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# A small Radon–Nikodým compact space from a parametrized diamond

by

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**Abstract.** A compact space $K$ is *Radon–Nikodým* if there is a lower semicontinuous metric fragmenting $K$. In this note, we show that, under $\diamondsuit(\mathrm{non}(\mathcal{M}))$, there is a Radon–Nikodým compact space of weight $\aleph_1$ with a continuous image that is not Radon–Nikodým, which partially answers a question posed by Avilés and Koszmider in 2013.

**1. Introduction.** The study of Radon–Nikodým compact spaces can be traced back to [Rey82], where they are introduced as a generalization of Eberlein compact spaces. These spaces were originally defined as weak$^*$ compact subspaces $K \subseteq X^*$, where every Radon probability measure on $K$ is supported on a countable union of norm compact sets. This definition has been studied extensively, and several connections have been found in Banach space theory and topology. A summary of some of these can be found in [Fab06].

Recall that metric $d$ *fragments* $K$ if for every $\varepsilon > 0$ and every non-empty $C \subseteq K$, there is an open set $U \subseteq K$ such that $U \cap C \neq \emptyset$ and $\mathrm{diam}(U \cap C) < \varepsilon$, and $d$ is *lower semicontinuous* if it is lower semicontinuous as a function on $K \times K$. In [Nam87], Namioka proved that a compact space $K$ is Radon–Nikodým if and only if there is a lower semicontinuous metric that fragments $K$. In the same work, the author asked whether the class of Radon–Nikodým compact spaces is closed under continuous functions. This question will be the central point of our work.

In [Arh96, p. 104], a weaker notion of fragmentability is considered, attributed to Reznichenko: A compact space $K$ is *quasi-Radon–Nikodým* (orig-

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inally called strongly fragmentable) if there is a metric $d$ that fragments $K$, and for any distinct $x, y \in K$, there are open sets $U, V$ such that $x \in U$, $y \in V$ and $d(U, V) > 0$. Clearly, all Radon–Nikodým compact spaces are quasi-Radon–Nikodým compact spaces, and, by the result of Namioka [Nam02], the class of quasi-Radon–Nikodým compact spaces is closed under continuous images.

Several results relating quasi-Radon–Nikodým and Radon–Nikodým compact spaces can be found in the literature: For example, Arvanitakis [Arv02] proved that totally disconnected quasi-Radon–Nikodým compact spaces are Radon–Nikodým, and Avilés [Avi05] proved that quasi-Radon–Nikodým spaces of weight less than $\mathfrak{b}$ are Radon–Nikodým. The reader can consult [AK10] for other related questions and results.

In 2013, Avilés and Koszmider answered Namioka's question in the negative.

Theorem 1.1 ([AK13]). *There are compact spaces $\mathbb{L}_0, \mathbb{L}_1$ of weight $\mathfrak{c}$ and a continuous surjection $\pi : \mathbb{L}_0 \to \mathbb{L}_1$ such that $\mathbb{L}_0$ is a Radon–Nikodým compact space, and $\mathbb{L}_1$ is not.*

In this work, Avilés and Koszmider asked about the possible weights of such spaces. In particular, they asked if $\mathfrak{b}$ is the smallest weight of these spaces, since, by Avilés' theorem of [Avi05], the weight of such spaces cannot be smaller than $\mathfrak{b}$, thus, under $\mathfrak{b} = \mathfrak{c}$, there are no such spaces of weight less than $\mathfrak{c}$. The purpose of this note is to provide a construction of such spaces of weight $\aleph_1$, under the guessing principle $\diamondsuit(\mathrm{non}(\mathcal{M}))$, which is consistent with the negation of the continuum hypothesis.

The guessing principles that we use in this work can be described using Borel relational systems. These relational systems were originally considered by Vojtáš [Voj92], and have been used as a tool to describe and compare many cardinal characteristics (for example, see [Bla10]): A *Borel relational system* is a triple $\mathbf{A} = \langle A_-, A_+, A\rangle$ where $A_-$ and $A_+$ are Borel subsets of a Polish space and $A \subseteq A_- \times A_+$ is a Borel relation. The *norm* of a relational system $\mathbf{A} = \langle A_-, A_+, A\rangle$ is the smallest cardinality of a set $D \subseteq A_+$ such that for every $a \in A_-$ there is a $d \in D$ such that $\langle a, d\rangle \in A$. Many of the most well-known cardinal invariants of the continuum can be stated as the norm of a relational system: For example, $\mathrm{non}(\mathcal{M})$ can be stated as the norm of $\mathbf{M} = \langle \mathcal{M}, \omega^\omega, \not\ni\rangle$, where $\mathcal{M}$ is the collection of all $F_\sigma$ meager sets of $\omega^\omega$, and $\mathfrak{b}$ can be stated as the norm of $\mathbf{B} = \langle \omega^\omega, \omega^\omega, \not\leq^*\rangle$. The reader can easily verify that both $\mathbf{M}$ and $\mathbf{B}$ are Borel relational systems.

