2, t_1, t_2 \in (-\infty, \infty)$ *be such that* $s_1 < t_1$ *and* $s_2 < t_2$. *Then the inclusion* $i : H^{t_1,t_2,p} \hookrightarrow H^{s_1,s_2,p}$ *is a compact operator.*

To prove Theorem 2.5, we recall pseudo-differential operators with symbols first introduced by Grushin [8]. Let $m \in (-\infty, \infty)$. Then we let S_0^m be the set of all functions σ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all multi-indices α and β, there exists a bounded function $C_{\alpha,\beta}$ on $\mathbb{R}^n$ for which

$$|(D_x^\alpha D_\xi^\beta \sigma)(x,\xi)| \le C_{\alpha,\beta}(x)(1+|\xi|)^{m-|\beta|}, \quad x, \xi \in \mathbb{R}^n,$$

and

$$\lim_{|x|\to\infty} C_{\alpha,\beta}(x) = 0$$

for $|\alpha| \neq 0$. For $\sigma \in S_0^m$, the pseudo-differential operator T_σ is defined as in (1.1). Then we have the following theorem proved in [17, 18]. The L^2-version of the theorem is in [8].

THEOREM 2.6. *Let $\sigma \in S_0^m$, $m \in (-\infty, \infty)$, be such that*

$$\lim_{|x|\to\infty} C_{\alpha,\beta}(x) = 0$$

for all multi-indices α and β. Then for every positive number ε, $T_\sigma : H^{s+m,p} \to H^{s-\varepsilon,p}$ is a compact operator for $-\infty < s < \infty$ and $1 < p < \infty$.

We need the following simple consequence of Theorem 2.6.

COROLLARY 2.7. *For every positive number ε, $J_{\varepsilon,\varepsilon} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a compact operator for $1 < p < \infty$.*

Proof of Theorem 2.5. Let ε be a positive number such that

$$t_1 - s_1 - \varepsilon > 0, \qquad t_2 - s_2 - \varepsilon > 0.$$

Since $J_{\varepsilon,\varepsilon}^{-1} J_{-s_1,-s_2}$ is an SG pseudo-differential operator of orders $s_1+\varepsilon, s_2+\varepsilon$, it follows that the composition $J_{\varepsilon,\varepsilon} i J_{\varepsilon,\varepsilon}^{-1} J_{-s_1,-s_2}$ of the mappings

$$\begin{aligned} J_{\varepsilon,\varepsilon}^{-1} J_{-s_1,-s_2} : H^{t_1,t_2,p} &\to H^{t_1-s_1-\varepsilon, t_2-s_2-\varepsilon, p}, \\ i : H^{t_1-s_1-\varepsilon, t_2-s_2-\varepsilon, p} &\hookrightarrow L^p(\mathbb{R}^n), \\ J_{\varepsilon,\varepsilon} : L^p(\mathbb{R}^n) &\to L^p(\mathbb{R}^n) \end{aligned}$$

is compact since $J_{\varepsilon,\varepsilon}^{-1} J_{-s_1,-s_2} : H^{t_1,t_2,p} \to H^{t_1-s_1-\varepsilon, t_2-s_2-\varepsilon, p}$ is a bounded linear operator by Theorem 2.3 and $J_{\varepsilon,\varepsilon} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a compact operator by Corollary 2.7. Thus, the linear operator

$$H^{t_1,t_2,p} \ni u \mapsto J_{\varepsilon,\varepsilon} i J_{\varepsilon,\varepsilon}^{-1} J_{-s_1,-s_2} u = J_{-s_1,-s_2} u \in L^p(\mathbb{R}^n)$$

is compact, and this completes the proof. ∎

3. Minimal and maximal operators. Let $\sigma \in S^{m_1,m_2}$, $m_1, m_2 > 0$. Then we can consider T_σ to be a linear operator from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ with dense domain $\mathcal{S}$ and we can easily prove that it is closable. Thus, the minimal operator $T_{\sigma,0}$ of T_σ exists. In fact, the domain $\mathcal{D}(T_{\sigma,0})$ of $T_{\sigma,0}$ consists of all functions u in $L^p(\mathbb{R}^n)$ for which there exists a sequence $\{\varphi_k\}_{k=1}^\infty$ in $\mathcal{S}$ such that $\varphi_k \to u$ in $L^p(\mathbb{R}^n)$ and $T_\sigma \varphi_k \to f$ for some f in $L^p(\mathbb{R}^n)$ as $k \to \infty$. Moreover, if $u \in \mathcal{D}(T_{\sigma,0})$, then it can be shown that the limit f does not depend on the choice of the sequence $\{\varphi_k\}_{k=1}^\infty$ in $\mathcal{S}$ and so we can define $T_{\sigma,0} u$ to be f.

