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Boundedness of differential transforms for Dunkl heat semigroups

by

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Abstract. We study the boundedness and the convergence of the differential transform

$$T_{N,\kappa}f(x) = \sum_{j=N_1}^{N_2} v_j(e^{a_{j+1}\Delta_\kappa} f(x) - e^{a_j\Delta_\kappa} f(x))$$

associated with the Dunkl heat semigroup $\{e^{t\Delta_\kappa} f\}_{t>0}$. Here $\{v_j\}_{j\in\mathbb{Z}}$ is a bounded sequence, $\{a_j\}_{j\in\mathbb{Z}}$ is an increasing sequence of positive real numbers , and $N = (N_1, N_2) \subset \mathbb{Z}^2$ with $N_1 < N_2$. We prove a Cotlar-type inequality for the differential maximal operator

$$T^*_\kappa f = \sup_{\substack{N=(N_1,N_2)\in\mathbb{Z}^2\\ N_1<N_2}} |T_{N,\kappa}f(x)|.$$

Using this, we obtain the boundedness of T^*_κ on weighted L^p spaces for $1 < p < \infty$. Moreover, we establish the boundedness of T^*_κ on the spaces of bounded mean oscillations in Dunkl settings. Finally, as an application, we show the pointwise convergence of $T_{N,\kappa}f$ as $N \to (-\infty, \infty)$ for $f \in L^p(\mathbb{R}^n, d\mu_\kappa)$, $1 < p < \infty$.

1. Introduction. Dunkl theory is a mathematical framework that generalizes the classical Fourier analysis to the settings with additional symmetries, particularly reflection. It was first introduced by Charles F. Dunkl [D92]. The motivation behind Dunkl's work can be traced back to the study of root systems, a fundamental concept in Lie theory and algebraic geometry. Dunkl recognized the importance of incorporating reflection symmetries associated with a root system in the framework of Fourier analysis and harmonic analysis. At the core of Dunkl theory are Dunkl operators, which are differential-difference operators associated with finite reflection groups. These operators play a fundamental role in Dunkl theory, much like ordinary derivatives in classical analysis. There has been a proliferation of research

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in Dunkl theory, with mathematicians exploring its theoretical foundations. This theory has wide applications in diverse areas such as quantum mechanics, probability theory, statistical mechanics, and representation theory.

Consider the Euclidean space $\mathbb{R}^n$ with the usual inner product $\langle \cdot, \cdot \rangle$. Let $0 \neq \lambda \in \mathbb{R}^n$. The reflection of $x \in \mathbb{R}^n$ with respect to the hyperplane $\lambda^\perp$ is defined by

$$\sigma_\lambda(x) = x - 2\frac{\langle x, \lambda \rangle}{\|\lambda\|^2}\lambda.$$

Let $R \subset \mathbb{R}^n \setminus \{0\}$ be finite. Then R is said to be a *root system* if

(i) $R \cap \mathbb{R}\lambda = \{\pm\lambda\}$, $\forall \lambda \in R$,
(ii) $\sigma_\lambda(R) = R$, $\forall \lambda \in R$.

The *reflection group* associated with the root system R is the finite group $G \subset O(n)$ generated by the orthogonal matrices $\{\sigma_\lambda : \lambda \in R\}$, $O(n)$ being the set of all $n \times n$ real orthogonal matrices. A *multiplicity function* associated to the root system R is a function $\kappa : R \to \mathbb{R}_{\geq 0}$ satisfying $\kappa(u) = \kappa(v)$ whenever σ_u is conjugate to σ_v. Let $\{e_i : i = 1, \dots, n\}$ be the standard unit vectors in $\mathbb{R}^n$. The *Dunkl operator* connected with the root system R and the multiplicity function κ is given by

$$T_\xi f(x) = \frac{\partial}{\partial \xi} f(x) + \sum_{\lambda \in R} \frac{\kappa(\lambda)}{2} \langle \lambda, \xi \rangle \frac{f(x) - f(\sigma_\lambda(x))}{\langle \lambda, x \rangle}, \quad \xi \in \mathbb{R}^n.$$

We note that if $\kappa(\lambda) = 0$ for $\lambda \in R$, then T_ξ is the classical directional differential operator $\frac{\partial}{\partial \xi}$. For $x \in \mathbb{R}^n$, consider the initial value problem

$$\begin{cases} T_\xi f = \langle \xi, x \rangle f, & \xi \in \mathbb{R}^n, \\ f(0) = 1. \end{cases} \tag{1.1}$$

The unique solution $y \mapsto E_\kappa(x, y)$ of (1.1) is known as the *Dunkl kernel.* The Dunkl kernel $E_\kappa(x, y)$ can be extended uniquely to a holomorphic function on $\mathbb{C}^n \times \mathbb{C}^n$. One can view $E_\kappa(x, y)$ as a generalization of the exponential function $e^{\langle x, y \rangle}$. The function E_κ has several interesting properties:

(i) E_κ is symmetric: $E_\kappa(x, y) = E_\kappa(y, x)$, $\forall x, y \in \mathbb{C}^n$.
(ii) $|E_\kappa(ix, y)| \leq 1$, $\forall x, y \in \mathbb{R}^n$ and $E_\kappa(0, y) = 1$, $\forall y \in \mathbb{C}^n$.
(iii) $E_\kappa(\lambda x, y) = E_\kappa(x, \lambda y)$, $\forall \lambda \in \mathbb{C}$, $\forall x, y \in \mathbb{C}^n$.
(iv) $E_\kappa(\sigma(x), \sigma(y)) = E_\kappa(x, y)$, $\forall \sigma \in G$, $\forall x, y \in \mathbb{C}^n$.

The *Dunkl Laplacian* associated with R and κ is defined by $\Delta_\kappa = \sum_{j=1}^n T_j^2$, where $T_j = T_{e_j}$. We have (see [D92])

$$\Delta_\kappa f(x) = \Delta f(x) + \sum_{\alpha \in R} \kappa(\alpha) \delta_\alpha f(x),$$

where

$$\delta_\alpha f(x) = \frac{\partial_\alpha f(x)}{\langle \alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle^2}$$

and $\Delta u = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2}$ is the Euclidean Laplacian operator. Let $\{v_j\}_{j\in\mathbb{Z}}$ be a bounded sequence and $\{a_j\}_{j\in\mathbb{Z}}$ be an increasing sequence of positive real numbers. For $N = (N_1, N_2) \in \mathbb{Z}^2$ with $N_1 < N_2$, we define

$$T_{N,\kappa} f(x) = \sum_{j=N_1}^{N_2} v_j (e^{a_{j+1}\Delta_\kappa} f(x) - e^{a_j \Delta_\kappa} f(x)) \tag{1.2}$$

and

$$T_\kappa^* f(x) = \sup_{\substack{N=(N_1,N_2)\in\mathbb{Z}^2 \\ N_1<N_2}} |T_{N,\kappa} f(x)|. \tag{1.3}$$

In this paper, we study the boundedness of $T_{N,\kappa}$ and T_κ^* on weighted L^p spaces and the BMO spaces induced by the Dunkl metric. Here, the operator $T_{N,\kappa}$ arises from the Dunkl heat semigroup $\{e^{t\Delta_\kappa} f\}_{t>0}$.

In classical case, the theory of the heat semigroup arises in the study of partial differential equations, specifically in the context of parabolic equations. The parabolic partial differential equations describe phenomena that evolve over time, such as heat diffusion. The heat equation is a classical example of a parabolic partial differential equation given by

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x), \end{cases}$$

where u represents the heat distribution, and t is time. The heat semigroup associated with the heat equation is expressed as

$$u(x,t) = e^{t\Delta} f(x) := \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{\|x-y\|^2}{4t}} f(y)\, dy.$$

The study of heat semigroups is important in understanding the qualitative behavior of the solution to the heat equation, including smoothing effects, regularity, convergence of numerical methods, and so on. These concepts are widely applied in various fields, such as mathematical analysis, numerical analysis, the theory of PDEs, etc. One of the interesting properties of the one-parameter family $\{e^{t\Delta} f\}_{t>0}$ is that $e^{t\Delta} f$ converges to f as $t \to 0^+$ uniformly for $f \in C_c^\infty(\mathbb{R}^n)$. However, the modes of convergence depend on the function f. An easy way of exploring the convergence of the semigroup is by analyzing the convergence of the differential-type transform

$$\sum_{j\in\mathbb{Z}} v_j (e^{a_{j+1}\Delta} f(x) - e^{a_j\Delta} f(x)).$$

In [JR02], Jones et al. first considered the above transform in order to study the rate of convergence of an ergodic averaging operator in the framework of a measure-preserving dynamical system. Let $(X, \mathcal{B}, \lambda, T)$ be a measure-preserving dynamical system and $A_k f(x) = \frac{1}{n_k}\sum_{j=0}^{n_k-1} f(T^j x)$ the corresponding ergodic averaging operator. It is well known that $A_k f$ converges to $\mathcal{P}_T f$ in $L^p(X, \mathcal{B}, \lambda)$, for $1 \leq p < \infty$, where $\mathcal{P}_T : L^p(X, \mathcal{B}, \lambda) \to L^p(X, \mathcal{F}_T, \lambda)$ is a projection, $\mathcal{F}_T$ being the σ-algebra of T-invariant sets in $\mathcal{B}$. However, there is no assertion about the convergence speed of $A_k f$. For certain f, the averaging operator $A_k f$ is asymptotic to $\sqrt{\frac{2\log\log k}{k}}$ as $k \to \infty$. In other words,

$$\limsup_{k\to\infty} \frac{A_k f}{\sqrt{\frac{2\log\log k}{k}}} = 1 \quad \text{for a.e. } \lambda \in X.$$

On the other hand, for a null sequence $\{\alpha_k\}$ of positive real numbers, there exists a continuous function f with integral 0 such that $\limsup_{k\to\infty} \frac{A_k f}{\alpha_k} = \infty$. In this connection, we refer to the works of Krengel and Petersen [K85, P83]. In the spirit of the rate of convergence, Jones et al. [JR02] considered the differential transform

$$\sum_{j\in\mathbb{Z}} v_j (A_j f(x) - A_{j-1} f(x))$$

and obtained the convergence results in $L^p(X, \lambda)$ spaces. In [BL$^+$06], these results were extended to BMO spaces and weighted L^p spaces.

Now, coming back to the case of the heat semigroup, recently Zhao and Torrea [CT21] analyzed the convergence of the differential transforms

$$\sum_{j\in\mathbb{Z}} v_j (e^{a_{j+1}\Delta} f(x) - e^{a_j\Delta} f(x)) \quad \text{and} \quad \sum_{j\in\mathbb{Z}} v_j (e^{-a_{j+1}\mathcal{L}} f(x) - e^{-a_j\mathcal{L}} f(x)),$$

where $\mathcal{L}$ denotes the Schrödinger operator in $\mathbb{R}^n$. For this purpose, they studied the partial sums

$$\begin{aligned} T_N f(x) &= \sum_{j=N_1}^{N_2} v_j (e^{a_{j+1}\Delta} f(x) - e^{a_j\Delta} f(x)), \\ T_N^{\mathcal{L}} f(x) &= \sum_{j=N_1}^{N_2} v_j (e^{-a_{j+1}\mathcal{L}} f(x) - e^{-a_j\mathcal{L}} f(x)), \end{aligned} \tag{1.4}$$

where $N = (N_1, N_2) \in \mathbb{Z}^2$ with $N_1 < N_2$. They obtained the boundedness of T_N, $T_N^{\mathcal{L}}$, and the corresponding maximal operators T^* and $T^{*,\mathcal{L}}$ in weighted $L^p(\mathbb{R}^n)$ and BMO spaces. Using these results, they were able to show the pointwise convergence of $T_N f(x)$ as $(N_1, N_2) \to (-\infty, \infty)$ for f in weighted L^p spaces. Furthermore, the growth of T^* was obtained near the origin. In [Z21], Zhao established the analogous results for the heat semigroup $\{e^{t\Delta^2}\}_{t>0}$ of the biharmonic operator Δ^2. In accomplishing these results, Calderón–Zygmund theory and Cotlar-type inequality played a crucial role.

Calderón–Zygmund theory has significant applications in harmonic analysis, partial differential equations, potential theory, and signal processing. Here, we recall some basics of the theory. A bounded linear operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ defined by

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy, \qquad x \notin \operatorname{supp} f,\ f \in \mathrm{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^n),$$

is said to be a *Calderón–Zygmund operator* if $\|T\phi\|_{L^2(\mathbb{R}^n)} \le C\|\phi\|_{L^2(\mathbb{R}^n)}$, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, and the kernel $K(x,y)$, which is defined on $\mathbb{R}^{2n} \setminus \{(x,x) : x \in \mathbb{R}^n\}$, satisfies the following conditions:

(i) There exists $A > 0$ such that

$$|K(x,y)| \le \frac{A}{|x-y|^n}.$$

(ii) There exists $\delta > 0$ such that

$$|K(x,y) - K(x',y)| \le A\frac{|x-x'|^{\delta}}{|x-y|^{n+\delta}} \quad \text{whenever } |x-x'| < \frac{1}{2}|x-y|$$

and

$$|K(x,y) - K(x,y')| \le A\frac{|y-y'|^{\delta}}{|x-y|^{n+\delta}} \quad \text{whenever } |y-y'| < \frac{1}{2}|x-y|.$$

The Calderón–Zygmund operator T has a unique extension on $L^2(\mathbb{R}^n)$. We have the following well-known theorem.

THEOREM 1.1 ([G09, Theorem 4.2.2]). *Let $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a Calderón–Zygmund operator with kernel K. Then the following hold:*

(i) *For $1 < p < \infty$, we have $\|Tf\|_{L^p(\mathbb{R}^n)} \le C_{p,n}\|f\|_{L^p(\mathbb{R}^n)}$.*
(ii) *For $p = 1$, we have $\|Tf\|_{L^{1,\infty}(\mathbb{R}^n)} \le C_n\|f\|_{L^1(\mathbb{R}^n)}$. In other words,*

$$\eta|\{x \in \mathbb{R}^n : |Tf(x)| > \eta\}| \le C\|f\|_{L^1(\mathbb{R}^n)} \quad \textit{for any } \eta \ge 0.$$

For further study of Calderón–Zygmund theory and Cotlar's inequality on $\mathbb{R}^n$, we refer to the books [G09, D01]. In this direction, the notion of Calderón–Zygmund operator in the Dunkl setting has been introduced by Tan et al. [THJ22, THL23]. The associated kernel involves the Euclidean metric and the Dunkl metric. Recently, Almeida et al. [AB+24] have studied the variational operator associated to $\{t^m\partial_t^m e^{t\Delta_\kappa}\}_{t>0}$ in the Dunkl setting. Dziubański et al. [DH23] have provided a considerable collection of singular integral operators of convolution type, whose kernels are the Dunkl–Calderón–Zygmund kernels. Dunkl theory is mainly based on the Dunkl transform (see (2.4) below). It is a generalization of the classical Fourier transform. Since the Dunkl transform enjoys properties similar to the classical Fourier transform, for the past few decades, Dunkl theory has become an enormous topic of interest for mathematicians from various fields such as harmonic analysis, quantum theory, probability theory, representation theory, and others.

This paper mainly focuses on the convergence issues for the differential transform in Dunkl settings. We study the differential operators $T_{N,\kappa}$ and T^*_κ, which are analogous to the Euclidean differential operators T_N and T^* respectively. We show that the $T_{N,\kappa}$ are Dunkl–Calderón–Zygmund operators. To do this, we need to carefully analyze the Dunkl heat kernel. Using the Dunkl–Calderón–Zygmund theorem, we obtain the weighted L^p-boundedness for $1 < p < \infty$ and the weak-type $(1,1)$ boundedness of these operators. Moreover, we show that $T_{N,\kappa}$'s are bounded from $\mathrm{BMO}_{d,\kappa}(\mathbb{R}^n)$ into BMO_κ (see (2.2) and (2.3) for the definitions). However, the boundedness of $T_{N,\kappa}$ on BMO_κ and $\mathrm{BMO}_{d,\kappa}$ remains an open problem; see Remark 3.6 for further discussion. We prove a Cotlar-type inequality in connection with the maximal operator T^*_κ. This helps us to obtain the weighted L^p-boundedness for $1 < p < \infty$ and the weak-type $(1,1)$ boundedness of T^*_κ. Furthermore, we show the boundedness of T^*_κ from $\mathrm{BMO}_{d,\kappa}$ into BMO_κ. We investigate the growth of T^*_κ near the origin. The growth is different from that of the classical maximal differential operator. We construct a function $f \in L^\infty(\mathbb{R})$ for which the maximal operator $T^*_\kappa f$ becomes unbounded on $\mathbb{R}$. Remarkably, the same example remains valid not only in the Dunkl framework but also in the classical case corresponding to $\kappa(1) = 0$. Moreover, our approach simplifies the argument presented in [CT21, Proposition 2.6], thereby offering an alternative and more transparent proof of the classical result.

The paper is organized as follows. In Section 2, we provide the essential background required for the remainder of this article. In Section 3, we investigate the boundedness of the differential transform $T_{N,\kappa}$ on weighted L^p spaces. We show that $T_{N,\kappa}$ is bounded from $\mathrm{BMO}_{d,\kappa}$ into BMO_κ. In Section 4, we prove the Cotlar-type inequality for $T_{N,\kappa}$ and obtain the boundedness of the maximal differential operator T^*_κ. Finally, in Section 5, we show the convergence of $T_{N,\kappa}f$ as $N \to (-\infty, \infty)$ for $f \in L^p(\mathbb{R}^n, d\mu_\kappa)$.

2. Preliminaries. Let $R \subset \mathbb{R}^n \setminus \{0\}$ be a root system and $G \subset O(n)$ be the corresponding finite reflection group. The orbit of $x \in \mathbb{R}^n$ with respect to G is $\mathcal{O}_G(x) := \{\sigma(x) : \sigma \in G\} \subset \mathbb{R}^n$. Once the underlying reflection group G is understood, we will denote the orbit of x simply as $\mathcal{O}(x)$. Since these orbits may contain more than one element, they induce the so-called Dunkl metric. For $x, y \in \mathbb{R}^n$, the distance between the orbits $\mathcal{O}(x)$ and $\mathcal{O}(y)$, denoted by $d(x,y)$, is defined by $d(x,y) = \min\{\|x - \sigma(y)\| : \sigma \in G\}$. This d is called the *Dunkl metric*. The orbit of a set $Q \subset \mathbb{R}^n$ is defined by

$$\mathcal{O}(Q) := \{\sigma(x) : \sigma \in G,\, x \in Q\} = \bigcup_{\sigma \in G} \sigma(Q).$$

Let κ be the multiplicity function associated to the root system R and consider the G-invariant homogeneous weight function

$$\omega_\kappa(x) = \prod_{\lambda\in R} |\langle x,\lambda\rangle|^{\kappa(\lambda)}, \qquad x\in\mathbb{R}^n.$$

Then the associated measure is defined by $d\mu_\kappa(x) = c_\kappa\omega_\kappa(x)dx$, where

$$c_\kappa^{-1} = \int_{\mathbb{R}^n} e^{-\|x\|^2/2}\omega_\kappa(x)\,dx.$$

Since ω_κ is G-invariant, the measure μ_κ is also G-invariant. In particular,

$$\mu_\kappa(\sigma(Q)) = \mu_\kappa(Q), \qquad \forall\sigma\in G,\ \forall Q\subset\mathbb{R}^n.$$

Moreover, $\mu_\kappa(Q)\le\mu_\kappa(\mathcal{O}(Q))\le\#(G)\mu_\kappa(Q)$, where $\#(G)$ denotes the cardinality of G. Here, we note that the measure μ_κ is G-invariant while the Lebesgue measure on $\mathbb{R}^n$ is translation invariant. Let $B(x,r)$ denote the Euclidean ball with center at $x\in\mathbb{R}^n$ and radius $r>0$. Then

$$\mu_\kappa(B(x,r)) \sim r^n \prod_{\lambda\in R}\left(|\langle x,\lambda\rangle| + r\right)^{\kappa(\lambda)}.$$

Here, unlike the Lebesgue measure, $\mu_\kappa(B(x,r))$ depends on the center $x\in\mathbb{R}^n$. This property of μ_κ makes the Dunkl setting a particular case of a homogeneous-type space. Using the Dunkl metric $d(x,y)$, one can rewrite $\mathcal{O}(B(x,r)) = \{y\in\mathbb{R}^n : d(x,y)\le r\}$. The triple $(\mathbb{R}^n,\|\cdot\|,\mu_\kappa)$ is a space of homogeneous type satisfying the doubling condition $\mu_\kappa(B(x,2r))\le \mathrm{c}\mu_\kappa(B(x,r))$ for some constant $\mathrm{c}>0$. Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$. Moreover,

$$\alpha^{-1}\left(\frac{\mathrm{r}_2}{\mathrm{r}_1}\right)^{n} \le \frac{\mu_\kappa(B(x_2,\mathrm{r}_2))}{\mu_\kappa(B(x_1,\mathrm{r}_1))} \le \alpha\left(\frac{\mathrm{r}_2}{\mathrm{r}_1}\right)^{n_\kappa}, \qquad \alpha\ge 1, \tag{2.1}$$

whenever $B(x_1,\mathrm{r}_1)\subset B(x_2,\mathrm{r}_2)$. Here $n_\kappa := n + \sum_{\lambda\in R}\kappa(\lambda)$ is called the *homogeneous dimension*. For $x,y\in\mathbb{R}^n$ and $t>0$, define $V(x,y,t) := \max\{\mu_\kappa(B(x,t)),\mu_\kappa(B(y,t))\}$.

