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Lattice points and Weyl's formula for the disc

by

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Dedicated to Henryk Iwaniec on his 75th birthday

Abstract. Following Kuznetsov and Fedosov and Colin de Verdière, we interpret counting eigenvalues of the Laplacian on the unit disc as a lattice point counting problem in analytic number theory. We obtain Weyl's eigenvalue counting theorem with an area term, a boundary term, and a remainder term as small as that currently known in the Gauss circle problem.

1. Introduction on lattice points. Let C be a convex closed curve in the plane. Then C encloses an area A which may be estimated by counting squares: choose a large integer R, and draw a lattice of squares of side $1/R$. Change coordinates so that these squares become the unit squares $Q(m,n)$, $m \le x \le m+1$, $n \le y \le n+1$, where m and n are integers. We count all the squares $Q(m,n)$ that fall inside the curve, and some squares $Q(m,n)$ cut by the curve. The Geography Rule counts the square $Q(m,n)$ when the point $(m+1/2, n+1/2)$ lies inside or on the curve. The Number Theory Rule counts the square $Q(m,n)$ when the point (m,n) lies inside or on the curve. To be inclusive, we consider Rule (α,β): having chosen real numbers α in $0 \le \alpha < 1$, and β in $0 \le \beta < 1$, we count the square $Q(m,n)$ when the point $(m+\alpha, n+\beta)$ lies inside or on the curve. Then N, the number of squares counted, is of the form

$$N = AR^2 + P, \tag{1.1}$$

where P, called the *lattice point remainder* (*Gitterrest*), has fewer digits to the left of the decimal point than R^2, so N/R^2 is an approximation to the

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area A. To indicate the choice of R, α, and β, we also write (1.1) as

$$N(R;\alpha,\beta) = AR^2 + P(R;\alpha,\beta). \tag{1.2}$$

By integrating over each unit square, we see that

$$\int_0^1\int_0^1 P(R,\alpha,\beta)\,d\alpha\,d\beta = 0. \tag{1.3}$$

Since $P(R;\alpha,\beta)$ is discontinuous as a function of α and β, (1.3) does not imply that some choice of α and β in Rule (α,β) makes $P(R;\alpha,\beta)$ zero. We can also regard $P(R;\alpha,\beta)$ as the remainder under the Number Theory rule when the curve C is translated by the vector (α,β). Translations by integer vectors do not change the lattice point remainder, so we may suppose that the origin $(0,0)$ lies inside C.

Suppose (for purposes of exposition) that the curve C surrounds the origin. Near the ends of the y-axis we describe C by Cartesian equations of the form $y = f(x)$, but near the ends of the x-axis, by equations of the form $x = g(y)$. In fact, we consider C to be partitioned into a finite number of arcs C_i such that on each arc there is a change of coordinates (rotation followed by dilation)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{a^2+b^2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{1.4}$$

where a and b are co-prime integers, after which there is a suitable equation $y' = f(x')$, called a *chart* of the arc C_i. The choices $(a,b) = (\pm1,0)$ and $(0,\pm1)$ preserve the set of integer points. In general integer points (m,n) for which the Gaussian integer $m+in$ lies in the ideal $J = \langle a+ib\rangle$ become integer points (m',n'). There are a^2+b^2 residue classes modulo J in the Gaussian integers, each of the form $J+j$ for some $j = 0,1,\dots,a^2+b^2-1$, and they map to points $(m'+\gamma_j, n'+\delta_j)$ in the (x',y') plane for some rational numbers γ_j and δ_j.

The lattice point remainder can be partitioned into contributions $P(C_i;R;\alpha,\beta)$ from each arc C_i, with

$$P(R;\alpha,\beta) = \sum_i (P(C_i;R;\alpha,\beta) + O(1)). \tag{1.5}$$

Let the arc C_i go to C_i' and the vector (α,β) to (α',β') under the linear map in (1.4). Then

$$P(C_i;R;\alpha,\beta) = \sum_{j=0}^{a^2+b^2-1}\left(P\left(C_i';\frac{R}{\sqrt{a^2+b^2}};\alpha'+\gamma_j,\beta'+\delta_j\right)+O(1)\right). \tag{1.6}$$

The correction term $O(1)$ in (1.5) and (1.6) is needed because the endpoints of the arcs C_i and C_i' need not be integer points.

The different charts on different arcs C_i are not merely a tedious necessity. They let us consider lattice points in regions such as the intersection of two ellipses, or the first quadrant of a circle, which have different equations on different arcs of the boundary. For the quadrant, two of the arcs C_i are straight line segments. When C_i is a line segment, the contribution $P(C_i; R; \alpha, \beta)$ to the lattice point remainder is known to great accuracy. The image C_i' under the chart map is some line segment whose gradient γ lies in $-1 \le \gamma \le 1$. The real number $|\gamma|$ has a continued fraction expansion, $0/1$ and $1/1$ at the ends of the range, and for $0 < |\gamma| < 1$, $1/a_1 + 1/a_2 + \cdots$ with convergents p_r/q_r obtained by truncating the expansion at a_r. Suppose that C_i' lies in some subinterval of $0 \le x \le L$, where L is a positive integer. Let $n = \max\{r \mid q_r \le L\}$. Then $P(C_i; R; \alpha, \beta)$ has terms of size $O(a_r)$ for $r = 1, \ldots, n$, and $O(L/q_n)$, which depend on the positions of C_i' and the point (α', β') with respect to the integer lattice. The contribution of the line segment C_i is zero if C_i' lies midway between two adjacent lattice lines of gradient γ, or if C_i' lies along a lattice line of gradient γ, but we count lattice points on C_i' with weight $1/2$.

Arcs C_i which are not straight line segments or approximate straight line segments give smaller contributions $P(C_i; R; \alpha, \beta)$ to the lattice point remainder. The various approaches to estimating $P(C_i; R; \alpha, \beta)$ when C_i is a curve each give an estimate with the minimum absolute values of some derivatives (or determinants of derivatives) in the denominator. Lattice point problems in number theory [8, 9, 10, 11, 12] usually require each arc C_i to be three times continuously differentiable. We have scaled up by a factor R, so the chart maps $y = f(x)$ satisfy

$$|f^{(r)}(x)| \le B_r/R^{r-1} \quad \text{for } r = 1, 2, 3 \tag{1.7}$$

with some absolute constants B_r. The second derivative always appears in the denominator, so we require

$$|f''(x)| \ge B_0/R \tag{1.8}$$

with some absolute positive constant B_0. These conditions become more elegant when expressed in terms of the intrinsic coordinates on the arc C_i: the tangent angle ψ, the arc length s (measured in the direction of ψ increasing), and the radius of curvature $\rho = ds/d\psi$. Then we require

$$c_0 R \le \rho \le c_1 R, \qquad \left|\frac{d\rho}{d\psi}\right| \le c_2 R \tag{1.9}$$

for some positive constants c_0, c_1, c_2.