In [MHD04], the authors introduced a plethora of combinatorial principles similar to $\diamondsuit$, called *parametrized diamond principles*: Given a Borel relational system $\mathbf{A} = \langle A_-, A_+, A\rangle$, the principle $\diamondsuit(\mathbf{A})$ is the statement that for every Borel $F : 2^{<\omega_1} \to A_-$ there is a function $g : \omega_1 \to A_+$ such that, for any $\gamma \in 2^{\omega_1}$, the set $\{\alpha \in \omega_1 : \langle g(\alpha), F(\gamma\restriction\alpha)\rangle \in A\}$ is stationary in $\omega_1$. One can easily verify that $\diamondsuit(\mathbf{A})$ is a consequence of $\diamondsuit$ and that $\diamondsuit(\mathbf{A})$ implies that the norm of $\mathbf{A}$ is at most $\aleph_1$. Here, a function $F$ with domain $2^{<\omega_1}$ is *Borel* if for each $\alpha < \omega_1$, the restrictions $F\restriction 2^\alpha$ are Borel functions between Polish spaces.

These principles have been used extensively, not only for set-theoretical purposes, but also in topology. Some applications of different kinds of parametrized diamonds can be found in [GA$^+$18, CM$^+$14, RST24, MDS10].

In our work, we will be using mostly $\diamondsuit(\mathbf{M})$, which is known as $\diamondsuit(\mathrm{non}(\mathcal{M}))$, and we will be using it in the following form:

DEFINITION 1.2. The *diamond of* $\mathrm{non}(\mathcal{M})$, denoted by $\diamondsuit(\mathrm{non}(\mathcal{M}))$, is the following combinatorial principle:

For every Borel $F : 2^{<\omega_1} \to \mathcal{M}$ there is a sequence $\Delta : \omega_1 \to (2^{<\omega})^{2^{<\omega}} \times \omega^{2^{<\omega}\times\omega}$ such that for any $\gamma \in 2^{\omega_1}$, the set $\{\alpha \in \omega_1 : \Delta(\alpha) \notin F(\gamma\restriction\alpha)\}$ is stationary in $\omega_1$.

Here, $\mathcal{M}$ is the collection of all $F_\sigma$ meager subsets in the range of $\Delta$. Clearly, this principle is equivalent to the principle $\diamondsuit(\mathbf{M})$ defined above.

The relationship between different parametrized diamonds and their consistency is widely discussed in [MHD04]. In particular, $\diamondsuit(\mathrm{non}(\mathcal{M}))$ follows from the classical $\diamondsuit$, but it also holds in Sacks' or Millers' model (see [BJ95, Chapter 7]), as both Sacks and Miller forcing fall into the scope of [MHD04, Theorem 6.6.]. This principle is also consistent with arbitrarily large continuum, by [MHD04, Theorem 6.1.] and the fact that the classical forcing to add $\kappa$ Cohen reals $\mathbb{C}_\kappa$ is forcing equivalent to the iteration $\mathbb{C}_\kappa * \mathbb{C}_{\omega_1}$.

The purpose of this note is to prove the following theorem.

THEOREM 3.2. *Under* $\diamondsuit(\mathrm{non}(\mathcal{M}))$ *there is a Radon–Nikodým compact space of weight* $\aleph_1$*, and a continuous function on this space whose image is not Radon–Nikodým.*

Therefore, it is consistent to have a Radon–Nikodým compact space of small weight, with a continuous image that is not Radon–Nikodým, for example, in any of the models mentioned above.

This paper is organized in the following way: Section 2 is devoted to briefly introducing the notion of a basic$^*$ space and some results that were originally proven in [AK13] which will be needed to explain why the existence of a basic$^*$ space implies our main theorem. Section 3 is the core of our construction and is dedicated to constructing a basic$^*$ space from $\diamondsuit(\mathrm{non}(\mathcal{M}))$, ending with a short discussion of the limitations of this method and some possible improvements.

Our notation is standard and mostly follows [AK16] and [Jec03]: If $X$ is a set, then $|X|$ denotes the cardinality of $X$. The ordinal $\omega = \{0, 1, \ldots\}$

is the first infinite cardinal, $\omega_1$ is the first uncountable ordinal, and $\mathfrak{c}$ is the cardinality of $\mathbb{R}$. If $X, Y$ are sets, then $X^Y$ is the collection of all functions $f : Y \to X$. In particular, if $n \in \omega$, then $2^n$ is the collection of all functions $s : n \to 2 = \{0, 1\}$, and $2^{<\omega} = \bigcup_{n\in\omega} 2^n$. If $f$ is a function and $A \subseteq \mathrm{dom}(f)$, then $f{\restriction}A$ is the usual restriction of $f$ to the set $A$. If $X$ is a topological space, then $w(X)$ is the *weight* of $X$, that is, the smallest cardinality of a basis for the topology of $X$. If $s, t \in 2^{<\omega}$, then $s^\frown t$ denotes the usual concatenation of sequences. The quantifiers $\forall^\infty n \in \omega$ and $\exists^\infty n \in \omega$ denote the quantifiers representing "for every, but finitely many $n \in \omega$", and "there are infinitely many $n \in \omega$", respectively.