Let u and f be functions in $L^p(\mathbb{R}^n)$. Then we say that $u \in \mathcal{D}(T_{\sigma,1})$ and $T_{\sigma,1} u = f$ if and only if

$$(u, T_\sigma^* \varphi) = (f, \varphi), \qquad \varphi \in \mathcal{S},$$

where

$$(u, v) = \int_{\mathbb{R}^n} u(x)\overline{v(x)}\,dx$$

for all measurable functions u and v on $\mathbb{R}^n$, provided that the integral exists. It can be proved that $T_{\sigma,1}$ is a closed linear operator from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ with domain $\mathcal{D}(T_{\sigma,1})$ and $\mathcal{S} \subset \mathcal{D}(T_{\sigma,1})$. In fact, we can also prove that $\mathcal{S}$ is contained in the domain $\mathcal{D}(T^t_{\sigma,1})$ of the true adjoint $T^t_{\sigma,1}$ of $T_{\sigma,1}$. Furthermore, $T_{\sigma,1}u = T_\sigma u$ for all u in $\mathcal{D}(T_{\sigma,1})$.

It is a simple fact that $T_{\sigma,1}$ is an extension of $T_{\sigma,0}$. Thus, $\mathcal{D}(T^t_{\sigma,1}) \subseteq \mathcal{D}(T^t_{\sigma,0})$ and hence $\mathcal{S} \subset \mathcal{D}(T^t_{\sigma,0})$. In fact, $T_{\sigma,1}$ is the largest closed extension of T_σ in the sense that if B is any closed extension of T_σ such that $\mathcal{S} \subset \mathcal{D}(B^t)$, then $T_{\sigma,1}$ is an extension of B. So, $T_{\sigma,1}$ is called the *maximal operator* of T_σ.

The main results that we want to prove in this section are given by the following theorems.

THEOREM 3.1. *If* $\sigma \in S^{m_1,m_2}$, $m_1, m_2 > 0$, *is elliptic*, *then* $T_{\sigma,0} = T_{\sigma,1}$.

THEOREM 3.2. *If* $\sigma \in S^{m_1,m_2}$, $m_1, m_2 > 0$, *is elliptic*, *then* $\mathcal{D}(T_{\sigma,0}) = H^{m_1,m_2,p}$.

We give a proof of Theorem 3.2 based on the following result that contains an analogue of the Agmon–Douglis–Nirenberg inequalities in [1] for SG pseudo-differential operators.

THEOREM 3.3. *Let* $\sigma \in S^{m_1,m_2}$, $m_1, m_2 > 0$, *be elliptic. Then there exist positive constants* C_1 *and* C_2 *such that*

$$C_1\|u\|_{m_1,m_2,p} \le \|T_\sigma u\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)} \le C_2\|u\|_{m_1,m_2,p}, \qquad u \in H^{m_1,m_2,p}.$$

Proof. By Theorem 2.2 on boundedness of SG pseudo-differential operators between Sobolev spaces and Theorem 2.3 on the boundedness of the Sobolev embedding, there exists a positive constant C' such that

$$\|T_\sigma u\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)} \le C'\|u\|_{m_1,m_2,p}, \qquad u \in H^{m_1,m_2,p}.$$

Since σ is elliptic, Theorem 1.4 ensures that there exists a symbol τ in $S^{-m_1,-m_2}$ such that

$$u = T_\tau T_\sigma u - Ru, \qquad u \in H^{m_1,m_2,p},$$

where R is an SG pseudo-differential operator with symbol in $\bigcap_{k_1,k_2\in\mathbb{R}} S^{k_1,k_2}$. So, by Theorem 2.2 again, there exists a positive constant C such that

$$\|u\|_{m_1,m_2,p} \le C(\|T_\sigma u\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)}), \qquad u \in H^{m_1,m_2,p}.$$

This completes the proof. ∎

We also need the following result.