The BMO space in the Dunkl setting is defined by

$$\text{(2.2)}\qquad \mathrm{BMO}_\kappa(\mathbb{R}^n) := \left\{ f\in L^1(\mathbb{R}^n,d\mu_\kappa) : \|f\|_{\mathrm{BMO}_\kappa} = \sup_{\substack{Q\subset\mathbb{R}^n\\ Q\,\text{a ball}}} \frac{1}{\mu_\kappa(Q)}\int_Q |f(x)-f_Q|\,d\mu_\kappa(x) < \infty \right\},$$

where

$$f_Q := \frac{1}{\mu_\kappa(Q)}\int_Q f(x)\,d\mu_\kappa(x).$$

Here, Q is a Euclidean ball. Using the Dunkl metric $d(x,y)$, we have the following BMO space:

$$\text{(2.3)} \quad \mathrm{BMO}_{d,\kappa}(\mathbb{R}^n) := \Big\{ f \in L^1(\mathbb{R}^n, d\mu_\kappa) : \\ \|f\|_{\mathrm{BMO}_{d,\kappa}} = \sup_{\substack{Q \subset \mathbb{R}^n \\ Q \text{ a ball}}} \frac{1}{\mu_\kappa(\mathcal{O}(Q))} \int_{\mathcal{O}(Q)} |f(x) - f_{\mathcal{O}(Q)}| \, d\mu_\kappa(x) < \infty \Big\}.$$

One can easily show that $L^\infty(\mathbb{R}^n) \subset \mathrm{BMO}_{d,\kappa}(\mathbb{R}^n) \subset \mathrm{BMO}_\kappa(\mathbb{R}^n)$ and the inclusions are strict. A function $f \in \mathrm{BMO}_\kappa(\mathbb{R}^n)$ belongs to $\mathrm{BMO}_{d,\kappa}(\mathbb{R}^n)$ if f is G-invariant. In general, $\mathrm{BMO}_{d,\kappa}(\mathbb{R}^n) \neq \mathrm{BMO}_\kappa(\mathbb{R}^n)$. In this regard, we refer to [JL23].

The *Dunkl transform* of $f \in L^1(\mathbb{R}^n, d\mu_\kappa)$ is defined by

$$(\mathcal{F}_\kappa f)(\xi) = \int_{\mathbb{R}^n} f(x) E_\kappa(-i\xi, x) \, d\mu_\kappa(x). \tag{2.4}$$

It can be shown that $\mathcal{F}_\kappa f \in C_0(\mathbb{R}^n)$ whenever $f \in L^1(\mathbb{R}^n, d\mu_\kappa)$. The Dunkl transform $\mathcal{F}_\kappa$ leaves the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ invariant. Like the classical Fourier transform, the Dunkl transform $\mathcal{F}_\kappa$ is an isometry on $L^2(\mathbb{R}^n, d\mu_\kappa)$. For $f, \mathcal{F}_\kappa f \in L^1(\mathbb{R}^n, d\mu_\kappa)$, the inversion formula $f(x) = (\mathcal{F}_\kappa^2 f)(-x)$ holds. For $x \in \mathbb{R}^n$, the Dunkl translation operator $\mathcal{T}_x : L^2(\mathbb{R}^n, d\mu_\kappa) \to L^2(\mathbb{R}^n, d\mu_\kappa)$ is defined by $\mathcal{F}_\kappa(\mathcal{T}_x f)(\xi) = E_\kappa(x, i\xi)\mathcal{F}_\kappa f(\xi)$ for $f \in L^2(\mathbb{R}^n, d\mu_\kappa)$. Using the properties of the Dunkl kernel E_κ and the isometry of the Dunkl transform, one can show that $\|\mathcal{T}_x f\|_{L^2(\mathbb{R}^n, d\mu_\kappa)} \leq \|f\|_{L^2(\mathbb{R}^n, d\mu_\kappa)}$ for $f \in L^2(\mathbb{R}^n, d\mu_\kappa)$. Assuming $f, \mathcal{F}_\kappa f \in L^1(\mathbb{R}^n, d\mu_\kappa)$, we have the pointwise equality

$$\mathcal{T}_x f(y) = \int_{\mathbb{R}^n} E_\kappa(ix, \xi) E_\kappa(iy, \xi) \mathcal{F}_\kappa f(\xi) \, d\mu_\kappa(\xi), \qquad y \in \mathbb{R}^n.$$

A function $f : \mathbb{R}^n \to \mathbb{C}$ is said to be *radial* if there exists $f_0 : [0, \infty) \to \mathbb{C}$ satisfying $f(x) = f_0(\|x\|)$ for $x \in \mathbb{R}^d$. Then, for $1 \leq p \leq \infty$, $L^p_{\mathrm{rad}}(\mathbb{R}^n, d\mu_\kappa)$ is defined to be the set of all radial functions $f \in L^p(\mathbb{R}^n, d\mu_\kappa)$. It has been proved in [TX05, GIT19] that

$$\|\mathcal{T}_x f\|_{L^p(\mathbb{R}^n, d\mu_\kappa)} \leq \|f\|_{L^p_{rad}(\mathbb{R}^n, d\mu_\kappa)} \tag{2.5}$$

for $1 \leq p \leq \infty$. The *Dunkl convolution* of $f, g \in L^2(\mathbb{R}^n, d\mu_\kappa)$ is defined by

$$f *_\kappa g(x) = \int_{\mathbb{R}^n} f(y) \mathcal{T}_x g(-y) \, d\mu_\kappa(y).$$

Taking the Dunkl transform of the Dunkl–Laplace operator, one can show that

$$\mathcal{F}_\kappa(\Delta_\kappa f)(\xi) = -\|\xi\|^2 \mathcal{F}_\kappa f(\xi), \tag{2.6}$$

where Δ_κ denotes the Dunkl–Laplace operator. Consider the *Dunkl heat* *equation*

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_\kappa u(x,t) & \text{on } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x). \end{cases} \tag{2.7}$$

For $x, y \in \mathbb{R}^n$ and $t > 0$, the *Dunkl heat kernel* $h_t(x,y)$ is defined by

$$h_t(x,y) := \mathcal{T}_x \mathbf{h}_t(-y) = \frac{1}{(2t)^{\frac{n_\kappa}{2}}} E_\kappa\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right) e^{-\frac{1}{4t}(\|x\|^2+\|y\|^2)},$$

where $\mathbf{h}_t$ is the radial function defined by

$$\mathbf{h}_t(x) := \widetilde{\mathbf{h}}_t(\|x\|) := (2t)^{-n_\kappa/2} e^{-\|x\|^2/(4t)}.$$

Then, for any $f \in \mathcal{S}(\mathbb{R}^n)$, the function $u(x,t) = e^{t\Delta_\kappa} f(x)$ solves (2.7), where

$$e^{t\Delta_\kappa} f(x) = \int_{\mathbb{R}^n} h_t(x,y) f(y)\, d\mu_\kappa(y) = f *_\kappa \mathbf{h}_t(x). \tag{2.8}$$

We have the following results in connection with the above solution.

THEOREM 2.1 ([R98, Theorem 4.7]). *Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then the following hold:*

(i) *For any $t > 0$, $e^{t\Delta_\kappa} f \in \mathcal{S}(\mathbb{R}^n)$.*
(ii) *For any $t, s \geq 0$, $e^{(t+s)\Delta_\kappa} f = e^{t\Delta_\kappa} e^{s\Delta_\kappa} f$.*
(iii) *$\|e^{t\Delta_\kappa} f - f\|_{L^\infty(\mathbb{R}^n)}$ converges to 0 as $t \to 0^+$.*

By explicit computation, one can show that

$$\mathcal{F}_\kappa \mathbf{h}_t(\xi) = e^{-t\|\xi\|^2}, \qquad t > 0,\ \xi \in \mathbb{R}^n. \tag{2.9}$$

Now, we record a few fundamental properties of the Dunkl heat kernel h_t.

(i) There exist $c, C > 0$ such that for all $x, y \in \mathbb{R}^n$, $t > 0$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$$\left|\frac{\partial^m}{\partial t^m} h_t(x,y)\right| \leq \frac{C}{t^m} V(x,y,\sqrt{t})^{-1} e^{-cd(x,y)^2/t},. \tag{2.10}$$

(ii) There exist $c, C > 0$ such that for all $x, y, y' \in \mathbb{R}^n$ and $t > 0$ with $\|y - y'\| < \sqrt{t}$, and all $m \in \mathbb{N}_0$,

$$\left|\frac{\partial^m}{\partial t^m} h_t(x,y) - \frac{\partial^m}{\partial t^m} h_t(x,y')\right| \leq \frac{C}{t^m} \frac{\|y-y'\|}{\sqrt{t}} V(x,y,\sqrt{t})^{-1} e^{-cd(x,y)^2/t}. \tag{2.11}$$

For a detailed study of the Dunkl heat kernel h_t, we refer to [ADH19, R98].

We know that the triple $(\mathbb{R}^n, \|\cdot\|, \mu_\kappa)$ is a space of homogeneous type. For $1 < p < \infty$, we say that a weight function ω belongs to the *Dunkl–Muckenhoupt class* $A_{p,\kappa}(\mathbb{R}^n, \|\cdot\|, \mu_\kappa)$ if

$$[\omega]_{A_{p,\kappa}} := \sup_{B \subset \mathbb{R}^n} \left(\frac{1}{\mu_\kappa(B)} \int_B \omega(x)\, d\mu_\kappa(x)\right)\left(\frac{1}{\mu_\kappa(B)} \int_B \omega(x)^{1-p'}\, d\mu_\kappa(x)\right)^{p-1} < \infty,$$

where the supremum is taken over all Euclidean balls $B \subset \mathbb{R}^n$ and p' denotes the conjugate exponent of p. For $p = 1$, a weight function ω belongs to the *Dunkl–Muckenhoupt class* $A_{1,\kappa}(\mathbb{R}^n, \|\cdot\|, \mu_\kappa)$ if

$$[\omega]_{A_{1,\kappa}} := \sup_{B \subset \mathbb{R}^n} \left(\frac{1}{\mu_\kappa(B)} \int_B \omega(x)\, d\mu_\kappa(x) \right) \left(\inf_{B \subset \mathbb{R}^d} \omega(x) \right)^{-1}.$$

For $p = \infty$, the *Dunkl–Muckenhoupt class* $A_{\infty,\kappa}(\mathbb{R}^n, \|\cdot\|, \mu_\kappa)$ is defined by $A_{\infty,\kappa} = \bigcup_{1 \le p < \infty} A_{p,\kappa}$. For further details on $A_{p,\kappa}$ weights on homogeneous type spaces, we refer to [ST89, GL$^+$14].

Let $T : \mathcal{S}(\mathbb{R}^n) \to (\mathcal{S}(\mathbb{R}^n))'$ be a linear operator which is explicitly represented as

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, d\mu_\kappa(y), \tag{2.12}$$

for $f \in C_{\mathrm{c}}(\mathbb{R}^n)$ and $x \notin \operatorname{supp} f$, where the kernel K is defined outside of the set $\{(x,y) \in \mathbb{R}^{2n} : x = \sigma(y) \text{ for some } \sigma \in G\}$. Then T is said to be a Dunkl–Calderón–Zygmund operator if it can be extended to a bounded linear operator on $L^2(\mathbb{R}^n, d\mu_\kappa)$ and for some $0 < \epsilon \le 1$ the kernel $K(x,y)$ satisfies

$$|K(x,y)| \lesssim \frac{1}{\mu_\kappa(B(x, d(x,y)))} \left(\frac{d(x,y)}{\|x-y\|} \right)^{\epsilon}, \tag{2.13}$$

$$\begin{aligned} &|K(x,y) - K(x,y')| \\ &\qquad \lesssim \frac{1}{\mu_\kappa(B(x, d(x,y)))} \left(\frac{\|y-y'\|}{\|x-y\|} \right)^{\epsilon} \quad \text{for } \|y-y'\| \le \frac{d(x,y)}{2}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} &|K(x,y) - K(x',y)| \\ &\qquad \lesssim \frac{1}{\mu_\kappa(B(x, d(x,y)))} \left(\frac{\|x-x'\|}{\|x-y\|} \right)^{\epsilon} \quad \text{for } \|x-x'\| \le \frac{d(x,y)}{2}. \end{aligned} \tag{2.15}$$

The size condition (2.13) ensures that the integral (2.12) is absolutely convergent for a.e. $x \in \mathbb{R}^n$.

Let Ψ_m be a multiplier operator with multiplier $m \in L^\infty(\mathbb{R}^n)$. In other words, assume that $\mathcal{F}_\kappa(\Psi_m f)(\xi) = m(\xi)\mathcal{F}_\kappa f(\xi)$. Dziubański et al. [DH23] provided a sufficient condition for Ψ_m to be a Calderón–Zygmund operator.

THEOREM 2.2 ([DH23, Corollary 4.9]). *Let Ψ_m be the multiplier integral operator given by $\mathcal{F}_\kappa(\Psi_m f)(\xi) = m(\xi)\mathcal{F}_\kappa f(\xi)$ and $K(x,y)$ be the corresponding kernel. Then $K(x,y)$ satisfies* (2.13)–(2.15) *if the following conditions hold:*

(i) $m \in C^{\mathbf{n}_\kappa}(\mathbb{R}^n \setminus \{0\})$, *where $\mathbf{n}_\kappa$ is the smallest integer satisfying $\mathbf{n}_\kappa > n_\kappa$.*
(ii) *For all $\beta \in \mathbb{N}_0^n$ with $|\beta| \le \mathbf{n}_\kappa$ there exists $C_\beta > 0$ such that*

$$\left| \frac{\partial^\beta}{\partial \xi^\beta} m(\xi) \right| \le \frac{C_\beta}{\|\xi\|^{|\beta|}}, \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}. \tag{2.16}$$

Recently, Mukherjee et al. [MP25] have studied the boundedness of multilinear Calderón–Zygmund operator and proved their weighted boundedness. Here we state their result in the linear case.

THEOREM 2.3 ([MP25, Theorem 5.3]). *Let T be a Dunkl–Calderón–Zygmund operator and let $\omega \in A_{p,\kappa}$ be G-invariant for some $1 \le p < \infty$. Then T is bounded on $L^p(\mathbb{R}^n, \omega d\mu_\kappa)$ if $1 < p < \infty$. Moreover, for $p = 1$, the operator T is bounded from $L^1(\mathbb{R}^n, \omega d\mu_\kappa)$ into $L^{1,\infty}(\mathbb{R}^n, \omega d\mu_\kappa)$.*

3. Boundedness of differential operators. Let $T_{N,\kappa}$ be the operator defined as in (1.2). First, we shall show that the operator $T_{N,\kappa}$ is an integral operator with some kernel $K_{N,\kappa}(x, y)$. Using (2.8) in (1.2), we get

$$\begin{aligned} T_{N,\kappa} f(x) &= \sum_{j=N_1}^{N_2} v_j (e^{a_{j+1}\Delta_\kappa} f(x) - e^{a_j \Delta_\kappa} f(x)) \\ &= \sum_{j=N_1}^{N_2} v_j \int_{\mathbb{R}^n} (h_{a_{j+1}}(x, y) - h_{a_j}(x, y)) f(y)\, d\mu_\kappa(y) \\ &= \int_{\mathbb{R}^n} \sum_{j=N_1}^{N_2} v_j (h_{a_{j+1}}(x, y) - h_{a_j}(x, y)) f(y)\, d\mu_\kappa(y) \\ &= \int_{\mathbb{R}^n} K_{N,\kappa}(x, y) f(y)\, d\mu_\kappa(y), \end{aligned} \tag{3.1}$$

where

$$K_{N,\kappa}(x, y) = \sum_{j=N_1}^{N_2} v_j (h_{a_{j+1}}(x, y) - h_{a_j}(x, y)),$$

which can be rewritten as

$$K_{N,\kappa}(x, y) = \sum_{j=N_1}^{N_2} v_j \int_{a_j}^{a_{j+1}} \frac{\partial}{\partial t}(h_t(x, y))\, dt. \tag{3.2}$$

In the following proposition, we establish the L^2-boundedness of $T_{N,\kappa}$.

PROPOSITION 3.1. *Let $\{v_j\}_{j\in\mathbb{Z}}$ be a bounded sequence of real numbers and $\{a_j\}_{j\in\mathbb{Z}}$ be an increasing sequence of positive real numbers. Let $T_{N,\kappa}$, where $N = (N_1, N_2) \in \mathbb{Z}^2$ with $N_1 < N_2$, be the operator defined in (1.2). Then there exists $C > 0$, depending upon $\|v\|_{\ell^\infty(\mathbb{Z})} := \|\{v_j\}\|_{\ell^\infty(\mathbb{Z})}$, such that*

$$\|T_{N,\kappa} f\|_{2,\kappa} \le C \|f\|_{2,\kappa}$$

for all $f \in L^2(\mathbb{R}^n, d\mu_\kappa)$.

Proof. By the Plancherel formula for the Dunkl transform and (2.6), we get

$$
\begin{aligned}
\|T_{N,\kappa}f\|^2_{L^2(\mathbb{R}^n,d\mu_\kappa)} &= \Big\| \sum_{j=N_1}^{N_2} v_j (e^{a_{j+1}\Delta_\kappa} f - e^{a_j\Delta_\kappa} f)\Big\|^2_{L^2(\mathbb{R}^n,d\mu_\kappa)} \\
&= \Big\| \sum_{j=N_1}^{N_2} v_j \mathcal{F}_\kappa (e^{a_{j+1}\Delta_\kappa} f - e^{a_j\Delta_\kappa} f)\Big\|^2_{L^2(\mathbb{R}^n,d\mu_\kappa)} \\
&= \Big\| \sum_{j=N_1}^{N_2} v_j \int_{a_j}^{a_{j+1}} \frac{\partial}{\partial t}(\mathcal{F}_\kappa(e^{t\Delta_\kappa} f))(\cdot)\, dt\Big\|^2_{L^2(\mathbb{R}^n,d\mu_\kappa)} \\
&= \int_{\mathbb{R}^n} \Big| \sum_{j=N_1}^{N_2} v_j \int_{a_j}^{a_{j+1}} \frac{\partial}{\partial t}(\mathcal{F}_\kappa(e^{t\Delta_\kappa} f))(\xi)\, dt\Big|^2 d\mu_\kappa(\xi) \\
&= \int_{\mathbb{R}^n} \Big| \sum_{j=N_1}^{N_2} v_j \int_{a_j}^{a_{j+1}} \frac{\partial}{\partial t}(e^{-t\|\xi\|^2} \mathcal{F}_\kappa f(\xi))\, dt\Big|^2 d\mu_\kappa(\xi) \\
&\le \|v\|^2_{\ell^\infty(\mathbb{Z})} \int_{\mathbb{R}^n} \Big(\int_0^\infty \|\xi\|^2 e^{-t\|\xi\|^2}\, dt\Big)^2 |\mathcal{F}_\kappa f(\xi)|^2\, d\mu_\kappa(\xi) \\
&= \|v\|^2_{\ell^\infty(\mathbb{Z})} \int_{\mathbb{R}^n} |\mathcal{F}_\kappa f(\xi)|^2\, d\mu_\kappa(\xi) = \|v\|^2_{\ell^\infty(\mathbb{Z})} \|f\|^2_{L^2(\mathbb{R}^n,d\mu_\kappa)},
\end{aligned}
$$

proving our assertion. ∎

Now, we shall show that the kernel $K_{N,\kappa}$ satisfies (2.13)–(2.15). This, in turn, will lead to the fact that $T_{N,\kappa}$ is a Dunkl–Calderón–Zygmund operator.