**2. Constructing Radon–Nikodým compact spaces from a basic$^*$ space.** In [AK13], the authors construct a Radon–Nikodým space $\mathbb{L}_0$ and a continuous function $\pi$ such that $\pi[\mathbb{L}_0] = \mathbb{L}_1$ is not Radon–Nikodým. In that work, both $\mathbb{L}_0$ and $\mathbb{L}_1$ are constructed following a recipe that requires what they call a basic space as an ingredient. It turns out that the existence of such a basic space of weight $\aleph_1$ implies the continuum hypothesis. Here, we will follow the same recipe, modifying the notion of basic space, allowing us to work without the need of the continuum hypothesis. The main purpose of this section is to show that this modification is tame enough to allow us to follow the recipe with minimal modifications and to see that the spaces have the proper weight.

Definition 2.1. A topological Hausdorff space $K = A \cup B \cup \{c\}$ is *pre-basic* if

(1) $A = \dot{\bigcup}_{t\in 2^{<\omega}} A_t$ and each $A_t$ and $B$ are infinite sets,
(2) $A$ consists of isolated points,
(3) for each $b \in B$ there is a countable set $C_b \subseteq A$ such that $\{(C_b \setminus F) \cup \{b\} : F \text{ finite}\}$ is a local basis of $b$,
(4) $K$ is the one-point compactification of $A \cup B$.

Additionally, if

(5) there is a function $D : B \to (2^{<\omega})^{2^{<\omega}}$ such that for every function $s : A \to 2^{<\omega}$ there is $b \in B$ such that there are infinitely many $n \in \omega$ such that for every $t \in 2^n$ the set
$$\{a \in A_t \cap C_b : D(b)(t) = s(a)\}$$
is infinite,

then $K$ is a *basic*$^*$ space.

Clearly, the first three conditions imply that the space is locally compact, so condition (4) is attainable. In fact, we have the following easy properties:

FACT 2.2. *Assume that $A \cup B$ satisfies conditions* (1)–(3). *Then*

- *for all $b, d \in B$ such that $b \neq d$, $C_b \cap C_d$ is finite,*
- *for all $b \in B$, $C_b \cup \{b\}$ is compact. Moreover, $\overline{C_b} = C_b \cup \{b\}$ is closed and open in both $A \cup B$ and $K$.*

*Proof.* The first point follows from the fact that the space is Hausdorff and from condition (3). The second one follows easily from (2) and (3). ∎

An easy consequence of this is that $K$ is a zero-dimensional space. The weight of these spaces is also easy to calculate.

LEMMA 2.3. *If $K$ is a pre-basic space, then $w(K) = |K|$.*

*Proof.* Since both $A$ and $B$ are discrete in their respective subspace topologies, we get

$$w(K) \geq \max\{w(A), w(B)\} = \max\{|A|, |B|\} = |K|.$$

For the other inequality, we only have to show that $A \cup B$ has a basis of size $|K|$. Notice that any set that contains an infinite subset of $B$ is not compact in $A \cup B$ and therefore any compact set in $A \cup B$ is contained in a set of the form $B_0 \cup \{C_b : b \in B_0\} \cup A_0$ for some finite $B_0 \subseteq B, A_0 \subseteq A$. Thus $\{K \setminus (B_0 \cup \{C_b : b \in B_0\} \cup A_0) : B_0 \subseteq B, A_0 \subseteq A \text{ finite}\}$ is a basis of neighborhoods of $c$ of size $|K|$ (the basis for other points is given by (2) and (3)). ∎

The only difference between a basic* space and the notion of basic space defined in [AK13, Section 3] is condition (5). In that work, they instead require the following property:

- There exists a function $\Psi : B \to \omega^{2^{<\omega}}$ such that, given any family $\{X^t_m : m \in \omega,\ t \in 2^{<\omega}\}$ of subsets such that, for every $t \in 2^{<\omega}$, $A_t = \bigcup_{m\in\omega} X^t_m$, there is an $\alpha \in B$ such that, for all $t \in 2^{<\omega}$, $C_\alpha \cap X^t_{\Psi(\alpha)(t)}$ is infinite.

Observe that $\Psi$ has to be onto: If $f \in \omega^{2^{<\omega}}$, and if we consider the family $\{X(f)^t_m : t \in 2^{<\omega}, m \in \omega\}$ where $X(f)^t_{f(t)} = A_t$, and $X(f)^t_m = \emptyset$ for any other $m \neq f(t)$, then there must be $b \in B$ such that, for all $t \in 2^{<\omega}$, $C_b \cap X(f)^t_{\Psi(b)(t)}$ is infinite. But then $C_b \subseteq X(f)^t_{\Psi(b)(t)}$ and therefore $f(t) = \Psi(b)(t)$ for all $t \in 2^{<\omega}$. This means that it is impossible to construct basic spaces of cardinality smaller than $\mathfrak{c}$.