PROPOSITION 3.4. *For* $-\infty < s_1, s_2 < \infty$ *and* $1 < p < \infty$, $\mathcal{S}$ *is dense in* $H^{s_1,s_2,p}$.

Proof. Let $u \in H^{s_1,s_2,p}$. Then $J_{-s_1,-s_2}u \in L^p(\mathbb{R}^n)$. Since $\mathcal{S}$ is dense in $L^p(\mathbb{R}^n)$, we can find a sequence $\{\varphi_k\}_{k=1}^\infty$ in $\mathcal{S}$ such that $\varphi_k \to u$ in $H^{s_1,s_2,p}$ as $k \to \infty$. For $k = 1, 2, \ldots$, let $\psi_k = J^{-1}_{-s_1,-s_2}\varphi_k$. Then $\psi_k \in \mathcal{S}$, $k = 1, 2, \ldots$, and

$$\|\psi_k - u\|_{s_1,s_2,p} = \|J_{-s_1,-s_2}\psi_k - J_{-s_1,-s_2}u\|_{L^p(\mathbb{R}^n)} = \|\varphi_k - J_{-s_1,-s_2}u\|_{L^p(\mathbb{R}^n)} \to 0$$

as $k \to \infty$, and this completes the proof of the theorem. ∎

Proof of Theorem 3.2. Let $u \in H^{m_1,m_2,p}$. By the density of $\mathcal{S}$ in $H^{m_1,m_2,p}$, we can find a sequence $\{\varphi_k\}_{k=1}^\infty$ in $\mathcal{S}$ such that $\varphi_k \to u$ in $H^{m_1,m_2,p}$ as $k \to \infty$. By the second of the Agmon–Douglis–Nirenberg inequalities in Theorem 3.3, we see that $\{T_\sigma\varphi_k\}_{k=1}^\infty$ and $\{\varphi_k\}_{k=1}^\infty$ are Cauchy sequences in $L^p(\mathbb{R}^n)$. Hence $\varphi_k \to u$ and $T_\sigma\varphi_k \to f$ for some u and f in $L^p(\mathbb{R}^n)$ as $k \to \infty$. Hence $u \in \mathcal{D}(T_{\sigma,0})$ and $T_{\sigma,0}u = f$. Now, let $u \in \mathcal{D}(T_{\sigma,0})$. Then there exists a sequence $\{\varphi_k\}_{k=1}^\infty$ in $\mathcal{S}$ such that $\varphi_k \to u$ in $L^p(\mathbb{R}^n)$ and $T_\sigma\varphi_k \to f$ for some f in $L^p(\mathbb{R}^n)$ as $k \to \infty$. So, $\{\varphi\}_{k=1}^\infty$ and $\{T_\sigma\varphi_k\}_{k=1}^\infty$ are Cauchy sequences in $L^p(\mathbb{R}^n)$. By the first of the Agmon–Douglis–Nirenberg inequalities in Theorem 3.3, we see that $\{\varphi\}_{k=1}^\infty$ is a Cauchy sequence in $H^{m_1,m_2,p}$. Since $H^{m_1,m_2,p}$ is complete, it follows that $\varphi_k \to v$ in $H^{m_1,m_2,p}$ for some v in $H^{m_1,m_2,p}$ as $k \to \infty$. So, by Theorem 2.3 on the boundedness of the Sobolev embedding, $\varphi_k \to v$ in $L^p(\mathbb{R}^n)$. Hence $u = v$ and $u \in H^{m_1,m_2,p}$. ∎

Proof of Theorem 3.1. Since $T_{\sigma,1}$ is an extension of $T_{\sigma,0}$ and $\mathcal{D}(T_{\sigma,0}) = H^{m_1,m_2,p}$, it is enough to prove that $\mathcal{D}(T_{\sigma,1}) \subseteq H^{m_1,m_2,p}$. Let $u \in \mathcal{D}(T_{\sigma,1})$. Since σ is elliptic, it follows from Theorem 1.4 that there exists a symbol τ in $S^{-m_1,-m_2}$ such that

$$u = T_\tau T_\sigma u - Ru,$$

where R is an SG pseudo-differential operator with symbol in $\bigcap_{k_1,k_2\in\mathbb{R}} S^{k_1,k_2}$. Since

$$T_\sigma u = T_{\sigma,1}u \in L^p(\mathbb{R}^n),$$

it follows from the boundedness of SG pseudo-differential operators between Sobolev spaces in Theorem 2.2 that $u \in H^{m_1,m_2,p}$. So, $\mathcal{D}(T_{\sigma,1}) \subseteq H^{m_1,m_2,p}$, as asserted. ∎