PROPOSITION 3.2. *Let $\{v_j\}_{j\in\mathbb{Z}}$ be a bounded sequence of real numbers and $\{a_j\}_{j\in\mathbb{Z}}$ be an increasing sequence of positive real numbers. Let $T_{N,\kappa}$, where $N = (N_1, N_2) \in \mathbb{Z}^2$ with $N_1 < N_2$ is defined in* (1.2). *Then $T_{N,\kappa}$ is a Dunkl–Calderón–Zygmund operator for each $N = (N_1, N_2) \in \mathbb{Z}^2$ with $N_1 < N_2$.*

Proof. We will make use of Theorem 2.2. First, we shall show that each $T_{N,\kappa}$ is a multiplier operator. Using (2.8) in (1.2), we can rewrite

$$
\begin{aligned}
T_{N,\kappa} f(x) &= \sum_{j=N_1}^{N_2} v_j (f *_\kappa \mathbf{h}_{a_{j+1}}(x) - f *_\kappa \mathbf{h}_{a_j}(x)) \\
&= \sum_{j=N_1}^{N_2} v_j (f *_\kappa (\mathbf{h}_{a_{j+1}} - \mathbf{h}_{a_j}))(x).
\end{aligned}
$$

Taking the Dunkl transform on both sides, we get

$$\begin{aligned}\mathcal{F}_\kappa\left(T_{N,\kappa}f\right)(\xi) &= \sum_{j=N_1}^{N_2} v_j \mathcal{F}_\kappa(f *_\kappa (\mathbf{h}_{a_{j+1}} - \mathbf{h}_{a_j}))(\xi)\\ &= \sum_{j=N_1}^{N_2} v_j \mathcal{F}_\kappa f(\xi) \mathcal{F}_\kappa(\mathbf{h}_{a_{j+1}} - \mathbf{h}_{a_j})(\xi)\\ &= \Big(\sum_{j=N_1}^{N_2} v_j (e^{-a_{j+1}\|\xi\|^2} - e^{-a_j\|\xi\|^2})\Big)\mathcal{F}_\kappa f(\xi)\\ &=: m_{N,\kappa}(\xi)\mathcal{F}_\kappa f(\xi),\end{aligned}$$

by applying (2.9). Here, m_κ can be rewritten as

$$m_{N,\kappa}(\xi) = \sum_{j=N_1}^{N_2} v_j \int\limits_{a_j}^{a_{j+1}} \frac{\partial}{\partial t}(e^{-t\|\xi\|^2})\,dt. \tag{3.3}$$

Hence

$$\begin{aligned}|m_{N,\kappa}(\xi)| &\le \|\{v_j\}\|_{\ell^\infty(\mathbb{Z})} \sum_{j=N_1}^{N_2} \int\limits_{a_j}^{a_{j+1}} \frac{\partial}{\partial t}(e^{-t\|\xi\|^2})\,dt\\ &\le \|\{v_j\}\|_{\ell^\infty(\mathbb{Z})} \int\limits_0^\infty \frac{\partial}{\partial t}(e^{-t\|\xi\|^2})\,dt\\ &= \|\{v_j\}\|_{\ell^\infty(\mathbb{Z})} < \infty.\end{aligned}$$

Thus, for each N, $T_{N,\kappa}$ is a multiplier operator with multiplier $m_{N,\kappa} \in L^\infty(\mathbb{R}^n)$. Moreover, from the definition of $m_{N,\kappa}$, it can be easily shown that $m_{N,\kappa} \in C^{\mathbf{n}_\kappa}(\mathbb{R}^n \setminus \{0\})$. For each $i \in \{1,\ldots,n\}$, from (3.3) we have

$$\begin{aligned}\left|\frac{\partial}{\partial \xi_i} m_{N,\kappa}(\xi)\right| &= \left|\sum_{j=N_1}^{N_2} v_j \int\limits_{a_j}^{a_{j+1}} \frac{\partial}{\partial t}\frac{\partial}{\partial \xi_i} e^{-t\|\xi\|^2}\,dt\right|\\ &= \left|\sum_{j=N_1}^{N_2} v_j \int\limits_{a_j}^{a_{j+1}} \frac{\partial}{\partial t}(-2t\xi_i e^{-t\|\xi\|^2})\,dt\right|\\ &= 2\Big|\sum_{j=N_1}^{N_2} \xi_i v_j \int\limits_{a_j}^{a_{j+1}} (1 - t\|\xi\|^2) e^{-t\|\xi\|^2}\,dt\Big|.\end{aligned}$$

Since $|\xi_i| \le \|\xi\|$, we get

$$\begin{aligned}\left|\frac{\partial}{\partial \xi_i} m_{N,\kappa}(\xi)\right| &\le \|\{v_j\}\|_{\ell^\infty(\mathbb{Z})}\|\xi\| \int\limits_0^\infty (1 + t\|\xi\|^2) e^{-t\|\xi\|^2}\,dt\\ &= \|\{v_j\}\|_{\ell^\infty(\mathbb{Z})}\|\xi\| \frac{1+\Gamma(2)}{\|\xi\|^2} =: \frac{c_{\kappa,v}}{\|\xi\|}, \qquad j = 1,\ldots,n.\end{aligned}$$

So $m_{N,\kappa}$ satisfies (2.16) for $|\beta|=1$. Let $\beta=(\beta_1,\dots,\beta_n)\in\mathbb{N}_0^n$. Then

$$\frac{\partial^\beta}{\partial\xi^\beta}(e^{-t\|\xi\|^2})=\prod_{j=1}^n\frac{\partial^{\beta_j}}{\partial\xi_j^{\beta_j}}e^{-t\xi_j^2}. \tag{3.4}$$

By explicit computation, we obtain polynomials $p_{\beta_j}^{(1)}$ and $p_{\beta_j}^{(2)}$ such that

$$\frac{\partial^{\beta_j}}{\partial\xi_j^{\beta_j}}e^{-t\xi_j^2}=\begin{cases}c_{\beta_j}^{(1)}\xi_j e^{-t\xi_j^2}t^{\frac{\beta_j+1}{2}}p_{\beta_j}^{(1)}(t\xi_j^2), & \deg(p_{\beta_j}^{(1)})=\frac{\beta_j-1}{2},\ \beta_j\in 2\mathbb{N}_0+1,\\ c_{\beta_j}^{(2)}e^{-t\xi_j^2}t^{\frac{\beta_j}{2}}p_{\beta_j}^{(2)}(t\xi_j^2), & \deg(p_{\beta_j}^{(2)})=\frac{\beta_j}{2},\ \beta_j\in 2\mathbb{N}_0,\end{cases}$$

where $c_{\beta_j}^{(1)},c_{\beta_j}^{(2)}\in\mathbb{R}$. Without loss of generality, we assume that $\beta=(\beta_1,\dots,\beta_l,\beta_{l+1},\dots,\beta_n)$ where $\beta_j\in 2\mathbb{N}_0$ for $j=1,\dots,l$ and $\beta_j\in 2\mathbb{N}_0+1$ for $j=l,\dots,n$. Set $\beta^{(l)}=(\beta_1,\dots,\beta_l)$ and $\beta^{(l+1)}=(\beta_{l+1},\dots,\beta_n)$. Then

$$\begin{aligned}&\Big(\prod_{j=1}^l p_{\beta_j}^{(1)}(t\xi_j^2)\Big)\Big(\prod_{j=l+1}^n p_{\beta_j}^{(2)}(t\xi_j^2)\Big)\\ &\quad=\Big[\sum_{j=0}^{|\beta^{(l)}|/2}\Big(\sum_{\substack{i_k=0\\ i_1+\cdots+i_l=j}}^{j}a_{\beta^{(l)},i_1,\dots,i_l}^{(j,1)}\xi_1^{2i_1}\cdots\xi_l^{2i_l}\Big)t^j\Big]\\ &\qquad\times\Big[\sum_{j=0}^{\frac{|\beta^{(l+1)}|-(n-l)}{2}}\Big(\sum_{\substack{i_k=0\\ i_{l+1}+\cdots+i_n=j}}^{j}a_{\beta^{(l+1)},i_{l+1},\dots,i_n}^{(j,2)}\xi_{l+1}^{2i_{l+1}}\cdots\xi_n^{2i_n}\Big)t^j\Big]\\ &\quad=\sum_{j=0}^{\frac{|\beta|-(n-l)}{2}}\Big(\sum_{\substack{i_k=0\\ i_1+\cdots+i_n=j}}^{j}a_{\beta,i_1,\dots,i_n}^{(j)}\xi_1^{2i_1}\cdots\xi_n^{2i_n}\Big)t^j,\end{aligned}$$

using the fact that $|\beta^{(l)}|+|\beta^{(l+1)}|=|\beta|$. Here $a_{\beta^{(l)},i_1,\dots,i_l}^{(j,1)}$ and $a_{\beta^{(l+1)},i_{l+1},\dots,i_n}^{(j,2)}$ are real coefficients. Consequently, (3.4) gives

$$\begin{aligned}\frac{\partial^\beta}{\partial\xi^\beta}(e^{-t\|\xi\|^2})&=\Big(\prod_{j=1}^l\frac{\partial^{\beta_j}}{\partial\xi_j^{\beta_j}}e^{-t\xi_j^2}\Big)\Big(\prod_{j=l+1}^n\frac{\partial^{\beta_j}}{\partial\xi_j^{\beta_j}}e^{-t\xi_j^2}\Big)\\ &=c_{\beta^{(l+1)}}^{(1)}c_{\beta^{(l)}}^{(2)}t^{\frac{|\beta|+(n-l)}{2}}e^{-t\|\xi\|^2}\xi_{l+1}\cdots\xi_n\sum_{j=0}^{\frac{|\beta|-(n-l)}{2}}\Big(\sum_{\substack{i_k=0\\ i_1+\cdots+i_n=j}}^{j}a_{\beta,i_1,\dots,i_n}^{(j)}\xi_1^{2i_1}\cdots\xi_n^{2i_n}\Big)t^j\\ &=c_{\beta^{(l+1)}}^{(1)}c_{\beta^{(l)}}^{(2)}e^{-t\|\xi\|^2}\xi_{l+1}\cdots\xi_n\sum_{j=0}^{\frac{|\beta|-(n-l)}{2}}\Big(\sum_{\substack{i_k=0\\ i_1+\cdots+i_n=j}}^{j}a_{\beta,i_1,\dots,i_n}^{(j)}\xi_1^{2i_1}\cdots\xi_n^{2i_n}\Big)t^{j+\frac{|\beta|+(n-l)}{2}},\end{aligned}$$

where $c^{(1)}_{\beta^{(l+1)}} := c^{(1)}_{\beta_{l+1}} \cdots c^{(1)}_{\beta_n}$ and $c^{(2)}_{\beta^{(l)}} := c^{(2)}_{\beta_1} \cdots c^{(2)}_{\beta_l}$. Hence (3.3) gives rise to

$$\begin{aligned}
\frac{\partial^\beta}{\partial\xi^\beta} m_{N,\kappa}(\xi) &= \sum_{p=N_1}^{N_2} v_p \int_{a_p}^{a_{p+1}} \frac{\partial}{\partial t} \frac{\partial^\beta}{\partial\xi^\beta} (e^{-t\|\xi\|^2})\,dt \\
&= c_\beta \xi_{l+1} \cdots \xi_n \Bigg(\sum_{p=N_1}^{N_2} v_p \sum_{j=0}^{\frac{|\beta|-(n-l)}{2}} \Bigg(\sum_{\substack{i_k=0 \\ i_1+\cdots+i_n=j}}^{j} a^{(j)}_{\beta,i_1,\ldots,i_n} \xi_1^{2i_1} \cdots \xi_n^{2i_n} \Bigg) \\
&\qquad\qquad \times \left(j + \frac{|\beta|+(n-l)}{2} \right) \int_{a_p}^{a_{p+1}} t^{j+\frac{|\beta|+(n-l)}{2}-1} e^{-t\|\xi\|^2}\,dt \\
&\quad - \|\xi\|^2 \sum_{p=N_1}^{N_2} v_p \sum_{j=0}^{\frac{|\beta|-(n-l)}{2}} \Bigg(\sum_{\substack{i_k=0 \\ i_1+\cdots+i_n=j}}^{j} a^{(j)}_{\beta,i_1,\ldots,i_n} \xi_1^{2i_1} \cdots \xi_n^{2i_n} \Bigg) \\
&\qquad\qquad \times \int_{a_p}^{a_{p+1}} t^{j+\frac{|\beta|+(n-l)}{2}} e^{-t\|\xi\|^2}\,dt \Bigg),
\end{aligned}$$

where $c_\beta := c^{(1)}_{\beta^{(l+1)}} c^{(2)}_{\beta^{(l)}}$. Consequently,

$$\begin{aligned}
&\left| \frac{\partial^\beta}{\partial\xi^\beta} m_{N,\kappa}(\xi) \right| \\
&\quad \le \|\{v_p\}\|_{\ell^\infty(\mathbb{Z})} |c_\beta|\, \|\xi\|^{n-l} \Bigg(\sum_{j=0}^{\frac{|\beta|-(n-l)}{2}} \|\xi\|^{2j} \Bigg(\sum_{\substack{i_k=0 \\ i_1+\cdots+i_n=j}}^{j} |a^{(j)}_{\beta,i_1,\ldots,i_n}| \Bigg) \\
&\qquad \times \left(j + \frac{|\beta|+(n-l)}{2} \right) \int_0^\infty t^{j+\frac{|\beta|+(n-l)}{2}-1} e^{-t\|\xi\|^2}\,dt \\
&\qquad + \|\xi\|^2 \sum_{j=0}^{\frac{|\beta|-(n-l)}{2}} \|\xi\|^{2j} \Bigg(\sum_{\substack{i_k=0 \\ i_1+\cdots+i_n=j}}^{j} |a^{(j)}_{\beta,i_1,\ldots,i_n}| \Bigg) \int_0^\infty t^{j+\frac{|\beta|+(n-l)}{2}} e^{-t\|\xi\|^2}\,dt \Bigg) \\
&\quad = \|\{v_p\}\|_{\ell^\infty(\mathbb{Z})} |c_\beta|\, \|\xi\|^{n-l} \Bigg(\sum_{j=0}^{\frac{|\beta|-(n-l)}{2}} \|\xi\|^{2j} \Bigg(\sum_{\substack{i_k=0 \\ i_1+\cdots+i_n=j}}^{j} |a^{(j)}_{\beta,i_1,\ldots,i_n}| \Bigg) \\
&\qquad \times \left(j + \frac{|\beta|+(n-l)}{2} \right) \frac{\Gamma\big(j + \frac{|\beta|+(n-l)}{2}\big)}{\|\xi\|^{2(j+\frac{|\beta|+(n-l)}{2})}} \\
&\qquad + \|\xi\|^2 \sum_{j=0}^{\frac{|\beta|-(n-l)}{2}} \|\xi\|^{2j} \Bigg(\sum_{\substack{i_k=0 \\ i_1+\cdots+i_n=j}}^{j} |a^{(j)}_{\beta,i_1,\ldots,i_n}| \Bigg) \frac{\Gamma\big(j + \frac{|\beta|+(n-l)}{2} + 1\big)}{\|\xi\|^{2(j+\frac{|\beta|+(n-l)}{2}+1)}} \Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\|\{v_p\}\|_{\ell^\infty(\mathbb{Z})}|c_\beta|}{\|\xi\|^{|\beta|}}\bigg(\sum_{j=0}^{\frac{|\beta|-(n-l)}{2}} \bigg(\sum_{\substack{i_k=0\\ i_1+\cdots+i_n=j}}^{j} |a^{(j)}_{\beta,i_1,\ldots,i_n}|\bigg) \\
&\quad \times \bigg(j+\frac{|\beta|+(n-l)}{2}\bigg)\Gamma\bigg(j+\frac{|\beta|+(n-l)}{2}\bigg) \\
&\quad + \sum_{j=0}^{\frac{|\beta|-(n-l)}{2}} \bigg(\sum_{\substack{i_k=0\\ i_1+\cdots+i_n=j}}^{j} |a^{(j)}_{\beta,i_1,\ldots,i_n}|\bigg)\Gamma\bigg(j+\frac{|\beta|+(n-l)}{2}+1\bigg)\bigg) \\
&\lesssim \frac{C_{\beta,v}}{\|\xi\|^{|\beta|}},
\end{aligned}
$$

for some $C_{\beta,v}>0$ and all $\xi\in\mathbb{R}^n\setminus\{0\}$. Now, using Theorem 2.2, we conclude that $T_{N,\kappa}$ is a Dunkl–Calderón–Zygmund operator. Moreover, for $0<\epsilon\le 1$, the kernel $K_{N,\kappa}$ satisfies

$$
|K_{N,\kappa}(x,y)| \lesssim \frac{1}{\mu_\kappa(B(x,d(x,y)))}\left(\frac{d(x,y)}{\|x-y\|}\right)^{\epsilon} \tag{3.5}
$$

$$
\begin{aligned}
&|K_{N,\kappa}(x,y)-K_{N,\kappa}(x,y')| \\
&\qquad \lesssim \frac{1}{\mu_\kappa(B(x,d(x,y)))}\left(\frac{\|y-y'\|}{\|x-y\|}\right)^{\epsilon} \quad \text{for } \|y-y'\|\le \frac{d(x,y)}{2},
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
&|K_{N,\kappa}(x,y)-K_{N,\kappa}(x',y)| \\
&\qquad \lesssim \frac{1}{\mu_\kappa(B(x,d(x,y)))}\left(\frac{\|x-x'\|}{\|x-y\|}\right)^{\epsilon} \quad \text{for } \|x-x'\|\le \frac{d(x,y)}{2}. \quad \blacksquare
\end{aligned} \tag{3.7}
$$

Now the $L^p(\mathbb{R}^n,\omega d\mu_\kappa)$ boundedness for $1<p<\infty$ and the weak-type $(1,1)$ boundedness of $T_{N,\kappa}$ are direct consequences of Theorem 2.3. We state the result without any proof.

Proposition 3.3. *Let $\{v_j\}_{j\in\mathbb{Z}}$ be a bounded sequence of real numbers and $\{a_j\}_{j\in\mathbb{Z}}$ be an increasing sequence of positive real numbers. Let $T_{N,\kappa}$, where $N=(N_1,N_2)\in\mathbb{Z}^2$ with $N_1<N_2$, be the operator defined in* (1.2). *Then $T_{N,\kappa}$ is bounded on $L^p(\mathbb{R}^n,\omega d\mu_\kappa)$ for all G-invariant functions $\omega\in A_{p,\kappa}$, where $1<p<\infty$. Moreover, $T_{N,\kappa}$ is bounded from $L^1(\mathbb{R}^n,\omega d\mu_\kappa)$ into $L^{1,\infty}(\mathbb{R}^n,\omega d\mu_\kappa)$ for all G-invariant functions $\omega\in A_{1,\kappa}$.*

Now, we establish that $T_{N,\kappa}$ is bounded from $L^\infty(\mathbb{R}^n)$ into $\mathrm{BMO}_\kappa(\mathbb{R}^n)$.

Theorem 3.4. *Let $\{v_j\}_{j\in\mathbb{Z}}$ be a bounded sequence of real numbers and $\{a_j\}_{j\in\mathbb{Z}}$ be an increasing sequence of positive real numbers. Let $T_{N,\kappa}$, where $N=(N_1,N_2)\in\mathbb{Z}^2$ with $N_1<N_2$, be the operator defined as in* (1.2). *Then there exists a constant $C>0$, depending upon N and $\|\{v_j\}\|_{\ell^\infty(\mathbb{Z})}$, such that $\|T_{N,\kappa}f\|_{\mathrm{BMO}_\kappa(\mathbb{R}^n)}\le C\|f\|_{L^\infty(\mathbb{R}^n)}$ for all $f\in L^\infty(\mathbb{R}^n)$.*

Proof. Let $f \in L^\infty(\mathbb{R}^n)$. Then $\|T_{N,\kappa}f\|_{\mathrm{BMO}_\kappa(\mathbb{R}^n)} \leq 2\|T_{N,\kappa}f\|_{L^\infty(\mathbb{R}^n)}$. Now, from (3.1), we obtain

$$\begin{aligned}|T_{N,\kappa}f(x)| &\leq \int_{\mathbb{R}^n} |K_{N,\kappa}(x,y)|\,|f(y)|\,d\mu_\kappa(y)\\ &\leq \|f\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |K_{N,\kappa}(x,y)|\,d\mu_\kappa(y)\\ &= \|f\|_{L^\infty(\mathbb{R}^n)}\|K_{N,\kappa}(x,\cdot)\|_{L^1(\mathbb{R}^n,d\mu_\kappa)}.\end{aligned}$$

But

$$\begin{aligned}(3.8)\qquad \|K_{N,\kappa}(x,\cdot)\|_{L^1(\mathbb{R}^n,d\mu_\kappa)} &= \int_{\mathbb{R}^n} |K_{N,\kappa}(x,y)|\,d\mu_\kappa(y)\\ \leq \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})} \sum_{j=N_1}^{N_2} &\Big[\int_{\mathbb{R}^n} h_{a_{j+1}}(x,y)\,d\mu_\kappa(y) + \int_{\mathbb{R}^n} h_{a_j}(x,y)\,d\mu_\kappa(y)\Big].\end{aligned}$$

Using (2.10), we get

$$\begin{aligned}(3.9)\qquad \int_{\mathbb{R}^n} |h_{a_{j+1}}(x,y)|\,d\mu_\kappa(y) &\leq \int_{\mathbb{R}^n} \frac{c_1}{V(x,y,\sqrt{a_{j+1}})} e^{-\frac{c}{a_{j+1}}d(x,y)^2}\,d\mu_\kappa(y)\\ &\leq c_1 \int_{\mathbb{R}^n} \frac{1}{\mu_\kappa(B(y,\sqrt{a_{j+1}}))} e^{-\frac{c}{a_{j+1}}d(x,y)^2}\,d\mu_\kappa(y) \leq c_1 M,\end{aligned}$$

where M (see [H20, (5.10)]) is independent of a_{j+1} and y. Now applying (3.9) in (3.8), we arrive at

$$\|K_{N,\kappa}(x,\cdot)\|_{L^1(\mathbb{R}^n,d\mu_\kappa)} \leq 2M(N_2-N_1+1)\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})} < \infty.$$

Hence, we get

$$\|T_{N,\kappa}f\|_{L^\infty(\mathbb{R}^n)} \leq 2M(N_2-N_1+1)\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\|f\|_{L^\infty(\mathbb{R}^n)},$$

proving our assertion. ∎

THEOREM 3.5. *Let $\{v_j\}_{j\in\mathbb{Z}}$ be a bounded sequence of real numbers and $\{a_j\}_{j\in\mathbb{Z}}$ be an increasing sequence of positive real numbers. Let $T_{N,\kappa}$, where $N=(N_1,N_2)\in\mathbb{Z}^2$ with $N_1<N_2$, be the operator defined in (1.2). Then there exists a constant $C>0$, depending upon $\|\{v_j\}\|_{\ell^\infty(\mathbb{Z})}$, such that*

$$\|T_{N,\kappa}f\|_{\mathrm{BMO}_\kappa} \leq C\|f\|_{\mathrm{BMO}_{d,\kappa}}$$

for all $f\in\mathrm{BMO}_{d,\kappa}$.