Complicated methods may require further lower bound conditions besides (1.8). For a given method, a problem where these conditions are satisfied is called *standard*, and a problem where one or more conditions fail is

called *non-standard*. For example, (1.8) fails for an arc C_i with an asymptote.

There are at least four approaches to estimating the contributions $P(C_i; R; \alpha, \beta)$ to the lattice point remainder:

(1) Approximating the arc C_i by a polygonal line with rational gradients.
(2) A real variable argument.
(3) Taking the Fourier transform of the plane region bounded by C.
(4) Taking the Fourier transform along the curve C.

Approach (3) leads to the same calculations as approach (4), after we use the divergence theorem. Nonetheless, approach (3) gives the tantalising mean value theorem, valid when each arc C_i is a standard curve,

$$\int_0^1\int_0^1 P(R;\alpha,\beta)^2\,d\alpha\,d\beta = O(R). \tag{1.10}$$

Our object is to find a bound of the form

$$|P(R;\alpha,\beta)| \le B(C,\epsilon)R^{\theta+\epsilon} \tag{1.11}$$

for any $\epsilon > 0$, where $B(C,\epsilon)$ is a positive constant determined by the curve C and the exponent ϵ. The bounds (1.11) are uniform under translation by (α,β), but (1.10) shows that for (α,β) in a subset containing at least half the unit square, (1.11) holds with $\theta = 1/2$ and $\epsilon = 0$ for some constant $B(C,0)$. Moreover, (1.10) comes from Parseval's identity, and there is a corresponding lower bound which shows that we can only take $\theta \ge 1/2$ in (1.11).

Approaches (1) to (4) all give (1.11) with $\theta = 2/3$, and, after careful estimation, with $\epsilon = 0$. Further progress entails truncating Fourier series, and using some non-trivial estimate for mean squares of short exponential sums. The Weyl Differencing Lemma (also called the van der Corput Lemma) leads to the van der Corput iteration (see Graham and Kolesnik [5]), with uncountably many branches even in its simplest form in which we reduce the length of the sum step by step, and finally we estimate trivially. Each use of the Weyl Lemma introduces an extra summand, and the lengths of these outside sums are not reduced. When the sum over the original summand becomes shorter than some outside sum, we would like to perform the iteration to shorten this sum instead. In practice, switching summands leads to an error explosion within two or three steps.

The Bombieri–Iwaniec method [1] combines approaches (1) and (4). The short sums correspond to sides of the approximating polygon. The mean square bound is the "large sieve inequality" for exponential sums of the form $\sum\sum e(\mathbf{x}_j\cdot\mathbf{y}_i)$ (where we have written $e(t)$ for $\exp 2\pi i t$). The vectors $\mathbf{x}_j$ are constructed from the summands in a multiple Fourier series. The vectors

$\mathbf{y}_i$ are constructed from the gradient of a polygon side and the function $y = f(x)$ of the chart map. The form of the upper bound leads to two "spacing problems", estimating the number of pairs of vectors in a neighbourhood of the diagonal. The First Spacing Problem counts solutions of a certain set of inequalities involving positive integers. The Second Spacing Problem counts Möbius transformations that map one polygon side and its short arc of C to another such. The free parameters of the method are chosen so that the diagonal terms dominate in both spacing problems.

The Bombieri–Iwaniec method gives values of θ in (1.11) with $\theta \geq 5/8$. The first application of the method to lattice point problems, by Iwaniec and Mozzochi [14], had

$$\theta = \frac{7}{11} = \frac{5}{8} + \frac{1}{88},$$

already better than the theoretical limit of the van der Corput iteration. The current best bound in Huxley [9] has

$$\theta = \frac{131}{208} = \frac{5}{8} + \frac{1}{208}.$$

Before he died, Bourgain was adapting his "decoupling method" to the various systems of inequalities met in the First Spacing Problem, replacing the mean squares of short sums by Hölder inequalities in some ranges. Dr. Watt (private communication) says that Bourgain and Watt have improved the bound for short mean squares of exponential sums (where the First Spacing Problem is intermediate between those from single exponential sums and from lattice point counting), but that the progress report [2] on arXiv, treating the Dirichlet divisor problem and the Gauss circle problem, has been criticised, and the proof must be considered incomplete. The improvement in the divisor problem would have extended to general lattice point problems, after more careful choice of the chart maps on arcs of the boundary curve.

2. The eigenvalues for the unit disc. Ivriĭ [13] has a precise version of Weyl's formula that counts the eigenvalues of the Laplacian on a bounded connected smooth plane domain D with Dirichlet boundary conditions satisfying a certain condition on periodic billiard paths. Let $N(M)$ be the number of eigenvalues λ with $(0 <)\ \lambda \leq M^2$. Then

$$N(M) = \frac{\text{area}(D)M^2}{4\pi} - \frac{\text{length}(\partial D)M}{4\pi} + R(M), \tag{2.1}$$

where $R(M)/M$ tends to 0 as $M \to \infty$. When D is the unit square, the eigenfunctions are $\sin \pi m x \cdot \sin \pi n y$ for all pairs of positive integers m and n so that $\lambda = \pi^2(m^2 + n^2)$, and

$$N(M) = \frac{M^2}{4\pi} - \frac{M}{\pi} + \frac{1}{4}P\left(\frac{M}{\pi}\right) + O(1),$$

where $P(r)$ is the remainder in the Gauss Circle Problem for the circle $x^2+y^2 \le r^2$. Non-trivial estimates are known for $P(r)$ of the form

$$P(r) = O(R^{\theta+\epsilon})$$

(for any $\epsilon > 0$) with various values of θ in

$$\frac{5}{8} < \theta \le \frac{2}{3}.$$

For the Laplacian with Neumann boundary conditions the eigenfunctions are $\cos \pi mx \cdot \cos \pi ny$ for non-negative integers m and n, and

$$N(M) = \frac{M^2}{4\pi} + \frac{M}{\pi} + \frac{1}{4}P\left(\frac{M}{\pi}\right) + O(1).$$

When D is the unit disc, the eigenfunctions of the Laplacian with Dirichlet boundary conditions are (in polar coordinates) $J_m(\mu r)\cos m\theta$ for $m \ge 0$ and $J_m(\mu r)\sin m\theta$ for $m \ge 1$, where we choose μ so that $J_m(\mu) = 0$. The eigenvalue is μ^2. We write $\mu(m,n)$ for the nth positive zero of the Bessel function $J_m(r)$. With Neumann boundary conditions we choose μ so that $J'_m(\mu) = 0$. The eigenvalue is μ^2. We write $\nu(m,n)$ for the nth positive zero of $J'_m(r)$. For $m \ge 1$, $\mu(m,n)^2$ and $\nu(m,n)^2$ are double eigenvalues of the disc. Siegel [17] showed that 0 is the only common zero of any two Bessel functions $J_m(r)$ and $J_{m'}(r)$. Hence the eigenvalue $\mu(m,n)$ has multiplicity exactly 2.