4. Fredholm SG pseudo-differential operators. Let us first recall that a closed linear operator A from a complex Banach space X into a complex Banach space Y with dense domain $\mathcal{D}(A)$ is said to be *Fredholm* if the range $R(A)$ of A is a closed subspace of Y, and the null space $N(A)$ of A and the null space $N(A^t)$ of the true adjoint A^t of A are finite-dimensional. For a Fredholm operator A, the *index* $i(A)$ of A is defined by

$$i(A) = \dim N(A) - \dim N(A^t).$$

The following criterion for a closed linear operator is usually attributed to Atkinson [2].

THEOREM 4.1. *Let A be a closed linear operator from a complex Banach space X into a complex Banach space Y with dense domain $\mathcal{D}(A)$. Then A is Fredholm if and only if we can find a bounded linear operator $B : Y \to X$, a compact operator $K_1 : X \to X$ and a compact operator $K_2 : Y \to Y$ such that $BA = I + K_1$ on $\mathcal{D}(A)$ and $AB = I + K_2$ on Y.*

Let X be a complex Banach space and let A be a closed linear operator from X into X with dense domain $\mathcal{D}(A)$. Then the *spectrum* $\Sigma(A)$ of A is defined in the usual way, i.e.,

$$\Sigma(A) = \mathbb{C} \setminus \varrho(A),$$

where $\varrho(A)$ is the *resolvent set* of A given by

$$\varrho(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is bijective}\}$$

and I is the identity operator on X. The *essential spectrum* $\Sigma_{\mathrm{w}}(A)$ *of A in the sense of Wolf* [16] is defined by

$$\Sigma_{\mathrm{w}}(A) = \mathbb{C} \setminus \Phi_{\mathrm{w}}(A), \quad \text{where} \quad \Phi_{\mathrm{w}}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is Fredholm}\}.$$

An important fact is that $i(A - \lambda I)$ is a constant for all λ in a connected component of $\Phi_{\mathrm{w}}(A)$. The *essential spectrum* $\Sigma_{\mathrm{s}}(A)$ *of A in the sense of Schechter* [13] is defined by

$$\Sigma_{\mathrm{s}}(A) = \mathbb{C} \setminus \Phi_{\mathrm{s}}(A), \quad \text{where} \quad \Phi_{\mathrm{s}}(A) = \{\lambda \in \Phi_{\mathrm{w}}(A) : i(A - \lambda I) = 0\}.$$

All the results on Fredholm operators hitherto described can be found in the books [14], [15] by Schechter.

The first main result in this section is the following theorem.

THEOREM 4.2. *Let $\sigma \in S^{m_1,m_2}$, $m_1, m_2 > 0$, be elliptic. Then for $1 < p < \infty$, $T_{\sigma,0}$ is a Fredholm operator on $L^p(\mathbb{R}^n)$ with domain $H^{m_1,m_2,p}$. Furthermore, if $\sigma \in S^{0,0}$ is elliptic, then the bounded linear operator $T_\sigma : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is Fredholm.*

Proof. Since σ is elliptic, it follows from Theorem 1.4 that there exists a symbol τ in $S^{-m_1,-m_2,p}$ such that

$$T_\tau T_\sigma = I + R \quad \text{and} \quad T_\sigma T_\tau = I + S,$$

where R and S are infinitely smoothing in the sense that they are SG pseudo-differential operators with symbols in $\bigcap_{k_1,k_2 \in \mathbb{R}} S^{k_1,k_2}$. So, for all positive numbers t_1 and t_2, the linear operator $R : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is the same as the composition of the linear operators $R : L^p(\mathbb{R}^n) \to H^{t_1,t_2,p}$ and $i : H^{t_1,t_2,p} \hookrightarrow L^p(\mathbb{R}^n)$. Since $R : L^p(\mathbb{R}^n) \to H^{t_1,t_2,p}$ is bounded by Theorem 2.2 and $i : H^{t_1,t_2,p} \hookrightarrow L^p(\mathbb{R}^n)$ is compact by Theorem 2.5, it follows that $R : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is compact. Similarly, $S : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is compact. So, by Theorem 4.1, $T_{\sigma,0}$ is a compact, as asserted. The proof for the Fredholmness of $T_\sigma : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ when $\sigma \in S^{0,0}$ is the same. ∎

The following theorem gives the essential spectrum in the sense of Wolf of an elliptic SG pseudo-differential operator.