Proof. Let $x_0\in\mathbb{R}^d$ and $r_0>0$. Set $B=B(x_0,r_0)$ and $B^*=B(x_0,2r_0)$, and write $f=f_1+f_2+f_3$, where $f_1=(f-f_{\mathcal{O}(B)})\chi_{\mathcal{O}(B^*)}$, $f_2=(f-f_{\mathcal{O}(B)})\chi_{\mathcal{O}((B^*)^c)}$ and $f_3=f_{\mathcal{O}(B)}$. We aim to show that $T_{N,\kappa}f(x)<\infty$ for a.e. $x\in\mathbb{R}^n$. By Proposition 3.1, $\|T_{N,\kappa}f_1\|_{L^2(\mathbb{R}^n,d\mu_\kappa)} \leq \|v\|_{\ell^\infty(\mathbb{Z})}\|f_1\|_{L^2(\mathbb{R}^n,d\mu_\kappa)}$.

Now

$$\begin{aligned}(3.10)\qquad \|f_1\|^2_{L^2(\mathbb{R}^n,d\mu_\kappa)} &= \int\limits_{\mathbb{R}^n} |f_1(x)|^2\, d\mu_\kappa(x) = \int\limits_{\mathcal{O}(B^*)} |f - f_{\mathcal{O}(B)}|^2\, d\mu_\kappa(x)\\ &\le 2\int\limits_{\mathcal{O}(B^*)} |f - f_{\mathcal{O}(B^*)}|^2\, d\mu_\kappa(x) + 2\int\limits_{\mathcal{O}(B^*)} |f_{\mathcal{O}(B^*)} - f_{\mathcal{O}(B)}|^2\, d\mu_\kappa(x)\\ &= 2\mu_\kappa(\mathcal{O}(B^*))(\|f\|^2_{\mathrm{BMO}_{d,\kappa}} + |f_{\mathcal{O}(B^*)} - f_{\mathcal{O}(B)}|^2).\end{aligned}$$

We see that

$$\begin{aligned}(3.11)\qquad |f_{\mathcal{O}(B)} - f_{\mathcal{O}(B^*)}| &= \left|\frac{1}{\mu_\kappa(\mathcal{O}(B))}\int\limits_{\mathcal{O}(B)} (f(x) - f_{\mathcal{O}(B^*)})\, d\mu_\kappa(x)\right|\\ &\le \frac{1}{\mu_\kappa(\mathcal{O}(B))}\int\limits_{\mathcal{O}(B^*)} |f(x) - f_{\mathcal{O}(B^*)}|\, d\mu_\kappa(x) \le \frac{\mu_\kappa(\mathcal{O}(B^*))}{\mu_\kappa(\mathcal{O}(B))}\|f\|_{\mathrm{BMO}_{d,\kappa}}.\end{aligned}$$

But $\mu_\kappa(\mathcal{O}(B^*)) \le \#(G)\mu_\kappa(B^*)$ and $\mu_\kappa(\mathcal{O}(B)) \ge \mu_\kappa(B)$ leads to

$$\frac{\mu_\kappa(\mathcal{O}(B^*))}{\mu_\kappa(\mathcal{O}(B))} \le \#(G)\frac{\mu_\kappa(B^*)}{\mu_\kappa(B)} \le 2^{n_\kappa}\alpha\#(G).$$

Hence (3.11) gives

$$(3.12)\qquad |f_{\mathcal{O}(B)} - f_{\mathcal{O}(B^*)}| \le 2^{n_\kappa}\alpha\#(G)\|f\|_{\mathrm{BMO}_{d,\kappa}}.$$

Now, using (3.12) in (3.10), we obtain

$$\begin{aligned}\|f_1\|^2_{2,\kappa} &\le 2\mu_\kappa(\mathcal{O}(B^*))(\|f\|^2_{\mathrm{BMO}_{d,\kappa}} + \alpha^2 2^{2n_\kappa}\#(G)^2\|f\|^2_{\mathrm{BMO}_{d,\kappa}})\\ &= 2\mu_\kappa(\mathcal{O}(B^*))(1 + \alpha^2 2^{2n_\kappa}\#(G)^2)\|f\|^2_{\mathrm{BMO}_{d,\kappa}}.\end{aligned}$$

Hence, we get

$$\begin{aligned}(3.13)\qquad &\|T_{N,\kappa}f_1\|_{L^2(\mathbb{R}^n,d\mu_\kappa)}\\ &\le \|v\|_{\ell^\infty(\mathbb{Z})}(2\mu_\kappa(\mathcal{O}(B^*)))^{1/2}(1 + \alpha^2 2^{2n_\kappa}\#(G)^2)^{1/2}\|f\|_{\mathrm{BMO}_{d,\kappa}} < \infty,\end{aligned}$$

showing that $T_{N,\kappa}f_1(x) < \infty$ for a.e. $x \in \mathbb{R}^n$.

Now, we aim to show $T_{N,\kappa}f_2(x) < \infty$ for a.e. $x \in \mathbb{R}^n$. For this, it is sufficient to show that $e^{t\Delta_\kappa}f_2(x) < \infty$ for all $x \in B$. Indeed, for $x \in B$ we have

$$\begin{aligned}|e^{t\Delta_\kappa}f_2(x)| &\le \int\limits_{\mathbb{R}^n} |h_t(x,y)|\,|f_2(y)|\, d\mu_\kappa(y)\\ &\le \int\limits_{\mathbb{R}^n} \frac{c_1}{V(x,y,\sqrt{t})}e^{-\frac{c}{t}d(x,y)^2}|f_2(y)|\, d\mu_\kappa(y).\end{aligned}$$

Since $V(x,y,t) := \max\{\mu_\kappa(B(x,t)), \mu_\kappa(B(y,t))\}$, we have

$$V(x,y,t) \ge \mu_\kappa(B(x,t)) \ge t^{n_\kappa/2}.$$

Hence,

$$
\begin{aligned}
(3.14)\qquad |e^{t\Delta_\kappa} f_2(x)| &\le \frac{c_1}{t^{n_\kappa/2}} \int_{\mathbb{R}^n} e^{-\frac{c}{t}d(x,y)^2}|f_2(y)|\,d\mu_\kappa(y)\\
&= \frac{c_1}{t^{n_\kappa/2}} \int_{\mathcal{O}((B^*)^c)} e^{-\frac{c}{t}d(x,y)^2}|f(y)-f_{\mathcal{O}(B)}|\,d\mu_\kappa(y)\\
&= \frac{c_1}{t^{n_\kappa/2}} \sum_{j=1}^{\infty} \int_{2^j r_0 < d(x_0,y)\le 2^{j+1}r_0} e^{-\frac{c}{t}d(x,y)^2}|f(y)-f_{\mathcal{O}(B)}|\,d\mu_\kappa(y)\\
&\le \frac{c_1}{t^{n_\kappa/2}} \sum_{j=1}^{\infty} e^{-(2^j-1)^2 r_0^2} \int_{\mathcal{O}(2^{j+1}B)} |f(y)-f_{\mathcal{O}(B)}|\,d\mu_\kappa(y)\\
&\le \frac{c_1}{t^{n_\kappa/2}} \sum_{j=1}^{\infty} e^{-(2^j-1)^2 r_0^2} \Big[\int_{\mathcal{O}(2^{j+1}B)} |f(y)-f_{\mathcal{O}(2^{j+1}B)}|\,d\mu_\kappa(y)\\
&\qquad\qquad + \int_{\mathcal{O}(2^{j+1}B)} |f_{\mathcal{O}(2^{j+1}B)}-f_{\mathcal{O}(B)}|\,d\mu_\kappa(y)\Big]\\
&\le \frac{c_1}{t^{n_\kappa/2}} \sum_{j=1}^{\infty} e^{-(2^j-1)^2 r_0^2} \big(\mu_\kappa(\mathcal{O}(2^{j+1}B))\|f\|_{\mathrm{BMO}_{d,\kappa}}\\
&\qquad\qquad + \mu_\kappa(\mathcal{O}(2^{j+1}B))|f_{\mathcal{O}(2^{j+1}B)}-f_{\mathcal{O}(B)}|\big)\\
&= \frac{c_1}{t^{n_\kappa/2}} \sum_{j=1}^{\infty} e^{-(2^j-1)^2 r_0^2} \mu_\kappa(\mathcal{O}(2^{j+1}B))(\|f\|_{\mathrm{BMO}_{d,\kappa}} + |f_{\mathcal{O}(2^{j+1}B)}-f_{\mathcal{O}(B)}|).
\end{aligned}
$$

But

$$
\begin{aligned}
|f_{\mathcal{O}(2^{j+1}B)}-f_{\mathcal{O}(B)}| &\le \frac{\mu_\kappa(\mathcal{O}(2^{j+1}B))}{\mu_\kappa(\mathcal{O}(B))}\|f\|_{\mathrm{BMO}_{d,\kappa}}\\
&\le \#(G)\frac{\mu_\kappa(2^{j+1}B)}{\mu_\kappa(B)}\|f\|_{\mathrm{BMO}_{d,\kappa}}\\
&\le 2^{(j+1)n_\kappa}\alpha\#(G)\|f\|_{\mathrm{BMO}_{d,\kappa}}.
\end{aligned}
$$

Hence (3.14) becomes

$$
\begin{aligned}
&|e^{t\Delta_\kappa} f_2(x)|\\
&\quad \le \frac{c_1}{t^{n_\kappa/2}} \sum_{j=1}^{\infty} e^{-(2^j-1)^2 r_0^2}\mu_\kappa(\mathcal{O}(2^{j+1}B))(1+2^{(j+1)d_\kappa}\#(G)\alpha)\|f\|_{\mathrm{BMO}_{d,\kappa}} < \infty,
\end{aligned}
$$

as claimed.

Moreover, we have $T_{N,\kappa}f_3 = T_{N,\kappa}f_{\mathcal{O}(B)} \equiv 0$. Therefore $T_{N,\kappa}f(x) < \infty$, for a.e. $x \in \mathbb{R}^d$. We choose $x_1 \in B$ with $T_{N,\kappa}f_2(x_1) < \infty$. Taking $c'_B :=$

$T_{N,\kappa}f_2(x_1)$, we obtain

$$\begin{aligned}(3.15)\quad &\frac{1}{\mu_\kappa(B)}\int_B |T_{N,\kappa}f(x)-c'_B|\,d\mu_\kappa(x)\\ &=\frac{1}{\mu_\kappa(B)}\int_B |T_{N,\kappa}(f_1+f_2+f_3)(x)-T_{N,\kappa}f_2(x_1)|\,d\mu_\kappa(x)\\ &\le \frac{1}{\mu_\kappa(B)}\Big[\int_B |T_{N,\kappa}f_1(x)|\,d\mu_\kappa(x)+\int_B |T_{N,\kappa}f_2(x)-T_{N,\kappa}f_2(x_1)|\,d\mu_\kappa(x)\Big]\\ &=: \frac{1}{\mu_\kappa(B)}(\mathcal{M}_1+\mathcal{M}_2).\end{aligned}$$

Employing the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\mathcal{M}_1 &\le \mu_\kappa(B)^{1/2}\|T_{N,\kappa}f_1\|_{L^2(\mathbb{R}^n,d\mu_\kappa)}\\ &\le \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\mu_\kappa(B)^{1/2}\mu_\kappa(\mathcal{O}(B^*))^{1/2}(1+\alpha^2 2^{2n_\kappa}\#(G)^2)^{1/2}\|f\|_{\mathrm{BMO}_{d,\kappa}},\end{aligned}$$

by using (3.13). Hence,

$$\begin{aligned}(3.16)\quad &\frac{1}{\mu_\kappa(B)}\mathcal{M}_1\\ &\le \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\left(\frac{\mu_\kappa(\mathcal{O}(B^*))}{\mu_\kappa(B)}\right)^{1/2}(1+\alpha^2 2^{2n_\kappa}\#(G)^2)^{1/2}\|f\|_{\mathrm{BMO}_{d,\kappa}}\\ &\le \#(G)^{1/2}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\alpha^{1/2}2^{n_\kappa/2}(1+\alpha^2 2^{2n_\kappa}\#(G)^2)^{1/2}\|f\|_{\mathrm{BMO}_{d,\kappa}}\\ &=: \eta^{(1)}_{v,\kappa}\|f\|_{\mathrm{BMO}_{d,\kappa}}.\end{aligned}$$

In order to estimate $\mathcal{M}_2$, we consider

$$\begin{aligned}&|T_{N,\kappa}f_2(x)-T_{N,\kappa}f_2(x_1)|\\ &\quad\le \int_{\mathbb{R}^n}|K_{N,\kappa}(x,y)-K_{N,\kappa}(x_1,y)|\,|f_2(y)|\,d\mu_\kappa(y)\\ &\quad\lesssim \int_{\mathbb{R}^n}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\frac{1}{\mu_\kappa(B(x,d(x,y)))}\left(\frac{\|x-x_1\|}{\|x-y\|}\right)^\epsilon |f_2(y)|\,d\mu_\kappa(y)\\ &\quad\le \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}(2r_0)^\epsilon\int_{\mathcal{O}((B^*)^c)}\frac{1}{\mu_\kappa(B(x,d(x,y)))}\,\frac{1}{\|x-y\|^\epsilon}\\ &\qquad\qquad\times|f(y)-f_{\mathcal{O}(B)}|\,d\mu_\kappa(y)\\ &\quad= \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}(2r_0)^\epsilon\sum_{j=1}^\infty\int_{2^jr_0<d(x_0,y)\le 2^{j+1}r_0}\frac{|f(y)-f_{\mathcal{O}(B)}|\,d\mu_\kappa(y)}{\|x-y\|^\epsilon\mu_\kappa(B(x,d(x,y)))}.\end{aligned}$$

Since $2^jr_0<d(x_0,y)\le\|x_0-y\|$ and $x\in B$, we have

$$d(x,y)\ge d(x_0,y)-d(x,x_0)\ge(2^j-1)r_0.$$

Consequently, $\|x-y\| \geq (2^j-1)r_0$ and $B(x,(2^j-1)r_0) \subset B(x,d(x,y))$. Thus,

$$
\begin{aligned}
(3.17)\quad & |T_{N,\kappa}f_2(x) - T_{N,\kappa}f_2(x_1)| \\
& \lesssim \sum_{j=1}^{\infty} \frac{\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}(2r_0)^\epsilon}{((2^j-1)r_0)^\epsilon \mu_\kappa(B(x,(2^j-1)r_0))} \int\limits_{2^j r_0 < d(x_0,y) \leq 2^{j+1} r_0} |f(y) - f_{\mathcal{O}(B)}|\, d\mu_\kappa(y) \\
& \leq \sum_{j=1}^{\infty} \frac{\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}(2r_0)^\epsilon}{((2^j-1)r_0)^\epsilon \mu_\kappa(B(x,(2^j-1)r_0))} \int\limits_{d(x_0,y) \leq 2^{j+1} r_0} |f(y) - f_{\mathcal{O}(B)}|\, d\mu_\kappa(y).
\end{aligned}
$$

But

(3.18)

$$
\begin{aligned}
\int\limits_{d(x_0,y)\leq 2^{j+1}r_0} |f(y) - f_{\mathcal{O}(B)}|\, d\mu_\kappa(y) &= \int\limits_{\mathcal{O}(2^{j+1}B)} |f(y) - f_{\mathcal{O}(B)}|\, d\mu_\kappa(y) \\
&\leq \int\limits_{\mathcal{O}(2^{j+1}B)} \Big(|f(y) - f_{\mathcal{O}(2^{j+1}B)}| + \sum_{l=1}^{j+1} |f_{\mathcal{O}(2^l B)} - f_{\mathcal{O}(2^{l-1}B)}| \Big)\, d\mu_\kappa(y) \\
&\leq \mu_\kappa(\mathcal{O}(2^{j+1}B)) \Big(\|f\|_{\mathrm{BMO}_{d,\kappa}} + \sum_{l=1}^{j+1} |f_{\mathcal{O}(2^l B)} - f_{\mathcal{O}(2^{l-1}B)}| \Big) \\
&\leq \mu_\kappa(\mathcal{O}(2^{j+1}B)) \left(\|f\|_{\mathrm{BMO}_{d,\kappa}} + \sum_{l=1}^{j+1} \frac{\mu_\kappa(\mathcal{O}(2^l B))}{\mu_\kappa(\mathcal{O}(2^{l-1}B))} \|f\|_{\mathrm{BMO}_{d,\kappa}} \right) \\
&\leq \mu_\kappa(\mathcal{O}(2^{j+1}B)) \left(1 + \sum_{l=1}^{j+1} \alpha \left(\frac{2^l r_0}{2^{l-1} r_0} \right)^{n_\kappa} \right) \|f\|_{\mathrm{BMO}_{d,\kappa}} \\
&= \mu_\kappa(\mathcal{O}(2^{j+1}B))(1 + \alpha 2^{n_\kappa}(j+1)) \|f\|_{\mathrm{BMO}_{d,\kappa}}.
\end{aligned}
$$

Using (3.18) in (3.17), we get

$$
\begin{aligned}
& |T_{N,\kappa}f_2(x) - T_{N,\kappa}f_2(x_1)| \\
& \quad \lesssim 2^\epsilon \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})} \sum_{j=1}^{\infty} \frac{1}{(2^j-1)^\epsilon} \frac{\mu_\kappa(\mathcal{O}(2^{j+1}B))}{\mu_\kappa(B(x,(2^j-1)r_0))} \\
& \qquad\qquad \times (1 + \alpha 2^{n_\kappa}(j+1)) \|f\|_{\mathrm{BMO}_{d,\kappa}} \\
& \quad \leq 2^\epsilon \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})} \sum_{j=1}^{\infty} \frac{\#(G)}{(2^j-1)^\epsilon} \frac{\mu_\kappa(B(x_0,2^{j+1}r_0))}{\mu_\kappa(B(x,(2^j-1)r_0))} \\
& \qquad\qquad \times (1 + \alpha 2^{n_\kappa}(j+1)) \|f\|_{\mathrm{BMO}_{d,\kappa}}.
\end{aligned}
$$

Since $B(x,(2^j-1)r_0) \subset B(x_0, 2^{j+1}r_0)$, using the doubling condition, we arrive at

$$
\begin{aligned}
&|T_{N,\kappa}f_2(x) - T_{N,\kappa}f_2(x_1)| \\
&\qquad\qquad \lesssim 2^{\epsilon}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\sum_{j=1}^{\infty}\frac{\alpha 2^{2n_\kappa}\#(G)}{(2^j-1)^{\epsilon}}(1+\alpha 2^{n_\kappa}(j+1))\|f\|_{\mathrm{BMO}_{d,\kappa}} \\
&\qquad\qquad \leq 2^{\epsilon}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\sum_{j=1}^{\infty}\frac{\alpha 2^{2n_\kappa}\#(G)}{2^{(j-1)\epsilon}}(1+\alpha 2^{n_\kappa}(j+1))\|f\|_{\mathrm{BMO}_{d,\kappa}} \\
&\qquad\qquad =: \eta^{(2)}_{v,\kappa}\|f\|_{\mathrm{BMO}_{d,\kappa}},
\end{aligned}
$$

where

$$
\eta^{(2)}_{v,\kappa} = 2^{\epsilon}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\sum_{j=1}^{\infty}\frac{\alpha 2^{2n_\kappa}\#(G)}{2^{(j-1)\epsilon}}(1+\alpha 2^{n_\kappa}(j+1)) < \infty.
$$

Hence

$$
\frac{1}{\mu_\kappa(B)}\mathcal{M}_1 \lesssim \frac{1}{\mu_\kappa(B)}\int_B \eta_{v,\kappa}\|f\|_{\mathrm{BMO}_{d,\kappa}}\,d\mu_\kappa(x) = \eta^{(2)}_{v,\kappa}\|f\|_{\mathrm{BMO}_{d,\kappa}}. \tag{3.19}
$$

Applying (3.16) and (3.19) in (3.15), we get

$$
\frac{1}{\mu_\kappa(B)}\int_B |T_{N,\kappa}f(x) - c'_B|\,d\mu_\kappa(x) \lesssim (\eta^{(1)}_{v,\kappa}+\eta^{(2)}_{v,\kappa})\|f\|_{\mathrm{BMO}_{d,\kappa}} =: \eta_{v,\kappa}\|f\|_{\mathrm{BMO}_{d,\kappa}},
$$

proving that $\|T_{N,\kappa}f\|_{\mathrm{BMO}_\kappa} \leq \eta_{v,\kappa}\|f\|_{\mathrm{BMO}_{d,\kappa}}$. ∎

Now, consider the ball $B = B(x_0, r_0)$ as in Theorem 3.5. Then, proceeding as in Theorem 3.5, we obtain the following two remarks in connection with the boundedness of $T_{N,\kappa}$ on each $\mathrm{BMO}_{d,\kappa}$ and BMO_κ.

Remark 3.6. Our method will provide neither boundedness of the operator $T_{N,\kappa}$ from BMO_κ into BMO_κ (or from $\mathrm{BMO}_{d,\kappa}$ into $\mathrm{BMO}_{d,\kappa}$), nor a counterexample. The main obstruction is that the associated measure of a Euclidean ball $B(x_0, r_0)$ depends upon the center x_0. In this connection, the authors in [AB+24] have studied the boundedness of the variational operator from $\mathrm{BMO}_{d,\kappa}$ into BMO_κ. Let $f \in \mathrm{BMO}_\kappa$ be G-invariant. Then $f \in \mathrm{BMO}_{d,\kappa}$ with $\|f\|_{\mathrm{BMO}_{d,\kappa}} \approx \|f\|_{\mathrm{BMO}_\kappa}$. Moreover $T_{N,\kappa}f$ is G-invariant with $\|T_{N,\kappa}f\|_{\mathrm{BMO}_\kappa} \approx \|T_{N,\kappa}f\|_{\mathrm{BMO}_{d,\kappa}}$. Using the above theorem, we have $\|T_{N,\kappa}f\|_{\mathrm{BMO}_\kappa} \leq \eta_{v,\kappa}\|f\|_{\mathrm{BMO}_\kappa}$.