The next step of the recipe is described as follows: Given a pre-basic space $K = A \cup B \cup \{c\}$, let

$$L = (A \times 2^\omega) \cup B \cup \{c\}.$$

To define the topology on $L$, pick a basis $\mathcal{U}$ of $K$. Then the collection

$$\{\{a\} \times [s] : a \in A,\ s \in 2^{<\omega}\} \cup \{((U \cap A) \times 2^\omega) \cup U \setminus A : U \in \mathcal{U}\}$$

forms a basis of the topology of $L$.

Lemma 2.4. *$L$ is a zero-dimensional compact space and $w(L) = |K|$.*

*Proof.* It is routine to check that $L$ is the one-point compactification of $(A \times 2^\omega) \cup B$, and that it is zero-dimensional. For the other part, notice that the size of the basis of $L$ depends only on the size of $\mathcal{U}$, so it follows directly from Lemma 2.3. ∎

For the next step of the recipe, we need to recall the lexicographic order: If $x, y \in 2^\omega \cup 2^{<\omega}$ and $x \neq y$, then $x <_{\mathrm{lex}} y$ if $x(i) = 0$, $y(i) = 1$ and $i = \min\{n \in \omega : x(n) \neq y(n)\}$.

For $s, t \in 2^{<\omega}$, define $\Gamma^s_t : 2^\omega \to 2^\omega$ by

$$\Gamma^s_t(\sigma) = \begin{cases} t^\frown(0,0,\dots) & \text{if } \sigma <_{\mathrm{lex}} s^\frown(0,0,\dots), \\ t^\frown\lambda & \text{if } \sigma = s^\frown\lambda, \\ t^\frown(1,1,\dots) & \text{if } \sigma >_{\mathrm{lex}} s^\frown(1,1,\dots). \end{cases}$$

Note that $\Gamma^s_t$ is a continuous function. Let $q : 2^\omega \to [0,1]$ be the standard continuous surjection given by calculating the value of a sequence as a base 2 number, that is, $q(x) = \sum_{i \in x^{-1}(1)} 2^{-(i+1)}$. From now on, assume that we have a pre-basic space $K = A \cup B \cup \{c\}$ and some arbitrary function $D : B \to (2^{<\omega})^{2^{<\omega}}$.

For $b \in B$ define $g_b : (L \setminus \{b\}) \to 2^\omega$ as follows:

- $g_b(x) = (0,0,0,\dots)$ if $x \notin C_b \times 2^\omega$, $x \neq b$,
- $g_b(a,\sigma) = \Gamma^{D(b)(t)}_t(\sigma)$ for $a \in A_t \cap C_b$, $\sigma \in 2^\omega$,

and let $h_b : (L \setminus \{b\}) \to [0,1]$ be defined by $h_b = q \circ g_b$.

Let

$$\mathbb{L}_0 = \{(u,v) \in L \times (2^\omega)^B : \forall_{b \in B} \text{ if } b \neq u \text{ then } v(b) = g_b(u)\},$$
$$\mathbb{L}_1 = \{(u,v) \in L \times [0,1]^B : \forall_{b \in B} \text{ if } b \neq u \text{ then } v(b) = h_b(u)\},$$

and equip $\mathbb{L}_0, \mathbb{L}_1$ with the subspace topology derived from $L \times (2^\omega)^B$ and $L \times [0,1]^B$, respectively. Clearly, $\mathbb{L}_1$ is an image of $\mathbb{L}_0$, by the continuous function $\pi$ that applies $q$ on each coordinate of $(2^\omega)^B$.

Lemma 2.5. $w(\mathbb{L}_0) = w(\mathbb{L}_1) = |K|$.

*Proof.* This follows from Lemma 2.4 and from the fact that $w((2^\omega)^B) = w([0,1]^B) = |B|$. ∎

To finish the recipe, we only need to show that $\mathbb{L}_0$ is Radon–Nikodým and that $\mathbb{L}_1$ is not Radon–Nikodým.

Theorem 2.6. *If $K$ is pre-basic and $D$ is arbitrary, then $\mathbb{L}_0$ is a Radon–Nikodým compact space.*

*Proof.* It is immediate to see that $\mathbb{L}_0$ is zero-dimensional, so by [Arv02, Theorem 3.6] we only need to check that it is quasi-Radon–Nikodým. The proof is exactly the same as the proof of [AK13, Proposition 4.1] as it depends only on the continuity of the functions $\Gamma_t^s$. ■

THEOREM 2.7. *If $K$ is basic* and $D$ witnesses that $K$ satisfies the condition* (5)*, then $\mathbb{L}_1$ is not a Radon–Nikodým compact space.*

*Proof.* The proof is almost identical to that of [AK13, Proposition 4.3], with a small modification to take into account the different property of the function $D$. First, notice that, for $u \in L$ such that $u = \langle a, x\rangle \in A \times 2^\omega$, there is a single function $v \in [0,1]^B$ (in this case, the function $v(b) = h_b(u)$) such that $\langle u, v\rangle \in \mathbb{L}_1$. Such a $\langle u, v\rangle$ will be denoted by $a + x$. Let $\vec{0}$ and $\vec{1}$ be the constant functions 0 and 1, respectively.