When D is a sector $0 \le \theta \le 2\pi/a$, $0 \le r \le 1$ of the unit disc, then the eigenfunctions are $J_{ak}(\mu r)\sin ak\theta$ for $k = 1,2,3,\ldots$. For $0 < a < 1/2$, the domain D can be embedded in a Riemann surface, but not in the Euclidean plane. We keep the notation $J_m(r)$ and $\mu(m,n)$ when $m \ge 0$ is considered to be a continuous real variable. Following [15] and [3], we also make n a continuous real variable

$$n = \frac{1}{\pi}\arg H_m(r) - \frac{1}{2},$$

where the exponential integral $H_m(r)$ is defined in (2.5) below.

The minimal nodal domains of the eigenfunction $J_0(\mu(0,n)r)$ are the annuli

$$\frac{\mu(0.k-1)}{\mu(0,n)} \le r \le \frac{\mu(0,k)}{\mu(0,n)}$$

and the central circle $r \le \mu(0,1)/\mu(0,n)$. For $m > 0$, the minimal nodal domains of $J_m(\mu(m,n)r)$ are sectors of annuli

$$\frac{2\pi b}{m} \le \theta \le \frac{2\pi(b+1)}{m}, \qquad \frac{\mu(m,k-1)}{\mu(m,n)} \le r \le \frac{\mu(m,k)}{\mu(m,n)}.$$

For m and k large, these sectors are approximately rectangles.

Lemma 1 (Comparison Lemma). *Let D and D' be bounded closed connected domains in the plane. Suppose that ℓ copies $D_1,\ldots,D_\ell$ of D fit in-*

side D', with $D_i \cap D_j \subset \partial D_i \cap \partial D_j$, so that they meet only at boundaries. Let $\lambda_1(D') \le \lambda_2(D') \le \cdots$ be the Dirichlet eigenvalues of D', counted with multiplicities. Let f be a Dirichlet eigenfunction of D with k minimal nodal domains $E_1, \ldots, E_k$ and eigenvalue λ. Then

$$\lambda \ge \lambda_{k\ell}(D').$$

COROLLARY. *The zeros $\mu(m,n)$ of the Bessel functions $J_m(r)$ for $m \ge 0$ satisfy*

$$\mu(m,1) \ge \mu(m',1) \qquad \text{for } m > m', \tag{2.2}$$

$$\mu(m,n) \ge \min(2m, \mu(m',n) \qquad \text{for } m > m', \tag{2.3}$$

and

$$\mu(m,1) \ge m, \qquad \mu(m,1) \ge m + \frac{26\pi^2}{3m} + O\left(\frac{1}{m^3}\right) \qquad \text{for large } m. \tag{2.4}$$

Proof of Lemma 1. There are rigid motions $T_1, \ldots, T_\ell$ with $D_j = T_j D$. Define $f_{ij}(x,y)$ to be $f(T_j(x,y))$ on $T_j E_i$, and 0 elsewhere. Then f_{ij} is continuous, differentiable, and lies in the space $L^2(D')$ with Rayleigh quotient λ. The functions f_{ij} are 0 outside D'. They generate a subspace of $L^2(D')$ of dimension $k\ell$ in which every function has Rayleigh quotient λ. Hence $\lambda \ge \lambda_{k\ell}(D')$. ∎

Proof of the Corollary. For (2.2) and (2.3) we take D and D' as sectors of the unit circle with angles $2\pi/m$ and $2\pi/m'$. For (2.4) we take D as a sector of angle $2\pi/m$, and D' as a rectangle divided by its diagonal into two right-angled triangles with one angle $2\pi/m$. We use (2.4) in the proof of (2.3) to ensure that the eigenvalue does not come from a zero of $J_{km}(r)$ with $k \ge 2$. ∎

The lemma remains true if we allow reflected copies of D as well as copies by rigid motions. But each nodal domain E_i for the unit disc has a symmetry axis through the origin, so a reflected copy of D is already a copy by a rigid motion.

More information about the zeros of Bessel functions comes from Bessel's integral representation

$$J_m(r) = \Re H_m(r) = \Re \frac{1}{\pi} \int_0^\pi e(F(t))\, dt, \tag{2.5}$$

where

$$e(u) = \exp 2\pi i u, \qquad F(t) = \frac{mt - r\sin t}{2\pi}. \tag{2.6}$$

We use the derivatives

(2.7)

$$F'(t) = \frac{m - r\cos t}{2\pi}, \quad F''(t) = \frac{r\sin t}{2\pi}, \quad F'''(t) = \frac{r\cos t}{2\pi}, \quad F''''(t) = -F''(t).$$

There is a unique stationary phase point at $t = \tau$, where

$$\cos\tau = m/r. \tag{2.8}$$

Lemma 2 (Stationary phase estimate). *For*

$$r > m \geq 0, \qquad 64r \leq (r-m)^3 \tag{2.9}$$

the Bessel function $J_m(r)$ satisfies

$$J_m(r) = \sqrt{\frac{2}{\pi r \sin\tau}}\cos\left(m\tau - r\sin\tau + \frac{\pi}{4}\right) + O\left(\frac{1}{r-m}\right), \tag{2.10}$$

$$J_m'(r) = \sqrt{\frac{2\sin\tau}{\pi r}}\sin\left(m\tau - r\sin\tau + \frac{\pi}{4}\right) + O\left(\frac{1}{\sqrt{r(r-m)}}\right), \tag{2.11}$$

where τ is the unique angle in $0 < \tau \leq \pi/2$ with $m = r\cos\tau$.

Proof. We divide the range of integration into three parts. On the central part $a \leq t \leq b$, we write $t = \tau + x$ and make the substitution

$$\lambda y^2 = F(\tau + x) - F(\tau), \quad \text{where} \quad \lambda = \frac{F''(\tau)}{2} = \frac{r\sin\tau}{4\pi}. \tag{2.12}$$

We can now explain the choices of a and b. We put

$$c = 1/r^{1/3}, \tag{2.13}$$

and take $t = a$ at $y = -c$, $t = b$ at $y = c$, with

$$\tau/2 \leq a < \tau < b \leq 3\tau/2. \tag{2.14}$$

We shall verify below that (2.14) follows from the choice (2.13) of c and the conditions (2.9) on r and m.