Let $f \in \mathrm{BMO}_\kappa(\mathbb{R}^n)$. Then an explicit computation gives rise to

$$
|T_{N,\kappa}f_2(x)| \leq C^{(1)}_{n,\kappa,x_0,r_0,N}\|f\|_{\mathrm{BMO}_\kappa},
$$

where

$$C^{(1)}_{n,\kappa,x_0,r_0,N} \lesssim \max\bigg\{\frac{(2\|x_0\|+4r_0)^n}{t^{n_\kappa/2}}\bigg(1+\alpha\bigg(\frac{2\|x_0\|+4r_0}{r_0}\bigg)^{n_\kappa}\bigg) \times \prod_{\lambda\in R^+}(|\langle x_0,\lambda\rangle|+2\|x_0\|+4r_0)^{2\kappa(\lambda)},$$

$$\frac{1}{t^{n_\kappa/2}}\sum_{j=1}^{\infty} e^{-\frac{c}{t}2^{2j}r_0^2}(2\|x_0\|+2r_0+2^{j+2}r_0)^n \times \prod_{\lambda\in R^+}(|\langle x_0,\lambda\rangle|+2\|x_0\|+2r_0+2^{j+2}r_0)^{2\kappa(\lambda)} \times \bigg(1+\alpha\bigg(\frac{2\|x_0\|+2r_0+2^{j+2}r_0}{r_0}\bigg)^{n_\kappa}\bigg)\bigg\}.$$

REMARK 3.7. Let $f\in \mathrm{BMO}_{d,\kappa}(\mathbb{R}^n)$. Then, there exists $C^{(2)}_{n,\kappa,r_0,N}>0$ such that $|T_{N,\kappa}f(x)|\le C^{(2)}_{n,\kappa,r_0,N}\|f\|_{\mathrm{BMO}_{d,\kappa}}<\infty$ for a.e. $x\in\mathbb{R}^n$. Moreover, there exists $x_1\in\mathcal{O}(B)$ such that $T_{N,\kappa}f_2(x_1)<\infty$ and

$$\frac{1}{\mu_\kappa(\mathcal{O}(B))}\int_{\mathcal{O}(B)}|T_{N,\kappa}f(x)-T_{N,\kappa}f_2(x_1)|\,d\mu_\kappa(x) \le \bigg(\eta^{(1)}_{v,r_0,n,\kappa}+\frac{\eta^{(2)}_{v,r_0,n,\kappa}}{\mu_\kappa(\mathcal{O}(B))}\int_{\mathcal{O}(B)}\|x-x_1\|\,d\mu_\kappa(x)\bigg)\|f\|_{\mathrm{BMO}_{d,\kappa}}$$

for some $\eta^{(1)}_{v,r_0,n,\kappa},\eta^{(2)}_{v,r_0,n,\kappa}>0$. On the other hand, in Theorem 3.5 we prove that $T_{N,\kappa}f$ is bounded from $\mathrm{BMO}_{d,\kappa}$ to BMO_κ.

4. Boundedness of the maximal differential operator. This section mainly deals with the maximal differential operator T^*_κ defined in (1.3). In the rest of this paper, we assume that $\{a_j\}_{j\in\mathbb{Z}}$ is a λ-lacunary sequence for $\lambda>1$. In other words, we assume that

$$1<\lambda\le a_{j+1}/a_j\le\lambda^2,\quad \forall j\in\mathbb{Z}.$$

For $M\in\mathbb{Z}^+$, we define the truncated maximal differential operator by

$$T^*_{M,\kappa}f(x)=\sup_{\substack{(N_1,N_2)\in\mathbb{Z}^2\\ -M\le N_1<N_2\le M}}|T_{(N_1,N_2),\kappa}f(x)|,\quad x\in\mathbb{R}^n.$$

Now, we aim to obtain a Cotlar-type inequality for $T^*_{M,\kappa}$.

THEOREM 4.1 (Cotlar-type inequality). *For each $q\in(1,\infty)$, there exists a constant $C>0$ such that for every $x\in\mathbb{R}^n$ and $M\in\mathbb{Z}^+$,*

$$T^*_{M,\kappa}f(x)\le C\Big\{\mathcal{M}(T_{(-M,M)}f)(x)+\sum_{\sigma\in G}\mathcal{M}f(\sigma x)+\Big(\sum_{\sigma\in G}(\mathcal{M}_qf(\sigma x))^q\Big)^{1/q}\Big\},$$

where

$$\mathcal{M}f(x) := \sup_{r>0} \frac{1}{\mu_\kappa(B(x,r))} \int_{B(x,r)} |f(y)|\, d\mu_\kappa(y),$$
$$\mathcal{M}_q f(x) := \sup_{r>0} \left(\frac{1}{\mu_\kappa(B(x,r))} \int_{B(x,r)} |f(y)|^q \, d\mu_\kappa(y) \right)^{1/q}.$$

Proof. Let $x_0 \in \mathbb{R}^n$ and $M \in \mathbb{Z}^+$. Assume that $N = (N_1, N_2)$ satisfies $-M \leq N_1 < N_2 \leq M$. Then

$$\begin{aligned} T_{N,\kappa} f(x_0) &= \sum_{j=N_1}^{N_2} v_j (e^{a_{j+1}\Delta_\kappa} f(x_0) - e^{a_j \Delta_\kappa} f(x_0)) \\ &= \sum_{j=N_1}^{M} v_j (e^{a_{j+1}\Delta_\kappa} f(x_0) - e^{a_j \Delta_\kappa} f(x_0)) \\ &\quad - \sum_{j=N_2+1}^{M} v_j (e^{a_{j+1}\Delta_\kappa} f(x_0) - e^{a_j \Delta_\kappa} f(x_0)) \\ &= T_{(N_1,M),\kappa} f(x_0) - T_{(N_2+1,M),\kappa} f(x_0). \end{aligned}$$

So, it is sufficient to find an estimate for $|T_{(m,M),\kappa} f(x_0)|$ when $|m| \leq M$. For each $l \in \mathbb{Z}$, we write $B_l = B(x_0, a_l^{1/2})$. We decompose f as

$$f = f\chi_{\mathcal{O}(B_m)} + f\chi_{\mathcal{O}(B_m^c)} =: f_1 + f_2.$$

Then

$$\begin{aligned} |T_{(m,M),\kappa} f(x_0)| &\leq |T_{(m,M),\kappa} f_1(x_0)| + |T_{(m,M),\kappa} f_2(x_0)| \\ &=: T_1 + T_2. \end{aligned} \tag{4.1}$$

Using (3.1), we get

$$T_1 \leq \int_{\mathbb{R}^n} |K_{(m,M),\kappa}(x_0, y)|\, |f_1(y)|\, d\mu_\kappa(y). \tag{4.2}$$

But

$$|K_{(m,M),\kappa}(x_0, y)| \leq \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})} \sum_{j=m}^{M} |h_{a_{j+1}}(x_0, y) - h_{a_j}(x_0, y)|. \tag{4.3}$$

By using the Mean-Value Theorem, there exists $a_j \leq \xi_j \leq a_{j+1}$ such that

$$\sum_{j=m}^{M} |h_{a_{j+1}}(x_0, y) - h_{a_j}(x_0, y)| \leq (a_{j+1} - a_j) \left| \frac{\partial}{\partial t} h_t(x_0, y) \Big|_{t=\xi_j} \right|.$$

Now, making use of (2.10), we obtain

$$\sum_{j=m}^{M} |h_{a_{j+1}}(x_0,y) - h_{a_j}(x_0,y)| \le (a_{j+1}-a_j)\frac{C}{\xi_j V(x_0,y,\sqrt{\xi_j})} e^{-\frac{c}{\xi_j}d(x_0,y)^2}.$$

Since $\{a_j\}_{j\in\mathbb{Z}}$ is a λ-lacunary sequence, we get

$$\sum_{j=m}^{M} |h_{a_{j+1}}(x_0,y) - h_{a_j}(x_0,y)| \le (\lambda^2-1)a_j\frac{C}{\xi_j V(x_0,y,\sqrt{\xi_j})} e^{-\frac{c}{\xi_j}d(x_0,y)^2}.$$

Thus (4.3) turns into

$$\begin{aligned}
(4.4)\qquad & |K_{(m,M),\kappa}(x_0,y)| \\
&\le \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})} \sum_{j=m}^{M} (\lambda^2-1)a_j \frac{C}{\xi_j V(x_0,y,\sqrt{\xi_j})} e^{-\frac{c}{\xi_j}d(x_0,y)^2} \\
&\le C\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}(\lambda^2-1)\sum_{j=m}^{M}\frac{1}{V(x_0,y,\sqrt{\xi_j})} \\
&\le C\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}(\lambda^2-1)\sum_{j=m}^{M}\frac{1}{\mu_\kappa(B(x_0,\sqrt{\xi_j}))} \\
&= C\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}(\lambda^2-1)\frac{1}{\mu_\kappa(B(x_0,\sqrt{a_m}))}\sum_{j=m}^{M}\frac{\mu_\kappa(B(x_0,\sqrt{a_m}))}{\mu_\kappa(B(x_0,\sqrt{\xi_j}))} \\
&\le C\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}(\lambda^2-1)\frac{1}{\mu_\kappa(B(x_0,\sqrt{a_m}))}\sum_{j=m}^{M}\alpha\left(\frac{a_m}{\xi_j}\right)^{n_\kappa/2} \\
&\le C\alpha\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}(\lambda^2-1)\frac{1}{\mu_\kappa(B(x_0,\sqrt{a_m}))}\sum_{j=m}^{M}\left(\frac{a_m}{a_j}\right)^{n_\kappa/2} \\
&= C\alpha\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}(\lambda^2-1)\frac{1}{\mu_\kappa(B(x_0,\sqrt{a_m}))}\sum_{j=m}^{M}\frac{1}{\lambda^{\frac{n_\kappa}{2}(j-m)}} \\
&\le C\alpha\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}(\lambda^2-1)\frac{1}{\mu_\kappa(B(x_0,\sqrt{a_m}))}\sum_{j=0}^{\infty}\frac{1}{\lambda^{\frac{jn_\kappa}{2}}} \\
&= C\alpha\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}(\lambda^2-1)\frac{\lambda^{\frac{n_\kappa}{2}}}{\lambda^{\frac{n_\kappa}{2}}-1}\frac{1}{\mu_\kappa(B(x_0,\sqrt{a_m}))} \\
&=: c_{v,\lambda,n,\kappa}\frac{1}{\mu_\kappa(B(x_0,\sqrt{a_m}))}, \qquad \forall y\in\mathbb{R}^n.
\end{aligned}$$

Hence (4.2) provides

$$\begin{aligned}T_1 &\le \frac{c_{v,\lambda,n,\kappa}}{\mu_\kappa(B(x_0,\sqrt{a_m}))}\int_{\mathbb{R}^d}|f_1(y)|\,d\mu_\kappa(y)=\frac{c_{v,\lambda,n,\kappa}}{\mu_\kappa(B(x_0,\sqrt{a_m}))}\int_{\mathcal{O}(B_m)}|f(y)|\,d\mu_\kappa(y)\\ &\le \frac{c_{v,\lambda,n,\kappa}}{\mu_\kappa(B(x_0,\sqrt{a_m}))}\sum_{\sigma\in G}\int_{\sigma(B_m)}|f(y)|\,d\mu_\kappa(y)\\ &\le c_{v,\lambda,n,\kappa}\frac{\mu_\kappa(B_m)}{\mu_\kappa(B(x_0,\sqrt{a_m}))}\sum_{\sigma\in G}\mathcal{M}f(\sigma(x_0))=c_{v,\lambda,n,\kappa}\sum_{\sigma\in G}\mathcal{M}f(\sigma(x_0)).\end{aligned}$$

On the other hand, we can rewrite $T_{(m,M),\kappa}f_2(x_0)$ as

$$\begin{aligned}&T_{(m,M),\kappa}f_2(x_0)\\ &=\frac{1}{\mu_\kappa\big(B\big(x_0,\tfrac12 a_{m-1}^{1/2}\big)\big)}\int_{B(x_0,\frac12 a_{m-1}^{1/2})}T_{(m,M),\kappa}f_2(x_0)\,d\mu_\kappa(z)\\ &=\frac{1}{\mu_\kappa\big(B\big(x_0,\tfrac12 a_{m-1}^{1/2}\big)\big)}\Big[\int_{B(x_0,\frac12 a_{m-1}^{1/2})}T_{(-M,M),\kappa}f(z)\,d\mu_\kappa(z)\\ &\quad-\int_{B(x_0,\frac12 a_{m-1}^{1/2})}T_{(-M,M),\kappa}f_1(z)\,d\mu_\kappa(z)-\int_{B(x_0,\frac12 a_{m-1}^{1/2})}T_{(-M,m-1),\kappa}f_2(z)\,d\mu_\kappa(z)\\ &\qquad+\int_{B(x_0,\frac12 a_{m-1}^{1/2})}(T_{(m,M),\kappa}f_2(x_0)-T_{(m,M),\kappa}f_2(z))\,d\mu_\kappa(z)\Big].\end{aligned}$$

Hence

$$\begin{aligned}&|T_{(m,M),\kappa}f_2(x_0)| \qquad (4.5)\\ &=\frac{1}{\mu_\kappa\big(B\big(x_0,\tfrac12 a_{m-1}^{1/2}\big)\big)}\Big[\int_{B(x_0,\frac12 a_{m-1}^{1/2})}|T_{(-M,M),\kappa}f(z)|\,d\mu_\kappa(z)\\ &+\int_{B(x_0,\frac12 a_{m-1}^{1/2})}|T_{(-M,M),\kappa}f_1(z)|\,d\mu_\kappa(z)+\int_{B(x_0,\frac12 a_{m-1}^{1/2})}|T_{(-M,m-1),\kappa}f_2(z)|\,d\mu_\kappa(z)\\ &\qquad+\int_{B(x_0,\frac12 a_{m-1}^{1/2})}|T_{(m,M),\kappa}f_2(x_0)-T_{(m,M),\kappa}f_2(z)|\,d\mu_\kappa(z)\Big]\\ &=: A_1+A_2+A_3+A_4.\end{aligned}$$

It is easy to see that

$$A_1\le\mathcal{M}(T_{(-M,M),\kappa}f)(x_0). \tag{4.6}$$

For A_2, employing Hölder's inequality, we get

$$\begin{aligned} A_2 &\le \frac{1}{\mu_\kappa\big(B\big(x_0, \frac{1}{2}a_{m-1}^{1/2}\big)\big)} \mu_\kappa\big(B\big(x_0, \tfrac{1}{2}a_{m-1}^{1/2}\big)\big)^{1/p} \\ &\quad \times \Big(\int\limits_{B(x_0, \frac{1}{2}a_{m-1}^{1/2})} |T_{(-M,M),\kappa} f_1(z)|^q \, d\mu_\kappa(z) \Big)^{1/q} \\ &\le \frac{1}{\mu_\kappa\big(B\big(x_0, \frac{1}{2}a_{m-1}^{1/2}\big)\big)^{1/q}} \|T_{(-M,M),\kappa} f_1\|_{L^q(\mathbb{R}^n, d\mu_\kappa)}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Now, making use of Proposition 3.3, we obtain

$$\begin{aligned} A_2 &\le \frac{c'}{\mu_\kappa\big(B\big(x_0, \frac{1}{2}a_{m-1}^{1/2}\big)\big)^{1/q}} \|f_1\|_{L^q(\mathbb{R}^n, d\mu_\kappa)} \\ &= c' \Big(\frac{1}{\mu_\kappa\big(B\big(x_0, \frac{1}{2}a_{m-1}^{1/2}\big)\big)} \int\limits_{\mathcal{O}(B_m)} |f(z)|^q \, d\mu_\kappa(z) \Big)^{1/q}, \qquad c' > 0. \end{aligned}$$

Hence,

$$\begin{aligned} A_2^q &\le c' \frac{1}{\mu_\kappa\big(B\big(x_0, \frac{1}{2}a_{m-1}^{1/2}\big)\big)} \int\limits_{\mathcal{O}(B_m)} |f(z)|^q \, d\mu_\kappa(z) \\ &\le c' \frac{1}{\mu_\kappa\big(B\big(x_0, \frac{1}{2}a_{m-1}^{1/2}\big)\big)} \sum_{\sigma \in G} \int\limits_{\sigma B_m} |f(z)|^q \, d\mu_\kappa(z) \\ &= c' \frac{\mu_\kappa(B_m)}{\mu_\kappa\big(B\big(x_0, \frac{1}{2}a_{m-1}^{1/2}\big)\big)} \sum_{\sigma \in G} \frac{1}{\mu_\kappa(\sigma B_m)} \int\limits_{\sigma B_m} |f(z)|^q \, d\mu_\kappa(z) \\ &\le \alpha c' 2^{n_\kappa} \Big(\frac{a_m}{a_{m-1}} \Big)^{n_\kappa/2} \sum_{\sigma \in G} (\mathcal{M}_q f(\sigma x_0))^q \le \alpha c' (2\lambda)^{n_\kappa} \sum_{\sigma \in G} (\mathcal{M}_q f(\sigma x_0))^q. \end{aligned}$$

Consequently,

$$A_2 \le (\alpha c' (2\lambda)^{n_\kappa})^{1/q} \Big(\sum_{\sigma \in G} (\mathcal{M}_q f(\sigma x_0))^q \Big)^{1/q}. \tag{4.7}$$

Now, for A_3, we see that

$$\begin{aligned} |T_{(-M,m-1),\kappa} f_2(z)| &\le \int\limits_{\mathbb{R}^n} |K_{(-M,m-1),\kappa}(z,y)| \, |f_2(y)| \, d\mu_\kappa(y) \\ &= \int\limits_{\mathcal{O}(B_m^c)} |K_{(-M,m-1),\kappa}(z,y)| \, |f_2(y)| \, d\mu_\kappa(y). \end{aligned} \tag{4.8}$$

Here we need to find an estimate for $|K_{(-M,m-1),\kappa}(z,y)|$ for $z \in \frac{1}{2}B_{m-1}$ and $y \in \mathcal{O}(B_m^c)$. Starting with the definition of the kernel, we get

$$\begin{aligned}|K_{(-M,m-1),\kappa}(z,y)| &\le \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})} \sum_{j=-M}^{m-1} |h_{a_{j+1}}(z,y) - h_{a_j}(z,y)| \\ &= \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})} \sum_{j=-M}^{m-1} (a_{j+1}-a_j) \left| \frac{\partial}{\partial t} h_t(z,y)\Big|_{t=\xi_j} \right|,\end{aligned}$$

by applying the Mean-Value Theorem, where $a_j \le \xi_j \le a_{j+1}$. Since $\{a_j\}_{j\in\mathbb{Z}}$ is a λ-lacunary sequence, we obtain

$$|K_{(-M,m-1),\kappa}(z,y)| \le \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})} \sum_{j=-M}^{m-1} (\lambda^2-1) a_j \left| \frac{\partial}{\partial t} h_t(z,y)\Big|_{t=\xi_j} \right|.$$

Employing (2.10), we have

$$\begin{aligned}&|K_{(-M,m-1),\kappa}(z,y)| \\ &\quad\le \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})} \sum_{j=-M}^{m-1} (\lambda^2-1) a_j \frac{C}{\xi_j V(z,y,\sqrt{\xi_j})} e^{-\frac{c}{\xi_j} d(z,y)^2} \\ &\quad\le C\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})} (\lambda^2-1) \sum_{j=-M}^{m-1} \frac{a_j}{\xi_j \mu_\kappa(B(z,\sqrt{\xi_j}))} e^{-\frac{c}{\xi_j} d(z,y)^2}.\end{aligned}$$

Seeing $a_j \le \xi_j \le a_{j+1}$ and $\frac{1}{\lambda^2 a_j} \le \frac{1}{a_{j+1}}$, we get

$$\begin{aligned}&|K_{(-M,m-1),\kappa}(z,y)| \\ &\quad\le C\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})} (\lambda^2-1) \sum_{j=-M}^{m-1} \frac{a_j}{a_j \mu_\kappa(B(z,\sqrt{a_j}))} e^{-\frac{c}{\lambda^2 a_j} d(z,y)^2}.\end{aligned} \tag{4.9}$$

Now, for $-M \le j \le m-1$, we wish to show that

$$\frac{1}{a_j \mu_\kappa(B(z,\sqrt{a_j}))} e^{-\frac{c}{\lambda^2 a_j} d(z,y)^2} \le \frac{\alpha 2^{n_\kappa}}{a_s \mu_\kappa(B_s)} \tag{4.10}$$

whenever

$$d(z,y)^2 \ge \frac{\lambda^2 a_j}{c} \left(\frac{n_\kappa}{2} + 1 \right) \log\left(\frac{a_s}{a_j} \right)$$

for $s \ge m$. Verifying (4.10) is the same as verifying

$$\frac{a_j \mu_\kappa(B(z,\sqrt{a_j}))}{a_s \mu_\kappa(B_s)} e^{\frac{c}{\lambda^2 a_j} d(z,y)^2} \ge \frac{1}{\alpha 2^{n_\kappa}}.$$