THEOREM 4.3. *Let* $\sigma \in S^{m_1,m_2}$, $m_1, m_2 > 0$, *be elliptic. Then*

$$\Sigma_{\mathrm{w}}(T_{\sigma,0}) = \emptyset.$$

Proof. We only need to prove that $\sigma - \lambda$ is elliptic for all λ in $\mathbb{C}$, for then by Theorem 4.2, $T_{\sigma,0} - \lambda I$ is a Fredholm operator on $L^p(\mathbb{R}^n)$ with domain $H^{m_1,m_2,p}$. By the ellipticity of σ, there exist positive constants C and R such that

$$|\sigma(x,\xi) - \lambda| \geq C\langle x\rangle^{m_2}\langle\xi\rangle^{m_1} = \langle x\rangle^{m_2}\langle\xi\rangle^{m_1}\left(C - \frac{|\lambda|}{\langle x\rangle^{m_2}\langle\xi\rangle^{m_1}}\right)$$

whenever $|x|^2 + |\xi|^2 \geq R$. Since $\langle x\rangle^{m_2}\langle\xi\rangle^{m_1} \to \infty$ as $|x|^2 + |\xi|^2 \to \infty$, it follows that there exists another positive constant R' such that

$$|\sigma(x,\xi) - \lambda| \geq \frac{C}{2}\langle x\rangle^{m_2}\,\langle\xi\rangle^{m_1}$$

whenever $|x|^2 + |\xi|^2 \geq R'$. Thus, $\sigma - \lambda$ is elliptic. ∎

The proof of Theorem 4.3 depends on the hypothesis that σ is a symbol with positive orders. If $\sigma \in S^{0,0}$, then by Theorem 2.2, $T_\sigma : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded linear operator and we have the following result on the essential spectra of T_σ.

THEOREM 4.4. *Let* $\sigma \in S^{0,0}$. *Then*

$$\Sigma_{\mathrm{s}}(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq L_{\mathrm{s}}\}, \qquad \Sigma_{\mathrm{w}}(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \geq L_{\mathrm{i}}\},$$

where

$$L_{\mathrm{i}} = \liminf_{|(x,\xi)|\to\infty} |\sigma(x,\xi)|, \qquad L_{\mathrm{s}} = \limsup_{|(x,\xi)|\to\infty} |\sigma(x,\xi)|.$$

REMARK 4.5. A proof for $\Sigma_{\mathrm{s}}(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq L_{\mathrm{s}}\}$ can be found in [11] by Nicola and Rodino using more advanced techniques. We give a completely elementary proof here.

Proof of Theorem 4.4. Let $\lambda \in \mathbb{C}$ be such that $|\lambda| > L_{\mathrm{s}}$. Let ε be a positive number such that

$$|\lambda| - \varepsilon > L_{\mathrm{s}}.$$

Then there exists a positive number R such that

$$\sup_{|x|^2+|\xi|^2\geq R} |\sigma(x,\xi)| < L_{\mathrm{s}} + \varepsilon/2.$$

So, for $|x|^2 + |\xi|^2 \geq R$,

$$|\sigma(x,\xi) - \lambda| \geq |\lambda| - |\sigma(x,\xi)| > L_{\mathrm{s}} + \varepsilon - L_{\mathrm{s}} - \varepsilon/2 = \varepsilon/2.$$

Therefore $\sigma - \lambda$ is elliptic and hence $T_\sigma - \lambda I : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is Fredholm by Theorem 2.2. Thus, $\{\lambda \in \mathbb{C} : |\lambda| > L_{\mathrm{s}}\} \subseteq \Phi_{\mathrm{w}}(T_\sigma)$, which is the same as

$$\Sigma_{\mathrm{w}}(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \le L_{\mathrm{s}}\}.$$