We compare the integral of $e(F(t))$ with

$$\int_{-\infty}^{\infty} e(\lambda y^2)\,dy = \frac{e(1/8)}{\sqrt{2\lambda}}. \tag{2.15}$$

We expand λy^2 and its derivative as Taylor series in x with remainder terms:

$$\lambda y^2 = F(\tau+x) - F(\tau) = \frac{x^2}{2}F''(\tau) + \frac{x^3}{6}F'''(\tau) + \frac{x^4}{24}F''''(\tau+z), \tag{2.16}$$

$$2\lambda y = F'(\tau+x)\frac{dx}{dy} = \left(xF''(\tau) + \frac{x^2}{2}F'''(\tau) + \frac{x^3}{6}F''''(\tau+w)\right)\frac{dx}{dy}, \tag{2.17}$$

for some points z and w between 0 and x. We substitute the values of the derivatives from (2.7) and (2.12). The fourth derivatives in (2.16) and (2.17) are $O(\lambda)$. After manipulating the power series, we have

$$\begin{aligned} y &= x\left(1 + \frac{x}{6}\cot\tau + O\left(\frac{x^2}{\tau^2}\right)\right), \\ \frac{dx}{dy} &= 1 - \frac{x}{3}\cot\tau + O\left(\frac{x^2}{\tau^2}\right) = 1 - \frac{y}{3}\cot\tau + O\left(\frac{y^2}{\tau^2}\right), \end{aligned} \tag{2.18}$$

and thus

$$\int_a^b e(F(t))\,dt = e(F(\tau)) \int_{-c}^{c} e(\lambda y^2)\frac{dx}{dy}\,dy = e(F(\tau)) \int_{-c}^{c} e(\lambda y^2)\,dy + O\left(\frac{c^3}{\tau^2}\right). \tag{2.19}$$

To extend the range of integration to that in (2.13), we add two equal integrals, which we treat by partial integration:

$$\begin{aligned}\int_{-\infty}^{-c} e(\lambda y^2)\,dy = \int_c^{\infty} e(\lambda y^2)\,dy &= \left[\frac{e(\lambda y^2)}{4\pi i\lambda y}\right]_c^{\infty} + \int_c^{\infty} \frac{e(\lambda y^2)}{4\pi i\lambda y^2}\,dy \\ &= -\frac{e(\lambda c^2)}{4\pi i\lambda c} + \left[\frac{e(\lambda y^2)}{(4\pi i\lambda)^2 y^3}\right]_c^{\infty} + \int_c^{\infty} \frac{3e(\lambda y^2)}{(4\pi i\lambda)^2 y^4}\,dy \\ &= -\frac{e(\lambda c^2)}{4\pi i\lambda c} + O\left(\frac{1}{c^3\lambda^2}\right).\end{aligned} \tag{2.20}$$

A slightly more complicated integration by parts gives

$$\begin{aligned}\int_0^a e(F(t))\,dt &= \left[\frac{e(F(t))}{2\pi i F'(t)}\right]_0^a + \int_0^a \frac{F''(t)e(F(t))}{2\pi i F'(t)^2}\,dt \\ &= \frac{e(F(a))}{2\pi i F'(a)} - \frac{e(F(0))}{2\pi i F'(0)} + \int_0^a \frac{r\sin t}{2\pi i(m - r\cos t)^2}\, e\left(\frac{mt - r\sin t}{2\pi}\right) dt \\ &= \frac{e(F(\tau) + \lambda c^2)}{2\pi i F'(a)} + \frac{1}{i(r-m)} + \left[\frac{r\sin t}{(2\pi i)^2(m - r\cos t)^3}\, e\left(\frac{mt - r\sin t}{2\pi}\right)\right]_0^a \\ &\quad + \int_0^a e\left(\frac{mt - r\sin t}{2\pi}\right)\frac{r}{2\pi}\,\frac{d}{dt}\,\frac{\sin t}{(m - r\cos t)^3}\,dt.\end{aligned} \tag{2.21}$$

Since the function $\sin t/(r\cos t - m)^3$ is increasing on $[0, a]$, we estimate the last integral in (2.21) as

$$\begin{aligned}O\left(\int_0^a \left|\frac{d}{dt}\,\frac{r\sin t}{(m - r\cos t)^3}\right| dt\right) &= O\left(\frac{r\sin a}{(r\cos a - m)^3}\right) \\ &= O\left(\frac{r\tau}{c^3\lambda^3}\right) = O\left(\frac{1}{c^3 r\tau^2}\right).\end{aligned} \tag{2.22}$$

By (2.8), (2.12), and (2.13), the remainder terms in (2.19), (2.20), and (2.22) are all

$$O\left(\frac{1}{r\tau^2}\right) = O\left(\frac{1}{r-m}\right). \tag{2.23}$$

Hence we have

$$\int_0^a e(F(t))\,dt = \frac{e(F(\tau)+\lambda c^2)}{2\pi i F'(a)} + O\Big(\frac{1}{r-m}\Big). \tag{2.24}$$

Similarly

$$\int_b^\pi e(F(t))\,dt = -\frac{e(F(\tau)+\lambda c^2)}{2\pi i F'(b)} + O\Big(\frac{1}{r-m}\Big). \tag{2.25}$$

Combining (2.15), (2.19), (2.20), (2.24), (2.25), and the error estimate (2.23), we have

$$\begin{aligned}\int_0^\pi e(F(t))\,dt \\ = \frac{e(F(\tau)+1/8)}{\sqrt{F''(\tau)}} + \frac{e(F(\tau)+\lambda c^2)}{2\pi i}\Big(\frac{1}{\lambda c} + \frac{1}{F'(a)} - \frac{1}{F'(b)}\Big) + O\Big(\frac{1}{r-m}\Big).\end{aligned} \tag{2.26}$$

Now by (2.18),

$$\frac{1}{F'(a)} = \frac{1}{2\lambda y}\,\frac{dx}{dy}\bigg|_{y=-c} = -\frac{1}{2\lambda c}\Big(1 + \frac{c}{3}\cot\tau + O\Big(\frac{c^2}{\tau^2}\Big)\Big), \tag{2.27}$$

$$\frac{1}{F'(b)} = \frac{1}{2\lambda y}\,\frac{dx}{dy}\bigg|_{y=c} = \frac{1}{2\lambda c}\Big(1 - \frac{c}{3}\cot\tau + O\Big(\frac{c^2}{\tau^2}\Big)\Big). \tag{2.28}$$

Thus the second term in (2.26) is

$$O\Big(\frac{c}{\lambda\tau^2}\Big) = O\Big(\frac{c}{r\tau^3}\Big) = O\Big(\frac{1}{r\tau^2}\Big) = O\Big(\frac{1}{r-m}\Big),$$

and

$$\int_0^\pi e(F(t))\,dt = \sqrt{\frac{2\pi}{r\sin\tau}}\,e\Big(F(\tau)+\frac{1}{8}\Big) + O\Big(\frac{1}{r-m}\Big), \tag{2.29}$$

which gives (2.10) of the lemma when we divide by π and take the real part. A similar but longer calculation gives (2.11).

