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# Uniqueness sets with angular density for spaces of entire functions, I. Basics

by

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**Abstract.** This is the first part of our work which is devoted to the uniqueness sets for spaces of entire functions. In this part we consider a set $\Lambda$ with angular density with respect to the order $\rho > 0$, satisfying the Lindelöf condition. We find the value of the critical zero set type for $\Lambda$ in geometrical terms. We give a necessary and sufficient condition for the coincidence of the critical zero set type and the critical uniqueness set type.

At the end of the paper we present an application of our results to random zero sets in Fock-type spaces.

**1. Introduction and main results.** This is the first part of a work which consists of three parts. These parts are essentially independent of each other.

Let $\mathcal{E}_{\rho,\sigma}$ be the class of entire functions of order $\rho$ and of type not exceeding $\sigma$. The latter means that for each $\varepsilon > 0$ there exists $C_\varepsilon$ such that

$$|f(z)| \le C_\varepsilon e^{(\sigma+\varepsilon)|z|^\rho}, \qquad z \in \mathbb{C}.$$

Let $\Lambda \subset \mathbb{C}$ be a discrete set, and $\mathcal{A}$ be some class of entire functions. We say that $\Lambda$ is a *uniqueness set for* $\mathcal{A}$ if

$$f \in \mathcal{A},\ f|_\Lambda = 0 \implies f \equiv 0.$$

Given $\rho > 0$ and a discrete set $\Lambda \subset \mathbb{C}$, we let

$$\begin{aligned}\sigma_U(\Lambda) &= \sup\{\sigma : \Lambda \text{ is a uniqueness set for } \mathcal{E}_{\rho,\sigma}\}\\ &= \inf\{\sigma : \exists f \in \mathcal{E}_{\rho,\sigma} \setminus \{0\},\ f|_\Lambda = 0\}.\end{aligned}$$

We call this quantity the *critical uniqueness set type* of $\Lambda$.

One of the reasons for the importance of this quantity is that, by a classical Markushevich duality argument [12, Lecture 3, Sect. 4], for $\rho = 1$,

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the uniqueness of $\Lambda$ in $\mathcal{E}_{1,\sigma}$ is equivalent to the completeness of the system $E_\Lambda = \{e^{\overline{\lambda} z} : \lambda \in \Lambda\}$ of exponential functions in the space of holomorphic functions in the disk $\sigma\mathbb{D} = \{\zeta : |\zeta| < \sigma\}$ with the topology of locally uniform convergence.

The importance of the case $\rho = 2$ is due to the fact that the exponential functions $e_w(z) = e^{z\overline{w}}$ are the reproducing kernels of the classical Fock–Bargmann space $\mathcal{B}$ of entire functions satisfying

$$\|f\|^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} \, \mathrm{d}m(z) < \infty,$$

where $m$ is the planar Lebesgue measure. That is, $f(w) = \langle f, e_w \rangle$ for $f \in \mathcal{B}$. In this case, $E_\Lambda$ is complete in the Fock–Bargmann space provided that $\sigma_U(\Lambda) > 1/2$, and incomplete for $\sigma_U(\Lambda) < 1/2$.

At present, in spite of many efforts [2, 4, 9], explicit expressions, which would allow one to *compute* $\sigma_U(\Lambda)$, are known only in very few cases. In this work we consider this problem only for regularly distributed sets $\Lambda$.

Note that our interest in this problem arose from the study of the uniqueness property in Fock-type spaces of a random set of points, which will be discussed in Section 9.

Remark. Given $p > 0$ and a set $M \subset [0, 2\pi p]$, we always consider it as a subset of the factor space $\mathbb{R}/(2\pi p\mathbb{Z})$.

We say that the set $\Lambda$ *has an angular density* (*with respect to the order* $\rho$) if for all $0 \le \alpha < \beta \le 2\pi$ except maybe countably many values, the limit

$$\varDelta_\Lambda(\alpha, \beta) = \lim_{R\to\infty} \frac{n_\Lambda(R; \alpha, \beta)}{R^\rho}$$

exists, where $n_\Lambda(R; \alpha, \beta) = \#\{\lambda_k \in \Lambda : |\lambda_k| < R, \arg \lambda_k \in (\alpha, \beta)\}$. The quantity $\varDelta_\Lambda(\alpha, \beta)$ is called the *angular density* of $\Lambda$. We will treat it as a non-negative measure on $[0, 2\pi]$, which will also be denoted by $\varDelta_\Lambda$.

A set $\Lambda$ is called *$\rho$-regular* if

(i) it has an angular density with respect to the order $\rho$;
(ii) if $\rho$ is an integer, then in addition the *Lindelöf condition* holds:

$$\lim_{R\to\infty} \sum_{0<|\lambda|\le R} \frac{1}{\lambda^\rho} \quad \text{exists and is finite.} \tag{1.1}$$

Note that (ii) implies that the measure $\varDelta_\Lambda$ has zero $\rho$th moment,

$$\int_0^{2\pi} e^{i\rho\theta} \, \mathrm{d}\varDelta_\Lambda(\theta) = 0 \tag{1.2}$$

(otherwise, the sum in (1.1) would grow like $\log R$ as $R \to \infty$).

On the other hand, given a finite measure $\varDelta$ satisfying (1.2), we can always construct a $\rho$-regular set $\Lambda$ such that $\varDelta_\Lambda = \varDelta$ [11, Chap. II, Sect. 4].

For $\rho$-regular sets $\Lambda$, the Levin–Pfluger theory of entire functions of completely regular growth [11] (for a streamlined presentation see [3]) reduces the problem of determining the value of $\sigma_U(\Lambda)$ to a question about $\rho$-trigonometrically convex functions.

In this part, we always assume that the set $\Lambda$ is $\rho$-regular.

A function $h : [\alpha, \beta] \to \mathbb{R}$ is called *$\rho$-trigonometrically convex* if, for any $\alpha \le \theta_1 \le \theta \le \theta_2 \le \beta$ with $\theta_2 - \theta_1 < \pi/\rho$, we have

$$h(\theta) \le \frac{h(\theta_1)\sin\rho(\theta_2-\theta) + h(\theta_2)\sin\rho(\theta-\theta_1)}{\sin\rho(\theta_2-\theta_1)}.$$

We denote by $TC_\rho$ the class of $\rho$-trigonometrically convex functions. For integer $\rho$,

$$T_\rho = \{A\cos\rho\theta + B\sin\rho\theta : A, B \in \mathbb{R}\} \subset TC_\rho$$

is the subclass of *$\rho$-trigonometric functions.*

For fundamental properties of $\rho$-trigonometrically convex functions see, for instance, [11, Chap. I, Sect. 16] and [12, Lecture 8].

With each measure $\Delta$ satisfying (1.2), we associate a $2\pi$-periodic $\rho$-trigonometrically convex function $h_\Delta$ such that

$$h''_\Delta + \rho^2 h_\Delta = 2\pi\rho\Delta \tag{1.3}$$

in the sense of distributions. For a non-integer $\rho$, the function $h_\Delta$ is uniquely defined, while for an integer $\rho$, it is defined up to a $\rho$-trigonometric term $k \in T_\rho$. Note that in the latter case the moment condition (1.2) guarantees the $2\pi$-periodicity of $h_\Delta$.

There are explicit expressions for $h_\Delta$ in terms of $\Delta$ [11, Chap. II, Theorems 1, 2]. Given a measure $\Delta$, we set

$$h_\Delta(\theta) = \begin{cases} \dfrac{\pi}{\sin\pi\rho} \displaystyle\int_{\theta-2\pi}^{\theta} \cos\rho(\theta-\varphi-\pi)\,\mathrm{d}\Delta(\varphi), & \rho \notin \mathbb{N}, \\ -\displaystyle\int_{\theta-2\pi}^{\theta} (\varphi-\theta)\sin\rho(\varphi-\theta)\,\mathrm{d}\Delta(\varphi), & \rho \in \mathbb{N}. \end{cases}$$

Note that, unlike the formulas from [11], we omit the $\rho$-trigonometric term in the case $\rho \in \mathbb{N}$, so that for all $A, \tau \in \mathbb{R}$, the function $h_\Delta(t) + A\cos\rho(t-\tau)$ also corresponds to the measure $\Delta$ and satisfies (1.3).

Given a $\rho$-regular set $\Lambda$ with angular distribution $\Delta_\Lambda$ we also use the notation $h_\Lambda = h_{\Delta_\Lambda}$.

We call a set $\Lambda$ a *zero set* for some class $\mathcal{A}$ of analytic functions if there exists $f \in \mathcal{A}$ such that $f^{-1}\{0\} = \Lambda$.

Given $\rho > 0$ and a discrete set $\Lambda \subset \mathbb{C}$, we define the *critical zero set type* of $\Lambda$ by

$$\sigma_Z(\Lambda) = \inf\{\sigma : \exists f \in \mathcal{E}_{\rho,\sigma},\ f^{-1}\{0\} = \Lambda\}.$$

Clearly,

$$\sigma_U(\Lambda) \le \sigma_Z(\Lambda). \tag{1.4}$$

Unlike $\sigma_U(\Lambda)$, the quantity $\sigma_Z(\Lambda)$ is easy to compute.

The following result is a consequence of the Levin–Pfluger theory of entire functions of completely regular growth [11].

THEOREM 1.1. *Suppose $\Lambda$ is a $\rho$-regular set with angular density $\Delta$.*

(i) *If $\rho$ is not an integer, then*

$$\sigma_Z(\Lambda) = \max_{t\in[0,2\pi]} h_\Delta(t).$$

(ii) *If $\rho$ is an integer, then*

$$\sigma_Z(\Lambda) = \min_{k\in T_\rho} \max_{t\in[0,2\pi]} [h_\Delta(t) + k(t)]. \tag{1.5}$$

For $\rho = 1$, the right-hand side of (1.5) has a simple geometric meaning [4, 6]. In this case, a trigonometrically convex function $h_\Delta$ is the supporting function of a planar convex compact set $I_\Delta$. Adding a trigonometric function

$$A\cos\theta + B\sin\theta = \mathrm{Re}(Ce^{i\theta}),$$

we shift $I_\Delta$ by $Ce^{i\theta}$. Thus, in this case, the right-hand side of (1.5) equals the circumradius $R_\Delta$ of the set $I_\Delta$, while the translation vector $Ce^{i\theta}$ is uniquely characterized by the condition that the origin lies in the convex hull of the set

$$\{e^{i\theta} : h_\Delta(\theta) + \mathrm{Re}(Ce^{i\theta}) = R_\Delta\}.$$

A similar interpretation of the right-hand side of (1.5) holds for all integer values of $\rho$. Given $h \in TC_\rho$, we associate $\Delta$ with it via (1.3). Assuming that $\rho$ is an integer, we let

$$h^*(\theta) = \max_{j=0,1,\dots,\rho-1} h\left(\frac{\theta + 2\pi j}{\rho}\right),$$

and observe that $h^*$ is 1-trigonometrically convex. We denote by $I^*_\Delta$ the planar convex compact set with supporting function $h^*$, and by $R^*_\Delta$ the circumradius of $I^*_\Delta$.