Aiming at a contradiction, assume that $\mathbb{L}_1$ is Radon–Nikodým, so there must be a lower semicontinuous metric $\delta$ fragmenting $\mathbb{L}_1$. For each $t \in 2^{<\omega}$ and each $a \in A_t$, consider the set $\{a + x : x \in [t]\}$, and apply the fragmentability of $\mathbb{L}_1$ to this set to find $s(a) \in 2^{<\omega}$ extending $t$ such that

$$(*)\qquad \delta(a + s(a)^\frown\vec{0}, a + s(a)^\frown\vec{1}) < 1/4^{|t|}.$$

Then, according to condition (5), there must be some $b \in B$ such that there is an infinite set $W \subseteq \omega$ such that for every $n \in W$ and every $t \in 2^n$, the set $\{a \in A_t \cap C_b : D(b)(t) = s(a)\}$ is infinite. Enumerate this set as $\{a_k^t : k \in \omega\}$.

For every $\xi \in [0,1]$ there is a unique function $v : B \to [0,1]$ such that $v(b) = \xi$ and $\langle b, v\rangle \in \mathbb{L}_1$ (note that for $b' \neq b$, $v(b') = h_b(b')$). Such a pair $\langle b, v\rangle$ will be denoted by $b \oplus \xi$. For $t \in 2^{<\omega}$, define $t^0 = q(t^\frown\vec{0})$ and $t^1 = q(t^\frown\vec{0})$.

Note that, for such $a_n^t$, and since $s(a) = D(b)(t)$ extends $t$, we have

$$\begin{aligned} h_b(\langle a_n^t, D(b)(t)^\frown\vec{0}\rangle) &= q(g_b(\langle a_n^t, D(b)(t)^\frown\vec{0}\rangle)) \\ &= q(\Gamma_t^{D(b)(t)}(D(b)(t)^\frown\vec{0})) = t^0. \end{aligned}$$

Similarly,

$$h_b(\langle a_n^t, D(b)(t)^\frown\vec{1}\rangle) = t^1.$$

Now, observe that, for every sequence $\{a_n^t : n \in \omega\}$ as before, $\{a_n^t\} \to b$ as $n \to \infty$, and therefore $a_n^t + D(b)(t)^\frown\vec{0} \to b \oplus t^0$ and $a_n^t + D(b)(t)^\frown\vec{1} \to b \oplus t^1$.

Using this, $(*)$, and the lower semicontinuity of $\delta$ we get

$$(**)\qquad \delta(b \oplus t^0, b \oplus t^1) \leq 1/4^{|t|}$$

Finally, notice that for every $n \in \omega$, there is a way to order $2^n = \{t_j : 1 \leq j \leq 2^{|n|}\}$ so that $t_0$ is the sequence of zeroes, $t_{2^{|n|}}$ is the sequence of ones, and for $j < 2^{|n|}$, $t_j^1 = t_{j+1}^0$. Using this, $(**)$, and the triangle inequality, we get, for all $n \in W$,

$$\delta(b \oplus \vec{0}, b \oplus \vec{1}) \leq 2^n/4^n = 1/2^n.$$

Since this happens for infinitely many $n$, we get $\delta(b\oplus\vec{0}, b\oplus\vec{1}) = 0$, which is the contradiction we were looking for, as $\delta$ is a metric and $\delta(b\oplus\vec{0}, b\oplus\vec{1}) > 0$. ∎

This finishes the recipe, so all that is left is to construct a basic* space of cardinality smaller than $\mathfrak{c}$.

**3. Constructing a small basic* space.** This section is the main part of this work, and it will be solely dedicated to constructing a basic* space of small weight, under $\diamondsuit(\mathrm{non}(\mathcal{M}))$. The important part of the construction is to show how the sequence sets $\langle C_b : b \in B\rangle$ and the function $D$ from condition (5) are obtained. Note that, for fixed $A, B$, the sequence $\langle C_b : b \in B\rangle$ and the function $D$ define a basic* space exactly if condition (5) is satisfied and $\langle C_b : b \in B\rangle$ is an *almost-disjoint family*, i.e., for any $b \neq d$, the set $C_b \cap C_d$ is finite.

THEOREM 3.1. *Under* $\diamondsuit(\mathrm{non}(\mathcal{M}))$ *there is a basic** *space of size* $\aleph_1$.

*Proof.* Let $B$ be the set of ordinals of $\omega_1$ that are limits of limit ordinals, and for each $t \in 2^{<\omega}$ let $A_t = \{t\} \times \omega_1$. Fix bijective $\varepsilon_\beta : \omega \to \lim(\omega_1) \cap \beta$ and $e_\beta : \omega \to \beta$ for each $\beta \in B$ and let $\pi_1, \pi_2$ be the projections on the first and second coordinates, respectively.

First, we are going to construct a suitable function $F$ for $\diamondsuit(\mathrm{non}(\mathcal{M}))$.