Finally, we verify that the choice (2.13) of c satisfies (2.14). We have to check that

$$F(2\tau) - F(\tau) \ge \lambda c^2, \qquad F(\tau) - F(\tau/2) \ge \lambda c^2. \tag{2.30}$$

From (2.9),

$$\frac{4}{c} = 4r^{1/3} \le r - m = r(1-\cos\tau) \le \frac{r\tau^2}{2} = \frac{\tau^2}{2c^3}, \qquad 8c^2 \le \tau^2. \tag{2.31}$$

Since $F''(t)$ is positive and increasing, and $F'(\tau)=0$, by (2.31) we have

$$\begin{aligned} F(\tau)-F\left(\frac{\tau}{2}\right) &\geq \frac{1}{2}\left(\frac{\tau}{2}\right)^2 F''\left(\frac{\tau}{2}\right) = \frac{\tau^2}{8}\,\frac{r}{2\pi}\sin\frac{\tau}{2} \\ &\geq \frac{\tau^2}{8}\,\frac{r}{2\pi}\sin\frac{\tau}{2} = \frac{\lambda\tau^2}{8} \geq \lambda c^2, \end{aligned}$$

which is the second inequality in (2.30); the first inequality follows similarly. ∎

We have not assumed that m is an integer in Lemma 2, so we can compare our result with the known values when $2m$ is an odd integer. They take the form

$$J_m(r)=\sqrt{\frac{2}{\pi r}}\left(P_m\left(\frac{1}{r}\right)\sin r+Q_m\left(\frac{1}{r}\right)\cos r\right), \tag{2.32}$$

where $P_m(x)$ and $Q_m(x)$ are polynomials of degree at most $2m-1$ with integer coefficients, and one of $P_m(x)$, $Q_m(x)$ is even with constant term ± 1, the other is odd. When we expand the main term in (2.10) as $\sin r$ times a power series in $1/r$, then only the constant term ± 1 agrees with (2.32). Hence for these values of m, (2.10) holds with a better error estimate $O(A_m/r^{3/2})$, where A_m is some constant.

LEMMA 3. *Let $N(M)$ denote the number of Dirichlet eigenvalues $\lambda=\mu^2$ of the unit disc with $0<\mu\leq M$. There is a constant A for which*

$$2N_1(M,A)\leq N(M)\leq 2N_2(M,A)+O(\sqrt{M}), \tag{2.33}$$

where $N_1(M,A)$ and $N_2(M,A)$ are the numbers of integer points (m,n) in the closed regions in the first quadrant bounded by the axes, the abscissa

$$x=M-M^{2/5}, \tag{2.34}$$

and two curves $C_1(M,A)$ and $C_2(M,A)$ respectively, with the convention that integer points on the x-axis are not counted, and integer points on the y-axis are counted with weight $1/2$. The curves $C_1(M,a)$ and $C_2(M,A)$ are given parametrically by

(2.35)
$$x=M\cos\tau,\quad y=\frac{M}{\pi}(\sin\tau-\tau\cos\tau)+\frac{1}{4}\pm\frac{A}{\sqrt{M\tau^3}},\qquad \textit{for } 0<\tau\leq\frac{\pi}{2},$$

where we take the minus sign for $C_1(M,A)$, and the plus sign for $C_2(M,A)$.

Similarly let $N'(M)$ denote the number of Neumann eigenvalues $\lambda=\nu^2$ of the unit disc with $0<\nu\leq M$. There is a constant A for which

$$2N_3(M,A)-1\leq N'(M)\leq 2N_4(M,A)+O(\sqrt{M}), \tag{2.36}$$

where $N_3(M,A)$ and $N_4(M,A)$ are defined similarly, with curves $C_3(M,A)$

and $C_4(M,A)$ given parametrically by

(2.37)
$$x = M\cos\tau, \quad y = \frac{M}{\pi}(\sin\tau - \tau\cos\tau) + \frac{3}{4} \pm \frac{A}{\sqrt{M\tau^3}}, \qquad \textit{for } 0 < \tau \le \frac{\pi}{2},$$

where we take the minus sign for $C_3(M,A)$, and the plus sign for $C_4(M,A)$.

Proof. By (2.29),

$$\begin{aligned} H_m(r) &= \frac{1}{\pi}\int_0^\pi e(F(t))\,dt \\ &= \sqrt{\frac{2}{\pi r\sin\tau}}\, e\left(\frac{m\tau - r\sin\tau}{2\pi} + \frac{1}{8}\right)\left(1 + O\left(\frac{1}{\sqrt{r\tau^3}}\right)\right), \end{aligned}$$

where τ is defined by $m = r\cos\tau$. We deduce that

$$\arg H_m(r) = \arg e(F(\tau)) + \frac{\pi}{4} + O\left(\frac{1}{\sqrt{r\tau^3}}\right).$$

Now $F(\tau)$ is continuous, and $F(\tau) \to 0$ as $t \to 0$ and $r \to 0$. Thus

$$\arg e(F(\tau)) = 2\pi F(\tau),$$

and

$$\arg H_m(r) = m\tau - r\sin\tau + \frac{\pi}{4} + O\left(\frac{1}{\sqrt{r\tau^3}}\right). \tag{2.38}$$

Similarly

$$\arg H'_m(r) = m\tau - r\sin\tau - \frac{\pi}{4} + O\left(\frac{1}{\sqrt{r\tau^3}}\right). \tag{2.39}$$

We choose A in the lemma so that when $r = M$, the absolute values of the error terms in (2.38) and (2.39) are at most $\pi A/\sqrt{M\tau^3}$ for $0 < \tau \le \pi/2$.