Set $M_h = \{\theta : h(\theta) = \max_{t\in[0,2\pi]} h(t)\}$.

Let $\operatorname{conv} X$ denote the convex hull of a plane set $X$.

THEOREM 1.2. *Let $\rho$ be a positive integer, and let $h \in TC_\rho$.*

(A) *The following two conditions are equivalent:*

(i) $0 \in \operatorname{conv}\{e^{it\rho} : t \in M_h\}$;

(ii) $\min\limits_{k \in T_\rho} \max\limits_{t \in [0,2\pi]} (h(t) + k(t)) = \max\limits_{t \in [0,2\pi]} h(t)$.

(B) $\min\limits_{k \in T_\rho} \max\limits_{t \in [0,2\pi]} (h(t) + k(t)) = R^*_\Delta$.

We call a function $h \in TC_\rho$ ($\rho$ an integer) *$\rho$-balanced* if it satisfies condition (i). Note that any function $h \in TC_\rho$ can be made $\rho$-balanced by adding a $\rho$-trigonometric function.

COROLLARY 1.1. *Suppose $\rho \in \mathbb{N}$, and $\Lambda$ is a $\rho$-regular set with angular density $\Delta$. Then*

$$\sigma_Z(\Lambda) = R^*_\Delta = \max_{t \in [0,2\pi]} \widehat{h}(t),$$

*where $\widehat{h}$ is the $\rho$-balanced function such that $\widehat{h} - h \in T_\rho$.*

*Moreover,*

$$\sigma_Z(\Lambda) = \min\{\sigma : \exists f \in \mathcal{E}_{\rho,\sigma},\ f^{-1}\{0\} = \Lambda\}.$$

Now, let us turn to uniqueness sets. The next theorem is also an immediate consequence of Levin–Pfluger theory:

THEOREM 1.3. *Suppose $\Lambda$ is a $\rho$-regular set. Then*

$$\sigma_U(\Lambda) = \inf_{k \in TC_\rho} \max_{t \in [0,2\pi]} (h_\Lambda(t) + k(t)).$$

*Furthermore, for $\rho \le 1/2$ and for $\rho = 1$,*

$$\sigma_U(\Lambda) = \sigma_Z(\Lambda).$$

It is not difficult to show that for other values of $\rho$ the quantities $\sigma_U(\Lambda)$ and $\sigma_Z(\Lambda)$ can be different:

THEOREM 1.4. *Given $\rho \in (1/2, \infty) \setminus \{1\}$, there exists a $\rho$-regular set $\Lambda$ with $\sigma_U(\Lambda) < \sigma_Z(\Lambda)$.*

Our next result characterizes the equality $\sigma_U(\Lambda) = \sigma_Z(\Lambda)$ for $\rho > 1/2$. Given a function $h \in TC_\rho$, we set

$$\widehat{h} = \begin{cases} h, & \rho \notin \mathbb{N}, \\ \rho\text{-balanced modification of } h, & \rho \in \mathbb{N}. \end{cases} \tag{1.6}$$

We call a set $M \subset [0, 2\pi]$ *locally $\rho$-balanced* if there exists $\theta^* \in [0, 2\pi]$ such that

$$0 \in \operatorname{conv}\{e^{it\rho} : t \in M \cap [\theta^*, \theta^* + 2\pi/\rho]\}. \tag{1.7}$$

We call the function $h \in TC_\rho$, $\rho > 1/2$, locally $\rho$-balanced when so is the set $M_{\widehat{h}} = \{\theta : \widehat{h}(\theta) = \max_{[0,2\pi]} \widehat{h}(t)\}$.

THEOREM 1.5. *Let $\rho > 1/2$, and let $\Lambda$ be a $\rho$-regular set. Then the following are equivalent:*

(i) $\sigma_U(\Lambda) = \sigma_Z(\Lambda)$;
(ii) *the function $\widehat{h}_\Lambda$ is locally $\rho$-balanced.*

Our next result gives sharp lower bounds for $\sigma_U(\Lambda)$, which might be useful when $\sigma_U(\Lambda) < \sigma_Z(\Lambda)$.

THEOREM 1.6. *Let $\rho > 1/2$, and let $\Lambda$ be a $\rho$-regular set. Put*

$$A_\Lambda := \tfrac{1}{2} \max_{\theta \in [0,2\pi]} (h_\Lambda(\theta) + h_\Lambda(\theta + \pi/\rho)).$$

*Then*

$$\sigma_U(\Lambda) \geq A_\Lambda. \tag{1.8}$$

*In addition,*

$$\sigma_U(\Lambda) \geq C \tag{1.9}$$

*provided that the set $\{t : \widehat{h}_\Lambda(t) \geq C\}$ is locally $\rho$-balanced.*

Our last two theorems are inspired by a result of Ascensi, Lyubarskii and Seip [1, Theorem 2]. Given a $\rho$-regular set $\Lambda \subset \mathbb{C}$ with angular density $\Delta_\Lambda$, these theorems focus on exploring possible densities

$$\mathcal{D}(\Lambda) = \Delta_\Lambda([0, 2\pi))$$

in some critical cases.

THEOREM 1.7. *Let $\mathcal{A}_\rho$ denote the class of all $\rho$-regular sets $\Lambda \subset \mathbb{C}$ such that*

(i) *$\Lambda$ is a zero set for $\mathcal{E}_{\rho,1}$;*
(ii) *$\Lambda$ is a uniqueness set for every $\mathcal{E}_{\rho,\sigma}$ with $\sigma < 1$.*

*Consider the density map*

$$\mathcal{D} : \mathcal{A}_\rho \to \mathbb{R}_+, \qquad \mathcal{D}(\Lambda) = \Delta_\Lambda([0, 2\pi)),$$

*where $\Delta_\Lambda$ is the angular density measure of $\Lambda$. Then*

$$\mathcal{D}(\mathcal{A}_\rho) = \begin{cases} \left[\dfrac{\sin \pi\rho}{\pi}, \rho\right], & \rho \in (0, 1/2], \\ \left[\dfrac{1 + |\cos \pi\rho|}{\pi}, \rho\right], & \rho \in (1/2, \infty). \end{cases}$$

THEOREM 1.8. *Let $\mathcal{B}_\rho$ denote the class of all $\rho$-regular sets $\Lambda \subset \mathbb{C}$ such that*

(i) *$\Lambda$ is a non-uniqueness set for $\mathcal{E}_{\rho,1}$;*
(ii) *for every $\rho$-regular set $\Lambda_1$ with non-zero angular density $\Delta_1$, the set $\Lambda \cup \Lambda_1$ is a uniqueness set for $\mathcal{E}_{\rho,1}$.*

*Consider the density map*

$$\mathcal{D} : \mathcal{B}_\rho \to \mathbb{R}_+, \qquad \mathcal{D}(\Lambda) = \Delta_\Lambda([0, 2\pi)),$$

*where $\Delta_\Lambda$ is the angular density measure of $\Lambda$. Then*

$$\mathcal{D}(\mathcal{B}_\rho) = \begin{cases} \left[\dfrac{\sin \pi\rho}{\pi}, \rho\right], & \rho \in (0, 1/2], \\ \left[\dfrac{1}{\pi}\left(\sin \dfrac{\pi\{2\rho\}}{2} + [2\rho]\right), \rho\right], & \rho \in (1/2, \infty). \end{cases}$$

Remark. Note that for $\rho = 2$ our bounds in Theorem 1.8 coincide with those in [1, Theorem 2], which is somewhat surprising, given the considerably more "massive" perturbation here.

At the end of the paper, in Section 9, we present an application of our results to random zero sets in Fock-type spaces.

**2. Proofs of Theorems 1.1 and 1.3.** By our conditions on $\Lambda$ and by [11, Chap. I, Sect. 10] we can define an entire function $W_\Lambda$:

- for $\rho \in \mathbb{N}$,

$$W_\Lambda(z) = \exp\left(-\frac{1}{\rho}\sum_k \frac{z^\rho}{\lambda_k^\rho}\right) \prod_{\lambda_k \in \Lambda} G\left(\frac{z}{\lambda_k}; \rho\right); \tag{2.1}$$

- for $\rho \notin \mathbb{N}$,

$$W_\Lambda(z) = \prod_{\lambda_k \in \Lambda} G\left(\frac{z}{\lambda_k}; [\rho]\right), \tag{2.2}$$

where $G(w; d) := (1 - w)\exp(w + w^2/2 + \cdots + w^d/d)$.

Let $f \in \mathcal{E}_{\rho,\sigma}$ be such that $f^{-1}\{0\} = \Lambda$. By the Hadamard factorization theorem, $f$ can be represented as an Hadamard product

$$f = e^{P_N} W_\Lambda,$$

where $P_N$ is a polynomial of degree $N \le [\rho]$.

Recall that the *indicator* of an entire function $f$ of order $\rho$ is defined by

$$h_f(t) := \limsup_{r\to\infty} \frac{\log|f(re^{it})|}{r^\rho}, \qquad t \in [0, 2\pi).$$

Moreover, by Levin–Pfluger theory [11, Chap. III, Sect. 3, Theorem 4], if the zero set of $f$ is $\rho$-regular, there exists an exceptional set $E$, a union of disks with zero linear density, such that

$$h_f(t) = \lim_{\substack{r\to\infty \\ re^{it}\notin E}} \frac{\log|f(re^{it})|}{r^\rho}.$$

One can think of the exceptional set $E$ as the union of small neighbourhoods of the zeros of $f$, where the supremum in each disk is governed by the maximum principle.

Furthermore, by [11, Chap. II, Sect. 1, Theorems 1, 2], outside $E$ the indicator $h_f$ satisfies

$$h_f(t) = \begin{cases} h_\Lambda(t), & \rho \notin \mathbb{N}, \\ h_\Lambda(t) + a\cos\rho t + b\sin\rho t,\ a, b \in \mathbb{R}, & \rho \in \mathbb{N}. \end{cases}$$

If $\rho \notin \mathbb{N}$, the type of the function $f$ is uniquely determined by the set $\Lambda$ and $\sigma_Z(\Lambda) = \max_{t\in[0,2\pi]} h_\Lambda(t)$.