Let $\Gamma$ be the set of triples $(\langle C_\alpha : \alpha < \beta,\ \alpha \in \lim(\omega_1)\rangle, \langle G_t : t \in 2^{<\omega}\rangle, H)$ with $\beta \in B$, $H \in (2^{<\omega})^{2^{<\omega}}$, $C_\alpha \subseteq 2^{<\omega} \times \alpha$ and $G_t \subseteq \{t\} \times \beta$. The *length* of $\gamma \in \Gamma$ is the ordinal $\beta$ as before. Given $\gamma \in \Gamma$ of length $\beta$ and $\delta \in B \cap \beta$ let $\gamma\restriction\delta = (\langle C_\alpha : \alpha < \delta,\ \alpha \in \lim(\omega_1)\rangle, \langle G_t \cap (2^{<\omega} \times \delta) : t \in 2^{<\omega}\rangle, H)$. An encoding $E : \Gamma \to 2^{<\omega_1}$ *preserves restriction* if for any $\gamma \in \Gamma$ of length $\beta \in B$ and any $\delta \in B \cap \beta$ we have $E(\gamma) \in 2^\beta$ and $E(\gamma\restriction\delta) = E(\gamma)\restriction\delta$. Such an encoding can be constructed inductively by sending $\langle C_\alpha : \beta \leq \alpha < \beta + \omega^2, \alpha \in \lim(\omega_1)\rangle$ and $\langle G_t \cap (2^{<\omega} \times [\beta, \beta + \omega^2) : t \in 2^{<\omega}\rangle$ into $2^{[\beta,\beta+\omega^2)}$. We leave the details of the construction to the reader and from now on we will implicitly use a fixed, restriction-preserving Borel encoding, thus assuming that the domain of $F$ is $\Gamma$.

We will say that $\gamma \in \Gamma$ is *suitable* if for any $\alpha, \alpha' \in \lim(\omega_1) \cap \beta$ with $\alpha \neq \alpha'$ and each $t \in 2^{<\omega}$,

(a) $C_\alpha \cap A_t$ is infinite,
(b) $\bigcup_{\alpha<\beta} C_\alpha = 2^{<\omega} \times \beta$ ,
(c) $G_t$ is cofinal in $\{t\} \times \beta$ (meaning that $\pi_2(G_t)$ is cofinal in $\beta$),
(d) $C_\alpha \cap C_{\alpha'}$ is finite.

For such suitable sequences, define the sets

$$\widehat{C}_n = C_{\varepsilon_\beta(n)} \setminus \bigcup_{k<n} C_{\varepsilon_\beta(k)} = C_{\varepsilon_\beta(n)} \setminus \bigcup_{k<n} \widehat{C}_k.$$

Note that, for each $t \in 2^{<\omega}$, the family $\{\widehat{C}_n \cap G_t : n \in \omega\}$ is a partition of $G_t$. Then define $f_t(n) = \min e_\beta^{-1}(\pi_2(\widehat{C}_n \cap G_t))$ whenever possible (otherwise $n \notin \mathrm{dom}(f_t)$). The properties (b) and (c) guarantee that each $f_t$ is an infinite partial function.

Let

$$\begin{aligned} F(\gamma) = \{\langle x, y\rangle \in (2^{<\omega})^{2^{<\omega}} \times \omega^{\,2^{<\omega}\times\omega} : \forall^\infty n \in \omega\ \exists t \in 2^n, \\ \text{either } H(t) \neq x(t) \text{ or } \forall^\infty \ell \in \mathrm{dom}(f_t)\,(f_t(\ell) \geq y(t,\ell))\}. \end{aligned}$$

It is routine to check that the following two sets are $F_\sigma$ meager, respectively in $(2^{<\omega})^{2^{<\omega}}$ and $\omega^{\,2^{<\omega}\times\omega}$: $J_\gamma = \{x \in (2^{<\omega})^{2^{<\omega}} : \forall^\infty n \in \omega\ \exists t \in 2^n, H(t) \neq x(t)\}$ and $I_\gamma = \{y \in \omega^{\,2^{<\omega}\times\omega} : \exists t \in 2^{<\omega}\ \forall^\infty \ell \in \mathrm{dom}(f_t),\ f_t(\ell) \geq y(t,\ell)\}$. Now, if $\langle x, y\rangle \in F(\gamma)$ then $x \in J_\gamma$ or $y \in I_\gamma$, hence $F(\gamma)$ is meager in $(2^{<\omega})^{2^{<\omega}} \times \omega^{2^{<\omega}\times\omega}$.

For a non-suitable sequence $\gamma$ let $F(\gamma) = \emptyset$. Since all the constructions carried out in order to build $F(\gamma)$ were dependent only on natural numbers, fixed bijections between countable sets, and the elements of the sequence $\gamma$, we deduce that $F$ is a Borel function.

Remark. The important thing to keep in mind about the construction of $F$ is that, if $\gamma = (\langle C_\alpha : \alpha < \beta, \alpha \in \lim(\omega_1)\rangle, \langle G_t : t \in 2^{<\omega}\rangle, H)$ is suitable, then $\langle x, y\rangle \notin F(\gamma)$ implies that, for infinitely many levels of $2^{<\omega}$, $x$ and $H$ will agree, and for $t$ in those levels, $y(t)(\ell) > f_t(\ell)$ for infinitely many $\ell \in \mathrm{dom}(f_t)$ for the function $f_t$ defined above.