In (2.38) and (2.39), $m\tau - r\sin\tau$ is negative, so the zeros $\mu(m,n)$ of $J_m(r)$ occur where

$$\begin{aligned} &\arg H_m(\mu(m,n)) = -n\pi + \frac{\pi}{2}, \\ &n = \frac{\mu(m,n)}{\pi}(\sin\tau - \tau\cos\tau) + \frac{1}{4} + O\left(\frac{1}{\sqrt{\mu(m,n)\tau^3}}\right). \end{aligned} \tag{2.40}$$

There are two cases for the zeros $\nu(m,n)$ of $J'_m(r)$. For $m \ge 1$, $\Re H_m(r)$ is positive for small r, and

$$n = \frac{\nu(m,n)}{\pi}(\sin\tau - \tau\cos\tau) + \frac{3}{4} + O\left(\frac{1}{\sqrt{\mu(m,n)\tau^3}}\right). \tag{2.41}$$

But $\Re H_0(r)$ is negative for small r. Since $\mu(0,n) < \nu(0,n) < \mu(0,n+1)$, we

take $\arg H_0'(r)$ to be close to $-\pi$ for small r, so that

$$\begin{aligned} &\arg H_0'(\nu(0,n)) = -(n+1)\pi + \frac{\pi}{2}, \\ &n = \frac{\nu(0,n)}{\pi}(\sin\tau - \tau\cos\tau) - \frac{1}{4} + O\biggl(\frac{1}{\sqrt{\mu(m,n)\tau^3}}\biggr). \end{aligned} \tag{2.42}$$

We check that these approximations are valid. For τ small we have

$$\sin\tau - \tau\cos\tau = \frac{\tau^3}{3} + O(\tau^5), \tag{2.43}$$

$$r - m = r(1-\cos\tau) = \frac{r\tau^2}{2} + O(r\tau^4). \tag{2.44}$$

Hence for $m < r \le M$, $m \le M - M^{2/5}$ we have

$$\begin{aligned} &\frac{r\tau^2}{2}(1+O(\tau^2)) \ge M^{2/5}, \\ &r\tau^3 \ge (2\sqrt{2}+O(\tau^2))\frac{M^{3/5}}{\sqrt{r}} \ge (2\sqrt{2}+O(\tau^2))M^{1/10}, \end{aligned} \tag{2.45}$$

and our approximations are valid indeed.

Now fix $m \le M - M^{2/5}$. Let $m = r\cos\tau$ as usual. If r increases, then τ decreases and $\arg H_m(r)$ decreases. At $r = M$,

$$\arg H_m(M) = 2\pi F(\tau_0) + \frac{\pi}{4} + \frac{\alpha}{\sqrt{M\tau_0^3}},$$

where $\cos\tau_0 = m/M$, for some α in $-\pi A \le \alpha \le \pi A$. If $\mu(m,n) \le M$, then

$$\begin{aligned} -n\pi + \frac{\pi}{2} &= \arg H_m(\mu(m,n)) \\ &\ge \arg H_m(M) \ge 2\pi F(\tau_0) + \frac{\pi}{4} - \frac{\pi A}{\sqrt{M\tau_0^3}}, \\ n &\le M\sin\tau_0 - m\tau_0 + \frac{1}{4} + \frac{A}{\sqrt{M\tau_0^3}}, \end{aligned} \tag{2.46}$$

so the point (m,n) lies on or below the upper curve $C_2(M,A)$. If $\mu(m,n) > M$, then similarly

$$n > M\sin\tau_0 - m\tau_0 + \frac{1}{4} - \frac{A}{\sqrt{M\tau_0^3}},$$

and the point (m,n) lies above the lower curve $C_1(M,A)$.

When $m = m_0$, the largest integer with $m_0 \le M - M^{2/5}$, then by (2.44),

$$\begin{aligned} &M\biggl(1 + O\biggl(\frac{1}{M^{3/5}}\biggr)\biggr)\frac{\tau_0^2}{2}(1+O(\tau_0^2)) = M^{2/5} + O(1), \\ &\tau_0 = \frac{\sqrt{2}}{M^{3/10}}\biggl(1 + O\biggl(\frac{1}{M^{3/5}}\biggr)\biggr), \end{aligned}$$

so by (2.46) and (2.43),

$$n \le M(\sin\tau_0 - \tau_0\cos\tau_0) + O(1) = O(M^{1/10}). \tag{2.47}$$

We can now estimate the number of eigenvalues $\mu^2(m,n)$ such that $M - M^{2/5} < m \le M$ and $\mu(m,n) \le M$. Let n_0 be the largest integer n with $\mu(m_0,n) \le M$. Then $n_0 < 2m-1$ by (2.47), provided that M is sufficiently large. By the Corollary to Lemma 1, $\mu(m,n_0+1)$ increases with m. So eigenvalues with $m > m_0$ have $n \le n_0$. There are at most $(M-m_0)n_0 = O(\sqrt{M})$ of them. Of course, if M is bounded, then $N(M)$ is bounded, and trivially $N(M) = O(\sqrt{M})$.

From the differential equation the zeros of $J_m(r)$ and $J'_m(r)$ interlace. Hence for $m > M - M^{2/5}$ we have $n = O(M^{1/10})$ for the zeros $\nu(m,n)$ of $J'_m(r)$ as well, and there are $O(\sqrt{M})$ zeros $\nu(m,n)$ in this region. This completes the proof of Lemma 3. ∎

Lemma 3 corresponds to [3, Lemma 5], which is an inequality between integer-valued counting functions, plus a remainder term which is usually less than 1. The curves in [3] corresponding to our C_1 and C_2 are closer together. Kuznetsov and Fedosov [15] used an asymptotic formula of Olver [16] for the nth zero of $J_m(r)$, with an error term at least $O(1/m)$; they had to treat zeros of Bessel functions with m small trivially. A paper [7] by Guo, Wang and Wang gains a small improvement on the exponent 2/3. However, the proof is incomplete: they ignore the error term in Lemma 2, so their equation (1.2) is our (2.35) with $A = 0$. Their result is weak, because they use the same chart map for all ranges.

Lemma 4. *The regions in the first quadrant bounded by the coordinate axes, the abscissa $x = M - M^{2/5}$ and the curves given parametrically by*

$$x = M\cos\tau, \quad y = \frac{M}{\pi}(\sin\tau - \tau\cos\tau) \pm \frac{A}{\sqrt{M\tau^3}} \quad \textit{for } 0 < \tau \le \frac{\pi}{2}$$

have area

$$\frac{M^2}{8} + O(\sqrt{M}). \tag{2.48}$$

Proof. First we note that

$$\int_0^{\pi/2} \frac{A}{\sqrt{M\tau^3}}\,|dx| = A\int_0^{\pi/2} \frac{M\sin\tau}{\sqrt{M\tau^3}}\,d\tau$$
$$\le A\int_0^{\pi/2} \sqrt{\frac{M}{\tau}}\,d\tau = A\sqrt{2\pi M} = O(\sqrt{M}),$$

so the contribution of the second term is absorbed by the error term in (2.48).