For $\rho \in \mathbb{N}$, the type of the function also depends on the exponential factor $e^{P_N(z)}$. Assume that $N = \rho$ (allowing the leading coefficient to be zero). Multiplying $W_\Lambda$ by an exponential factor $e^{(A-iB)z^\rho}$ we do not change its set of zeros, while its indicator changes by a $\rho$-trigonometric function $k(t) = A\cos\rho t + B\sin\rho t$. Hence, for every $k \in T_\rho$ we obtain a function $\widetilde{f} \in \mathcal{E}_{\rho,\sigma}$ such that $\widetilde{f}^{-1}\{0\} = \Lambda$ and $h_{\widetilde{f}} = h_\Lambda + k$, varying the leading coefficient of the polynomial $P_N(z)$. Therefore,

$$\sigma_Z(\Lambda) \le \inf_{k\in T_\rho} \max_{t\in[0,2\pi]} [h_\Lambda(t) + k(t)].$$

On the other hand, by compactness there exists $k_\Lambda \in T_\rho$ such that

$$\max_{t\in[0,2\pi]} [h_\Lambda(t) + k_\Lambda(t)] = \min_{k\in T_\rho} \max_{t\in[0,2\pi]} [h_\Lambda(t) + k(t)].$$

Consequently,

$$\sigma_Z(\Lambda) = \min_{k\in T_\rho} \max_{t\in[0,2\pi]} [h_\Lambda(t) + k(t)].$$

Theorem 1.1 is proved. ∎

We now proceed to the proof of Theorem 1.3.

Let $\rho > 0$. By the definition, the set $\Lambda$ is *not* a uniqueness set for $\mathcal{E}_{\rho,\sigma}$ if and only if there exists an entire function $g$ (a multiplier) such that $g \cdot W_\Lambda \in \mathcal{E}_{\rho,\sigma}$. Note that if such a function exists, then it has a finite type with respect to the order $\rho$, and, by Levin's theorem on the indicator of the product of two entire functions [11, Chap. III, Sect. 4, Theorem 5], we have

$$h_{g\cdot W_\Lambda} = h_g + h_{W_\Lambda}.$$

So, the existence of a function $g$ such that

$$\max_{t\in[0,2\pi]} h_{g\cdot W_\Lambda}(t) \le \sigma$$

is necessary and sufficient for $\Lambda$ to be a non-uniqueness set for $\mathcal{E}_{\rho,\sigma}$. If we can find a $\rho$-trigonometrically convex function $k$ such that

$$\max_{t\in[0,2\pi]} (k(t) + h_{W_\Lambda}(t)) \le \sigma,$$

then we can take any function $g$ of completely regular growth with indicator $h_g = k$. Thus, we replace the problem of finding a multiplier $g$ with the problem of finding a $\rho$-trigonometrically convex function $k$. It follows that

$$\sigma_U(\Lambda) = \inf_{k\in TC_\rho} \max_{t\in[0,2\pi]} [h_\Lambda(t) + k(t)].$$

Note that for $\rho \le 1/2$ all $\rho$-trigonometrically convex functions are non-negative, so we cannot lower the maximal value of $h_\Lambda$ by adding a $\rho$-trigonometrically convex function. Hence, for $\rho \le 1/2$ we have

$$\sigma_U(\Lambda) = \sigma_Z(\Lambda).$$

Let us show that this equality also holds for $\rho = 1$. Suppose $\sigma_U(\Lambda) < \sigma_Z(\Lambda)$. That is, there exists $k^* \in TC_1$ such that

$$\max_{t\in[0,2\pi]} (h^*_\Lambda(t) + k^*(t)) < \sigma_Z(\Lambda) = \max_{t\in[0,2\pi]} h^*_\Lambda(t), \tag{2.3}$$

where $h^*_\Lambda$ is the 1-balanced modification of $h_\Lambda$.

As a continuous function, $k^*$ has at least one maximum point on the interval $[0, 2\pi]$. Then, due to the basic properties of trigonometrically convex functions [11, Chap. I, Sect. 16] there is a closed interval $J$ of length $\pi$ such that $k^*(t) \ge 0$ for all $t \in J$.

It follows from the definition of a 1-balanced function that every closed subinterval of $[0, 2\pi]$ of length $\pi$ contains at least one maximum point of $h^*_\Lambda$, in particular, there exists $t_M \in J$ such that

$$h^*_\Lambda(t_M) = \max_{t\in[0,2\pi]} h^*_\Lambda(t).$$

Then, by (2.3),

$$h^*_\Lambda(t_M) = \max_{t\in[0,2\pi]} h^*_\Lambda(t) > \max_{t\in[0,2\pi]} (h^*_\Lambda(t)+k^*(t)) \ge h^*_\Lambda(t_M)+k^*(t_M) \ge h^*_\Lambda(t_M),$$

and we get a contradiction.

Theorem 1.3 is proved. ■

**3. Proofs of Theorem 1.2 and Corollary 1.1.** We start with the proof of Theorem 1.2 and prove Corollary 1.1 at the end of this section.

*Part* (A). (i)⇒(ii) Let $h \in TC_\rho$ be $\rho$-balanced. In other words, for any $\alpha \in \mathbb{R}$ each of the sets

$$A_1(\alpha) := \bigcup_{k\in\mathbb{Z}} [\alpha + 2\pi k/\rho, \alpha + \pi(2k + 1)/\rho)$$

and

$$A_2(\alpha) := \bigcup_{k\in\mathbb{Z}} [\alpha + \pi(2k - 1)/\rho, \alpha + 2\pi k/\rho)$$

intersects $M_h$.

Suppose that condition (ii) does not hold. Then there is a function $k \in T_\rho$, $k(t) = C \sin \rho(t - t_0)$, where $C, t_0 \in \mathbb{R}$, such that

$$\max_{t\in[0,2\pi]} (h(t) + k(t)) < \max_{t\in[0,2\pi]} h(t).$$

Consider now two points $\alpha_1 \in A_1(t_0) \cap M_h$ and $\alpha_2 \in A_2(t_0) \cap M_h$. Then

$$k(\alpha_1) \cdot k(\alpha_2) = C^2 \sin \rho(\alpha_1 - t_0) \sin \rho(\alpha_2 - t_0) \le 0.$$

Therefore, either

$$h(\alpha_1) + k(\alpha_1) \ge h(\alpha_1) = \max_{t\in[0,2\pi]} h(t),$$

or

$$h(\alpha_2) + k(\alpha_2) \ge h(\alpha_2) = \max_{t\in[0,2\pi]} h(t).$$

In either case, we arrive at a contradiction.

(ii)⇒(i) Now, suppose that $h$ is not $\rho$-balanced, that is, (i) is not satisfied. Then there exists an $\alpha$ such that $M_h \subset A_1(\alpha)$. Without loss of generality we can suppose that $\alpha = 0$, and hence $M_h \subset A_1(0)$. Moreover, since $M_h$ is compact, we can suppose that for some $\varepsilon > 0$,

$$M_h \subset A_1^\varepsilon := \bigcup_{k\in\mathbb{Z}} (2\pi k/\rho + \varepsilon, \pi(2k + 1)/\rho - \varepsilon).$$

Define also $A_2^\varepsilon := \mathbb{R} \setminus A_1^\varepsilon$. Put

$$a := \max_{t\in\mathbb{R}} h(t), \quad b := \sup_{A_2^\varepsilon} h(t).$$

From the construction, it follows that $a > b$. Consider the function

$$\widetilde{h}(t) := h(t) - \frac{a - b}{2} \sin \rho t.$$

- If $t \in A_1^\varepsilon$, then

$$\widetilde{h}(t) \le a - \frac{a - b}{2} \sin \rho\epsilon < a,$$

- If $t \in A_2^\varepsilon$, then

$$\widetilde{h}(t) \le b + \frac{a - b}{2} = \frac{a + b}{2} < a.$$

So, condition (ii) is not satisfied.

*Part* (B). Recall the definition of the function $h^*$ given in the introduction:

$$h^*(t) = \max_{j=0,1,\ldots,\rho-1} h\left(\frac{t + 2\pi j}{\rho}\right).$$

It is a $2\pi$-periodic 1-trigonometrically convex function that serves as the support function for the convex set $I^*$ with circumradius $R^*$ and circumcenter $C^* \in \mathbb{C}$. By taking the geometrical sum of the set $I^*$ and the point $-C^*$

(that is, shifting the set $I^*$ so that its center coincides with the origin), we obtain a convex set with supporting function $h_0^*(t) = h^*(t) - \operatorname{Re}(C^* e^{it})$. Note that

$$h_0^*(t) = \max_{j=0,1,\dots,\rho-1} \widetilde{h}\left(\frac{t+2\pi j}{\rho}\right),$$

where

$$\widetilde{h}(t) = h(t) - \operatorname{Re}(C^* e^{i\rho t}).$$

Furthermore, $\widetilde{h}$ is $\rho$-balanced, since $h_0^*$ is 1-balanced, and $\{e^{i\rho t} : t \in M_{\widetilde{h}}\}$ coincides with the set $\{e^{it} : t \in M_{h_0^*}\}$. Hence, by part (A), we have

$$\min_{k\in T_\rho} \max_{t\in[0,2\pi]} (h(t) + k(t)) = \max_{t\in[0,2\pi]} \widetilde{h}(t) = R^*. \ \blacksquare$$

EXAMPLE 1. Let $\rho \in \mathbb{N}$. Consider the following family of probability measures on $[0, 2\pi]$:

$$\varDelta_1^n := \frac{1}{n} \sum_{j=1}^{n} \delta_{(2j+1)\pi/n}, \qquad n \in \mathbb{N},\ n \geq 2\rho,$$

and let $\varDelta_1^\infty$ be the normalized Lebesgue measure on $[0, 2\pi]$. Let $\varLambda_n$ and $\varLambda_\infty$ be $\rho$-regular sets such that $\varDelta_{\varLambda_n} = \varDelta_1^n$ and $\varDelta_{\varLambda_\infty} = \varDelta_1^\infty$. Then

$$h_{\varDelta_1^n}(t) = \frac{\pi}{n \sin\frac{\rho\pi}{n}} \max\{\cos\rho(t - 2\pi j/n) : j = 0, \dots, n-1\},$$

and $h_{\varDelta_1^\infty}(t) = 1/\rho$. It is easy to see that all these functions are $\rho$-balanced, and hence

$$\sigma_Z(\varLambda_n) = \frac{\pi}{n \sin\frac{\rho\pi}{n}}.$$

As $n \to \infty$ and the set $\varLambda$ spreads evenly over $[0, 2\pi]$, approaching the uniform distribution $\varDelta_1^\infty$, we obtain

$$\lim_{n\to\infty} \sigma_Z(\varLambda_n) = 1/\rho = \sigma_Z(\varLambda_\infty).$$

We now proceed to the proof of Corollary 1.1. The first statement of the corollary is an immediate consequence of Theorems 1.1 and 1.2.