Now $d(z,y)^2 \geq \frac{\lambda^2 a_j}{c}\left(\frac{n_\kappa}{2}+1\right)\log\left(\frac{a_s}{a_j}\right)$ if and only if $e^{\frac{c}{\lambda^2 a_j}d(z,y)^2} \geq \left(\frac{a_s}{a_j}\right)^{\frac{n_\kappa}{2}+1}$. Consequently,

$$\begin{aligned}\frac{a_j\mu_\kappa(B(z,\sqrt{a_j}))}{a_s\mu_\kappa(B_s)}e^{\frac{c}{\lambda^2 a_j}d(z,y)^2} &\geq \frac{\mu_\kappa(B(z,\sqrt{a_j}))}{\mu_\kappa(B_s)}\left(\frac{a_s}{a_j}\right)^{n_\kappa/2} \\ &\geq \frac{\mu_\kappa(B(z,\frac{1}{2}\sqrt{a_j}))}{\mu_\kappa(B_s)}\left(\frac{a_s}{a_j}\right)^{n_\kappa/2}.\end{aligned} \tag{4.11}$$

We intend to show that $B(z,\frac{1}{2}\sqrt{a_j}) \subset B_s = B(x_0,\sqrt{a_s})$ whenever $s \geq m$. Assume that $u \in B(z,\frac{1}{2}\sqrt{a_j})$. Then $\|u-x_0\| \leq \|u-z\|+\|z-x_0\| \leq \frac{1}{2}\sqrt{a_j}+\|z-x_0\|$. As $z \in \frac{1}{2}B_{m-1}$, we have $\|z-x_0\| \leq \frac{1}{2}\sqrt{a_{m-1}}$. So, using the fact that $\{a_j\}_{j\in\mathbb{Z}}$ is λ-lacunary, we have

$$\begin{aligned}\|u-x_0\| &\leq \frac{1}{2}\sqrt{a_j}+\frac{1}{2}\sqrt{a_{m-1}} = \frac{1}{2}\sqrt{a_s}\left(\left(\frac{a_j}{a_s}\right)^{1/2}+\left(\frac{a_{m-1}}{a_s}\right)^{1/2}\right) \\ &\leq \frac{1}{2}\sqrt{a_s}\left(\frac{1}{\lambda^{(s-j)/2}}+\frac{1}{\lambda^{(s-m+1)/2}}\right) \\ &= \frac{\sqrt{a_s}}{2\lambda^{(s-m+1)/2}}\left(1+\frac{1}{\lambda^{\frac{(m-1)-j}{2}}}\right) \leq \sqrt{a_s}\end{aligned}$$

for $s \geq m$ and $-M \leq j \leq m-1$, since $0 < 1/\lambda < 1$, proving our claim. Utilizing the doubling condition, we get

$$\frac{1}{\alpha 2^{n_\kappa}}\left(\frac{a_j}{a_s}\right)^{n_\kappa/2} \leq \frac{\mu_\kappa(B(z,\frac{1}{2}\sqrt{a_j}))}{\mu_\kappa(B_s)} \leq \frac{\alpha}{2^n}\left(\frac{a_j}{a_s}\right)^{n/2}. \tag{4.12}$$

Employing (4.12) in (4.11), we find that

$$\frac{a_j\mu_\kappa(B(z,\sqrt{a_j}))}{a_s\mu_\kappa(B_s)}e^{\frac{c}{\lambda^2 a_j}d(z,y)^2} \geq \frac{1}{\alpha 2^{n_\kappa}}\left(\frac{a_j}{a_s}\right)^{n_\kappa/2}\left(\frac{a_s}{a_j}\right)^{n_\kappa/2} = \frac{1}{\alpha 2^{n_\kappa}},$$

proving (4.10). Now, making use of (4.10) in (4.9), we obtain

$$\begin{aligned}|K_{(-M,m-1),\kappa}(z,y)| &\leq C\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}(\lambda^2-1)\sum_{j=-M}^{m-1}\alpha 2^{n_\kappa}\frac{1}{\mu_\kappa(B_s)}\frac{a_j}{a_s} \\ &\leq C\alpha 2^{n_\kappa}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}(\lambda^2-1)\frac{1}{\mu_\kappa(B_s)}\sum_{j=-M}^{m-1}\frac{1}{\lambda^{s-j}} \\ &= \frac{C\alpha 2^{n_\kappa}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}(\lambda^2-1)}{\mu_\kappa(B_s)}\left(\frac{1}{\lambda^{s-m+1}}+\frac{1}{\lambda^{s-m+2}}+\cdots+\frac{1}{\lambda^{s+M}}\right) \\ &\leq \frac{C\alpha 2^{n_\kappa}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}(\lambda^2-1)}{\mu_\kappa(B_s)}\frac{1}{\lambda^{s-m+1}}\left(1+\frac{1}{\lambda}+\cdots\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{C\alpha 2^{n_\kappa}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}(\lambda^2-1)}{\mu_\kappa(B_s)}\,\frac{1}{\lambda^{s-m+1}}\,\frac{\lambda}{\lambda-1}\\
&= \frac{C\alpha 2^{n_\kappa}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\lambda(\lambda+1)}{\mu_\kappa(B_s)}\,\frac{1}{\lambda^{s-m+1}}.
\end{aligned}
$$

Hence (4.8) gives

$$
\begin{aligned}
&|T_{(-M,m-1),\kappa}f_2(z)| \leq \sum_{s=m}^{\infty}\int\limits_{a_s^{1/2}<d(x_0,y)\leq a_{s+1}^{1/2}} |K_{(-M,m-1),\kappa}(z,y)|\,|f(y)|\,d\mu_\kappa(y)\\
&\leq \sum_{s=m}^{\infty}\int\limits_{a_s^{1/2}<d(x_0,y)\leq a_{s+1}^{1/2}} \frac{C\alpha 2^{n_\kappa}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\lambda(\lambda+1)}{\mu_\kappa(B_s)}\,\frac{1}{\lambda^{s-m+1}}|f(y)|\,d\mu_\kappa(y)\\
&\leq C\alpha 2^{n_\kappa}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\lambda(\lambda+1)\sum_{s=m}^{\infty}\frac{1}{\lambda^{s-m+1}\mu_\kappa(B_s)}\int\limits_{d(x_0,y)\leq a_{s+1}^{1/2}}|f(y)|\,d\mu_\kappa(y)\\
&\leq C\alpha 2^{n_\kappa}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\lambda(\lambda+1)\sum_{s=m}^{\infty}\frac{1}{\lambda^{s-m+1}\mu_\kappa(B_s)}\sum_{\sigma\in G}\int\limits_{\sigma B_{s+1}}|f(y)|\,d\mu_\kappa(y)\\
&\leq C\alpha 2^{n_\kappa}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\lambda(\lambda+1)\sum_{\sigma\in G}\sum_{s=m}^{\infty}\frac{1}{\lambda^{s-m+1}}\,\frac{\mu_\kappa(\sigma B_{s+1})}{\mu_\kappa(B_s)}\mathcal{M}f(\sigma(x_0))\\
&= C\alpha 2^{n_\kappa}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\lambda(\lambda+1)\sum_{\sigma\in G}\sum_{s=m}^{\infty}\frac{1}{\lambda^{s-m+1}}\,\frac{\mu_\kappa(B_{s+1})}{\mu_\kappa(B_s)}\mathcal{M}f(\sigma(x_0))\\
&\leq C\alpha^2 2^{n_\kappa}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\lambda(\lambda+1)\sum_{s=m}^{\infty}\frac{1}{\lambda^{s-m+1}}\lambda^{n_\kappa}\sum_{\sigma\in G}\mathcal{M}f(\sigma(x_0))\\
&= C\alpha^2 2^{n_\kappa}\lambda^{n_\kappa}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}(\lambda+1)\left(\sum_{s=0}^{\infty}\frac{1}{\lambda^s}\right)\sum_{\sigma\in G}\mathcal{M}f(\sigma(x_0))\\
&= \frac{C\alpha^2 2^{n_\kappa}\lambda^{n_\kappa}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\lambda(\lambda+1)}{\lambda-1}\sum_{\sigma\in G}\mathcal{M}f(\sigma(x_0)).
\end{aligned}
$$

Therefore, we get

$$
A_3 \leq \frac{C\alpha^2 2^{n_\kappa}\lambda^{n_\kappa}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\lambda(\lambda+1)}{\lambda-1}\sum_{\sigma\in G}\mathcal{M}f(\sigma(x_0)). \tag{4.13}
$$

Now, we aim to estimate A_4. For $z\in\frac{1}{2}B_{m-1}$, we have

$$
\begin{aligned}
&|T_{(m,M),\kappa}f_2(z)-T_{(m,M),\kappa}f_2(x_0)| \\
&\qquad\leq \int\limits_{\mathbb{R}^d}|K_{(m,M),\kappa}(z,y)-K_{(m,M),\kappa}(x_0,y)|\,|f_2(y)|\,d\mu_\kappa(y)
\end{aligned}
\tag{4.14}
$$

$$\lesssim \int_{\mathbb{R}^d} \|\{v_j\}_{j\in\mathbb{Z}}\| \frac{1}{\mu_\kappa(B(z,d(z,y)))} \frac{\|z-x_0\|}{\|z-y\|} |f_2(y)|\, d\mu_\kappa(y)$$

$$\leq \|\{v_j\}_{j\in\mathbb{Z}}\| \frac{a_{m-1}^{1/2}}{2} \int_{\mathcal{O}((B_m)^c)} \frac{1}{\mu_\kappa(B(z,d(z,y)))} \frac{1}{\|z-y\|} |f(y)|\, d\mu_\kappa(y)$$

$$= \frac{a_{m-1}^{1/2}}{2} \|\{v_j\}_{j\in\mathbb{Z}}\| \sum_{j=m}^{\infty} \int_{a_j^{1/2}<d(x_0,y)\leq a_{j+1}^{1/2}} \frac{1}{\mu_\kappa(B(z,d(z,y)))} \frac{1}{\|z-y\|} |f(y)|\, d\mu_\kappa(y).$$

For $j \geq m$, we notice that

$$\begin{aligned} d(z,y) &\geq d(x_0,y) - d(x_0,z) \geq a_j^{1/2} - \frac{1}{2} a_{m-1}^{1/2} = a_j^{1/2} \left(1 - \frac{1}{2}\left(\frac{a_{m-1}}{a_j}\right)^{1/2}\right) \\ &\geq a_j^{1/2} \left(1 - \frac{1}{2\lambda^{\frac{j-m+1}{2}}}\right) \geq \frac{1}{2} a_j^{1/2}. \end{aligned}$$

So $B\big(z, \frac{1}{2}a_j^{1/2}\big) \subset B(z,d(z,y))$ and $\|z-y\| \geq \frac{1}{2}a_j^{1/2}$. Hence (4.14) gives

$$\begin{aligned} &|T_{(m,M),\kappa} f_2(z) - T_{(m,M),\kappa} f_2(x_0)| \\ &\quad\leq \frac{a_{m-1}^{1/2}}{2} \|\{v_j\}_{j\in\mathbb{Z}}\| \sum_{j=m}^{\infty} \frac{2}{a_j^{1/2} \mu_\kappa\big(B\big(z, \frac{1}{2}a_j^{1/2}\big)\big)} \int_{a_j^{1/2}<d(x_0,y)\leq a_{j+1}^{1/2}} |f(y)|\, d\mu_\kappa(y) \\ &\quad\leq \frac{a_{m-1}^{1/2}}{2} \|\{v_j\}_{j\in\mathbb{Z}}\| \sum_{j=m}^{\infty} \frac{2}{a_j^{1/2} \mu_\kappa\big(B\big(z, \frac{1}{2}a_j^{1/2}\big)\big)} \int_{\mathcal{O}(B_{j+1})} |f(y)|\, d\mu_\kappa(y) \\ &\quad\leq \frac{a_{m-1}^{1/2}}{2} \|\{v_j\}_{j\in\mathbb{Z}}\| \sum_{\sigma\in G} \sum_{j=m}^{\infty} \frac{2}{a_j^{1/2}} \frac{\mu_\kappa(B(\sigma(x_0), a_{j+1}^{1/2}))}{\mu_\kappa\big(B\big(z, \frac{1}{2}a_j^{1/2}\big)\big)} \mathcal{M}f(\sigma(x_0)) \\ &\quad= \frac{a_{m-1}^{1/2}}{2} \|\{v_j\}_{j\in\mathbb{Z}}\| \sum_{\sigma\in G} \sum_{j=m}^{\infty} \frac{2}{a_j^{1/2}} \frac{\mu_\kappa(B(x_0, a_{j+1}^{1/2}))}{\mu_\kappa\big(B\big(z, \frac{1}{2}a_j^{1/2}\big)\big)} \mathcal{M}f(\sigma(x_0)). \end{aligned}$$

A straightforward computation shows that $B(z, \frac{1}{2}a_j^{1/2}) \subset B(x_0, a_{j+1}^{1/2})$. Hence, using the doubling condition, we obtain

$$\begin{aligned} &|T_{(m,M),\kappa} f_2(z) - T_{(m,M),\kappa} f_2(x_0)| \\ &\qquad\leq \frac{a_{m-1}^{1/2}}{2} \|\{v_j\}_{j\in\mathbb{Z}}\| \sum_{j=m}^{\infty} \frac{2}{a_j^{1/2}} 2^{n_\kappa} \alpha \left(\frac{a_{j+1}}{a_j}\right)^{n_\kappa/2} \sum_{\sigma\in G} \mathcal{M}f(\sigma(x_0)) \\ &\qquad= \frac{a_{m-1}^{1/2}}{2} \|\{v_j\}_{j\in\mathbb{Z}}\| 2^{n_\kappa+1} \alpha \lambda^{n_\kappa} \left(\sum_{j=m}^{\infty} \frac{1}{a_j^{1/2}}\right) \sum_{\sigma\in G} \mathcal{M}f(\sigma(x_0)) \end{aligned}$$

$$= \frac{a_{m-1}^{1/2}}{2}\|\{v_j\}_{j\in\mathbb{Z}}\|2^{n_\kappa+1}\alpha\lambda^{n_\kappa}\frac{\sqrt{\lambda}}{a_m^{1/2}(\sqrt{\lambda}-1)}\sum_{\sigma\in G}\mathcal{M}f(\sigma(x_0))$$
$$\le \frac{1}{\lambda}\|\{v_j\}_{j\in\mathbb{Z}}\|2^{n_\kappa}\alpha\lambda^{n_\kappa}\frac{\sqrt{\lambda}}{\sqrt{\lambda}-1}\sum_{\sigma\in G}\mathcal{M}f(\sigma(x_0)).$$

Consequently,

$$A_4 \le \frac{1}{\lambda}\|\{v_j\}_{j\in\mathbb{Z}}\|2^{n_\kappa}\alpha\lambda^{n_\kappa}\frac{\sqrt{\lambda}}{\sqrt{\lambda}-1}\sum_{\sigma\in G}\mathcal{M}f(\sigma(x_0)). \tag{4.15}$$

Applying (4.6), (4.7), (4.13) and (4.15) in (4.1), we get

$$|T_{(m,M),\kappa}f(x_0)|$$
$$\le C_{n,\kappa,v,\lambda}\Big(\mathcal{M}(T_{(-M,M),\kappa}f)(x_0)+\sum_{\sigma\in G}\mathcal{M}f(\sigma(x_0))+\Big(\sum_{\sigma\in G}(\mathcal{M}_qf(\sigma x_0))^q\Big)^{1/q}\Big).$$

This yields the required result since the constant $C_{n,\kappa,v,\lambda}$ does not depend on the point x_0. ∎

THEOREM 4.2. *Let $\{a_j\}$ be a λ-lacunary sequence with $\lambda > 1$. Then the following hold:*

(i) *For $1<p<\infty$ and every G-invariant function $\omega\in A_{p,\kappa}$, there exists a constant $C>0$, depending upon n, κ, v and λ, such that $\|T_\kappa^* f\|_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)} \le C\|f\|_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)}$ for all $f\in L^p(\mathbb{R}^n,\omega d\mu_\kappa)$.*
(ii) *For $p=1$ and every G-invariant function $\omega\in A_{1,\kappa}$, there exists a constant $C>0$, depending upon n, κ, v and λ, such that $\|T_\kappa^* f\|_{L^{1,\infty}(\mathbb{R}^n,\omega d\mu_\kappa)} \le C\|f\|_{L^1(\mathbb{R}^n,\omega d\mu_\kappa)}$ for all $f\in L^1(\mathbb{R}^n,\omega d\mu_\kappa)$.*

Proof. For $M\in\mathbb{Z}^+$, using Theorem 4.1, we have

$$T_{M,\kappa}^* f(x)$$
$$\le C\Big\{\mathcal{M}(T_{(-M,M),\kappa}f)(x)+\sum_{\sigma\in G}\mathcal{M}f(\sigma x)+\Big(\sum_{\sigma\in G}(\mathcal{M}_qf(\sigma x))^q\Big)^{1/q}\Big\}.$$

Hence, for $1<p<\infty$,

$$\|T_{M,\kappa}^* f\|_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)}$$
$$\le C\Big\{\|\mathcal{M}(T_{(-M,M),\kappa}f)\|_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)}$$
$$+\sum_{\sigma\in G}\|\mathcal{M}f(\sigma\,\cdot)\|_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)}+\Big\|\Big(\sum_{\sigma\in G}(\mathcal{M}_qf(\sigma\,\cdot))^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)}\Big\}. \tag{4.16}$$

For $x\in\mathbb{R}^n$, consider $\Phi(x)=\{\Phi_\sigma(x)\}_{\sigma\in G}$, where $\Phi_\sigma(x)=\mathcal{M}_qf(\sigma x)$. Then a straightforward application of Minkowski's integral inequality shows that

$\big\|\|\Phi(\cdot)\|_{\ell^q(G)}\big\|_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)} \le \big\|\|\Phi(\cdot)\|_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)}\big\|_{\ell^q(G)}$. In other words,

$$\Big\|\Big(\sum_{\sigma\in G}(\mathcal{M}_q f(\sigma\,\cdot))^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)} \le \Big(\sum_{\sigma\in G}\|\mathcal{M}_q f(\sigma\,\cdot)\|^q_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)}\Big)^{1/q}.$$

Hence (4.16) becomes

$$\begin{aligned}\|T^*_{M,\kappa}f\|_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)} \le C\Big\{&\|\mathcal{M}(T_{(-M,M),\kappa}f)\|_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)}\\ &+\sum_{\sigma\in G}\|\mathcal{M}f(\sigma\,\cdot)\|_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)} + \Big(\sum_{\sigma\in G}\|\mathcal{M}_q f(\sigma\,\cdot)\|^q_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)}\Big)^{1/q}\Big\}.\end{aligned}$$

Since the triple $(\mathbb{R}^n,\|\cdot\|,\mu_\kappa)$ is a space of homogeneous type, the maximal operators $\mathcal{M}$ and $\mathcal{M}_q$ are bounded on $L^p(\mathbb{R}^n,\omega d\mu_\kappa)$ for $1<p<\infty$ (see [GL$^+$14]). So, we get

$$\begin{aligned}&\|T^*_{M,\kappa}f\|_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)}\\ &\quad\le C\Big\{\|T_{(-M,M),\kappa}f\|_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)} + \sum_{\sigma\in G}\|f(\sigma\,\cdot)\|_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)}\\ &\qquad\qquad + \Big(\sum_{\sigma\in G}\|f(\sigma\,\cdot)\|^q_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)}\Big)^{1/q}\Big\}\\ &\quad= C\big\{\|T_{(-M,M),\kappa}f\|_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)} + (\#(G)+\#(G)^{1/q})\|f\|_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)}\big\},\end{aligned}$$

as the measure μ_κ and the weight function ω are both G-invariant. Now, using Proposition 3.3, we obtain

$$\|T^*_{M,\kappa}f\|_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)} \lesssim \|f\|_{L^p(\mathbb{R}^n,\omega d\mu_\kappa)}$$

for $1<p<\infty$, proving part (i) of the theorem.