Let $\Delta : \omega_1 \to (2^{<\omega})^{2^{<\omega}} \times \omega^{\,2^{<\omega}\times\omega}$ be the sequence given by $\diamondsuit(\mathrm{non}(\mathcal{M}))$ for that function $F$. Let $D = \pi_1(\Delta)$. Recursively, we are going to construct a sequence $\langle C_\alpha : \alpha \in \lim(\omega_1)\rangle$ such that, for each $\alpha \in \lim(\omega_1)$,

(i) $C_\alpha \subseteq 2^{<\omega} \times \alpha$,
(ii) for each $t \in 2^{<\omega}$, $C_\alpha \cap A_t$ is infinite,
(iii) for each $\alpha' < \alpha$, $C_\alpha \cap C_{\alpha'}$ is finite,
(iv) if $\alpha \in B$, then $\bigcup_{\alpha'<\alpha} C_{\alpha'} = 2^{<\omega} \times \alpha$,
(v) if $\langle C_{\alpha'} : \alpha' < \alpha\rangle$ forms a suitable triple $\gamma$ with some $\langle G_t : t \in 2^{<\omega}\rangle$ and $H \in (2^{<\omega})^{2^{<\omega}}$ and if $\Delta(\alpha) \notin F(\gamma)$, then there are infinitely many $n \in \omega$ such that for every $t \in 2^n$ the set $C_\alpha \cap G_t$ is infinite.

For $\beta \in \lim(\omega_1) \setminus B$ let $\alpha$ be the maximal element of $\lim(\omega_1) \cap \beta$ and set $C_\beta = 2^{<\omega} \times [\alpha, \beta)$. Assume that $\beta \in B$ and that the set $C_\alpha \subseteq A \cap (2^{<\omega} \times \alpha)$ has been constructed for $\alpha < \beta$ with $\alpha \in \lim(\omega_1)$. Exactly as before, let

$$\widehat{C}_n = C_{\varepsilon_\beta(n)} \setminus \bigcup_{k<n} C_{\varepsilon_\beta(k)} = C_{\varepsilon_\beta(n)} \setminus \bigcup_{k<n} \widehat{C}_k.$$

Let

$$C_\beta = \{\langle t,a\rangle \in \widehat{C}_n : n\in\omega,\ t\in 2^{\le n} \text{ and } e_\beta^{-1}(a) \le \pi_2(\Delta(\beta))(t,n)\} \cup \bigcup_{t\in 2^{<\omega}} M_t,$$

where

$$M_t = \{\langle t,a\rangle \in \widehat{C}_n : n\in\omega,\ n\ge |t|, \text{ and } e_\beta^{-1}(a) \text{ is the smallest such that } \langle t,a\rangle \in \widehat{C}_n \cap A_t\}.$$

Since $\widehat{C}_n \cap A_t \ne \emptyset$ for all $n\in\omega$, $t\in 2^{<\omega}$, each $M_t$ is infinite. Note that, for each $n$, $C_\beta$ takes at most $\pi_2(\Delta(\beta))(t,n)+1$ elements from $\widehat{C}_n \cap A_t$, and only for finitely many $t$, so, for each $\alpha<\beta$, $C_\alpha \cap C_\beta$ is finite, and properties (i)–(iv) are satisfied.

To see that our construction satisfies property (v), assume that $\Delta(\beta) \notin F(\gamma)$ for some suitable $\gamma$ containing $\langle C_\alpha : \alpha<\beta\rangle$, $\langle G_t : t\in 2^{<\omega}\rangle$ and $H \in (2^{<\omega})^{2^{<\omega}}$. Then there are infinitely many $n\in\omega$ such that for all $t\in 2^{<\omega}$, both $H(t) = \pi_1(\Delta(\beta))(t) = D(\beta)(t)$ and $f_t(\ell) < \pi_2(\Delta(\beta))(t)(\ell)$ for infinitely many $\ell \in \mathrm{dom}(f_t)$, where $f_t(\ell) = \min e_\beta^{-1}(\pi_2(\widehat{C}_\ell \cap G_t))$ (see the Remark above). For $t\in 2^{<\omega}$ and $\ell \ge |t|$ such that $f_t(\ell) < \pi_2(\Delta(\beta))(t)(\ell)$, we have at least $e_\beta(\min e_\beta^{-1}(\pi_2(\widehat{C}_\ell \cap G_t))) \in \pi_2(C_\beta \cap G_t)$, and, in particular, the set $C_\beta \cap G_t$ is infinite, which is the conclusion we wanted.

Properties (1)–(4) follow easily from (i)–(iii). We will show that property (5) is satisfied: Let $s : A \to 2^{<\omega}$. For each $t\in 2^{<\omega}$, pick an element $H(t)\in 2^{<\omega}$ such that $H(t) = s(t,a)$ for uncountably many $\langle t,a\rangle \in A_t$. Let $G_t = \{\langle t,a\rangle \in A_t : H(t) = s(t,a)\}$. Notice that, for each $t\in 2^{<\omega}$, the set of all $\delta\in B$ such that $G_t$ is cofinal in $\{t\}\times\delta$ is a club and therefore the set $\mathcal{C} = \{\delta\in B : (\forall t\in 2^{<\omega})\ G_t \text{ is cofinal in } \{t\}\times\delta\}$ is a club.