To show that in the definition of the critical zero set type we can replace infimum by minimum, note that $\widehat{h}$ is the indicator of the entire function $f_\varLambda(z) = e^{Cz^\rho} W_\varLambda(z)$ with an appropriate value of $C \in \mathbb{C}$. Since $\widehat{h}(t) = h_{f_\varLambda}(t) \leq R_\Delta^* = \sigma_Z(\varLambda)$, we have $f_\varLambda \in \mathcal{E}_{\rho,\sigma_Z(\varLambda)}$, while $f^{-1}\{0\} = \varLambda$. $\blacksquare$

**4. Proof of Theorem 1.4.** Here, given $\rho \in (1/2, \infty) \setminus \{1\}$, we will provide some examples of non-negative measures $\varDelta$ such that for any $\rho$-regular set $\varLambda$ with $\varDelta_\varLambda = \varDelta$ we have $\sigma_U(\varLambda) < \sigma_Z(\varLambda)$.

EXAMPLE 2. Let $\rho \in \mathbb{N}_{\geq 3}$. Put

$$\varDelta_2 := \delta_{\pi/\rho} + \delta_{2\pi/\rho} + \delta_{4\pi/\rho} + \delta_{5\pi/\rho}.$$

Then (see Fig. 1)

$$h_{\Delta_2}(t) = \begin{cases} 2\pi|\sin\rho t|, & t \in [\pi/\rho, 2\pi/\rho] \cup [4\pi/\rho, 5\pi/\rho], \\ 0, & \text{elsewhere.} \end{cases}$$

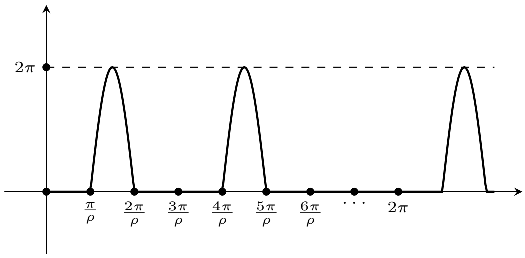


Fig. 1. Illustration to Example 2: $y = h_{\Delta_2}(t)$

The function $h_{\Delta_2}$ is $\rho$-balanced, since $0 \in \operatorname{conv}\{e^{\frac{3\pi}{2}i}, e^{\frac{9\pi}{2}i}\}$. Hence,

$$\sigma_Z(\Lambda) = \max_{[0,2\pi]} h_{\Delta_2}(t) = 2\pi$$

for every $\rho$-regular set $\Lambda$ such that $\Delta_\Lambda = \Delta_2$.

Let

$$k(t) = \begin{cases} \pi\sin\rho t & \text{if } t \in [0, 3\pi/\rho], \\ -\pi\sin\rho t & \text{if } t \in [3\pi/\rho, 6\pi/\rho]. \end{cases}$$

Then $\max_{[0,2\pi]}(h_{\Delta_2}(t) + k(t)) = \pi < \sigma_Z(\Lambda)$. Hence,

$$\sigma_U(\Lambda) \le \pi < \sigma_Z(\Lambda) = 2\pi.$$

Example 3. Let $\rho = 2$. The construction from the previous example does not work in this case. Let $\Delta_3 = \delta_0 + \delta_{2\pi/3} + \delta_{4\pi/3}$. Then (see Fig. 2)

$$h_{\Delta_3}(t) = \frac{2\pi}{\sqrt{3}}\cos 2(t - \pi/3) \text{ if } t \in [0, 2\pi/3], \quad h_{\Delta_3}(t + 2\pi/3) = h_{\Delta_3}(t).$$

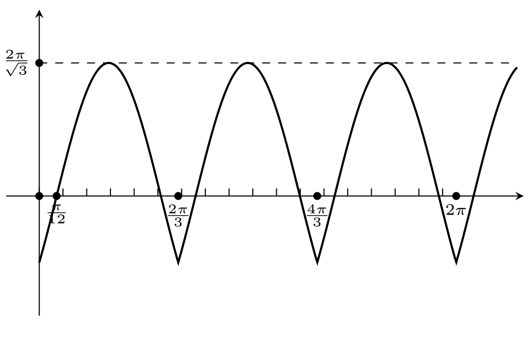


Fig. 2. Illustration to Example 3: $y = h_{\Delta_3}(t)$

This function is 2-balanced. So, $\sigma_Z(\Lambda) = 2\pi/\sqrt{3}$ for every 2-regular set $\Lambda$ with $\varDelta_\Lambda = \varDelta_3$. Now take

$$k(t) = \tfrac{1}{2} h_{\varDelta_3}(t + \pi/3).$$

Then (see Fig. 3)

$$\begin{aligned}\max_{t\in[0,2\pi]} (h_{\varDelta_3}(t) + k(t)) &= \frac{\pi}{\sqrt{3}} \max_{t\in[0,\pi/3]} (2\cos 2(t-\pi/3) + \cos 2t)\\ &= \frac{\pi}{\sqrt{3}} \max_{t\in[0,\pi/3]} (\sqrt{3}\sin 2t) = \pi.\end{aligned}$$

Hence, for every 2-regular set $\Lambda$ with $\varDelta_\Lambda = \varDelta_3$ we have

$$\sigma_U(\Lambda) \le \pi < \sigma_Z(\Lambda) = 2\pi/\sqrt{3}.$$

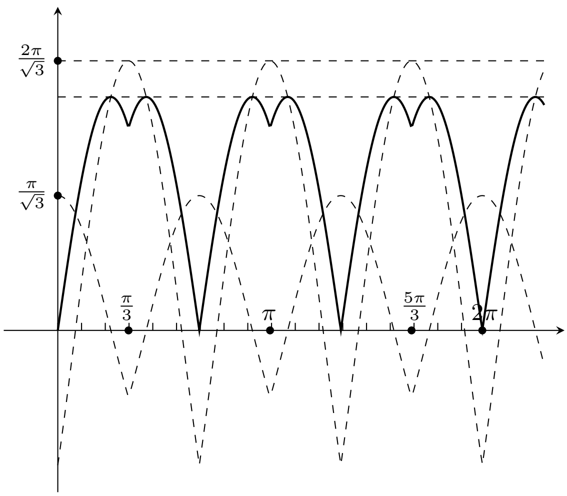


Fig. 3. Illustration to Example 3: $y = h_{\varDelta_3}(t) + k(t)$ (bold line)

Example 4. Let $\rho \in (1,\infty) \setminus \mathbb{N}$ and $\varDelta_4 = \delta_{-\frac{\pi}{2\rho}} + \delta_{\frac{\pi}{2\rho}}$. Then

$$h_{\varDelta_4}(t) = 2\pi \cos \rho t \cdot \mathbf{1}_{[-\frac{\pi}{2\rho},\frac{\pi}{2\rho}]}.$$

Put

$$k(t) := -\pi \cos \rho t \cdot \mathbf{1}_{[-\tau,\tau]},$$

where $\tau = \min\big(\frac{3\pi}{2\rho}, \pi\big)$. Then

$$\max_{t\in[0,2\pi]} (h_{\varDelta_4}(t) + k(t)) = \pi < \max_{t\in[0,2\pi]} h_{\varDelta_4}(t) = 2\pi.$$

Hence, $\sigma_U(\Lambda) \le \pi < \sigma_Z(\Lambda)$ for every $\Lambda$ with $\varDelta_\Lambda = \varDelta_4$.

Example 5. Let $\rho \in (1/2, 1)$ and $\varDelta_5 = \delta_\pi$. Then

$$h_{\varDelta_5}(t) = \frac{\pi \cos \rho t}{\sin \pi\rho}, \qquad t \in [-\pi, \pi].$$

Put

$$k(t) := -\pi \cdot \frac{\cos \pi\rho}{\sin \pi\rho} \cdot \cos \rho(\pi - |t|), \quad t \in [-\pi, \pi].$$

Then

$$\max_{t\in[0,2\pi]} (h_{\Delta_5}(t) + k(t)) = \frac{\pi}{\sin \pi\rho} \cdot \max_{t\in[0,\pi]} \big(\cos \rho t - \cos \pi\rho \cdot \cos \rho(\pi - t)\big) = \pi.$$

Hence, in this case we get

$$\sigma_U(\Lambda) \le \pi < \sigma_Z(\Lambda) = \frac{\pi}{\sin \pi\rho}$$

for every $\Lambda$ with $\Delta_\Lambda = \Delta_5$.

As we will see later, all the estimates of $\sigma_U$ we obtained in this section for $\Delta_\Lambda = \Delta_n, n = 1, 2, 3, 4, 5$, turn out to be sharp. We return to these examples in Section 6.

**5. Proof of Theorem 1.5.** (ii)⇒(i) From the definition of being locally $\rho$-balanced (1.7) it follows that a set $M$ is locally $\rho$-balanced if and only if there exists a triple $\{\alpha, \beta, \gamma\} \subset M$ such that

(5.1) $$0 < \beta - \alpha \le \pi/\rho, \quad 0 \le \gamma - \beta < \pi/\rho, \quad \gamma - \alpha \ge \pi/\rho$$

(here we use the simple observation that 0 lies in the convex hull of a subset of the unit circle if and only if this subset contains three points whose convex hull contains 0). In particular, in the degenerate case when $\gamma = \beta = \alpha + \pi/\rho$ the set $M$ is locally $\rho$-balanced.

Note that for any function $k \in TC_\rho$ with $\rho > 1/2$ the set $\{t : k(t) < 0\}$ is not locally $\rho$-balanced.

Suppose that (i) does not hold. Then there exists a function $k \in TC_\rho$ such that

(5.2) $$\max_{t\in[0,2\pi]} (\widehat{h}_\Lambda(t) + k(t)) < \max_{t\in[0,2\pi]} \widehat{h}_\Lambda(t).$$

Therefore, for every $t \in M_{\widehat{h}_\Lambda}$ we have $k(t) < 0$. Hence, the function $\widehat{h}_\Lambda$ is not locally $\rho$-balanced.

(i)⇒(ii) Note that for $\rho = 1$ a set is locally 1-balanced if and only if it is 1-balanced, so it suffices to consider the case $\rho \in (1/2, 1) \cup (1, \infty)$ only.

Suppose now that $\widehat{h}_\Lambda$ is not locally $\rho$-balanced. We will show how to construct a $\rho$-trigonometrically convex function $k$ such that

$$\max_{t\in[0,2\pi]} (\widehat{h}_\Lambda(t) + k(t)) < \max_{t\in[0,2\pi]} \widehat{h}_\Lambda(t).$$

Note that a set $M$ is not locally $\rho$-balanced if and only if it can be covered by a finite union $\bigcup I_j$ of open disjoint intervals $I_j \subset [0, 2\pi]$, each of length less than $\pi/\rho$, while the distance between neighboring intervals is greater than $\pi/\rho$.

Put $M = M_{\widehat{h}_\Lambda} \subset [0, 2\pi]$ and denote by $\widetilde{M}$ its $2\pi$-periodization. Since $M$ is not locally $\rho$-balanced, there exists an open cover $\bigcup_{j=1}^{L} I_j =: \mathcal{I}$ of $\widehat{M}$ such that

- for each $j$,
$$I_j = \bigcup_{k\in\mathbb{Z}} (\alpha_j + 2\pi k, \beta_j + 2\pi k),$$
where $0 < \beta_j - \alpha_j < \pi/\rho$;
- for each $j = 1, \dots, L$,
$$\alpha_{j+1} - \beta_j > \pi/\rho,$$
where $\alpha_{L+1} := \alpha_1 + 2\pi$.