Since $i(T_\sigma - \lambda I)$ is constant for all λ in $\{\lambda \in \mathbb{C} : |\lambda| > L_{\mathrm{s}}\}$ and

$$\varrho(T_\sigma) \cap \{\lambda \in \mathbb{C} : |\lambda| > L_{\mathrm{s}}\} \neq \emptyset,$$

it follows that $i(T_\sigma - \lambda I) = 0$ for all λ in $\{\lambda \in \mathbb{C} : |\lambda| > L_{\mathrm{s}}\}$. Thus,

$$\Sigma_{\mathrm{s}}(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \le L_{\mathrm{s}}\}.$$

Now, let $\lambda \in \mathbb{C}$ be such that $|\lambda| < \lambda_i$. Let ε be a positive number such that

$$|\lambda| + \varepsilon < L_{\mathrm{i}}.$$

Then there exists a positive number R such that

$$\inf_{|x|^2+|\xi|^2 \ge R} |\sigma(x,\xi)| > L_{\mathrm{i}} - \varepsilon/2.$$

So, for $|x|^2 + |\xi|^2 \ge R$,

$$|\sigma(x,\xi) - \lambda| \ge |\sigma(x,\xi)| - |\lambda| > L_{\mathrm{i}} - \varepsilon/2 - L_{\mathrm{i}} + \varepsilon = \varepsilon/2.$$

Therefore $\sigma - \lambda$ is elliptic and hence $T_\sigma - \lambda I : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is Fredholm by Theorem 4.2. Thus, $\{\lambda \in \mathbb{C} : |\lambda| < L_{\mathrm{i}}\} \subseteq \Phi_{\mathrm{w}}(T_\sigma)$, or $\Sigma_{\mathrm{w}}(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \ge L_{\mathrm{i}}\}$. ∎

As a consequence of Theorem 4.4, we have the following *spectral alternative* for a class of elliptic SG pseudo-differential operators with symbols in $S^{0,0}$.

THEOREM 4.6. *Let $\sigma \in S^{0,0}$ be such that*

$$\lim_{|(x,\xi)| \to \infty} |\sigma(x,\xi)| = L > 0.$$

Then

$$\Sigma_{\mathrm{w}}(T_\sigma) = \{\lambda \in \mathbb{C} : |\lambda| = L\} \quad \textit{or} \quad \Sigma_{\mathrm{s}}(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = L\}.$$

Proof. By Theorem 4.4,

$$\Sigma_{\mathrm{w}}(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = L\}.$$

Suppose that $\Sigma_{\mathrm{w}}(T_\sigma) \neq \{\lambda \in \mathbb{C} : |\lambda| = L\}$. Then there exists a complex number λ_0 such that $|\lambda_0| = L$ and $\lambda_0 \in \Phi_{\mathrm{w}}(T_\sigma)$. So, using the first conclusion in Theorem 4.4, the fact that $\Phi_{\mathrm{w}}(T_\sigma)$ is an open set and that $i(T_\sigma - \lambda I)$ is constant on every connected component of $\Phi_{\mathrm{w}}(T_\sigma)$, we see that $i(T_\sigma - \lambda I) = 0$ for all λ in $\mathbb{C}$ with $|\lambda| \neq L$. So,

$$\Sigma_{\mathrm{s}}(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = L\},$$

as asserted. ∎

Remark 4.7. Suppose that

$$\Sigma_{\mathrm{w}}(T_\sigma) = \{\lambda \in \mathbb{C} : |\lambda| = L\}.$$

Then the best that we can say about $\Sigma_{\mathrm{s}}(T_\sigma)$ is given by the first conclusion of Theorem 4.4 to the effect that

$$\Sigma_{\mathrm{s}}(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \le L\}.$$

To see this by means of an example, let σ be the symbol in $S^{0,0}$ given by

$$\sigma(x, \xi) = Le^{i \arg(x+i\xi)}$$

for all x and ξ in $\mathbb{R}$ such that $x^2 + \xi^2 \ge 1$. Then $T_\sigma : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a Fredholm operator with nonzero index. In other words, for $p = 2$, $0 \in \Sigma_{\mathrm{s}}(T_\sigma) \setminus \Sigma_{\mathrm{w}}(T_\sigma)$. So,

$$\Sigma_{\mathrm{s}}(T_\sigma) \nsubseteq \{\lambda \in \mathbb{C} : |\lambda| = L\}.$$

See, for instance, Theorem 2.3 in Chapter 5 of the book [9] by Kumano-go for more details.

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