To obtain (ii), we define $\mathbf{T}_\kappa : L^p(\mathbb{R}^n,\omega d\mu_\kappa)\to L^p(\mathbb{R}^n,\ell^\infty(\mathbb{Z}^2),\omega d\mu_\kappa)$ by $\mathbf{T}_\kappa f(x) := \{T_{N,\kappa}f(x)\}_{N\in\mathbb{Z}^2}$, $1<p<\infty$. Then it is obvious from (i) that $\mathbf{T}_\kappa$ is bounded from $L^p(\mathbb{R}^n,\omega d\mu_\kappa)$ into $L^p(\mathbb{R}^n,\ell^\infty(\mathbb{Z}^2),\omega d\mu_\kappa)$. We notice that $\mathbf{T}_\kappa$ is an integral operator with kernel $\mathbf{K}_\kappa(x,y) := \{K_{N,\kappa}(x,y)\}_{N\in\mathbb{Z}^2}$. Since each $K_{N,\kappa}$ is a Dunkl–Calderón–Zygmund kernel, by vector-valued Dunkl–Calderón–Zygmund theory, the operator $\mathbf{T}_\kappa$ is bounded from $L^1(\mathbb{R}^n,\omega d\mu_\kappa)$ into $L^{1,\infty}(\mathbb{R}^n,\ell^\infty(\mathbb{Z}^2),\omega d\mu_\kappa)$ for all G-invariant $\omega\in A_{1,\kappa}$, proving (ii). ∎

THEOREM 4.3. *Let $\{a_j\}$ be a λ-lacunary sequence with $\lambda>1$. Then there exists a constant $C>0$, depending upon n, v, κ, such that $\|T^*_\kappa f\|_{\mathrm{BMO}_\kappa} \le C\|f\|_{\mathrm{BMO}_{d,\kappa}}$ for all $f\in\mathrm{BMO}_{d,\kappa}(\mathbb{R}^n)$.*

Proof. Let $f\in\mathrm{BMO}_{d,\kappa}(\mathbb{R}^n)$. Suppose there exists $x_0\in\mathbb{R}^n$ such that $T^*_\kappa f(x_0)<\infty$. We intend to show $T^*_\kappa f(x)<\infty$, for a.e. $x\in\mathbb{R}^n$. We denote $B=B(x_0,4\|x-x_0\|)$ with $x_0\ne x\in\mathbb{R}^n$. We break down f in the following way:

$$f = (f - f_{\mathcal{O}(B)})\chi_{\mathcal{O}(B)} + (f - f_{\mathcal{O}(B)})\chi_{(\mathcal{O}(B))^c} + f_{\mathcal{O}(B)} =: f_1 + f_2 + f_3.$$

Using Theorem 4.2, we obtain $\|T_\kappa^* f_1\|_{2,\kappa} \leq C\|f_1\|_{2,\kappa}$. But

$$\|f_1\|_{2,\kappa}^2 = \int_{\mathcal{O}(B)} |f(x) - f_{\mathcal{O}(B)}|^2 \, d\mu_\kappa(x) \leq \mu_\kappa(\mathcal{O}(B))\|f\|^2_{\mathrm{BMO}_{d,\kappa}}.$$

Hence,

$$\|T_\kappa^* f_1\|_{2,\kappa} \leq C\mu_\kappa(\mathcal{O}(B))^{1/2}\|f\|_{\mathrm{BMO}_{d,\kappa}} < \infty. \tag{4.17}$$

So $T_\kappa^* f_1(x) < \infty$ for a.e. $x \in \mathbb{R}^n$. Now, by definition of the maximal operator,

$$T_\kappa^* f_3(x) = \sup_{\substack{N=(N_1,N_2)\in\mathbb{Z}^2 \\ N_1<N_2}} |T_{N,\kappa} f_3(x)| = 0, \qquad \forall x \in \mathbb{R}^n,$$

since $T_{N,\kappa} f_3 \equiv 0$. Furthermore, using (3.6), we get

$$\begin{aligned} &|T_{N,\kappa} f_2(x) - T_{N,\kappa} f_2(x_0)| \\ &\lesssim \int_{\mathbb{R}^n} \|\{v_j\}_{j\in\mathbb{Z}}\| \frac{1}{\mu_\kappa(B(x, d(x,y)))} \frac{\|x - x_0\|}{\|x - y\|} |f_2(y)| \, d\mu_\kappa(y) \\ &= \|\{v_j\}_{j\in\mathbb{Z}}\| \int_{(\mathcal{O}(B))^c} \frac{1}{\mu_\kappa(B(x, d(x,y)))} \frac{\|x - x_0\|}{\|x - y\|} |f_2(y)| \, d\mu_\kappa(y) \\ &= \|\{v_j\}_{j\in\mathbb{Z}}\| \sum_{j=1}^\infty \int_{2^{j+1}\|x-x_0\|<d(x_0,y)\leq 2^{j+2}\|x-x_0\|} \frac{1}{\mu_\kappa(B(x, d(x,y)))} \frac{\|x - x_0\|}{\|x - y\|} \\ &\qquad \times |f(y) - f_{\mathcal{O}(B)}| \, d\mu_\kappa(y). \end{aligned} \tag{4.18}$$

Since $d(x_0, y) > 2^{j+1}\|x - x_0\|$, we have

$$\begin{aligned} d(x,y) &\geq d(x_0,y) - d(x_0,x) \geq 2^{j+1}\|x - x_0\| - \|x - x_0\| \\ &= (2^{j+1} - 1)\|x - x_0\| \\ &> 2^j\|x - x_0\|, \qquad \forall j \in \mathbb{N}. \end{aligned} \tag{4.19}$$

Consequently, $B(x, 2^j\|x - x_0\|) \subset B(x, d(x,y))$ and $\|x - y\| > 2^j\|x - x_0\|$ for all $j \in \mathbb{N}$. Hence (4.18) becomes

$$\begin{aligned} T_{N,\kappa} f_2(x) - T_{N,\kappa} f_2(x_0)| &\lesssim \|\{v_j\}_{j\in\mathbb{Z}}\| \sum_{j=1}^\infty \frac{1}{2^j} \frac{1}{\mu_\kappa(B(x, 2^j\|x - x_0\|))} \\ &\qquad \times \int_{d(x_0,y)\leq 2^{j+2}\|x-x_0\|} |f(y) - f_{\mathcal{O}(B)}| \, d\mu_\kappa(y) \end{aligned} \tag{4.20}$$

$$
\begin{aligned}
&= \|\{v_j\}_{j\in\mathbb{Z}}\| \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{1}{\mu_\kappa(B(x,2^j\|x-x_0\|))} \\
&\qquad\qquad \times \int_{\mathcal{O}(B(x_0,2^{j+2}\|x-x_0\|))} |f(y)-f_{\mathcal{O}(B)}|\, d\mu_\kappa(y) \\
&\le \|\{v_j\}_{j\in\mathbb{Z}}\| \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{1}{\mu_\kappa(B(x,2^j\|x-x_0\|))} \Big[\int_{\mathcal{O}(2^jB)} |f(y)-f_{\mathcal{O}(2^jB)}|\, d\mu_\kappa(y) \\
&\qquad\qquad + \mu_\kappa(\mathcal{O}(2^jB)) \sum_{l=0}^{j-1} |f_{\mathcal{O}(2^lB)} - f_{\mathcal{O}(2^{l+1}B)}| \Big] \\
&\le \|\{v_j\}_{j\in\mathbb{Z}}\| \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\mu_\kappa(\mathcal{O}(2^jB))}{\mu_\kappa(B(x,2^j\|x-x_0\|))} \\
&\qquad\qquad \times \Big(\|f\|_{\mathrm{BMO}_{d,\kappa}} + \sum_{l=0}^{j-1} |f_{\mathcal{O}(2^lB)} - f_{\mathcal{O}(2^{l+1}B)}| \Big) \\
&\le \|\{v_j\}_{j\in\mathbb{Z}}\| \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\mu_\kappa(\mathcal{O}(2^jB))}{\mu_\kappa(B(x,2^j\|x-x_0\|))} \\
&\qquad\qquad \times \left(\|f\|_{\mathrm{BMO}_{d,\kappa}} + \sum_{l=0}^{j-1} \frac{\mu_\kappa(\mathcal{O}(2^{l+1}B))}{\mu_\kappa(\mathcal{O}(2^lB))} \|f\|_{\mathrm{BMO}_{d,\kappa}} \right) \\
&\le \|\{v_j\}_{j\in\mathbb{Z}}\| \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\mu_\kappa(\mathcal{O}(2^jB))}{\mu_\kappa(B(x,2^j\|x-x_0\|))} \left(1 + \sum_{l=0}^{j-1} \frac{\mu_\kappa(\mathcal{O}(2^{l+1}B))}{\mu_\kappa(\mathcal{O}(2^lB))} \right) \|f\|_{\mathrm{BMO}_{d,\kappa}}.
\end{aligned}
$$

We have

$$\mu_\kappa(\mathcal{O}(2^jB)) = \mu_\kappa\Big(\bigcup_{\sigma\in G} \sigma(2^jB) \Big) \le \sum_{\sigma\in G} \mu_\kappa(\sigma(2^jB)) = \#(G)\mu_\kappa(2^jB).$$

So

$$
\begin{aligned}
\frac{\mu_\kappa(\mathcal{O}(2^jB))}{\mu_\kappa(B(x,2^j\|x-x_0\|))} &= \#(G) \frac{\mu_\kappa(B(x_0,2^{j+2}\|x-x_0\|))}{\mu_\kappa(B(x,2^j\|x-x_0\|))} \\
&\le \#(G)\alpha 2^{2n_\kappa},
\end{aligned}
\tag{4.21}
$$

and

$$\frac{\mu_\kappa(\mathcal{O}(2^{l+1}B))}{\mu_\kappa(\mathcal{O}(2^lB))} = \frac{\mu_\kappa((B(x_0,2^{l+3}\|x-x_0\|)))}{\mu_\kappa((B(x_0,2^{l+2}\|x-x_0\|)))} \le \alpha 2^{n_\kappa}. \tag{4.22}$$

Employing (4.21) and (4.22) in (4.20), we get

$$
\begin{aligned}
&|T_{N,\kappa} f_2(x) - T_{N,\kappa} f_2(x_0)| \\
&\qquad \lesssim \|\{v_j\}_{j\in\mathbb{Z}}\| \sum_{j=1}^{\infty} \frac{1}{2^j} \#(G)\alpha 2^{2n_\kappa}(1 + j\alpha 2^{n_\kappa}) \|f\|_{\mathrm{BMO}_{d,\kappa}} \\
&\qquad = C_{n,\kappa,v} \|f\|_{\mathrm{BMO}_{d,\kappa}},
\end{aligned}
\tag{4.23}
$$

showing that $T_{N,\kappa}f_2(x) < \infty$ for a.e. $x \in \mathbb{R}^n$. Accordingly,

$$\begin{aligned} T_\kappa^* f(x) &= \sup_{\substack{N=(N_1,N_2)\in\mathbb{Z}^2\\ N_1<N_2}} |T_{N,\kappa}f(x)| = \|T_{N,\kappa}f(x)\|_{\ell^\infty(\mathbb{Z}^2)} \\ &\le \|T_{N,\kappa}f_1(x)\|_{\ell^\infty(\mathbb{Z}^2)} + \|T_{N,\kappa}f_2(x)\|_{\ell^\infty(\mathbb{Z}^2)} + \|T_{N,\kappa}f_3(x)\|_{\ell^\infty(\mathbb{Z}^2)} \\ &= \|T_{N,\kappa}f_1(x)\|_{\ell^\infty(\mathbb{Z}^2)} + \|T_{N,\kappa}f_2(x)\|_{\ell^\infty(\mathbb{Z}^2)} \\ &< \infty \quad \text{for a.e. } x \in \mathbb{R}^n. \end{aligned}$$

Consider $\alpha_B := T_\kappa^* f_2(x') < \infty$ for some $x' \in B$. Then

$$\begin{aligned} &\frac{1}{\mu_\kappa(B)} \int_B |T_\kappa^* f(y) - c'_B| \, d\mu_\kappa(y) \\ &\le \frac{1}{\mu_\kappa(B)} \Big[\int_B |T_\kappa^* f_1(y)| \, d\mu_\kappa(x) + \int_B |T_\kappa^* f_2(y) - T_\kappa^* f_2(x')| \, d\mu_\kappa(x) \Big] \\ &=: \frac{1}{\mu_\kappa(B)} (\Omega_1 + \Omega_2). \end{aligned} \tag{4.24}$$

Using (4.17), we get

$$\Omega_1 \le \mu_\kappa(B)^{1/2} \|T_\kappa^* f_1\|_{2,\kappa} \le C \mu_\kappa(B)^{1/2} \mu_\kappa(\mathcal{O}(B))^{1/2} \|f\|_{\mathrm{BMO}_{d,\kappa}}.$$

Consequently, we obtain

$$\begin{aligned} \frac{1}{\mu_\kappa(B)} \Omega_1 &\le C \bigg(\frac{\mu_\kappa(\mathcal{O}(B))}{\mu_\kappa(B)} \bigg)^{1/2} \|f\|_{\mathrm{BMO}_{d,\kappa}} \\ &\le C \#(G)^{1/2} \bigg(\frac{\mu_\kappa(B)}{\mu_\kappa(B)} \bigg)^{1/2} \|f\|_{\mathrm{BMO}_{d,\kappa}} \\ &= C \#(G)^{1/2} \|f\|_{\mathrm{BMO}_{d,\kappa}}. \end{aligned} \tag{4.25}$$

On the other hand, for Ω_2, we have

$$\begin{aligned} |T_\kappa^* f_2(y) - T_\kappa^* f_2(x')| &= \Big| \sup_{\substack{N=(N_1,N_2)\in\mathbb{Z}^2\\ N_1<N_2}} (|T_{N,\kappa}f_2(y)| - |T_{N,\kappa}f_2(x')|) \Big| \\ &\le \sup_{\substack{N=(N_1,N_2)\in\mathbb{Z}^2\\ N_1<N_2}} |T_{N,\kappa}f_2(y) - T_{N,\kappa}f_2(x')|. \end{aligned} \tag{4.26}$$

Now

$$\begin{aligned} &|T_{N,\kappa}f_2(y) - T_{N,\kappa}f_2(x')| \\ &\lesssim \|\{v_j\}_{j\in\mathbb{Z}}\| \int_{(\mathcal{O}(B))^c} \frac{1}{\mu_\kappa(B(y, d(y,z)))} \frac{\|y - x'\|}{\|y - z\|} |f_2(z)| \, d\mu_\kappa(z) \\ &\le \|\{v_j\}_{j\in\mathbb{Z}}\| \sum_{j=1}^{\infty} \int_{2^{j+1}\|x-x_0\| < d(x_0,z) \le 2^{j+2}\|x-x_0\|} \frac{1}{\mu_\kappa(B(y, d(y,z)))} \frac{8\|x - x_0\|}{\|y - z\|} \\ &\qquad \times |f(z) - f_{\mathcal{O}(B)}| \, d\mu_\kappa(z), \end{aligned} \tag{4.27}$$

as $y, x' \in B$. Employing (4.19) in the above equation, we get

$$
\begin{aligned}
&|T_{N,\kappa} f_2(y) - T_{N,\kappa} f_2(x')| \\
&\lesssim 8\|\{v_j\}_{j\in\mathbb{Z}}\| \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{1}{\mu_\kappa(B(y, 2^j\|x-x_0\|))} \\
&\qquad\qquad \times \int\limits_{d(x_0,z)\le 2^{j+2}\|x-x_0\|} |f(z) - f_{\mathcal{O}(B)}|\, d\mu_\kappa(z) \\
&= 8\|\{v_j\}_{j\in\mathbb{Z}}\| \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{1}{\mu_\kappa(B(y, 2^j\|x-x_0\|))} \\
&\qquad\qquad \times \int\limits_{\mathcal{O}(B(x_0, 2^{j+2}\|x-x_0\|))} |f(z) - f_{\mathcal{O}(B)}|\, d\mu_\kappa(z) \\
&\le 8C_{n,\kappa,v} \|f\|_{\mathrm{BMO}_{d,\kappa}},
\end{aligned}
$$

using similar calculations to (4.23). Hence (4.26) reduces to

$$|T_\kappa^* f_2(y) - T_\kappa^* f_2(x')| \lesssim 8C_{n,\kappa,v}\|f\|_{\mathrm{BMO}_{d,\kappa}},$$

and accordingly

$$\frac{1}{\mu_\kappa(B)} \Omega_2 \lesssim 8C_{n,\kappa,v}\|f\|_{\mathrm{BMO}_{d,\kappa}}. \tag{4.28}$$

Finally, using (4.25) and (4.28) in (4.24), we arrive at

$$\frac{1}{\mu_\kappa(B)} \int\limits_B |T_\kappa^* f(y) - c_B'|\, d\mu_\kappa(y) \lesssim (C\#(G)^{1/2} + 8C_{n,\kappa,v})\|f\|_{\mathrm{BMO}_{d,\kappa}},$$

proving that $\|T_\kappa^* f\|_{\mathrm{BMO}_\kappa} \le C\|f\|_{\mathrm{BMO}_{d,\kappa}}$ for some constant $C > 0$. ∎

5. Pointwise convergence of differential transforms. In this section, we show the existence of the pointwise limit of $T_{N,\kappa} f$ as $N \to (-\infty, \infty)$. Here, we use the relationship between the weak-type inequalities and almost everywhere convergence. Moreover, we study the growth of the maximal Dunkl transform T_κ^* near the origin.

THEOREM 5.1. *Let $1 \le p < \infty$. Then, for any $f \in L^p(\mathbb{R}^n, d\mu_\kappa)$, the limit*

$$\lim_{N=(N_1,N_2)\to(-\infty,\infty)} T_{N,\kappa} f(x) \text{ exists for a.e. } x \in \mathbb{R}^n. \tag{5.1}$$

Proof. First, we show that (5.1) holds for $f \in C_c^\infty(\mathbb{R}^n)$. So, let $f \in C_c^\infty(\mathbb{R}^n)$ and $0 < L < M$. It is sufficient to show that $T_{(L,M),\kappa} f(x)$ and $T_{(-M,-L),\kappa} f(x)$ both converge to 0 as $L \to \infty$. We have

$$|T_{(L,M),\kappa} f(x)| \le \int\limits_{\mathbb{R}^n} |K_{(L,M),\kappa}(x,y)|\, |f(y)|\, d\mu_\kappa(y). \tag{5.2}$$

Using (4.4), we get $|K_{(L,M),\kappa}(x,y)| \leq c_{v,\lambda,n,\kappa}\frac{1}{\mu_\kappa(B(x,\sqrt{a_L}))} \leq c_{v,\lambda,n,\kappa}\frac{1}{a_L^{n_\kappa/2}}$ for all $y \in \mathbb{R}^n$. Hence (5.2) gives

$$|T_{(L,M),\kappa}f(x)| \leq c_{v,\lambda,n,\kappa}\frac{1}{a_L^{n_\kappa/2}}\|f\|_{1,\kappa}.$$

Since $\|f\|_{1,\kappa} < \infty$ and $\{a_j\}_{j\in\mathbb{Z}}$ is an increasing sequence of positive real numbers with $a_L \to \infty$ as $L \to \infty$, we obtain $T_{(L,M),\kappa}f(x) \to 0$ as $L \to \infty$ for all $x \in \mathbb{R}^n$. From the definition of $T_{(-M,-L),\kappa}$, we have

$$\begin{aligned}T_{(-M,-L),\kappa}f(x) &= \sum_{j=-M}^{-L} v_j(e^{a_{j+1}\Delta_\kappa}f(x) - e^{a_j\Delta_\kappa}f(x))\\ &= \sum_{j=-M}^{-L} v_j[(e^{a_{j+1}\Delta_\kappa}f(x) - f(x)) - (e^{a_j\Delta_\kappa}f(x) - f(x))].\end{aligned}$$

Hence,

$$\begin{aligned}(5.3)\qquad &|T_{(-M,-L),\kappa}f(x)|\\ &\leq \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\sum_{j=-M}^{-L}\big(|e^{a_{j+1}\Delta_\kappa}f(x) - f(x)| + |e^{a_j\Delta_\kappa}f(x) - f(x)|\big).\end{aligned}$$

For any $f \in C_c^\infty(\mathbb{R}^n)$, we know that $\|e^{t\Delta_\kappa}f - f\|_{L^\infty(\mathbb{R}^n)} \to 0$ as $t \to 0$ (see Theorem 2.1(iii)). So, there exists $L_j > 0$ such that $|e^{a_j\Delta_\kappa}f(x) - f(x)| \leq 2^j$ whenever $j \leq -L_j$. Consider $L'_j := \max\{L_j, L_{j+1}\}$. Then (5.3) gives

$$\begin{aligned}&|T_{(-M,-L'_j),\kappa}f(x)|\\ &\leq \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\sum_{j=-M}^{-L'_j}(2^{j+1} + 2^j) \leq 2\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\sum_{j=-M}^{-L'_j}2^{j+1}\\ &= 4\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty(\mathbb{Z})}\sum_{j=L'_j}^{M}\frac{1}{2^j} \to 0 \quad \text{as } L'_j \to \infty.\end{aligned}$$

For $1 \leq p < \infty$, let $\Omega_p := \{f \in L^p(\mathbb{R}^n, d\mu_\kappa) : \lim_{N=(N_1,N_2)\to(-\infty,\infty)} T_{N,\kappa}f(x)$ exists for a.e. $x \in \mathbb{R}^n\}$. Clearly $C_c^\infty(\mathbb{R}^n) \subset \Omega_p$. Since T_κ^* is weak type (p,p), the set Ω_p is closed in $L^p(\mathbb{R}^n, d\mu_\kappa)$. Therefore, we arrive at $\Omega_p = L^p(\mathbb{R}^n, d\mu_\kappa)$ for all $1 \leq p < \infty$, proving our result. ∎

Now we analyze the growth of T_κ^* near the origin. Unlike the classical case, it turns out that the growth is not of the same order as that of T^*.