We denote by $J_j := [\beta_j, \alpha_{j+1}]$ the complementary intervals to $\mathcal{I}$.

Put
$$a := \max_{t\in[0,2\pi]} \widehat{h}_\Lambda(t), \qquad b := \sup_{t\notin\mathcal{I}} \widehat{h}_\Lambda(t).$$
From the definition of a locally $\rho$-balanced set it follows that $b < a$.

Let us introduce the auxiliary function
$$s_\rho(t) := \sin\rho t \cdot \mathbf{1}_{[0,\pi/\rho]}.$$
Put
$$k(t) = \begin{cases} \dfrac{a-b}{2} \max\big(\sin\rho(t-\beta_j), \sin\rho(\alpha_j - t)\big), & t \in I_j, \\ \dfrac{a-b}{2} \max\big(s_\rho(t-\beta_j), s_\rho(\alpha_{j+1} - t)\big), & t \in J_j. \end{cases}$$
Since $0 < \beta_j - \alpha_j < \pi/\rho$, it follows that $k$ is *strictly negative* on $\mathcal{I}$. On the other hand, $\max_{t\in[0,2\pi]} k(t) = (a-b)/2$.

Recall [11] that a $2\pi$-periodic function $h$ is $\rho$-trigonometrically convex if and only if for all $\alpha < \beta$,
$$h'_+(\beta) - h'_-(\alpha) + \rho^2 \int_\alpha^\beta h \ge 0.$$
Since $k$ is continuous, $2\pi$-periodic and piecewise $\rho$-trigonometric, to justify that $k \in TC_\rho$ it is sufficient to examine the set $\mathcal{S}$ of all singular points of $k$ (that is, the points where $k$ is not differentiable), and to show that for every $s \in \mathcal{S}$,
$$k'_+(s) \ge k'_-(s). \tag{5.3}$$

First of all, note that all the points $\alpha_j, \beta_j$ are regular points of $k$.

Next, for each $j = 1, \dots, L$ the point $\gamma_j := (\alpha_j + \beta_j)/2$ is the only singular point in the interval $I_j$, and since $k'_+(\gamma_j) > k'_-(\gamma_j)$, the function $k$ is $\rho$-trigonometrically convex on each $I_j$.

Further, if $|J_j| > 2\pi/\rho$, there are two singular points on $J_j$: $\beta_j + \pi/\rho$ and $\alpha_{j+1} - \pi/\rho$. It is easy to see that in this case condition (5.3) also holds.

Finally, if $|J_j| \le 2\pi/\rho$, there is one singularity on $J_j$ at the point $\delta_j = (\alpha_{j+1} + \beta_j)/2$, and $k'_+(\delta_j) > k'_-(\delta_j)$. Hence, $k$ is $\rho$-trigonometrically convex.

Thus, we have constructed a $\rho$-trigonometrically convex function $k$ such that

- if $t \in \mathcal{I}$, then $k(t) < 0$, and therefore

$$\widehat{h}_\Lambda(t) + k(t) < a;$$

- if $t \notin \mathcal{I}$, then

$$\widehat{h}_\Lambda(t) + k(t) \le b + \frac{a-b}{2} = \frac{a+b}{2} < a.$$

Hence

$$\max_{t\in[0,2\pi]} (\widehat{h}_\Lambda(t) + k(t)) < \max_{t\in[0,2\pi]} \widehat{h}_\Lambda(t).$$

Theorem 1.5 is proved. ∎

Example 1 revisited. Note that all the functions $h_{\Delta_1^n}$ and $h_{\Delta_1^\infty}$ are locally $\rho$-balanced for $n \ge 2\rho$, hence

$$\sigma_U(\Lambda_n) = \sigma_Z(\Lambda_n) = \frac{\pi}{n \sin \frac{\rho\pi}{n}},$$

while $\sigma_U(\Lambda_\infty) = \sigma_Z(\Lambda_\infty) = 1/\rho$.

**6. Proof of Theorem 1.6.** We will prove inequalities (1.8) and (1.9) by contradiction. Suppose that

$$\varepsilon := A_\Lambda - \sigma_U(\Lambda) > 0.$$

Then there exists $k \in TC_\rho$ such that

$$\max_{t\in[0,2\pi]} (h_\Lambda(t) + k(t)) < A_\Lambda - \varepsilon/2.$$

Using the fact that $k(t) + k(t + \pi/\rho) \ge 0$, we get

$$h_\Lambda(t) + h_\Lambda\left(t + \frac{\pi}{\rho}\right) \le h_\Lambda(t) + h_\Lambda\left(t + \frac{\pi}{\rho}\right) + k(t) + k\left(t + \frac{\pi}{\rho}\right) < 2A_\Lambda - \varepsilon;$$

this contradiction proves (1.8).

Suppose now that $\sigma_U(\Lambda) < C$. Then there exists $k \in TC_\rho$ such that

$$\widehat{h}_\Lambda(t) + k(t) < C, \quad \forall t.$$

Therefore,

$$k(t) < 0, \quad \forall t : \widehat{h}_\Lambda(t) \ge C.$$

Since $k$ is $\rho$-trigonometrically convex, it follows that the set of its negative values cannot contain a locally $\rho$-balanced set, leading to a contradiction.

Theorem 1.6 is proved. ∎

Examples 2–5 revisited.

$2'$ Returning to Example 2, we note that if $\Delta_\Lambda = \Delta_2$ for some $\rho$-regular set $\Lambda$, then $A_\Lambda = \pi$. Now, due to (1.8), we know that $\sigma_U(\Lambda) \geq \pi$, and since it has been shown in Example 2 that this level is achievable, we can conclude that

$$\sigma_U(\Lambda) = \pi.$$

$3'$ In the same way we can find the value of $\sigma_U(\Lambda)$ for the $\rho$-regular set $\Lambda$ with $\Delta_\Lambda = \Delta_3$. Here we have

$$h_{\Delta_3}\left(\frac{5\pi}{12}\right) + h_{\Delta_3}\left(\frac{5\pi}{12} + \frac{\pi}{2}\right) = \frac{4\pi}{\sqrt{3}} \cos\frac{\pi}{6} = 2\pi \leq 2A_\Lambda,$$

hence $\sigma_U(\Lambda) = \pi$. The same result can be obtained by applying the second part of Theorem 1.6: put $C = \pi$ and consider the set

$$\{t : h_{\Delta_3}(t) \geq \pi\} = [\pi/4, 5\pi/12] \cup [11\pi/12, 13\pi/12] \cup [19\pi/12, 21\pi/12].$$

Since this set is locally 2-balanced, by (1.9) we get $\sigma_U(\Lambda) \geq \pi$, and hence $\sigma_U(\Lambda) = \pi$.

$4'$ Returning to Example 4, we also see that for the set $\Lambda$ with $\Delta_\Lambda = \Delta_4$, we have $A_\Lambda = \pi$, hence $\sigma_U(\Lambda) = \pi$.

$5'$ For $\Delta_\Lambda = \Delta_5$ we have

$$\begin{aligned} h_{\Delta_5}\left(\pi - \frac{\pi}{2\rho}\right) &+ h_{\Delta_5}\left(\left(\pi - \frac{\pi}{2\rho}\right) + \frac{\pi}{\rho}\right) \\ &= h_{\Delta_5}\left(\pi - \frac{\pi}{2\rho}\right) + h_{\Delta_5}\left(-\pi + \frac{\pi}{2\rho}\right) \\ &= \frac{2\pi}{\sin \pi\rho} \cos(\pi\rho - \pi/2) = 2\pi \leq 2A_\Lambda. \end{aligned}$$

Hence, $\sigma_U(\Lambda) = \pi$.

A natural question arises: do estimates (1.8) and/or (1.9) always provide an exact value for $\sigma_U(\Lambda)$? Unfortunately, this is not the case, as the following examples show.

Example $2''$. As shown in Example $2'$, inequality (1.8) provides an exact bound on $\sigma_U(\Lambda)$, while from (1.9) we cannot derive any meaningful bound, as the only value of $C \geq 0$ for which the set $\{t : h_{\Delta_2}(t) \geq C\}$ is locally $\rho$-balanced is $C = 0$.

Example 6. Let now $\rho = 1$. Consider some *Reuleaux triangle* $\mathcal{R}$, i.e., a convex body of constant width bounded by three congruent arcs, with width $W = h(t) + h(t + \pi)$ and circumradius $R > W/2$. Suppose that its circumcenter is located at the origin. All three vertices of $\mathcal{R}$ lie on the circumcircle.

Let $h_6$ be the support function of $\mathcal{R}$, and denote by $\varDelta_6$ the corresponding measure. Then, if $\varDelta_\varLambda = \varDelta_6$, we have $\sigma_Z(\varLambda) = R$, and $A_\varLambda := W/2$, so there is a strict inequality in (1.8), while (1.9) gives us a sharp bound, since the convex hull of the vertices of the Reuleaux triangle contains the origin, hence the function $h_6$ is 1-balanced, and for $\rho = 1$ it is locally 1-balanced.

**7. Proof of Theorem 1.7.** Fix some $\varLambda \in \mathcal{A}_\rho$ and put $\varDelta := \varDelta_\varLambda$. Note that conditions (i) and (ii) from the definition of $\mathcal{A}_\rho$ are equivalent to

$$\sigma_U(\varLambda) = \sigma_Z(\varLambda) = 1.$$

Hence, $\max_{t\in[0,2\pi]} \widehat{h}_\varLambda(t) = 1$, where, as before, $\widehat{h}_\varLambda$ denotes the $\rho$-balanced modification of $h_\varDelta$ defined by (1.6).

The results in [11, Chap. IV, Sect. 1] (or just formula (1.3)) imply that

$$\mathcal{D}(\varLambda) = \frac{\rho}{2\pi} \int_{-\pi}^{\pi} \widehat{h}_\varDelta(t)\,\mathrm{d}t, \tag{7.1}$$

which immediately gives the upper bound:

$$\mathcal{D}(\varLambda) \le \rho = \frac{\rho}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}t.$$

This estimate is achieved for any $\varLambda$ whose angular distribution is uniform, i.e. $\varDelta_\varLambda = \varDelta^*([\alpha,\beta]) = \frac{\rho}{2\pi}(\beta-\alpha)$ for all $0 \le \alpha < \beta \le 2\pi$. The corresponding $\rho$-trigonometrically convex function is

$$h^*(t) = 1.$$

Put

$$M := \{t \in \mathbb{R} : \widehat{h}_\varDelta(t) = 1\};$$

this set is non-empty.