Next,

$$\int_0^{\pi/2} \frac{M}{\pi}(\sin\tau - \tau\cos\tau)\,|dx| = \frac{M^2}{\pi}\int_0^{\pi/2}(\sin^2\tau - \tau\sin\tau\cos\tau)\,d\tau$$
$$= \frac{M^2}{2\pi}\int_0^{\pi/2}(1-\cos 2\tau - \tau\sin 2\tau)\,d\tau = \frac{M^2}{8}.$$

We have integrated from 0 to M, but, as in the proof of Lemma 3, the subregion to the right of the abscissa $x = M - M^{2/5}$ has area $O(\sqrt{M})$. ∎

For our next lemma, a restatement of [9, Theorem 3], we use the notation $[t]$ for the largest integer n with $n \le t$.

LEMMA 5 (Counting integer points). *Let $C_1, \ldots, C_5$ be real numbers with $C_i \ge 1$. Let M and N be large real parameters, and let $T = MN$. Let $F(x)$ be a real function three times continuously differentiable for $1 - 1/M \le x \le 2 + 1/M$, with*

$$|F^{(r)}(x)| \le C_r \qquad \text{for } r = 1, 2, 3, \tag{2.49}$$

$$|F^{(r)}(x)| \ge 1/C_r \qquad \text{for } r = 1, 2. \tag{2.50}$$

Let $\kappa = 3/10$ and $\lambda = 57/140$. Suppose that

$$C_5^{-1}T^{67\kappa-6}(\log T)^{(45\kappa-4)\lambda} \le M^{156\kappa-14} \le C_5 T^{89\kappa-8}(\log T)^{-(45\kappa-4)\lambda}. \tag{2.51}$$

Let M_2 be an integer in the range $M \le M_2 \le 2M - 1$. Then

$$\left|\sum_{m=M}^{M_2}\left[NF\left(\frac{m}{M}\right)\right] - \int_{M-1/2}^{M_2+1/2}\left(NF\left(\frac{x}{M}\right) - \frac{1}{2}\right)dx\right| \le BT^{\theta/2}(\log T)^{\phi},$$

where

$$\theta = \frac{67\kappa - 7}{106\kappa - 11}, \qquad \phi = \frac{459\kappa - 45 + (39\kappa - 4)\lambda}{212\kappa - 22},$$

and B is a positive constant constructed from $C_1, C_2, C_3, C_5, \kappa, \lambda$.

Every regular polygon inscribed in the unit circle is a periodic billiard path, so there are many closed trajectories. Nevertheless, Lemma 3 leads to two-term asymptotics for the number of eigenvalues. Kuznetsov and Fedosov [15] used van der Corput's elementary treatment of the lattice point remainder [4] to get $R(M) = O(M^{2/3})$ in (2.1). Colin de Verdière [3] gave a longer account, which still lacks detail for the case of small zeros of Bessel functions with large index. More recent results [9] let us estimate the remainder $R(M)$ with the same accuracy as for eigenvalues of the square.

THEOREM. *For M large, the number of eigenvalues $\lambda = \mu^2$ of the Laplacian on the unit disc with $\mu \le M$ is*

$$\frac{M^2}{4} - \frac{M}{2} + O(M^\theta (\log M)^\phi) \quad \textit{under Dirichlet boundary conditions,}$$
$$\frac{M^2}{4} + \frac{M}{2} + O(M^\theta (\log M)^\phi) \quad \textit{under Neumann boundary conditions,}$$

with

$$\theta = \frac{131}{208} = \frac{5}{8} + \frac{1}{208}, \qquad \phi = \frac{18627}{8320}. \tag{2.52}$$

COROLLARY. *The Minakshisundaram–Pleijel zeta function $\sum 1/\lambda^s$ has a meromorphic continuation to $\Re s > \theta/2$, with poles at $s = 1$ and $s = 1/2$.*

Proof of the Theorem. Lemma 3 reduces counting eigenvalues to counting integer points in plane regions bounded by three straight lines and a curve $C_i(M, A)$. There are four boundary contributions. Integer points on the y-axis are counted with weight $1/2$, so the y-axis contributes $O(1)$. The line $x = M - M^{2/5}$ is $O(M^{1/10})$ long, so it contributes $O(M^{1/10})$ trivially. Integer points on the x-axis are not counted, so the x-axis contributes $-(M - M^{2/5})/2 + O(1)$.

Lemma 5 treats the contribution of the curves $C_i(M, A)$ after we prepare the sums and align the notation. We divide the range for τ into subintervals I_q, $[\pi/(4q), \pi/(2q)]$, where q runs through the powers of 2, $q = 1, 2, 4, 8, \ldots$. Let M^* be the smallest integer with $M^* \ge M$. We put

$$k = M^* - m = M - m + O(1) = M(1 - \cos\tau) + O(1).$$

We subdivide the sum over τ in I_q again by $k = \ell q + a$, $a = 0, 1, \ldots, q-1$, so that

$$\ell = \frac{2M}{q} \sin^2 \frac{\tau}{2} + O(1), \qquad \frac{M}{8q^3} + O(1) \le \ell \le \frac{\pi^2 M}{8q^3} + O(1).$$

We cover the range for ℓ with four intervals of the form $L \le \ell \le L_2$ $(\le 2L-1)$.

The curves C_i have the parametric form

$$x = qu + a = M(1 - \cos\tau), \qquad y = \frac{M}{\pi}(\sin\tau - \tau\cos\tau) + \frac{B}{\sqrt{M\tau^3}} + C,$$

giving

$$\begin{aligned} \frac{dy}{du} &= \frac{q\tau}{\pi} - \frac{3Bq}{2M^{3/2}} \cdot \frac{1}{\tau^{5/2}\sin\tau} \\ &= \frac{q\tau}{\pi}\left(1 + O\left(\left(\frac{q^3}{M}\right)^{3/2}\right)\right) = \frac{q\tau}{\pi}\left(1 + O\left(\frac{1}{L^{3/2}}\right)\right). \end{aligned}$$

Further differentiations give

$$\frac{d^2y}{du^2} = \frac{q^2}{\pi M \sin\tau}\left(1 + O\left(\frac{1}{L^{3/2}}\right)\right),$$
$$\frac{d^3y}{du^3} = -\frac{q^3\cos\tau}{\pi M^2 \sin^3\tau}\left(1 + O\left(\frac{1}{L^{3/2}}\right)\right).$$