THEOREM 5.2. *Let* $1 < p < \infty$, $\{v_j\}_{j\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$, *and* $\{a_j\}_{j\in\mathbb{Z}}$ *be an increasing sequence of positive real numbers. Let* $B = B(0,1) \subset \mathbb{R}^n$ *and*

$B_s = B(0,s) \subset \mathbb{R}^n$ for $s > 0$. Then for any $f \in L^\infty(\mathbb{R}^n)$ with $\operatorname{supp} f \subset B$ *and any $B_r \subset B$ with $2r < 1$, we have*

$$\frac{1}{\mu_\kappa(B_r)} \int\limits_{B_r} |T_\kappa^* f(x)|\, d\mu_\kappa(x) \lesssim \frac{1}{r^{d_\kappa/q}} \|f\|_{L^\infty(\mathbb{R}^d)},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We set $f = f\chi_{\mathcal{O}(B_{2r})} + f\chi_{\mathcal{O}(B)\setminus\mathcal{O}(B_{2r})} =: f_1 + f_2$, where $2r < 1$. Then, by the Cauchy–Schwarz inequality and Theorem 4.2, we get

$$\begin{aligned}
(5.4)\qquad & \frac{1}{\mu_\kappa(B_r)} \int\limits_{B_r} |T_\kappa^* f_1(x)|\, d\mu_\kappa(x) \le \Big(\frac{1}{\mu_\kappa(B_r)} \int\limits_{B_r} |T_\kappa^* f_1(x)|^2\, d\mu_\kappa(x)\Big)^{1/2} \\
&= \frac{1}{\mu_\kappa(B_r)^{1/2}} \|T_\kappa^* f_1\|_{L^2(\mathbb{R}^d, d\mu_\kappa)} \le \frac{C}{\mu_\kappa(B_r)^{1/2}} \|f_1\|_{L^2(\mathbb{R}^d, d\mu_\kappa)} \\
&= \frac{C}{\mu_\kappa(B_r)^{1/2}} \Big(\int\limits_{\mathcal{O}(B_{2r})} |f(x)|^2\, d\mu_\kappa(x)\Big)^{1/2} \le C\Big(\frac{\mu_\kappa(\mathcal{O}(B_{2r}))}{\mu_\kappa(B_r)}\Big)^{1/2} \|f\|_{L^\infty(\mathbb{R}^n)} \\
&\le C\#(G)^{1/2} \Big(\frac{\mu_\kappa(B_{2r})}{\mu_\kappa(B_r)}\Big)^{1/2} \|f\|_{L^\infty(\mathbb{R}^n)} \le C\alpha 2^{n_\kappa/2} \#(G)^{1/2} \|f\|_{L^\infty(\mathbb{R}^n)}.
\end{aligned}$$

For $x \in \mathbb{R}^n$,

$$\begin{aligned}
(5.5)\qquad |T_{N,\kappa} f_2(x)| &\le \sum_{j=N_1}^{N_2} |v_j|\, |e^{a_{j+1}\Delta_\kappa} f_2(x) - e^{a_j \Delta_\kappa} f_2(x)| \\
&\le \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^p(\mathbb{Z})} \Big(\sum_{j=N_1}^{N_2} |e^{a_{j+1}\Delta_\kappa} f_2(x) - e^{a_j \Delta_\kappa} f_2(x)|^q\Big)^{1/q},
\end{aligned}$$

by applying the Cauchy–Schwarz inequality. Then

$$\begin{aligned}
(5.6)\qquad & |e^{a_{j+1}\Delta_\kappa} f_2(x) - e^{a_j \Delta_\kappa} f_2(x)|^q \\
&= \Big| \int\limits_{\mathbb{R}^n} \big(h_{a_{j+1}}(x,y) - h_{a_j}(x,y)\big) f_2(y)\, d\mu_\kappa(y)\Big|^q \\
&\le \Big(\int\limits_{\mathbb{R}^n} \big(|h_{a_{j+1}}(x,y) - h_{a_j}(x,y)|^{1/p}\big) \\
&\qquad\qquad \times \big(|h_{a_{j+1}}(x,y) - h_{a_j}(x,y)|^{1/q} |f_2(y)|\big)\, d\mu_\kappa(y)\Big)^q \\
&\le \Big(\int\limits_{\mathbb{R}^n} |h_{a_{j+1}}(x,y) - h_{a_j}(x,y)|\, d\mu_\kappa(y)\Big)^{q/p} \\
&\qquad\qquad \times \int\limits_{\mathbb{R}^n} |h_{a_{j+1}}(x,y) - h_{a_j}(x,y)|\, |f_2(y)|^q\, d\mu_\kappa(y).
\end{aligned}$$

But, from (3.9), we have

$$\int_{\mathbb{R}^n} |h_{a_{j+1}}(x,y) - h_{a_j}(x,y)|\, d\mu_\kappa(y) \le 2M.$$

Hence, (5.6) becomes

$$\begin{aligned} (5.7)\qquad & |e^{a_{j+1}\Delta_\kappa} f_2(x) - e^{a_j\Delta_\kappa} f_2(x)|^q \\ &\qquad \le (2M)^{q/p} \int_{\mathbb{R}^n} |h_{a_{j+1}}(x,y) - h_{a_j}(x,y)|\, |f_2(y)|^q\, d\mu_\kappa(y). \end{aligned}$$

Using (5.7) in (5.5), we get

$$\begin{aligned} (5.8)\qquad & |T_{N,\kappa} f_2(x)| \\ & \le (2M)^{1/p} \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^p(\mathbb{Z})} \Big(\int_{\mathbb{R}^n} \Big(\sum_{j=N_1}^{N_2} |h_{a_{j+1}}(x,y) - h_{a_j}(x,y)| \Big) \\ &\qquad\qquad \times |f_2(y)|^q\, d\mu_\kappa(y) \Big)^{1/q} \\ & \lesssim (2M)^{1/p} \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^p(\mathbb{Z})} \Big(\int_{\mathbb{R}^n} \frac{1}{\mu_\kappa(B(x,d(x,y)))} \frac{d(x,y)}{\|x-y\|} |f_2(y)|^q\, d\mu_\kappa(y) \Big)^{1/q} \\ & \le (2M)^{1/p} \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^p(\mathbb{Z})} \Big(\int_{\mathbb{R}^n} \frac{1}{\mu_\kappa(B(x,d(x,y)))} |f_2(y)|^q\, d\mu_\kappa(y) \Big)^{1/q} \\ & \le (2M)^{1/p} \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^p(\mathbb{Z})} \|f\|_{L^\infty(\mathbb{R}^n)} \\ &\qquad \times \Big(\int_{\mathcal{O}(B)\setminus\mathcal{O}(B_{2r})} \frac{1}{\mu_\kappa(B(x,d(x,y)))}\, d\mu_\kappa(y) \Big)^{1/q}, \end{aligned}$$

by applying (3.5). Now, for $x \in B_r$ and $y \in \mathcal{O}(B)\setminus\mathcal{O}(B_{2r})$, we have $r \le d(x,y) \le \frac{3}{2}$. Thus, (5.8) gives

$$\begin{aligned} |T_{N,\kappa} f_2(x)| &\le (2M)^{1/p} \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^p(\mathbb{Z})} \|f\|_{L^\infty(\mathbb{R}^n)} \\ &\qquad \times \Big(\int_{\mathcal{O}(B)} \frac{1}{\mu_\kappa(B(x,r))}\, d\mu_\kappa(y) \Big)^{1/q} \\ &\le (2M)^{1/p} \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^p(\mathbb{Z})} \|f\|_{L^\infty(\mathbb{R}^n)} \Big(\frac{\mu_\kappa(\mathcal{O}(B))}{\mu_\kappa(B(x,r))} \Big)^{1/q}. \end{aligned}$$

Since $r < \frac{1}{2}$ and $x \in B_r$, we have $B(x,r) \subset B$. So $\frac{\mu_\kappa(\mathcal{O}(B))}{\mu_\kappa(B(x,r))} \le \frac{\#(G)}{r^{n_\kappa}}$. Hence, we obtain

$$|T_{N,\kappa} f_2(x)| \le (2M)^{1/p} \|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^p(\mathbb{Z})} \|f\|_{L^\infty(\mathbb{R}^n)} \Big(\frac{\#(G)}{r^{n_\kappa}} \Big)^{1/q}.$$

Hence,

$$\frac{1}{\mu_\kappa(B_r)}\int_{B_r}|T_{N,\kappa}f_2(x)|\,d\mu_\kappa(x)\le(2M)^{1/p}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^p(\mathbb{Z})}\|f\|_{L^\infty(\mathbb{R}^n)}\left(\frac{\#(G)}{r^{n_\kappa}}\right)^{1/q}$$

Now, from the definition of the operator T^*_κ, we get

$$\begin{aligned}\frac{1}{\mu_\kappa(B_r)}\int_{B_r}&|T^*_\kappa f_2(x)|\,d\mu_\kappa(x)\\&\le(2M)^{1/p}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^p(\mathbb{Z})}\|f\|_{L^\infty(\mathbb{R}^n)}\left(\frac{\#(G)}{r^{n_\kappa}}\right)^{1/q}.\end{aligned}\tag{5.9}$$

Combining (5.4) and (5.9), we arrive at

$$\begin{aligned}&\frac{1}{\mu_\kappa(B_r)}\int_{B_r}|T^*_\kappa f(x)|\,d\mu_\kappa(x)\\&\le\left(C2^{n_\kappa/2}\#(G)^{1/2}+(2M)^{1/p}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^p(\mathbb{Z})}\left(\frac{\#(G)}{r^{n_\kappa}}\right)^{1/q}\right)\|f\|_{L^\infty(\mathbb{R}^n)}\\&\le(2M)^{1/p}\|\{v_j\}_{j\in\mathbb{Z}}\|_{\ell^p(\mathbb{Z})}\left(\frac{\#(G)}{r^{n_\kappa}}\right)^{1/q}\|f\|_{L^\infty(\mathbb{R}^n)}\lesssim\frac{1}{r^{n_\kappa/q}}\|f\|_{L^\infty(\mathbb{R}^n)}.\ \blacksquare\end{aligned}$$

For $n=1$, (2.8) can be rewritten as

$$e^{t\Delta_\kappa}f(x)=\int_{\mathbb{R}}h_t(x,y)f(y)y^{2\kappa(1)}\,dy,\qquad x\in\mathbb{R}.$$

Using $h_t(0,y)=\frac{1}{(2t)^{(1+2\kappa(1))/2}}e^{-\frac{y^2}{4t}}$ in the above equation, we get

$$e^{t\Delta_\kappa}f(0)=\frac{1}{(2t)^{(1+2\kappa(1))/2}}\int_{\mathbb{R}}y^{2\kappa(1)}e^{-\frac{y^2}{4t}}f(y)\,dy.\tag{5.10}$$

Now, applying a change of variable, we get

$$\int_0^\infty y^{2\kappa(1)}e^{-y^2/4}\,dy=2^{2\kappa(1)}\Gamma(\kappa(1)+1/2)=:A.$$

In other words, we have

$$\frac{1}{A}\int_0^\infty y^{2\kappa(1)}e^{-y^2/4}\,dy=1.\tag{5.11}$$

Now we prove the following lemma.

Lemma 5.3. *Assume that $\eta\in(0,1/2)$.*

(i) *There exists $\epsilon>0$ satisfying*

$$\frac{1}{A\epsilon^{1+2\kappa(1)}}\int_0^1 y^{2\kappa(1)}e^{-\frac{y^2}{4\epsilon^2}}\,dy=\frac{1}{2}+\eta.\tag{5.12}$$

(ii) *There exists* $\epsilon > 0$ *as in* (5.12) *and* $a > 1$, *large enough, such that*

$$\frac{\eta+1}{2} \leq \frac{1}{A\epsilon^{1+2\kappa(1)}} \int_{1/a}^{1} y^{2\kappa(1)} e^{-\frac{y^2}{4\epsilon^2}}\, dy. \tag{5.13}$$

Proof. From (5.11), we can choose $\alpha > 0$ such that

$$\frac{1}{A}\int_0^{\alpha} y^{2\kappa(1)} e^{-y^2/4}\, dy = \frac{1}{2} + \eta.$$

Making the change of variable $y = \alpha z$, we get

$$\frac{\alpha^{1+2\kappa(1)}}{A}\int_0^{1} y^{2\kappa(1)} e^{-\alpha^2 y^2/4}\, dy = \frac{1}{2} + \eta.$$

Considering $\alpha = 1/\epsilon$, we obtain (5.12). This proves (i). For (ii), we start with $a > 1$. We can rewrite (5.12) as

$$\frac{1}{A\epsilon^{1+2\kappa(1)}}\Big[\int_0^{1/a} y^{2\kappa(1)} e^{-\frac{y^2}{4\epsilon^2}}\, dy + \int_{1/a}^{1} y^{2\kappa(1)} e^{-\frac{y^2}{4\epsilon^2}}\, dy\Big] = \frac{\eta+1}{2} + \frac{\eta}{2} \tag{5.14}$$

We can choose a very large in such a way that

$$\frac{1}{A\epsilon^{1+2\kappa(1)}} \int_0^{1/a} y^{2\kappa(1)} e^{-\frac{y^2}{4\epsilon^2}}\, dy \leq \frac{\eta}{2}. \tag{5.15}$$

Hence (5.14) gives

$$\frac{1}{A\epsilon^{1+2\kappa(1)}} \int_{1/a}^{1} y^{2\kappa(1)} e^{-\frac{y^2}{4\epsilon^2}}\, dy \geq \frac{\eta+1}{2},$$

proving (5.13). ∎

The following example shows that for some $f \in L^\infty(\mathbb{R})$, $T_\kappa^* f(x)$ is infinite for all $x \in \mathbb{R}$.

EXAMPLE 5.4. Let $a > 1$ and $\epsilon > 0$ be as in Lemma 5.3. Consider the function (see [CT21, Proposition 2.6])

$$f(x) = \sum_{j\in\mathbb{Z}} (-1)^{j+1} \chi_{(a^j, a^{j+1}]}(x).$$

Then $f \in L^\infty(\mathbb{R})$. We choose the lacunary sequence $\{a_j\}_{j\in\mathbb{Z}}$ with $a_j := \epsilon^2 a^{2j}$. From (5.10), we obtain

$$e^{a_j \Delta_\kappa} f(0) = \frac{1}{2^{(1+2\kappa(1))/2}(\epsilon a^j)^{1+2\kappa(1)}} \int_{\mathbb{R}} y^{2\kappa(1)} e^{-\frac{y^2}{4\epsilon^2 a^{2j}}} f(y)\, dy.$$

Making a change of variable, we can show that

$$\begin{aligned} e^{a_j\Delta_\kappa}f(0) &= \frac{1}{2^{(1+2\kappa(1))/2}\epsilon^{1+2\kappa(1)}}\int_{\mathbb{R}} y^{2\kappa(1)}e^{-\frac{y^2}{4\epsilon^2}}f(a^jy)\,dy \\ &= \frac{(-1)^j}{2^{(1+2\kappa(1))/2}\epsilon^{1+2\kappa(1)}}\int_0^\infty y^{2\kappa(1)}e^{-\frac{y^2}{4\epsilon^2}}f(y)\,dy, \end{aligned}$$

using $f(a^jy) = (-1)^jf(y)$. But, applying $f \geq -1$, we have

$$\begin{aligned} &\int_0^\infty y^{2\kappa(1)}e^{-\frac{y^2}{4\epsilon^2}}f(y)\,dy \qquad (5.16) \\ &= \int_{1/a}^1 y^{2\kappa(1)}e^{-\frac{y^2}{4\epsilon^2}}f(y)\,dy + \Bigg[\int_0^{1/a} y^{2\kappa(1)}e^{-\frac{y^2}{4\epsilon^2}}f(y)\,dy + \int_1^\infty y^{2\kappa(1)}e^{-\frac{y^2}{4\epsilon^2}}f(y)\,dy\Bigg] \\ &\geq \int_{1/a}^1 y^{2\kappa(1)}e^{-\frac{y^2}{4\epsilon^2}}\,dy - \Bigg[\int_0^{1/a} y^{2\kappa(1)}e^{-\frac{y^2}{4\epsilon^2}}\,dy + \int_1^\infty y^{2\kappa(1)}e^{-\frac{y^2}{4\epsilon^2}}\,dy\Bigg]. \end{aligned}$$

Using a change of variable in (5.11), we have

$$\frac{1}{A\epsilon^{1+2\kappa(1)}}\int_0^\infty y^{2\kappa(1)}e^{-\frac{y^2}{4\epsilon^2}}\,dy = 1.$$

This gives

$$\begin{aligned} \frac{1}{A\epsilon^{1+2\kappa(1)}}\int_1^\infty y^{2\kappa(1)}e^{-\frac{y^2}{4\epsilon^2}}\,dy &= 1 - \frac{1}{A\epsilon^{1+2\kappa(1)}}\int_0^1 y^{2\kappa(1)}e^{-\frac{y^2}{4\epsilon^2}}\,dy \\ &= 1 - \left(\frac{1}{2}+\eta\right) = \frac{1}{2}-\eta. \end{aligned} \tag{5.17}$$

Now, employing (5.13), (5.15), and (5.17) in (5.16), we obtain

$$\frac{1}{A\epsilon^{1+2\kappa(1)}}\int_0^\infty y^{2\kappa(1)}e^{-\frac{y^2}{4\epsilon^2}}f(y)\,dy \geq \left[\frac{\eta+1}{2} - \left(\frac{\eta}{2}+\frac{1}{2}-\eta\right)\right] = \eta. \tag{5.18}$$

For $j \in \mathbb{Z}$, we have

$$\begin{aligned} e^{a_j\Delta_\kappa}f(x) &= \int_{\mathbb{R}} y^{2\kappa(1)}h_{a_j}(x,y)f(y)\,dy \\ &= \int_{\mathbb{R}} \frac{y^{2\kappa(1)}}{(2a_j)^{\frac{1+2\kappa(1)}{2}}}E_\kappa\left(\frac{x}{\sqrt{2a_j}},\frac{y}{\sqrt{2a_j}}\right)e^{-\frac{1}{4a_j}(x^2+y^2)}f(y)\,dy \\ &= \int_{\mathbb{R}} \frac{y^{2\kappa(1)}}{2^{\frac{1+2\kappa(1)}{2}}(\epsilon a^j)^{1+2\kappa(1)}}E_\kappa\left(\frac{x}{\sqrt{2}\epsilon a^j},\frac{y}{\sqrt{2}\epsilon a^j}\right)e^{-\frac{1}{4\epsilon^2a^{2j}}(x^2+y^2)}f(y)\,dy. \end{aligned}$$

Making the change of variable $y = a^j z$, we arrive at

$$\begin{aligned} e^{a_j \Delta_\kappa} f(x) &= \int_{\mathbb{R}} \frac{z^{2\kappa(1)}}{2^{\frac{1+2\kappa(1)}{2}} \epsilon^{1+2\kappa(1)}} E_\kappa\left(\frac{x}{\sqrt{2}\epsilon a^j}, \frac{z}{\sqrt{2}\epsilon}\right) e^{-\frac{1}{4\epsilon^2 a^{2j}}(x^2+a^{2j}z^2)} f(a^j z)\, dz \\ &= \int_{\mathbb{R}} \frac{(-1)^j z^{2\kappa(1)}}{2^{\frac{1+2\kappa(1)}{2}} \epsilon^{1+2\kappa(1)}} E_\kappa\left(\frac{x}{\sqrt{2}\epsilon a^j}, \frac{z}{\sqrt{2}\epsilon}\right) e^{-\frac{1}{4\epsilon^2 a^{2j}}(x^2+a^{2j}z^2)} f(z)\, dz. \end{aligned}$$

Hence,

$$\begin{aligned} &e^{a_{j+1}\Delta_\kappa} f(x) - e^{a_j \Delta_\kappa} f(x) \\ &= \frac{(-1)^{j+1}}{2^{\frac{1+2\kappa(1)}{2}} \epsilon^{1+2\kappa(1)}} \Bigg(\int_{\mathbb{R}} z^{2\kappa(1)} E_\kappa\left(\frac{x}{\sqrt{2}\epsilon a^{j+1}}, \frac{z}{\sqrt{2}\epsilon}\right) \\ &\qquad\qquad \times e^{-\frac{1}{4\epsilon^2 a^{2(j+1)}}(x^2+a^{2(j+1)}z^2)} f(z)\, dz \\ &\qquad + \int_{\mathbb{R}} z^{2\kappa(1)} E_\kappa\left(\frac{x}{\sqrt{2}\epsilon a^j}, \frac{z}{\sqrt{2}\epsilon}\right) e^{-\frac{1}{4\epsilon^2 a^{2j}}(x^2+a^{2j}z^2)} f(z)\, dz \Bigg) \\ &=: \frac{(-1)^{j+1}}{2^{\frac{1+2\kappa(1)}{2}} \epsilon^{1+2\kappa(1)}} (I_{j+1}(x) + I_j(x)). \end{aligned} \tag{5.19}$$

Using the dominated convergence theorem, we get

$$\begin{aligned} \lim_{h\to 0} \int_{\mathbb{R}} z^{2\kappa(1)} E_\kappa\left(\frac{h}{\sqrt{2}\epsilon}, \frac{z}{\sqrt{2}\epsilon}\right) e^{-\frac{h^2+z^2}{4\epsilon^2}} f(z)\, dz &= \int_0^\infty z^{2\kappa(1)} e^{-\frac{z^2}{4\epsilon^2}} f(z)\, dz \\ &\geq A\eta\epsilon^{1+2\kappa(1)}, \end{aligned}$$

by applying (5.18). Hence, we can find $\delta \in (0,1)$ such that for $|h| < \delta$ we have

$$\int_{\mathbb{R}} z^{2\kappa(1)} E_\kappa\left(\frac{h}{\sqrt{2}\epsilon}, \frac{z}{\sqrt{2}\epsilon}\right) e^{-\frac{h^2+z^2}{4\epsilon^2}} f(z)\, dz \geq \frac{1}{2} A\eta\epsilon^{1+2\kappa(1)}.$$

Thus, there are infinitely many $j \in \mathbb{Z}$ such that for $|x/a^j| < \delta$ we have $I_{j+1}(x) + I_j(x) \geq A\eta\epsilon^{1+2\kappa(1)}$. Now choosing $v_j = (-1)^{j+1}$, from (5.19) we have

$$\begin{aligned} T_\kappa^* f(x) &\geq \sum_{|x/a^j|<\delta} (-1)^{j+1} (e^{a_{j+1}\Delta_\kappa} f(x) - e^{a_j\Delta_\kappa} f(x)) \\ &\geq \sum_{|x/a^j|<\delta} \frac{1}{2^{\frac{1+2\kappa(1)}{2}} \epsilon^{1+2\kappa(1)}} A\eta\epsilon^{1+2\kappa(1)} = \sum_{|x/a^j|<\delta} \frac{A\eta}{2^{\frac{1+2\kappa(1)}{2}}} = \infty. \end{aligned}$$

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