We now proceed to the proof of the lower bound, which is more involved. We subdivide it into three parts depending on the value of $\rho$.

- $\rho \in (0, 1/2]$. Basic properties of $\rho$-trigonometrically convex functions imply that if $\theta_0 \in M$, then for every $t$ such that $|t - \theta_0| \le \pi/\rho$, we have

$$\widehat{h}_\varDelta(t) \ge \cos\rho(t-\theta_0). \tag{7.2}$$

Without loss of generality, we assume that $0 \in M$. Applying (7.2), we get

$$\int_{-\pi}^{\pi} \widehat{h}_\varDelta(t)\,\mathrm{d}t \ge \int_{-\pi}^{\pi} \cos\rho t\,\mathrm{d}t = \frac{2\sin\pi\rho}{\rho}.$$

This bound is achieved for $h_*(t) = \cos\rho t$, and the corresponding measure is

$$\varDelta_* = \frac{\sin\pi\rho}{\pi}\cdot\delta_\pi.$$

• $\rho \in (1/2, 1]$. Since $\sigma_U(\Lambda) = \sigma_Z(\Lambda)$, by Theorem 1.5, the set $M$ is locally $\rho$-balanced (see (5.1)). Therefore, without loss of generality, we can assume that there exists $\alpha \in [\pi - \pi/(2\rho), \pi/(2\rho)]$ such that $\pm\alpha \in M$. It follows that

$$\widehat{h}_\Delta(t) \geq \cos\rho(|t| - \alpha), \qquad |t| \leq \pi.$$

The auxiliary function

$$\begin{aligned} f(\alpha) &= \int_{-\pi}^{\pi} \cos\rho(|t| - \alpha)\,\mathrm{d}t = \frac{2}{\rho}(\sin\rho(\pi - \alpha) + \sin\alpha\rho) \\ &= \frac{4}{\rho}\sin\frac{\pi\rho}{2}\cos\left(\frac{\pi\rho}{2} - \alpha\rho\right) \end{aligned}$$

is concave on $[\pi - \pi/(2\rho), \pi/(2\rho)]$, therefore its minimal value is attained on the boundary of this interval. Hence,

$$f(\alpha) \geq \frac{4}{\rho}\sin^2\frac{\pi\rho}{2},$$

and therefore

$$\mathcal{D}(\Lambda) \geq \frac{\rho}{2\pi}\int_{-\pi}^{\pi}\widehat{h}_\Delta(t)\,\mathrm{d}t \geq \frac{\rho}{2\pi}f(\alpha) \geq \frac{2}{\pi}\sin^2\frac{\pi\rho}{2} = \frac{1 + |\cos\pi\rho|}{\pi}.$$

The bound is achieved for the function $h_*(t) = \sin\rho|t|$, and the corresponding angular density is

$$\Delta_* = \frac{1}{\pi}(\delta_0 + |\cos\rho\pi| \cdot \delta_\pi).$$

• $\rho > 1$. Put

$$N = [\rho - 1/2] = \begin{cases} [\rho] - 1, & \{\rho\} \in [0, 1/2], \\ [\rho], & \{\rho\} \in (1/2, 1), \end{cases}$$

and $\delta := \rho - N \in [1/2, 3/2]$.

As in the previous case, since $\sigma_U(\Lambda) = \sigma_Z(\Lambda)$, by Theorem 1.5 the set $M$ is locally $\rho$-balanced. Recall that in this case at least one of the following conditions is satisfied (see (5.1)):

(i) $\exists\alpha : \{\alpha, \alpha + \pi/\rho\} \subset M$;
(ii) $\exists\{\alpha, \beta, \gamma\} \subset M : 0 < \beta - \alpha < \pi/\rho;\ 0 < \gamma - \beta < \pi/\rho;\ \gamma - \alpha > \pi/\rho$.

Let us start with (i). Without loss of generality we suppose that $\alpha = -\frac{\pi}{2\rho}$.

Basic properties of $\rho$-trigonometrically convex functions imply that if $\theta_0 \in M$, then for every $t$ such that $|t - \theta_0| \leq \pi/\rho$, we have

$$\widehat{h}_\Delta(t) \geq \cos\rho(t - \theta_0). \tag{7.3}$$

It follows that

$$\widehat{h}_\Delta(t) \geq \sin\rho|t|, \qquad |t| \leq \frac{3\pi}{2\rho}.$$

The next lemma follows immediately from standard properties of $\rho$-trigonometrically convex functions [11, Chap. I, Sect. 16].

LEMMA 7.1. *If $h \in TC_\rho$, then for each $x$ we have*

$$\int_x^{x+2\pi/\rho} h \geq 0.$$

Now, by Lemma 7.1, we get (recall that $\rho = N + \delta$)

$$\begin{aligned}\int_{-\pi}^{\pi} \widehat{h}_\Delta(t)\,\mathrm{d}t &= \int_{-\delta\pi/\rho}^{2\pi\frac{N+\delta}{\rho}-\frac{\delta\pi}{\rho}} \widehat{h}_\Delta(t)\,\mathrm{d}t \geq \int_{-\delta\pi/\rho}^{\delta\pi/\rho} \widehat{h}_\Delta(t)\,\mathrm{d}t \\ &\geq \int_{-\delta\pi/\rho}^{\delta\pi/\rho} \sin\rho|t|\,\mathrm{d}t = \frac{2}{\rho}(1-\cos\delta\pi).\end{aligned}$$

Recalling that $\rho - N = \delta \in [1/2, 3/2]$, we obtain

$$\int_{-\pi}^{\pi} \widehat{h}_\Delta(t)\,\mathrm{d}t \geq \frac{2}{\rho}(1+|\cos\rho\pi|). \tag{7.4}$$

In case (ii), we introduce the auxiliary function

$$h_{\alpha,\beta,\gamma}(t) := \begin{cases} \cos\rho(t-\alpha), & \alpha \leq t \leq \frac{\beta+\alpha}{2}, \\ \cos\rho(t-\beta), & \frac{\alpha+\beta}{2} \leq t \leq \frac{\beta+\gamma}{2}, \\ \cos\rho(t-\gamma), & \frac{\beta+\gamma}{2} \leq t \leq \gamma, \\ \widehat{h}_\Delta, & \text{elsewhere.} \end{cases} \tag{7.5}$$

Considering $\beta$ as a parameter, we will show that replacing $\beta$ by $\beta^* = \alpha + \pi/\rho$ decreases the integral of $h_{\alpha,\beta,\gamma}$. This result will also be used in the next section, so we formulate it as a lemma.

LEMMA 7.2. *Given $h_\Delta \in TC_\rho$ with $\max_{t\in[-\pi,\pi]} \widehat{h}_\Delta(t) = 1$, let $\{\alpha,\beta,\gamma\} \subset M = \{t : \widehat{h}_\Lambda(t) = 1\}$ be such that*

$$\alpha < \gamma - \pi/\rho < \beta < \alpha + \pi/\rho < \gamma.$$

*Put $\beta^* := \alpha + \pi/\rho$. Then*

$$\int_{-\pi}^{\pi} h_{\alpha,\beta,\gamma}(t)\,\mathrm{d}t \geq \int_{-\pi}^{\pi} h_{\alpha,\beta^*,\gamma}(t)\,\mathrm{d}t.$$

*Proof.* Since

$$h_{\alpha,\beta,\gamma}(t) = h_{\alpha,\beta^*,\gamma}(t) \quad \forall t \in [-\pi,\pi] \setminus [\alpha,\gamma],$$

it is sufficient to consider the integral on $[\alpha,\gamma]$. We have

$$\begin{aligned}
&\int_{\alpha}^{\gamma} h_{\alpha,\beta,\gamma}(t)\,\mathrm{d}t \\
&\qquad = \int_{\alpha}^{\frac{\beta+\alpha}{2}} \cos\rho(t-\alpha)\,\mathrm{d}t + \int_{\frac{\beta+\alpha}{2}}^{\frac{\gamma+\beta}{2}} \cos\rho(t-\beta)\,\mathrm{d}t + \int_{\frac{\gamma+\beta}{2}}^{\gamma} \cos\rho(t-\gamma)\,\mathrm{d}t \\
&\qquad = 2\int_{\frac{\beta+\alpha}{2}}^{\frac{\gamma+\beta}{2}} \cos\rho(t-\beta)\,\mathrm{d}t = \frac{2}{\rho}\Big(\sin\rho\frac{\gamma-\beta}{2} - \sin\rho\frac{\alpha-\beta}{2}\Big) \\
&\qquad = \frac{4}{\rho}\sin\rho\frac{\gamma-\alpha}{4}\cos\rho\frac{\alpha+\gamma-2\beta}{4}.
\end{aligned}$$

Recalling the inequality

$$\alpha < \gamma - \pi/\rho < \beta < \alpha + \pi/\rho < \gamma,$$

we see that

$$-\pi/\rho < \gamma - \alpha - 2\pi/\rho < \alpha + \gamma - 2\beta < \alpha - \gamma + 2\pi/\rho < \pi/\rho.$$

Hence

$$\rho\frac{\alpha+\gamma-2\beta}{4} \in (-\pi/2, \pi/2).$$

Therefore, by the elementary properties of the cosine function, we have

$$\cos\rho\frac{\alpha+\gamma-2\beta}{4} > \cos\rho\frac{\alpha-\gamma+2\pi/\rho}{4} = \cos\rho\frac{\alpha+\gamma-2\beta^*}{4},$$

and finally

$$\int_{-\pi}^{\pi} h_{\alpha,\beta,\gamma}(t)\,\mathrm{d}t \geq \int_{-\pi}^{\pi} h_{\alpha,\beta^*,\gamma}(t)\,\mathrm{d}t. \ \blacksquare$$

Now, since $h_{\alpha,\beta^*,\gamma}$ is of type (i), applying (7.4) we get

$$\int_{-\pi}^{\pi} \widehat{h}_{\Delta}(t)\,\mathrm{d}t \geq \int_{-\pi}^{\pi} h_{\alpha,\beta,\gamma}(t)\,\mathrm{d}t \geq \int_{-\pi}^{\pi} h_{\alpha,\beta^*,\gamma}(t)\,\mathrm{d}t \geq \frac{2}{\rho}(1+|\cos\rho\pi|),$$

and therefore

$$\mathcal{D} \geq \frac{\rho}{2\pi}\int_{-\pi}^{\pi} \widehat{h}_{\Delta}(t)\,\mathrm{d}t \geq \frac{1+|\cos\pi\rho|}{\pi}.$$

To show that this estimate is sharp, we introduce the following $\rho$-trigonometrically convex function:

$$k(t) = \begin{cases} \sin\rho|t|, & |t| \leq \delta\pi/\rho, \\ \sin\rho t, & \delta\pi/\rho \leq t \leq 2\pi - \delta\pi/\rho. \end{cases}$$