Consider a triple $\gamma = (\langle C_\alpha : \alpha \in \lim(\omega_1)\rangle, \langle G_t : t\in 2^{<\omega}\rangle, H)$. Since $\Delta$ was the function witnessing $\diamondsuit(\mathrm{non}(\mathcal{M}))$, the set $\mathcal{S} = \{\beta\in\omega_1 : \Delta(\beta) \notin F(\gamma\restriction\beta)\}$ is stationary in $\omega_1$ and therefore $\mathcal{S}\cap\mathcal{C}$ is uncountable. For each $\beta \in \mathcal{S}\cap\mathcal{C}$, we have $\Delta(\beta)\notin F(\gamma\restriction\beta)$, which means that there are infinitely many $n\in\omega$ such that for all $t\in 2^n$ we have $D(\beta)(t) = \pi_1(\Delta(\beta))(t) = H(t)$ and, by (v), the set $C_\beta \cap (G_t \cap \{t\}\times\beta)$ is infinite. Since the set $G_t$ is exactly the set of $\langle t,a\rangle \in A_t$ such that $H(t) = s(t,a)$, the conclusion follows. ∎

Combining the above result with Theorems 2.6, 2.7, and Lemma 2.5 we get the main result of our work:

THEOREM 3.2. *Under $\diamondsuit(\mathrm{non}(\mathcal{M}))$ there is a Radon–Nikodým compact space of weight $\aleph_1$, and a continuous function on this space whose image is not Radon–Nikodým.*

There are several models where $\diamondsuit(\mathrm{non}(\mathcal{M}))$ holds and the continuum is large. Adding $\kappa$ Cohen reals for $\kappa$ of uncountable cofinality yields such a model. Other well-known models where $\diamondsuit(\mathrm{non}(\mathcal{M}))$ holds are Sacks', Miller's and Silver's model. In fact, in several *canonical* models where $\mathrm{non}(\mathcal{M}) = \aleph_1$, the principle $\diamondsuit(\mathrm{non}(\mathcal{M}))$ also holds. The reader can consult [MHD04] to see in which canonical models this principle holds.

We would like to finish this work by looking at the limitations of our construction. First, we will need a characterization of $\mathrm{non}(\mathcal{M})$ due to Bartoszyński.

THEOREM 3.3 ([Bar87]).

$$\mathrm{non}(\mathcal{M}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \omega^\omega,\ \forall f \in \omega^\omega\ \exists g \in \mathcal{A}\ \exists^\infty n,\ g(n) = f(n)\}.$$

Using this, it is easy to show the following result, presenting a limitation of this method.

PROPOSITION 3.4. *If $K$ is a basic* space, then $w(K) \geq \mathrm{non}(\mathcal{M})$.*

*Proof.* Let $D$ be a function witnessing condition (5). Notice that by renaming $\omega^\omega$, we can easily see that

$$\mathrm{non}(\mathcal{M}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq (2^{<\omega})^{2^{<\omega}},\ \forall f \in (2^{<\omega})^{2^{<\omega}}\ \exists g \in \mathcal{A}\ \exists^\infty n\ \forall t \in 2^n,\ g(t) = f(t)\}.$$

If $f \in (2^{<\omega})^{2^{<\omega}}$, we define $s_f : A \to 2^{<\omega}$ by $s_f(a) = f(t)$ whenever $a \in A_t$. For some $b \in B$ we have $D(b)(t) = f(t)$ for each $t \in 2^n$ for infinitely many $n$, i.e., the image of $D$ is a family satisfying the conditions for $\mathrm{non}(\mathcal{M})$. The conclusion follows from this and from Lemma 2.3. ∎

This shows that our construction has the smallest weight possible, so if one wants to construct one of these spaces in a similar way, it would be necessary to change condition (5) from the definition of the basic* space.

Recall Avilés' result from [Avi05], that all quasi-Radon–Nikodým spaces of weight less than $\mathfrak{b}$ are Radon–Nikodým. Our spaces have weight $\mathrm{non}(\mathcal{M})$ and it is known that $\mathfrak{b} \leq \mathrm{non}(\mathcal{M})$ (see [Bla10] for a proof), which leads to the following question.

QUESTION 1. *Is it consistent to have* $\mathrm{non}(\mathcal{M}) > \aleph_1$*, but have a Radon–Nikodým compact space of weight $\aleph_1$ with a continuous image that is not Radon–Nikodým?*

One approach to a possible answer to this question would be to, in some way, modify condition (5) to allow such a space to be constructed using $\diamondsuit(\mathfrak{b})$ ($\diamondsuit(\mathfrak{b})$ is the principle $\diamondsuit(\mathbf{B})$, where $\mathbf{B}$ was defined in the introduction).

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