Now we have

$$\frac{1}{4} \le \frac{q\tau}{\pi} \le \frac{1}{2},$$
$$\frac{1}{4\pi^2 L} + O\left(\frac{1}{L^2}\right) \le \frac{2q^3}{\pi^2 M} \le \frac{q^2}{\pi M \sin\tau} \le \frac{2q^3}{\pi M} \le \frac{4}{\pi L} + O\left(\frac{1}{L^2}\right),$$
$$\frac{q^3\cos\tau}{\pi M^2 \sin^3\tau} \le \frac{64q^6}{\pi^4 M^2} \le \frac{1}{L^2} + O\left(\frac{1}{L^3}\right).$$

We may now apply Lemma 5 with $\ell, L, L_2, M^2/\pi$ for m, M, M_2, T, and with $C_1 = 2$, $C_2 = 16$, $C_3 = 4$, $C_5 = 1$. The bounds (2.49), (2.50), and (2.51) hold for L sufficiently large. Since L has order of magnitude at least $M^{1/10}$, we require M to be sufficiently large in terms of the constant A. The function F is given by $y' = F(x')$, defined implicitly by

$$qLx' + a = M^* - M\cos\tau, \qquad Ly' = M(\sin\tau - \tau\cos\tau) + \frac{B\pi}{\sqrt{M\tau^3}} + C\pi.$$

On the interval I_q with τ small we have

$$y' = \sqrt{\frac{2q^3L}{9M}}\, x'^{3/2}\left(1 + O(\tau^2) + O\left(\frac{1}{\sqrt{M\tau^3}}\right)\right).$$

The expression under the first square root sign is bounded above and below.

By Lemma 5 applied to each subinterval $L \le \ell \le L_2$, the contribution of each curve $C_i(M, A)$ to the remainder term is

$$\begin{aligned} &\le \sum_q 4qB\left(\frac{L^2}{\pi}\right)^{\theta/2}(\log M)^\phi \le \sum_q 4qB\left(\frac{\pi^3 M}{8q^3}\right)^\theta(\log M)^\phi \\ &\le \frac{4\sqrt{2}}{\sqrt{2}-1}B\left(\frac{\pi^3}{8}\right)^\theta M^\theta(\log M)^\phi = O(M^\theta(\log M)^\phi), \end{aligned} \tag{2.53}$$

where we have given L the maximum value possible on I_q, and then summed q through powers of 2. The constant B is constructed from the known constants $C_1, C_2, C_3, C_5, \kappa, \lambda$, so (2.53) holds with an absolute constant in the O-sign.

We can now estimate the counting numbers $N_i(M, A)$ in Lemma 3. The area below the curves $C_i(M, A)$ is given by Lemma 4, plus the quantity $(M - M^{2/5})/4$ for $C_1(M, A)$ and $C_2(M, A)$, which are shifted by $1/4$, or plus $3(M - M^{2/5})/4$ for $C_3(M, A)$ and $C_4(M, A)$, which are shifted by $3/4$. The boundary correction for the x-axis is $-(M - M^{2/5})/2$. The other three

boundary corrections have size (2.53) or smaller. Substituting into (2.33) and (2.36) of Lemma 3 gives the Theorem with the error term exponents in (2.52).

The Corollary follows by the Mellin transform. ∎

A recent paper [6] by Guo, Müller, Wang and Wang considers the more difficult problem of the Dirichlet eigenvalues of an annulus for which the ratio r/R of the radii is a rational number, a/q in lowest terms. Using chart maps well, those authors obtain the result corresponding to our Theorem. They claim that the estimate for Dirichlet eigenvalues of the disc follows by letting r tend to zero. But the height q of the ratio r/R must occur in the error terms, and $q \to \infty$ as $r \to 0$, so no deduction is possible for the eigenvalues of the disc.

References

[1] E. Bombieri and H. Iwaniec, *On the order of* $\zeta(\frac{1}{2}+it)$, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 13 (1986), 449–472.

[2] J. Bourgain and N. Watt, *Mean square of zeta function, circle problem and divisor problem revisited*, arXiv:1709.04340v1 (2017).

[3] Y. Colin de Verdière, *On the remainder in the Weyl formula for the Euclidean disk*, in: Séminaire de Théorie Spectrale et Géométrie 29 (2010-2011), Inst. Fourier, Univ. Grenoble I, Saint-Martin-d'Hères, 2011, 1–13.

[4] J. G. van der Corput, *Over roosterpunten in het platte vlak*, Noordhoff, Groningen, 1919.

[5] S. W. Graham and G. Kolesnik, *Van der Corput's Method of Exponential Sums*, London Math. Soc. Lecture Note Ser. 126, Cambridge Univ. Press, Cambridge, 1991.

[6] J. Guo, W. Müller, W. Wang and Z. Wang, *The Weyl formula for planar annuli*, J. Funct. Anal. 281 (2021), no. 4, art. 109063, 38 pp.

[7] J. Guo, W. Wang and Z. Wang, *An improved remainder estimate in the Weyl formula for the planar disk*, J. Fourier Anal. Appl. 25 (2019), 1553–1579.

[8] M. N. Huxley, *Area, Lattice Points, and Exponential Sums*, Clarendon Press, Oxford Univ. Press, 1996.

[9] M. N. Huxley, *Exponential sums and lattice points III*, Proc. London Math. Soc. (3) 87 (2003), 591–609.

[10] M. N. Huxley, *The tiled circle problem*, Mathematika 60 (2014), 321–346.

[11] M. N. Huxley, *The convex hull of the lattice points inside a curve*, Period. Math. Hungar. 68 (2014), 100–118.

[12] M. N. Huxley and J. Žunić, *The number of configurations in lattice point counting I*, Forum Math. 22 (2010), 127–152.

[13] V. Ya. Ivriĭ, *Second term of the spectral asymptotic expansion of the Laplace–Beltrami operator on manifolds with boundary*, Funct. Anal. Appl. 14 (1980), 98–106.

[14] H. Iwaniec and C. J. Mozzochi, *On the divisor and circle problems*, J. Number Theory 29 (1988), 60–93.

[15] N. V. Kuznetsov and B. V. Fedosov, *An asymptotic formula for eigenvalues of a circular membrane*, Differential Equations 1 (1965), 1326–1329.

[16] F. W. J. Olver, *The asymptotic expansion of Bessel functions of large order*, Philos. Trans. Roy. Soc. London Ser. A 247 (1954), 328–368.

[17] C. L. Siegel, *Über einige Anwendungen diophantischer Approximationen*, Abh. Preuss Akad. Wiss. Phys.-Math. Kl. 1929, Nr. 1.

[18] H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen*, Math. Ann. 71 (1912), 441–479.

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