Since $k$ is piecewise $\rho$-trigonometric, to ensure that $k \in TC_\rho$, it is sufficient to show that at the points $t = 0, \pm\delta\pi/\rho$ we have $k'_+(t) \geq k'_-(t)$. Indeed, this obviously holds at 0 and $\delta\pi/\rho$, and at $-\delta\pi/\rho$ we have

$$k'_+(-\delta\pi/\rho) = -\rho\cos\delta\pi \geq 0$$

and

$$k'_-(-\delta\pi/\rho) = k'_-(2\pi - \delta\pi/\rho) = \rho\cos(2\pi\rho - \delta\pi) = \rho\cos\delta\pi \leq 0.$$

Hence, $k \in TC_\rho$. Moreover, $k$ is locally $\rho$-balanced and

$$\begin{aligned}\int_{-\pi}^{\pi} k(t)\,\mathrm{d}t &= \Big(\int_{-\delta\pi/\rho}^{\delta\pi/\rho} + \int_{\delta\pi/\rho}^{\delta\pi/\rho + N2\pi/\rho}\Big) k(t)\,\mathrm{d}t \\ &= \int_{-\delta\pi/\rho}^{\delta\pi/\rho} k(t)\,\mathrm{d}t = 2\int_0^{\delta\pi/\rho} \sin\rho t\,\mathrm{d}t = \frac{2}{\rho}(1-\cos\delta\pi) = \frac{2}{\rho}(1+|\cos\rho\pi|).\end{aligned}$$

So, the estimate is achieved for $h_* = k$, and the corresponding angular density is

$$\varDelta_* = \frac{1}{\pi}(\delta_0 + |\cos\rho\pi| \cdot \delta_{-\delta\pi/\rho}).$$

Thus, in each case we have constructed two $\rho$-trigonometrically convex functions $h_*$ and $h^*$, that correspond to the angular densities $\varDelta_*$ and $\varDelta^*$, such that for any $\varLambda \in \mathcal{A}_\rho$ with $\varDelta_\varLambda = \varDelta_*$ the density map $\mathcal{D}$ attains its minimum value, and for any $\varLambda \in \mathcal{A}_\rho$ with $\varDelta_\varLambda = \varDelta^*$, the map $\mathcal{D}$ attains its maximal value.

Now, for every $d \in [-1, 1]$ put

$$h_d := \max(h_*, d).$$

Then $h_d$ is $\rho$-trigonometrically convex, $h_{-1} = h_*$ and $h_1 = 1 = h^*$. The corresponding angular density is denoted by $\varDelta_d$. Then for any $\varLambda_d \in \mathcal{A}_\rho$ with $\varDelta_{\varLambda_d} = \varDelta_d$ we have

$$w(d) := \mathcal{D}(\varLambda_d) = \frac{\rho}{2\pi}\int_{-\pi}^{\pi} \widehat{h}_d(t)\,\mathrm{d}t = \frac{\rho}{2\pi}\int_{-\pi}^{\pi} \max(\widehat{h}_*(t), d)\,\mathrm{d}t.$$

The function $w$ depends continuously on $d$, hence $\mathcal{D}(\mathcal{A}_\rho)$ fills the whole interval. ∎

**8. Proof of Theorem 1.8.** The proof of this theorem uses an idea from the proof of [1, Theorem 2].

Lemma 8.1. *If $\varLambda \in \mathcal{B}_\rho$ then*

$$\sigma_U(\varLambda) = \sigma_Z(\varLambda) = 1.$$

*Proof.* Put $A := \sigma_U(\Lambda) \le \sigma_Z(\Lambda) =: B$ and $\Delta := \Delta_\Lambda$.

Recall that $B = \max_{t\in[-\pi,\pi]} \widehat{h}_\Delta(t)$, where $\widehat{h}_\Delta$ is the $\rho$-balanced modification of $h_\Delta$ defined by (1.6).

The first condition of the definition of the set $\mathcal{B}_\rho$ implies that $A \le 1$. In other words, there exists $k^* \in TC_\rho$ such that

$$\widehat{h}_\Delta(t) + k^*(t) \le \frac{A+1}{2} \le 1, \qquad \forall t \in [-\pi,\pi].$$

Suppose that $A < 1$. Then for $\varepsilon$ small enough, we have

$$\widehat{h}_\Delta(t) + k^*(t) + \varepsilon < 1, \qquad \forall t \in [-\pi,\pi],$$

which contradicts the second condition of the definition of $\mathcal{B}_\rho$. Therefore, $A = 1$.

Assume now that $k^* \in TC_\rho \setminus T_\rho$. Note that in this case the corresponding density $\Delta^*$ is non-zero. Then it follows from the second condition of the definition of $\mathcal{B}_\rho$ that

$$\max_{t\in[-\pi,\pi]} (k^*(t) + \widehat{h}_\Delta(t)) > 1,$$

and we get a contradiction.

Now, let $k^* \in T_\rho$. Since $\widehat{h}_\Delta$ is $\rho$-balanced, we obtain

$$1 = A \le B \le \max_{t\in[-\pi,\pi]} (k^*(t) + \widehat{h}_\Delta(t)) \le 1.$$

The lemma is proved. ∎

Fix $\Lambda \in \mathcal{B}_\rho$ and put $\Delta := \Delta_\Lambda$. From the assumptions of the theorem, by Lemma 8.1 and Theorem 1.5, it follows that $\widehat{h}_\Delta \in TC_\rho$ is a locally $\rho$-balanced function with $\max_{t\in[-\pi,\pi]} \widehat{h}_\Delta(t) = 1$. The upper bound follows immediately from (7.1):

$$\mathcal{D}(\Lambda) \le \rho = \frac{\rho}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}t.$$

The estimate is achieved for any $\Lambda$ with uniform angular density $\Delta^*$:

$$\Delta^*([\alpha,\beta]) = \frac{\rho}{2\pi} \cdot (\beta - \alpha), \quad 0 \le \alpha < \beta \le 2\pi.$$

The corresponding $\rho$-trigonometrically convex function is $h^*(t) = 1$.

Turning to the proof of the lower bound, let us start with the following lemma.

LEMMA 8.2. *Let $\Lambda \in \mathcal{B}_\rho$ and $\Delta_\Lambda = \Delta$. Then the set*

$$M = \{\theta : \widehat{h}_\Delta(\theta) = 1\}$$

*has no gap of length greater than $\pi/\rho$.*

*Proof.* First of all, from Lemma 8.1 it follows that $M \neq \emptyset$. Hence, for $\rho \leq 1/2$ the lemma is trivial.

Suppose that $\rho > 1/2$. We use the fact that $M$ is locally $\rho$-balanced. Assume that $J := [-\pi/(2\rho), \pi/(2\rho)] \subset \mathbb{R} \setminus M$, and put

$$m := \max_{t \in J} \widehat{h}(t) < 1.$$

Let $\Lambda_1$ be a $\rho$-regular set with angular density $\Delta_1$ that corresponds to the $\rho$-trigonometrically convex function

$$h_{\Delta_1}(t) = \begin{cases} (1-m)\cos\rho t, & |t| \leq \pi/(2\rho), \\ 0, & \pi/(2\rho) \leq |t| \leq \pi. \end{cases}$$

Consider the set $\Lambda' = \Lambda \cup \Lambda_1$. It is $\rho$-regular with density

$$\Delta' = \Delta + \Delta_1,$$

which corresponds to the $\rho$-trigonometrically convex function

$$h_{\Delta'}(t) = \widehat{h}_{\Delta}(t) + h_{\Delta_1}(t) \leq 1.$$

Moreover, we have

$$M' = \{\theta : h_{\Delta'}(t) = 1\} \supset M,$$

and hence $M'$ is locally $\rho$-balanced. Therefore, by Theorem 1.5 and Corollary 1.1, $\sigma_U(\Lambda') = \sigma_Z(\Lambda') = 1$. So, we have found a $\rho$-regular set with positive density such that $\Lambda \cup \Lambda_1$ is a zero set for $\mathcal{E}_{\rho,1}$. This contradicts the second condition of the definition of $\mathcal{B}_\rho$, concluding the proof of the lemma. ∎

Returning to the proof of the theorem, let us start with the case $\rho \leq 1/2$.

- $\rho \in (0, 1/2]$. Without loss of generality, we assume that $0 \in M$. Then, applying inequality (7.2), we get

$$\mathcal{D}(\Lambda) \geq \frac{\rho}{2\pi} \int_{-\pi}^{\pi} \widehat{h}_\Delta \,\mathrm{d}t \geq \frac{\rho}{2\pi} \int_{-\pi}^{\pi} \cos\rho t \,\mathrm{d}t = \frac{1}{\pi}\sin\pi\rho,$$

and the minimum of the density is attained for $h_*(t) = \cos\rho t$, which corresponds to the angular density $\Delta_* = \frac{\sin\pi\rho}{\pi}\delta_\pi$.

- $\rho > 1/2$. Without loss of generality we can assume that $0 \in M$. From Lemma 8.2 it follows that there exists a finite set $\Gamma := \{\gamma_k\}_{k=0}^N \subset M$, where

$$0 = \gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_N = 2\pi,$$

such that $0 \leq \gamma_k - \gamma_{k-1} \leq \pi/\rho$. Let us define a $\rho$-trigonometrically convex function $H$ as follows: for $k = 0, \ldots, N-1$ we put

$$H_\Gamma(t) = \max\bigl(\cos\rho(t-\gamma_k), \cos\rho(t-\gamma_{k+1})\bigr), \qquad \gamma_k \leq t \leq \gamma_{k+1}.$$

Then, by (7.2), we have $\widehat{h}_\Delta(t) \geq H_\Gamma(t)$.

Applying Lemma 7.2, we observe that increasing the distance between neighboring points $\gamma_k$ until it equals $\pi/\rho$ will decrease the integral of the function $H_\Gamma$. Hence, the optimal configuration that minimizes the integral is achieved for

$$\Gamma^* := \{\gamma_k^*\}_{k=0}^{N^*} \subset M,$$

where $N^* = [2\rho]$ and

$$\gamma_0^* = 0, \quad \gamma_1^* = \frac{\{2\rho\}\pi}{\rho}, \quad \gamma_{k+1}^* = \gamma_k^* + \frac{\pi}{\rho}, \quad k = 1, \dots, N^* - 1.$$

It follows that

$$\begin{aligned}\int_{-\pi}^{\pi} H_\Gamma(t)\,\mathrm{d}t &\geq \int_{-\pi}^{\pi} H_{\Gamma^*}(t)\,\mathrm{d}t \\ &= 2\int_0^{\gamma_1^*} \cos\rho t\,\mathrm{d}t + 2N^* \int_0^{\pi/2\rho} \cos\rho t\,\mathrm{d}t = \frac{2}{\rho}\left(\sin\frac{\{2\rho\}\pi}{2} + [2\rho]\right).\end{aligned}$$

Finally, we get

$$\mathcal{D}(\Lambda) \geq \frac{\rho}{2\pi}\int_{-\pi}^{\pi} \widehat{h}_\Delta\,\mathrm{d}t \geq \frac{\rho}{2\pi}\int_{-\pi}^{\pi} H_\Gamma(t)\,\mathrm{d}t \geq \frac{1}{\pi}\left(\sin\frac{\{2\rho\}\pi}{2} + [2\rho]\right).$$

The measure where the minimal value of the density is achieved is

$$\Delta_* = \frac{1}{\pi}\left(\sin\frac{\{2\rho\}\pi}{2}\delta_{\frac{\{2\rho\}\pi}{2\rho}} + \sum_{k=1}^{[2\rho]-1} \delta_{\gamma_1^* + \frac{(2k+1)\pi}{2\rho}}\right).$$

The remainder of the proof is analogous to the final part of the proof of Theorem 1.7. For every $\rho > 0$ we have constructed two angular densities, $\Delta_*$ and $\Delta^*$, such that for any $\Lambda \in \mathcal{B}_\rho$ with $\Delta_\Lambda = \Delta_*$ the density map $\mathcal{D}$ attains its minimum value on $\mathcal{B}_\rho$, and for any $\Lambda \in \mathcal{B}_\rho$ with $\Delta_\Lambda = \Delta^*$ the map $\mathcal{D}$ attains its maximal value. The corresponding $\rho$-trigonometrically convex functions will be denoted by $h_*$ and $h^*$.

Put

$$h_d := \max(h_*, d).$$

Then $h_d$ is $\rho$-trigonometrically convex, $h_{-1} = h_*$ and $h_1 = 1 = h^*$. The corresponding angular density will be denoted by $\Delta_d$. Then for any $\Lambda_d \in \mathcal{B}_\rho$ with $\Delta_{\Lambda_d} = \Delta_d$ we have

$$w(d) := \mathcal{D}(\Lambda_d) = \frac{\rho}{2\pi}\int_{-\pi}^{\pi} \widehat{h}_d(t)\,\mathrm{d}t = \frac{\rho}{2\pi}\int_{-\pi}^{\pi} \max(\widehat{h}_*(t), d)\,\mathrm{d}t.$$

The function $w$ depends continuously on $d$, hence $\mathcal{D}(\mathcal{B}_\rho)$ is the interval $[\mathcal{D}(\Lambda_*), \mathcal{D}(\Lambda^*)]$.

**9. Application to random zero sets in Fock-type spaces.** An entire function $f$ belongs to the *Fock-type space* $\mathcal{F}_\rho$, $\rho > 0$, if

$$\|f\|_\rho^2 := \int_{\mathbb{C}} (|f(z)|e^{-|z|^\rho})^2 \, \mathrm{d}m(z) < \infty.$$

Note that

$$\bigcup_{\sigma<1} \mathcal{E}_{\rho,\sigma} \subset \mathcal{F}_\rho \subset \mathcal{E}_{\rho,1}.$$

The question we are concerned with in this section is whether or not a kind of randomization of a given sequence of points is almost surely a zero set (or a uniqueness set) of the Fock-type space $\mathcal{F}_\rho$.

Let us fix a non-decreasing positive sequence

$$\Lambda_{\mathbb{R}} = \{l_k\}_{k\in\mathbb{N}},$$

and suppose that it has a finite density

$$\mathcal{D} = \lim_{R\to\infty} \frac{n_{\Lambda_{\mathbb{R}}}(R)}{R^\rho} < \infty.$$

To determine the type of randomization of $\Lambda_{\mathbb{R}}$, we fix a probability measure $\Delta$ on $[0, 2\pi)$, and for $\rho \in \mathbb{N}$ we require in addition that it has zero $\rho$th moment,

$$\int_0^{2\pi} e^{i\rho t} \, d\Delta(t) = 0. \tag{9.1}$$

We define a *$\Delta$-randomization* of $\Lambda_{\mathbb{R}}$ to be a random set

$$\widetilde{\Lambda}_\Delta := \{\lambda_k = l_k e^{i\theta_k} : l_k \in \Lambda_{\mathbb{R}}\},$$

where $\theta_k$ are independent random variables equally distributed with distribution $\Delta$.

The case of uniform randomization was considered earlier in [8, 10].

Note that each of the events "$\widetilde{\Lambda}_\Delta$ is a zero set of $\mathcal{F}_\rho$" and "$\widetilde{\Lambda}_\Delta$ is a uniqueness set of $\mathcal{F}_\rho$" is a tail event, and so by the Kolmogorov zero-one law each of them either happens almost surely or does not happen almost surely, for a fixed sequence $\Lambda_{\mathbb{R}}$.

Let us show that the random set $\widetilde{\Lambda}_\Delta$ almost surely satisfies the Lindelöf condition (1.1). Indeed, first of all, due to condition (9.1), $\mathbb{E}(\lambda_k^{-\rho}) = 0$ for each $k$. Furthermore,

$$\sum_{\lambda_k\in\widetilde{\Lambda}_\Delta} \mathrm{Var}(\lambda_k^{-\rho}) = \sum_{l_k\in\Lambda_{\mathbb{R}}} l_k^{-2\rho} < \infty,$$

hence, by the Khinchin–Kolmogorov theorem [5, Theorem 22.6] (also known as Kolmogorov's two-series theorem), the series $\sum \lambda_k^{-\rho}$ converges almost surely. Thus, for the random set $\widetilde{\Lambda}_{\Delta}$ the Lindelöf condition (1.1) holds almost surely.

Our next step is to show that $\widetilde{\Lambda}_{\Delta}$ almost surely has angular density.

LEMMA 9.1. *The random sequence $\widetilde{\Lambda}_{\Delta}$ has angular density $\mathcal{D}\Delta$ almost surely.*

*Proof.* Even though this lemma may seem obvious to specialists, we nevertheless prefer to prove it, relying on the following generalization of the Glivenko–Cantelli theorem, which is due to Varadarajan.

VARADARAJAN'S THEOREM ([13], [7, Theorem 11.4.1]). *Given a separable metric space $(S,d)$ with a Borel $\sigma$-algebra $\Sigma \subset 2^S$ and a probability measure $\mu : \Sigma \to [0,1]$, define the probability space $(\Omega, \mathbb{P})$, where $\Omega := S^{\mathbb{N}}$ and $\mathbb{P}$ is the corresponding infinite product probability* (*uniquely defined,* [7, Theorem 8.2.2]). *For $\omega = (\xi_1^\omega, \xi_2^\omega, \ldots) \in \Omega$ define an empirical distribution $\mu_n^\omega$ on the $\sigma$-algebra $\Sigma$ by*

$$\mu_n^\omega := \frac{1}{n}(\delta_{\xi_1^\omega} + \cdots + \delta_{\xi_n^\omega}).$$

*Then $\mu_n^\omega$ weakly converges to $\mu$ almost surely:*

$$\mathbb{P}(\{\omega \in S^{\mathbb{N}} : \mu_n^\omega \xrightarrow{weak} \mu\}) = 1.$$

The weak convergence of the empirical distributions $\mu_n^\omega$ to $\mu$ implies that for all arcs $(\alpha,\beta)$ such that $\mu(\{\alpha\}) = \mu(\{\beta\}) = 0$, we have

$$\lim_{n\to\infty} \mu_n^\omega(\alpha,\beta) = \mu(\alpha,\beta),$$

where $\mu$ is some probability measure on the metric space $S$, and $\mu_n^\omega$ are the corresponding empirical measures.

Now, if we take $S = [0,2\pi)$, $\mu = \Delta$, and $\xi_k = \theta_k$, then for all $\alpha,\beta \in [0,2\pi)$, and for any $R \in (\lambda_n, \lambda_{n+1}]$,

$$\mu_n^\omega(\alpha,\beta) = \frac{n_\Lambda(R;\alpha,\beta)}{n} = \frac{n_\Lambda(R;\alpha,\beta)}{n_{\Lambda_{\mathbb{R}}}(R)}.$$

Hence, with probability 1, for all but a countable set of angles,

$$\lim_{R\to\infty} \frac{n_\Lambda(R;\alpha,\beta)}{R^\rho} = \lim_{R\to\infty} \frac{\mu_{n(R)}^\omega(\alpha,\beta)\, n_{\Lambda_{\mathbb{R}}}(R)}{R^\rho} = \mathcal{D}\cdot\Delta(\alpha,\beta). \ \blacksquare$$

It follows that a random sequence $\widetilde{\Lambda}_{\Delta}$ is almost surely $\rho$-regular, hence the results of the previous sections can be applied. Thus, all theorems lead to corollaries that hold almost surely for $\widetilde{\Lambda}_{\Delta}$. In particular, the following holds.

COROLLARY 9.1. *Let $\rho \in \mathbb{N}$, and let a probability measure $\Delta$ on $[0,2\pi)$ have zero $\rho$th moment. Given a positive sequence $\Lambda_{\mathbb{R}}$ with density $\mathcal{D}$, its randomization $\widetilde{\Lambda}_{\Delta}$ is almost surely*

- *a zero set of $\mathcal{F}_\rho$ if $\mathcal{D}R^*_\Delta < 1$;*
- *not a zero set of $\mathcal{F}_\rho$ if $\mathcal{D}R^*_\Delta > 1$.*

The critical case $\mathcal{D}R^*_\Delta = 1$ here is more subtle and requires a special consideration. For uniform randomization, some examples of a.s. zero sets of critical density and a.s. uniqueness sets of critical density for $\mathcal{F}_2$ can be found in [10].

Now, shifting focus to the difference between $\sigma_U$ and $\sigma_Z$ and referring to Theorem 1.4 and Examples 2–4, we arrive at the following corollary.

COROLLARY 9.2. *For every $\rho \in (1/2, \infty) \setminus \{1\}$, there exists a probability measure $\Delta$ satisfying condition* (1.2)*, and a non-empty open interval $J_{\rho,\Delta}$ such that for any positive sequence $\Lambda_\mathbb{R}$ with density $\mathcal{D} \in J_{\rho,\Delta}$, its randomization $\widetilde{\Lambda}_\Delta$ is almost surely a non-zero set and a non-uniqueness set for the Fock space $\mathcal{F}_\rho$.*

EXAMPLE 7. Put $\Lambda_\mathbb{R} = \{\sqrt{n/D}\}$, and let $\Delta = \frac{1}{3}\Delta_3$, where $\Delta_3$ is defined as in Example 3. Then the random set $\widetilde{\Lambda}_\Delta$ is almost surely

- a zero set of $\mathcal{F}_2$ if $D < 3\sqrt{3}/(2\pi)$;
- a non-zero and non-uniqueness set of $\mathcal{F}_2$ if $3\sqrt{3}/(2\pi) < D < 3/\pi$;
- a uniqueness set of $\mathcal{F}_2$ if $D > 3/\pi$.

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